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ORIGINAL PAPER

Exact Null Controllability of Sobolev-Type Hilfer Fractional Stochastic Differential Equations with Fractional Brownian Motion and Poisson Jumps

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Abstract In this paper, we establish sufficient conditions for exact null controllability of Sobolev type stochastic differential equations with fractional Brownian motion and Poisson jumps in Hilbert spaces, where the time fractional derivative is the Hilfer derivative. The exact null controllability result is derived by using fractional calculus, compact semigroup, fixed point theorem and stochastic analysis. Finally, an example is given to show the application of our results.

Keywords Sobolev type stochastic differential equations · Fractional Brownian motion · Poisson jumps · Hilfer fractional derivative · Exact null controllability.

Mathematics Subject Classification 34A08 · 60H10 · 93B05 · 60G22 · 60J75

1 Introduction

The noises THAT arise in mathematical finance, physics, telecommunication networks, hydrology and medicine etc., can be modeled by fractional Brownian motions

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(see [1–3]). Recently, fractional differential equations and stochastic fractional differential equations driven by fractional Brownian motion have been considered greatly by the research community in various aspects due to its salient features for real-world problems (see [4–15]). In addition, controllability problems for different kinds of dynamical systems have been studied by several authors (see [16–25]), and references therein. Moreover, Hilfer proposed a generalized Riemann–Liouville fractional derivative for short, Hilfer fractional derivative, which includes Riemann–Liouville fractional derivative and Caputo fractional derivative (see [26,27]). Subsequently, many authors studied the existence of solutions and controllability for fractional differential equations involving Hilfer fractional derivatives (see [28–31]). Few authors have studied the stochastic fractional differential equations with Poisson jumps (see [32–35]). On the other hand, the Sobolev-type (fractional) equation appears in a variety of physical problems such as flow of fluid through fissured rocks, thermodynamics, propagation of long waves of small amplitude and shear in second-order fluids and so on. There are many interesting results on the existence and uniqueness of mild solutions and approximate controllability for a class of Sobolev-type fractional evolution equations. Revathi et al (see [36]) studied the local existence of mild solution for a class of stochastic functional differential equations of Sobolev type with infinite delay. Benchaabanea and Sakthivel (see [37]) investigated the existence and uniqueness of mild solutions for a class of nonlinear fractional Sobolev-type stochastic differential equations in Hilbert spaces with non-Lipschitz coefficients. Sakthivel et al (see [38]) investigated the approximate controllability of fractional stochastic differential inclusions with nonlocal conditions. However, no work has been reported in the literature regarding the null controllability of Sobolev-type Hilfer fractional stochastic differential equations with fractional Brownian motion and Poisson jumps. Motivated by these facts, the purpose of this paper is to investigate the exact null controllability of Sobolev-type stochastic differential equations with fractional Brownian motion and Poisson jumps in Hilbert spaces, where the time fractional derivative is the Hilfer derivative, of the form

$$\begin{cases} D_{0+}^{\nu,\mu} Gx(t) = Ax(t) + Bu(t) + F(t, x(t)) + \sigma(t, x(t))\frac{\mathrm{d}B^H(t)}{\mathrm{d}t} \\ + \int_Z h(t, x(t), z)\tilde{N}(\mathrm{d}t, \mathrm{d}z),\ t \in J = (0, b], \\ I_{0+}^{(1-\nu)(1-\mu)} x(0) = x_0, \end{cases} \tag{1}$$

where $D_{0+}^{\nu,\mu}$ is the Hilfer fractional derivative, $0 \le \nu \le 1$, $\frac{1}{2} < \mu < 1$, $x(\cdot)$ takes values in a Hilbert space X with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and B^H is a fractional Brownian motion (fBm) on a real and separable Hilbert space Y with Hurst parameter $H \in (\frac{1}{2}, 1)$. The symbol A and G are linear operators on X. The control function $u(\cdot)$ is given in $L_2(J, U)$, and the Hilbert space of admissible control functions with U as a separable Hilbert space. The symbol B stands for a bounded linear operator from U into X. The mappings $F : J \times X \to X$, $h : J \times X \times Z \to X$ and $\sigma : J \times X \to L_2^0(Y, X)$ are nonlinear functions.

The rest of the paper is organized as follows. In Sect. 2, we collect some notations, definitions and lemmas of fractional operators, fractional Brownian motion, stochastic analysis and exact null controllability. In Sect. 3, we study the exact null controllability of the system (1). We present an example to illustrate the theoretical result in the final section.

2 Preliminaries

To study the null controllability of Sobolev-type stochastic Hilfer fractional differential equations with fractional Brownian motion and Poisson jumps, we need the following basic definitions and lemmas.

Definition 2.1 (*see* [39,40]) The fractional integral operator of order $\mu > 0$ for a function f can be defined as $I^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t \frac{f(s)}{(t-s)^{1-\mu}} \mathrm{d}s, \quad t > 0$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (*see* [26]) The Hilfer fractional derivative of order $0 \leq \nu \leq 1$ and $0 < \mu < 1$ is defined as $D_{0+}^{\nu,\mu} f(t) = I_{0+}^{\nu(1-\mu)} \frac{d}{\mathrm{d}t} I_{0+}^{(1-\nu)(1-\mu)} f(t)$.

Fix a time interval $[0, b]$ and let (Ω, η, P) be a complete probability space furnished with complete family of right continuous increasing sub-σ-algebras $\{\eta_t : t \in [0, b]\}$ satisfying $\eta_t \subset \eta$. Let $(Z, \Psi, \lambda(\mathrm{d}z))$ be a σ-finite measurable space. We are given stationary Poisson point process $(p_t)_{t \geq 0}$, which is defined on (Ω, η, P) with values in Z and with characteristic measure λ. We will denote by $N(t, \mathrm{d}z)$ the counting measure of p_t such that $\tilde{N}(t, \Theta) := E(N(t, \Theta)) = t\lambda(\Theta)$ for $\Theta \in \Psi$. Define $\tilde{N}(t, \mathrm{d}z) := N(t, \mathrm{d}z) - t\lambda(\mathrm{d}z)$, the Poisson martingale measure generated by p_t.

Suppose that $\{\beta^H(t), t \in [0, b]\}$ is the one-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. That is, β^H is a centered Gaussian process with covariance function $R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H})$ (see [1]). Moreover β^H has the following Wiener integral representation: $\beta^H(t) = \int_0^t K_H(t, s) d\beta(s)$, where $\beta = \{\beta(t), t \in [0, b]\}$ is a Wiener process, and $K_H(t, s)$ is the kernel given by $K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_s^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du$ for $s < t$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and $\beta(p, q) = \int_0^1 t^{p-1}(1 - t)^{q-1}, \ p > 0, \ q > 0$. We put $K_H(t, s) = 0$ if $t \leq s$. We will denote by ζ the reproducing kernel Hilbert space of the fBm. In fact, ζ is the closure of set of indicator functions $\{1_{[0,t]}, t \in [0, b]\}$ with respect to the scalar product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\zeta} = R_H(t, s)$.

The mapping $1_{[0,t]} \to \beta^H(t)$ can be extended to an isometry from ζ onto the first Wiener chaos and we will denote by $\beta^H(\varphi)$ the image of φ under this isometry. We recall that for $\psi, \varphi \in \zeta$ their scalar product in ζ is given by

$$\langle \psi, \varphi \rangle_{\zeta} = H(2H - 1) \int_0^b \int_0^b \psi(s)\varphi(t)|t - s|^{2H-2} \mathrm{d}s\mathrm{d}t.$$

Let us consider that the operator K^* from ζ to $L^2([0, b])$ is defined by

$$(K_H^* \varphi)(s) = \int_s^b \varphi(r) \frac{\partial K}{\partial r}(r, s)\mathrm{d}r.$$

Moreover for any $\varphi \in \zeta$, we have

$$\beta^H(\varphi) = \int_0^b (K_H^* \varphi_)(t) \mathrm{d}\beta(t).$$

Let X and Y be two real, separable Hilbert spaces and let $L(Y, X)$ be the space of bounded linear operators from Y to X. For the sake of convenience, we shall use the same notation to denote the norms in X, Y and $L(Y, X)$. Let $Q \in L(Y, Y)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $trQ = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \geq 0$ $(n = 1, 2, \ldots)$ are non-negative real numbers and $\{e_n\}$ $(n = 1, 2, \ldots)$ is a complete orthonormal basis in Y.

We define the infinite dimensional fBm on Y with covariance Q as

$$B^H(t) = B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n \beta_n^H(t),$$

where β_n^H are real, independent fBms. The Y-valued process is Gaussian, starts from 0, and has mean zero and covariance:

$$E\langle B^H(t), x\rangle \langle B^H(s), y\rangle = R(s, t)\langle Q(x), y\rangle, \text{for all } x,\ y \in Y \text{ and } t,\ s \in [0, b].$$

To define Wiener integrals with respect to Q-fBm, we introduce the space $L_2^0 := L_2^0(Y, X)$ of all Q-Hilbert–Schmidt operators $\psi : Y \to X$. We recall that $\psi \in L(Y, X)$ is called a Q-Hilbert–Schmidt operator, if $\|\psi\|_{L_2^0}^2 := \sum_{n=1}^{\infty} \|\sqrt{\lambda_n} \psi e_n\|^2 < \infty$ and that the space L_2^0 equipped with the inner product $\langle \vartheta, \psi\rangle_{L_2^0} = \sum_{n=1}^{\infty} \langle \vartheta e_n, \psi e_n\rangle$ is a separable Hilbert space. Let $\phi(s);\ s \in [0, b]$ be a function with values in $L_2^0(Y, X)$; the Wiener integral of ϕ with respect to B^H is defined by

$$\int_0^t \phi(s) dB^H(s) = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} \phi(s) e_n \mathrm{d}\beta_n^H = \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda} K^*(\phi e_n)(s) \mathrm{d}\beta_n(s), \quad (2)$$

where β_n is the standard Brownian motion.

Lemma 2.3 (see [14, Lemma 2]) *If* $\psi : [0, b] \to L_2^0(Y, X)$ *satisfies* $\int_0^b \|\psi(s)\|_{L_2^0}^2 < \infty$, *then the above sum in* (2) *is well defined as* X*-valued random variable and we have*

$$E\left\|\int_0^t \psi(s) \mathrm{d}B^H(s)\right\|^2 \leq 2Ht^{2H-1} \int_0^t \|\psi(s)\|_{L_2^0}^2 \mathrm{d}s.$$

Let $C(J, L_2(\Omega, X))$ be the Banach space of all continuous maps from J into $L_2(\Omega, X)$ satisfying the condition $\sup_{t \in J} E\|x(t)\|^2 < \infty$. Define $\bar{C} = \{x : \cdot^{(1-\nu)(1-\mu)} x(\cdot) \in C(J, L_2(\Omega, X))\}$, with norm $\|\cdot\|_{\bar{C}}$ defined by

$$\|\cdot\|_{\bar{C}} = (\sup_{t\in J} E\|t^{(1-\nu)(1-\mu)}x(t)\|^2)^{\frac{1}{2}}.$$

Obviously, $\bar{C}$ is a Banach space. Like [41], we denote $\bar{C}^{\nu+\mu-\nu\mu} = \{x : x \in \bar{C}, D_{0+}^{\nu+\mu-\nu\mu}x \in \bar{C}\}$ $\bar{C}^{\nu,\mu} = \{x : x \in \bar{C}, D_{0+}^{\nu,\mu}x \in \bar{C}\}$. Obviously, $\bar{C}^{\nu+\mu-\nu\mu} \subseteq \bar{C}^{\nu,\mu}$.

The operators $A : D(A) \subset X \to X$ and $G : D(G) \subset X \to X$ satisfy the following hypotheses:

(I) A and G are closed linear operators.
(II) $D(G) \subset D(A)$ and G is bijective.
(III) $G^{-1} : X \to D(G)$ is continuous.

Here, (I) and (II) together with the closed graph theorem imply the boundedness of the linear operator $AG^{-1} : X \to X$.

(IV) For each $t \in J$ and for $\lambda \in (\rho(AG^{-1}))$, the resolvent of AG^{-1}, the resolvent $R(\lambda, AG^{-1})$ is the compact operator.

Lemma 2.4 (see [42, Theorem 3.3]) *Let $T(t)$ be a uniformly continuous semigroup. If the resolvent set $R(\lambda, A)$ of A is compact for every $\lambda \in \rho(A)$, then $T(t)$ is a compact semigroup.*

From the above fact, AG^{-1} generates a compact semigroup $\{S(t), t > 0\}$ in X, which means that there exists $M > 1$ such that $\sup_{t\in J} \|S(t)\| \leq M$.

For $x \in X$, we define two families of operators $\{S_{\nu,\mu}(t) : t > 0\}$ and $\{P_\mu(t) : t > 0\}$ by

$$S_{\nu,\mu}(t) = I_{0+}^{\nu(1-\mu)}P_\mu(t), \quad P_\mu(t) = t^{\mu-1}T_\mu(t), \quad T_\mu(t) = \int_0^\infty \mu\theta\Psi_\mu(\theta)S(t^\mu\theta)\mathrm{d}\theta, \tag{3}$$

where

$$\Psi_\mu(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-n\mu)}, \quad 0 < \mu < 1, \ \theta \in (0,\infty) \tag{4}$$

is a function of Wright type which satisfies

$$\int_0^\infty \theta^\Psi \Psi_\mu(\theta)\mathrm{d}\theta = \frac{\Gamma(1+\Psi)}{\Gamma(1+\mu\Psi)} \text{ for } \theta \geq 0. \tag{5}$$

Lemma 2.5 (see [29, Propositions 2.15,2.16,2.17]) *The operator $S_{\nu,\mu}$ and P_μ have the following properties:*

(i) $\{P_\mu(t) : t > 0\}$ is continuous in the uniform operator topology.
(ii) For any fixed $t > 0$, $S_{\nu,\mu}(t)$ and $P_\mu(t)$ are linear and bounded operators, and

$$\|P_\mu(t)x\| \leq \frac{Mt^{\mu-1}}{\Gamma(\mu)}\|x\|, \quad \|S_{\nu,\mu}(t)x\| \leq \frac{Mt^{(\nu-1)(1-\mu)}}{\Gamma(\nu(1-\mu)+\mu)}\|x\|. \tag{6}$$

(iii) $\{P_\mu(t) : t > 0\}$ and $\{S_{\nu,\mu}(t) : t > 0\}$ are strongly continuous.

To study the exact null controllability of (1), we consider the fractional stochastic linear system

$$\begin{cases} D_{0+}^{\nu,\mu} Gy(t) = Ay(t) + F(t) + Bu(t) + \sigma(t)\frac{\mathrm{d}B^H(t)}{\mathrm{d}t}, \ t \in J = (0, b], \\ I_{0+}^{(1-\nu)(1-\mu)} y(0) = y_0, \end{cases} \tag{7}$$

associated with the system (1). Consider

$$L_0^b u = \int_0^b G^{-1} P_\mu(b-s) Bu(s)\mathrm{d}s \ : L_2(J, U) \to X,$$

where $L_0^b u$ has a bounded inverse operator $(L_0)^{-1}$ with values in $L_2(J, U)/ker(L_0^b)$, and

$$N_0^b(y, F, \sigma) = G^{-1} S_{\nu,\mu}(b) Gy + \int_0^b G^{-1} P_\mu(b-s) F(s)\mathrm{d}s \\ + \int_0^b G^{-1} P_\mu(b-s)\sigma(s)\mathrm{d}B^H(s) : X \times L_2(J, X) \to X.$$

Definition 2.6 (*see*[23, Definition 2.2 and Remark 2.3]) The system (7) is said to be exactly null controllable on J if $Im L_0^b \supset Im N_0^b$ or there exists a $\gamma > 0$ such that $\|(L_0^b)^* y\|^2 \geq \gamma \|(N_0^b)^* y\|^2$ for all $y \in X$.

By [43, Lemma 3], we have the following result.

Lemma 2.7 *Suppose that the linear system* (7) *is exactly null controllable on* J. *Then the linear operator* $(L_0)^{-1} N_0^b : X \times L_2(J, X) \to L_2(J, U)$ *is bounded and the control*

$$u(t) = -(L_0)^{-1}\Big[G^{-1} S_{\nu,\mu}(b) G y_0 + \int_0^b G^{-1} P_\mu(b-s) F(s)\mathrm{d}s \\ + \int_0^b G^{-1} P_\mu(b-s)\sigma(s)\mathrm{d}B^H(s)\Big](t)$$

transfers the system (7) *from* y_0 *to* 0, *where* L_0 *is the restriction of* L_0^b *to* $[\ker L_0^b]^\perp$ *and* $F \in L_2(J, X), \sigma \in L_2^0(J, L(Y, X))$.

3 Exact Null Controllability

In this section, we formulate sufficient conditions for exact null controllability for the system (1).

First, we give the definitions of mild solution and exact null controllability.

Definition 3.1 We say $x \in \bar{C}$ is a mild solution to (1) if it satisfies that

$$x(t) = G^{-1}S_{\nu,\mu}(t)Gx_0 + \int_0^t G^{-1}P_\mu(t-s)[F(s,x(s)) + Bu(s)]\mathrm{d}s$$
$$+ \int_0^t G^{-1}P_\mu(t-s)\sigma(s,x(s))\mathrm{d}B^H(s)$$
$$+ \int_0^t G^{-1}P_\mu(t-s)\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z),\ t \in J.$$

Definition 3.2 The system (1) is said to be exact null controllable on the interval J if there exists a stochastic control $u \in L_2(J,U)$ such that the solution $x(t)$ of the system (1) satisfies $x(b) = 0$.

To prove the main result, we need the following hypotheses:

(H1) The fractional linear system (7) is exactly null controllable on J.
(H2) The function $F : J \times X \to X$ satisfies the following two conditions:
  (i) The function $F : J \times X \to X$ is continuous. Assume $F \in \bar{C}^{\mu(1-\nu)}$ for any $x \in \bar{C}^{\mu(1-\nu)}$, which guarantees $D_{0+}^{\nu,\mu}x \in \bar{C}$ exists.
  (ii) For each positive number $q \in N$, there is a positive function $f_q(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le q} E\|F(t,x)\|^2 \le f_q(t),$$

the function $s \to (t-s)^{\mu-1}f_q(s) \in L^1([0,t],R^+)$, and there exists a $\delta > 0$ such that

$$\lim_{q\to\infty}\inf \frac{\int_0^t (t-s)^{\mu-1}f_q(s)\mathrm{d}s}{q} = \delta < \infty, \quad t \in J.$$

(H3) The function $\sigma : J \times X \to L_2^0(K,X)$ satisfies the following two conditions:
  (i) The function $\sigma : J \times X \to L_2^0(K,X)$ is continuous.
  (ii) for each positive number $q \in N$, there is a positive function $g_q(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le q} E\|\sigma(t,x)\|_{L_2^0}^2 \le g_q(t),$$

the function $s \to (t-s)^{\mu-1}g_q(s) \in L^1([0,t],R^+)$, and there exists a $\delta > 0$ such that

$$\lim_{q\to\infty}\inf \frac{\int_0^t (t-s)^{\mu-1}g_q(s)\mathrm{d}s}{q} = \delta < \infty, \quad t \in J.$$

(H4) The function $h : J \times X \times Z \to X$ satisfies the following two conditions:

(i) The function $h : J \times X \times Z \to X$ is continuous.

(ii) For each positive number $q \in N$, there is a positive function $\chi_q(\cdot) : J \to R^+$ such that

$$\sup_{\|x\|^2 \le q} \int_Z E\|h(t, x, z)\|^2 \lambda(\mathrm{d}z) \le \chi_q(t),$$

the function $s \to (t-s)^{\mu-1}\chi_q(s) \in L^1([0, t], R^+)$, and there exists a $\delta > 0$ such that

$$\lim_{q\to\infty} \inf \frac{\int_0^t (t-s)^{\mu-1}\chi_q(s)\mathrm{d}s}{q} = \delta < \infty, \quad t \in J.$$

Theorem 3.3 *If the hypotheses* $(H1)$, $(H2)$, $(H3)$ *and* $(H4)$ *are satisfied, then the system* (1) *is exactly null controllable on* J *provided that*

$$\left[\frac{10\delta b^{(1-\mu)(1-2\nu)} M^2 \|G^{-1}\|^2}{\mu\Gamma^2(\mu)} + \frac{10\delta M^4 \|(L_0)^{-1}\|^2 \|B\|^2 \|G^{-1}\|^4 b^{2\nu(\mu-1)+\mu}}{\mu(2\mu-1)\Gamma^2(\mu)}\right] \times [b + b^{2H}] < 1. \tag{8}$$

Proof For an arbitrary $x(\cdot)$, define the operator Φ on $\bar{C}$ as follows:

$$\begin{aligned}(\Phi x)(t) = {} & G^{-1}S_{\nu,\mu}(t)Gx_0 + \int_0^t G^{-1}P_\mu(t-s)[F(s, x(s)) + Bu(s)]\mathrm{d}s \\ & + \int_0^t G^{-1}P_\mu(t-s)\sigma(s, x(s))\mathrm{d}B^H(s) \\ & + \int_0^t G^{-1}P_\mu(t-s)\int_Z h(s, x(s), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z), t \in J,\end{aligned} \tag{9}$$

where

$$\begin{aligned}u(t) = {} & -(L_0)^{-1}\{G^{-1}S_{\nu,\mu}(b)Gx_0 + \int_0^b G^{-1}P_\mu(b-s)F(s, x(s))\mathrm{d}s \\ & + \int_0^b G^{-1}P_\mu(b-s)\sigma(s, x(s))\mathrm{d}B^H(s) \\ & + \int_0^b G^{-1}P_\mu(b-s)\int_Z h(s, x(s), z)\tilde{N}(\mathrm{d}s, \mathrm{d}z)\}.\end{aligned}$$

It will be shown that the operator Φ from $\bar{C}$ into itself has a fixed point. For each positive integer q, set $B_q = \{v \in \bar{C}, \|v\|^2_{\bar{C}} \le q\}$. We claim that there exists a positive number q such that $\Phi(B_q) \subseteq B_q$. If it is not true, then for each positive number q, there is a function $x_q(\cdot) \in B_q$. But $\Phi(x_q) \notin B_q$, that is, $\|\Phi(x_q))(t)\|^2_{\bar{C}} > q$ for some $t = t(q) \in J$, where $t(q)$ denotes that t is dependent on q.

From $(H2)$ and Lemma 2.5 together with Hölder inequality, we obtain

$$\begin{aligned}
&E\left\| \int_0^t G^{-1}P_\mu(t-s)F(s,x(s))\mathrm{d}s \right\|^2 \\
&\leq \frac{M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} E\left[\int_0^t \|(t-s)^{\mu-1}F(s,x(s))\|\mathrm{d}s\right]^2 \\
&\leq \frac{M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\mathrm{d}s \int_0^t (t-s)^{\mu-1}E\|F(s,x(s))\|^2\mathrm{d}s \\
&\leq \frac{b^\mu M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}f_q(s)\mathrm{d}s. \qquad (10)
\end{aligned}$$

Also from $(H3)$, Lemmas 2.3 and 2.5 together with Burkholder–Gungy's inequality, yields

$$\begin{aligned}
&E\left\| \int_0^t G^{-1}P_\mu(t-s)\sigma(s,x(s))\mathrm{d}B^H(s) \right\|^2 \\
&\leq \frac{2Hb^{2H-1}M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} E\left[\int_0^t \|(t-s)^{\mu-1}\sigma(s,x(s))\|_{L_2^0}\mathrm{d}s\right]^2 \\
&\leq \frac{2Hb^{2H-1}M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\mathrm{d}s \int_0^t (t-s)^{\mu-1}E\|\sigma(s,x(s))\|_{L_2^0}^2\mathrm{d}s \\
&\leq \frac{2Hb^{2H+\mu-1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}g_q(s)\mathrm{d}s. \qquad (11)
\end{aligned}$$

Similarly from $(H3)$ and Lemma 2.5 together with Hölder inequality, we obtain

$$\begin{aligned}
&E\left\| \int_0^t G^{-1}P_\mu(t-s)\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \right\|^2 \\
&\leq \frac{M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} E\left[\int_0^t \left\|(t-s)^{\mu-1}\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|\right]^2 \\
&\leq \frac{M^2\|G^{-1}\|^2}{\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\mathrm{d}s \int_0^t (t-s)^{\mu-1}\int_Z E\|h(s,x(s),z)\|^2\lambda(\mathrm{d}z)\mathrm{d}s \\
&\leq \frac{b^\mu M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)} \int_0^t (t-s)^{\mu-1}\chi_q(s)\mathrm{d}s. \qquad (12)
\end{aligned}$$

However, from (10), (11) and (12), we have

$$\begin{aligned}
q &\leq \|\Phi(x_q)(t)\|_{\tilde{C}}^2 = \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|\Phi(x_q)(t)\|^2 \\
&\leq 5\sup_{t\in J} t^{2(1-\nu)(1-\mu)}\left\{E\|G^{-1}S_{\nu,\mu}(t)Gx_0\|^2 + E\left\|\int_0^t G^{-1}P_\mu(t-s)F(s,x(s))\mathrm{d}s\right\|^2\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ 5\sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\left\| \int_0^t G^{-1}P_\mu(t-s)Bu(s)\mathrm{d}s \right\|^2 \\
&+ 5\sup_{t\in J} t^{2(1-\nu)(1-\mu)} \left\{ E\left\| \int_0^t G^{-1}P_\mu(t-s)\sigma(s,x(s))\mathrm{d}B^H(s) \right.\right. \\
&\left.\left. + \int_0^b G^{-1}P_\mu(t-s)\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z) \right\|^2 \right\} \\
\le\ & \frac{5M^2\|G^{-1}\|^2\|G\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu)+\mu)} + \frac{5b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}f_q(s)\mathrm{d}s \\
&+ \frac{10Hb^{(1-\mu)(1-2\nu)+2H}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}g_q(s)\mathrm{d}s \\
&+ \frac{5b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^t (t-s)^{\mu-1}\chi_q(s)\mathrm{d}s \\
&+ \frac{M^2b^{2\mu-1}\|(L_0)^{-1}\|^2\|B\|^2\|G^{-1}\|^2}{2\mu-1}\left\{ \frac{5M^2\|G^{-1}\|^2\|G\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu)+\mu)} \right. \\
&+ \frac{5b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b (b-s)^{\mu-1}f_q(s)\mathrm{d}s \\
&+ \frac{10Hb^{(1-\mu)(1-2\nu)+2H}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b (b-s)^{\mu-1}g_q(s)\mathrm{d}s \\
&\left. + \frac{5b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b (b-s)^{\mu-1}\chi_q(s)\mathrm{d}s \right\}. \qquad (13)
\end{aligned}
$$

Dividing both sides of (13) by q and taking the lower limit $q \to +\infty$, we get

$$
\left[\frac{10\delta b^{(1-\mu)(1-2\nu)}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)} + \frac{10\delta M^4\|(L_0)^{-1}\|^2\|B\|^2\|G^{-1}\|^4 b^{2\nu(\mu-1)+\mu}}{\mu(2\mu-1)\Gamma^2(\mu)} \right] \times[b+b^{2H}] \ge 1.
$$

This contradicts (8). Hence, for positive q, $\Phi(B_q) \subseteq B_q$ for positive number q.

In fact, the operator Φ maps B_q into a compact subset of B_q. To prove this, we first show that the set $V_q(t) = \{(\Phi x)(t) : x \in B_q\}$ is a precompact in X, for every fixed $t \in J$. This is trivial for $t = 0$, since $V_q(0) = \{x_0\}$. Let t, $0 < t \le b$, be fixed. For $0 < \epsilon < t$ and arbitrary $\kappa > 0$, take

$$
\begin{aligned}
&(\Phi^{\epsilon,\kappa}x)(t) \\
=\ & \frac{\mu}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S(s^\mu\theta)Gx_0\mathrm{d}\theta\mathrm{d}s \\
&+ \mu\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)F(s,x(s))\mathrm{d}\theta\mathrm{d}s \\
&+ \mu\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)\sigma(s,x(s))\mathrm{d}\theta\mathrm{d}B^H(s) \\
&+ \mu\int_0^{t-\epsilon}\int_\kappa^\infty\int_Z G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)h(s,x(s),z)\mathrm{d}\theta\tilde{N}(\mathrm{d}s,\mathrm{d}z)
\end{aligned}
$$

$$- \mu \int_0^{t-\epsilon} \int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)B(L_0)^{-1}\{G^{-1}S_{\nu,\mu}(b)Gx_0$$

$$+ \int_0^b G^{-1}P_\mu(b-\tau)F(\tau,x(\tau))\mathrm{d}\tau + \int_0^b G^{-1}P_\mu(b-\tau)\sigma(\tau,x(\tau))\mathrm{d}B^H(\tau)$$

$$+ \int_0^b G^{-1}P_\mu(t-\tau)\int_Z h(\tau,x(\tau),z)\tilde{N}(\mathrm{d}\tau,\mathrm{d}z)\}(s)\mathrm{d}\theta\mathrm{d}s$$

$$= \frac{\mu S(\epsilon^\mu\kappa)}{\Gamma(\nu(1-\mu))}\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S(s^\mu\theta-\epsilon^\mu\kappa)Gx_0\mathrm{d}\theta\mathrm{d}s$$

$$+ \mu S(\epsilon^\mu\kappa)\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta-\epsilon^\mu\kappa)F(s,x(s))\mathrm{d}\theta\mathrm{d}s$$

$$+ \mu S(\epsilon^\mu\kappa)\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta-\epsilon^\mu\kappa)\sigma(s,x(s))\mathrm{d}\theta\mathrm{d}B^H(s)$$

$$+ \mu S(\epsilon^\mu\kappa)\int_0^{t-\epsilon}\int_\kappa^\infty\int_Z G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta-\epsilon^\mu\kappa)h(s,x(s),z)\mathrm{d}\theta\tilde{N}(\mathrm{d}s,\mathrm{d}z)$$

$$-\mu S(\epsilon^\mu\kappa)\int_0^{t-\epsilon}\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta-\epsilon^\mu\kappa)B(L_0)^{-1}\{G^{-1}S_{\nu,\mu}(b)Gx_0$$

$$+ \int_0^b G^{-1}P_\mu(b-\tau)F(\tau,x(\tau))\mathrm{d}\tau + \int_0^b G^{-1}P_\mu(b-\tau)\sigma(\tau,x(\tau))\mathrm{d}B^H(\tau)$$

$$+ \int_0^b G^{-1}P_\mu(t-\tau)\int_Z h(\tau,x(\tau),z)\tilde{N}(\mathrm{d}\tau,\mathrm{d}z)\}(s)\mathrm{d}\theta\mathrm{d}s.$$

Since $S(\epsilon^\mu\kappa)$, $\epsilon^\mu\kappa > 0$ is a compact operator, the set $V^{\epsilon,\kappa}(t) = \{(\Phi^{\epsilon,\kappa}x)(t) : x \in B_q\}$ is a precompact set in X for every ϵ, $0 < \epsilon < t$, and for all $\kappa > 0$. Moreover, for $x \in B_q$, we have

$$\|(\Phi x)(t) - (\Phi^{\epsilon,\kappa}x)(t)\|_{\tilde{C}}^2 = \sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\|(\Phi x)(t) - (\Phi^{\epsilon,\kappa}x)(t)\|^2$$

$$\leq \frac{5\mu^2}{\Gamma^2(\nu(1-\mu))}\sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\left\|\int_0^t\int_0^\kappa G^{-1}\theta(t-s)^{\nu(1-\mu)-1}\right.$$

$$\left.\times s^{\mu-1}\Psi_\mu(\theta)S(s^\mu\theta)Gx_0\mathrm{d}\theta\mathrm{d}s\right\|^2 + \frac{5\mu^2}{\Gamma^2(\nu(1-\mu))}\sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\left\|\int_{t-\epsilon}^t\int_\kappa^\infty G^{-1}\theta(t-s)^{\nu(1-\mu)-1}s^{\mu-1}\Psi_\mu(\theta)S(s^\mu\theta)Gx_0\mathrm{d}\theta\mathrm{d}s\right\|^2$$

$$+ 5\mu^2\sup_{t\in J} t^{2(1-\nu)(1-\mu)}E\left\|\int_0^t\int_0^\kappa G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)\right.$$

$$\left.\times S((t-s)^\mu\theta)F(s,x(s))\mathrm{d}\theta\mathrm{d}s\right\|^2 + 5\mu^2\sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\left\|\int_{t-\epsilon}^t\int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)\right.$$

$$\left.\times S((t-s)^\mu\theta)F(s,x(s))\mathrm{d}\theta\mathrm{d}s\right\|^2 + 5\mu^2\sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\Big\| \int_0^t \int_0^\kappa G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)$$

$$\times S((t-s)^\mu\theta)\sigma(s,x(s))\mathrm{d}\theta\mathrm{d}B^H(s)\Big\|^2 + 5\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\Big\| \int_{t-\epsilon}^t \int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)\sigma(s,x(s))\mathrm{d}\theta\mathrm{d}B^H(s)\Big\|^2$$

$$+ 5\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\Big\| \int_0^t \int_0^\kappa \int_Z G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)$$

$$\times S((t-s)^\mu\theta)h(s,x(s),z)\mathrm{d}\theta\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big\|^2 + 5\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\Big\| \int_{t-\epsilon}^t \int_\kappa^\infty \int_Z G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)$$

$$\times S((t-s)^\mu\theta)h(s,x(s),z)\mathrm{d}\theta\tilde{N}(\mathrm{d}s,\mathrm{d}z)\Big\|^2 + 5\|B\|^2\|(L_0)^{-1}\|^2\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times E\Big\| \int_0^t \int_0^\kappa G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)S((t-s)^\mu\theta)\{G^{-1}S_{\nu,\mu}(b)Gx_0$$

$$+ \int_0^b G^{-1}P_\mu(b-\tau)F(\tau,x(\tau))\mathrm{d}\tau + \int_0^b G^{-1}P_\mu(b-\tau)\sigma(\tau,x(\tau))\mathrm{d}B^H(\tau)$$

$$+ \int_0^b G^{-1}P_\mu(t-\tau)\int_Z h(\tau,x(\tau),z)\tilde{N}(\mathrm{d}\tau,\mathrm{d}z)\}(s)\mathrm{d}\theta\mathrm{d}s\Big\|^2$$

$$+5\|B\|^2\|(L_0)^{-1}\|^2\mu^2 \sup_{t\in J} t^{2(1-\nu)(1-\mu)} E\Big\| \int_{t-\epsilon}^t \int_\kappa^\infty G^{-1}\theta(t-s)^{\mu-1}\Psi_\mu(\theta)$$

$$\times S((t-s)^\mu\theta)\{G^{-1}S_{\nu,\mu}(b)Gx_0 + \int_0^b G^{-1}P_\mu(b-\tau)F(\tau,x(\tau))\mathrm{d}\tau$$

$$+ \int_0^b G^{-1}P_\mu(b-\tau)\sigma(\tau,x(\tau))\mathrm{d}B^H(\tau)$$

$$+ \int_0^b G^{-1}P_\mu(t-\tau)\int_Z h(\tau,x(\tau),z)\tilde{N}(\mathrm{d}\tau,\mathrm{d}z)\}(s)\mathrm{d}\theta\mathrm{d}s\Big\|^2$$

$$\le \frac{5M^2\mu^2\|G\|^2\|G^{-1}\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu))} \sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times \int_0^t \left((t-s)^{\nu(1-\mu)-1}s^{\mu-1}\int_0^\kappa \theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2 \mathrm{d}s$$

$$+\frac{5M^2\mu^2\|G\|^2\|G^{-1}\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu))} \sup_{t\in J} t^{2(1-\nu)(1-\mu)}$$

$$\times \int_{t-\epsilon}^t \left((t-s)^{\nu(1-\mu)-1}s^{\mu-1}\int_\kappa^\infty \theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2 \mathrm{d}s$$

$$
\begin{aligned}
&+5\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times\int_0^t(t-s)^{\mu-1}\left(\int_0^\kappa\theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2\int_0^t(t-s)^{\mu-1}f_q(s)\mathrm{d}s\\
&+5\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times\int_{t-\epsilon}^t(t-s)^{\mu-1}\left(\int_\kappa^\infty\theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2\int_{t-\epsilon}^t(t-s)^{\mu-1}f_q(s)\mathrm{d}s\\
&+10Hb^{2H-1}\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times\int_0^t(t-s)^{\mu-1}\left(\int_0^\kappa\theta\Psi_\mu(\theta)d\theta\right)^2\int_0^t(t-s)^{\mu-1}g_q(s)\mathrm{d}s\\
&+10Hb^{2H-1}\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\int_{t-\epsilon}^t(t-s)^{\mu-1}\left(\int_\kappa^\infty\theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2\\
&\times\int_{t-\epsilon}^t(t-s)^{\mu-1}g_q(s)\mathrm{d}s+5\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\\
&\times\int_0^t(t-s)^{\mu-1}\left(\int_0^\kappa\theta\Psi_\mu(\theta)d\theta\right)^2\int_0^t(t-s)^{\mu-1}\chi_q(s)\mathrm{d}s\\
&+5\mu^2M^2\|G^{-1}\|^2\sup_{t\in J}t^{2(1-\nu)(1-\mu)}\int_{t-\epsilon}^t(t-s)^{\mu-1}\\
&\times\left(\int_\kappa^\infty\theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2\int_{t-\epsilon}^t(t-s)^{\mu-1}\chi_q(s)\mathrm{d}s\\
&+5M^2\|G^{-1}\|^2\|B\|^2\|(L_0)^{-1}\|^2\mu^2\int_0^t(t-s)^{2\mu-2}\\
&\times\left(\int_0^\kappa\theta\Psi_\mu(\theta)\mathrm{d}\theta\right)^2\left\{\frac{\|G^{-1}\|^2M^2\|G\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu)+\mu)}\right.\\
&+\frac{b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}f_q(\tau)\mathrm{d}\tau\\
&+\frac{2Hb^{(1-\mu)(1-2\nu)+2H}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}g_q(\tau)\mathrm{d}\tau\\
&\left.+\frac{b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}\chi_q(\tau)\mathrm{d}\tau\right\}\mathrm{d}s\\
&+5M^2\|G^{-1}\|^2\|B\|^2\|(L_0)^{-1}\|^2\mu^2\int_{t-\epsilon}^t(t-s)^{2\mu-2}\left(\int_\kappa^\infty\theta\Psi_\mu(\theta)\right)^2\\
&\times\left\{\frac{\|G^{-1}\|^2M^2\|G\|^2E\|x_0\|^2}{\Gamma^2(\nu(1-\mu)+\mu)}\right.\\
&+\frac{b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}f_q(\tau)\mathrm{d}\tau
\end{aligned}
$$

$$+\frac{2Hb^{(1-\mu)(1-2\nu)+2H}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}g_q(\tau)\mathrm{d}\tau$$
$$+\frac{b^{(1-\mu)(1-2\nu)+1}M^2\|G^{-1}\|^2}{\mu\Gamma^2(\mu)}\int_0^b(b-\tau)^{\mu-1}\chi_q(\tau)\mathrm{d}\tau\}\mathrm{d}s.$$

We see that for each $x \in B_q$, $\|(\Phi x)(t) - (\Phi^{\epsilon,\kappa}x)(t)\|^2_{\bar{C}} \to 0$ as $\epsilon \to 0^+$, $\kappa \to 0^+$. Therefore, there are precompact sets arbitrarily close to the set $V_q(t)$ and so $V_q(t)$ is precompact in X.

Next, we prove that the family $\{\Phi x : x \in B_q\}$ is an equicontinuous family of functions. Let $x \in B_q$ and $t_1, t_2 \in J$ such that $0 < t_1 < t_2$, then

$$\begin{aligned}
&\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^2_{\bar{C}}\\
&\leq 5\|G^{-1}S_{\nu,\mu}(t_2)Gx_0 - G^{-1}S_{\nu,\mu}(t_1)Gx_0\|^2_{\bar{C}}\\
&+5\left\|\int_0^{t_1} G^{-1}P_\mu[(t_2-s)-(t_1-s)]F(s,x(s))\mathrm{d}s\right\|^2_{\bar{C}}\\
&+5\left\|\int_{t_1}^{t_2} G^{-1}P_\mu(t_2-s)F(s,x(s))\mathrm{d}s\right\|^2_{\bar{C}}\\
&+5\left\|\int_0^{t_1} G^{-1}P_\mu[(t_2-s)-(t_1-s)]\sigma(s,x(s))\mathrm{d}B^H(s)\right\|^2_{\bar{C}}\mathrm{d}s\\
&+5\left\|\int_{t_1}^{t_2} G^{-1}P_\mu(t_2-s)\sigma(s,x(s))\mathrm{d}B^H(s)\right\|^2_{\bar{C}}\\
&+5\left\|\int_0^{t_1} G^{-1}P_\mu[(t_2-s)-(t_1-s)]Bu(s)\mathrm{d}s\right\|^2_{\bar{C}}\\
&+5\left\|\int_{t_1}^{t_2} G^{-1}P_\mu(t_2-s)Bu(s)\mathrm{d}s\right\|^2_{\bar{C}}\\
&+5\left\|\int_0^{t_1} G^{-1}P_\mu[(t_2-s)-(t_1-s)]\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2_{\bar{C}}\\
&+5\left\|\int_{t_1}^{t_2} G^{-1}P_\mu(t_2-s)\int_Z h(s,x(s),z)\tilde{N}(\mathrm{d}s,\mathrm{d}z)\right\|^2_{\bar{C}}.
\end{aligned}$$

From the above fact, we see that $\|(\Phi x)(t_2) - (\Phi x)(t_1)\|^2_{\bar{C}}$ tends to zero independently of $x \in B_q$ as $t_2 \to t_1$. The compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, $\Phi(B_q)$ is both equicontinuous and bounded. By the Arzela–Ascoli theorem, $\Phi(B_q)$ is precompact in X. Hence, Φ is a completely continuous operator on X. From the Schauder fixed point theorem, Φ has a fixed point in B_q. Any fixed point of Φ is a mild solution of (1) on J. Therefore, the system (1) is exact null controllable on J. □

4 An Example

In this section, we present an example to illustrate our main result. Let us consider the Sobolev-type Hilfer fractional stochastic partial differential equation with fractional Brownian motion and Poisson jump in the following form:

$$\begin{cases} D_{0+}^{\nu,\frac{3}{5}}[(1-\frac{\partial^2}{\partial\xi^2})x(t,\xi)] = \frac{\partial^2}{\partial\xi^2}x(t,\xi)+u(t,\xi)+F(t,x(t,\xi))+\sigma(t,x(t,\xi))\frac{\mathrm{d}B^H(t)}{\mathrm{d}t} \\ +\int_Z h(t,x(t,\xi),z)\tilde{N}(\mathrm{d}t,\mathrm{d}z), \quad t\in J,\ 0<\xi<1, \\ x(t,0)=x(t,1)=0,\ t\in J, \\ I_{0+}^{\frac{2}{5}(1-\nu)}x(0,\xi)=x_0(\xi), \quad 0\le\xi\le 1, \end{cases} \tag{14}$$

where $D_{0+}^{\nu,\frac{3}{5}}$ is the Hilfer fractional derivative, $0\le\nu\le 1$, $\mu=\frac{3}{5}$ and B^H is a fractional Brownian motion. The functions $x(t)(\xi)=x(t,\xi)$, $F(t,x(t))(\xi)=F(t,x(t,\xi))$, $\sigma(t,x(t))(\xi)=\sigma(t,x(t,\xi))$ and $h(t,x(t),z)(\xi)=h(t,x(t,\xi),z)$.

The bounded linear operator B is defined by $Bu=u(t,\xi),\ 0\le\xi\le 1,\ u\in U$.

To study this system, let $X=Y=U=L^2([0,1])$, and the operators $A:D(A)\subset X\to X$ and $G:D(A)\subset X\to X$, $t\ge 0$ be given by $A=\frac{\partial^2}{\partial\xi^2}$ and $G=1-A$ with $D(A)=D(G)=\{x\in X; x,\frac{\partial x}{\partial\xi}$ be absolutely continuous, $\frac{\partial^2 x}{\partial\xi^2}\in X,\ x(0)=x(1)=0\}$. Then, A and G can be written as

$$Ax=\sum_{n=1}^{\infty}n^2(x,x_n)x_n, \quad x\in D(A), \quad Gx=\sum_{n=1}^{\infty}(1+n^2)(x,x_n)x_n,\ x\in D(G).$$

Furthermore, for $x\in X$ we have

$$G^{-1}x=\sum_{n=1}^{\infty}\frac{1}{1+n^2}(x,x_n)x_n, \quad AG^{-1}x=\sum_{n=1}^{\infty}\frac{n^2}{1+n^2}(x,x_n)x_n.$$

It is known that AG^{-1} is self-adjoint and has the eigenvalues $\lambda_n=-n^2\pi^2,\ n\in N$, with the corresponding normalized eigenvectors $e_n(\xi)=\sqrt{2}\,\sin(n\pi\xi)$. Furthermore, AG^{-1} generates a uniformly strongly continuous semigroup of bounded linear operators $S(t)$, $t>0$, on a separable Hilbert space X which is given by

$$S(t)y=\sum_{n=1}^{\infty}(y_n,e_n)e_n=\sum_{n=1}^{\infty}2e^{-n^2\pi^2 t}\sin(n\pi\xi)\int_0^1\sin(n\pi\tau)y(\tau)\mathrm{d}\tau,\ y\in X.$$

If $u\in L_2(J,U)$, then $B=I,\ B^*=I$. We consider the fractional linear system

$$\begin{cases} D_{0+}^{\nu,\frac{3}{5}}[(1-\frac{\partial^2}{\partial\xi^2})y(t,\xi)] = \frac{\partial^2}{\partial\xi^2}y(t,\xi)+u(t,\xi)+F(t,\xi)+\sigma(t,\xi)d\omega(t), t\in J,\ 0<\xi<1, \\ y(t,0)=y(t,1)=0, t\in J, \\ I_{0+}^{\frac{2}{5}(1-\nu)}(y(0,\xi))=y_0(\xi), 0\le\xi\le 1, \end{cases} \tag{15}$$

The system (15) is exact null controllable if there is a $\gamma > 0$, such that

$$\int_0^b \|G^{-1}B^* P_\mu^*(b-s)y\|^2 \mathrm{d}s \geq \gamma \left[\|G^{-1}S_{\mu,\nu}^*(b)Gy\|^2 + \int_0^b \|G^{-1}P_\mu^*(b-s)y\|^2 \mathrm{d}s\right]$$

or equivalently

$$\int_0^b \|G^{-1}P_\mu(b-s)y\|^2 \mathrm{d}s \geq \gamma \left[\|G^{-1}S_{\mu,\nu}(b)Gy\|^2 + \int_0^b \|G^{-1}P_\mu(b-s)y\|^2 \mathrm{d}s\right].$$

If $F = 0$ and $\sigma = 0$ in (15), then the fractional linear system is exactly null controllable if

$$\int_0^b \|G^{-1}P_\mu(b-s)y\|^2 \mathrm{d}s \geq b[\|G^{-1}S_{\mu,\nu}(b)Gy\|^2.$$

Therefore,

$$\int_0^b \|G^{-1}P_\mu(b-s)y\|^2 \mathrm{d}s \geq \frac{b}{1+b}\Bigg[\|G^{-1}S_{\mu,\nu}(b)Gy\|^2 + \int_0^b \|G^{-1}P_\mu(b-s)y\|^2 \mathrm{d}s\Bigg].$$

Hence, the linear fractional system (15) is exactly null controllable on J. So the hypothesis (H_1) is satisfied. Hence, all the hypotheses of Theorem 3.3 are satisfied and

$$\left[\frac{10\delta b^{(1-\mu)(1-2\nu)} M^2 \|G^{-1}\|^2}{\mu\Gamma^2(\mu)} + \frac{10\delta M^4 \|(L_0)^{-1}\|^2 \|B\|^2 \|G^{-1}\|^4 b^{2\nu(\mu-1)+\mu}}{\mu(2\mu-1)\Gamma^2(\mu)}\right][b + b^{2H}] < 1,$$

so the Sobolev-type Hilfer fractional stochastic partial differential equation with fractional Brownian motion and Poisson jump (14) is exact null controllable on J.

5 Conclusion

This paper dealt with a class of Sobolev-type Hilfer fractional stochastic differential equations with fractional Brownian motion and Poisson jumps in Hilbert spaces. By using fractional calculus, compact semigroup, fixed point theorem and stochastic analysis, we established sufficient conditions for exact null controllability of Sobolev-type stochastic differential equations with fractional Brownian motion and Poisson jumps in Hilbert spaces, where the time fractional derivative is the Hilfer derivative. Finally, an example is given to illustrate our results.

Our future work will be focused on investigating the approximate and null controllability for Sobolev-type nonlinear Hilfer fractional stochastic delay differential equations with impulsive condition in Hilbert space.

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