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ORIGINAL PAPER

Vanishing Theorem for p-Harmonic 1-Forms on Complete Submanifolds in Spheres

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Abstract In this paper, we study a complete submanifold M^m in a sphere S^{m+l}. We obtain that there is no nontrivial $L^{2\beta}$ p-harmonic 1-forms on M^m if the total curvature is bounded from above by a constant depending only on m, and we also obtain that M^m has only one p-nonparabolic end.

Keywords p-Harmonic 1-form · p-Nonparabolic end

Mathematics Subject Classification Primary 53C21 · 53C25

1 Introduction

The topological properties and vanishing theorems of submanifolds in various ambient spaces have been studied extensively during the past few years. In [4], Cao, Shen and Zhu showed that a complete connected stable minimal hypersurface in Euclidean space must have exactly one end. Its strategy was to utilize a result of Schoen–Yau asserting that a complete stable minimal hypersurface in Euclidean space cannot admit a non-constant harmonic function with finite integral [22]. Later, Ni [17] proved that if n-dimensional complete minimal submanifold M in Euclidean space has sufficient small total scalar curvature (i.e.$\int_M |A|^n < C_1$), then M has only one end. In [21], Seo improved the upper bound C_1. In [8], Fu and Xu proved that a complete submanifold M^m with finite total curvature and some conditions on mean curvaute in an $(n+p)$-dimensional simply connected space form $M^{m+p}(c)$ must have finitely many ends. In

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[5], Cavalcante, Mirandola, Vitório proved that a complete submanifold M^m with finite total curvature and some conditions on the first eigenvalue of the Laplace–Beltrami operator of M in an Hadamard manifold must have finitely many ends. In [14], Lin proved some vanishing theorems for L^2 harmonic forms under the assumptions on the second fundamental forms. In [15], Lin proved some vanishing and finiteness theorems for L^2 harmonic forms under the assumptions on Schrödinger operators involving the squared norm of the traceless second fundamental form. In [16], Lin obtained some vanishing theorems for L^2 forms on hypersurfaces in sphere. In [25,26], Zhu and Fang obtained some vanishing and finiteness theorems for L^2 harmonic 1-forms on submanifold in sphere. In [9], the author investigates complete noncompact submanifolds in sphere; we obtained some vanishing and finiteness theorems for L^2 harmonic forms.

For p-harmonic 1-forms, Zhang [24] obtained vanishing results for p-harmonic 1-form. Chang [6] obtained the compactness for any bounded set of p-harmonic 1-forms. In [11], the author and Pan investigated L^p p-harmonic 1-forms on complete noncompact submanifolds in Hadamard manifolds, and obtained some vanishing and finiteness theorems under finite total curvature and the first eigenvalues of Laplace-Beltrami operator. In [12], the author, Zhang and Liang obtained some vanishing and finiteness theorems under the conditions of the scalar curvature and Ricci curvature. In [10], the author obtained some vanishing and finiteness theorems for p-harmonic forms on complete submanifolds in spheres. In [18], Dung and Seo obtained some vanishing results for p-harmonic forms. In [19] Dung obtained some vanishing results for p-harmonic l-forms, for $2 \leq l \leq n-2$ on Riemannian manifolds with a weighted Poincaré inequality.

Let (M^m, g) be a Riemannian manifold and let u be a real C^∞ function on M^m. Fix $p \in R$, $p \geq 2$ and consider a compact domain $\Omega \subset M^m$. The p-energy of u on Ω is defined to be

$$E_p(\Omega, u) = \frac{1}{p} \int_\Omega |\nabla u|^p.$$

The function u is said to be p-harmonic on M^m if u is a critical point of $E_p(\Omega, *)$ for every compact domain $\Omega \subset M^m$. Equivalently, u satisfies the Euler–Lagrange equation.

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0.$$

Thus, the concept of p-harmonic function is a natural generalization of that of harmonic function, that is, of a critical point of the 2-energy functional.

Definition 1.1 A p-harmonic 1-form is a differentiable 1-form on M^m satisfying the following properties:

$$\begin{cases} \mathrm{d}\omega = 0, \\ \delta(|\omega|^{p-2}\omega) = 0, \end{cases}$$

where δ is the codifferential operator. It is easy to see that the differential of a p-harmonic function is a p-harmonic 1-form.

In this paper, we investigate the properties of p-harmonic 1-forms on noncompact submanifolds with finite total curvature. We assume that M^m is a complete noncompact manifold and define the space of the L^p p-harmonic 1-forms on M by

$$H^{1,p}(L^{2\beta}(M)) = \left\{\omega | \int_M |\omega|^{2\beta} \mathrm{d}v < \infty, \mathrm{d}\omega = 0 \quad \text{and} \quad \delta(|\omega|^{p-2}\omega) = 0\right\}$$

where $p \geq 2$ and $\beta > 0$.

In this paper, we obtain the following results.

Theorem 1.2 *Let $x : M^m \to S^{m+l}$, $m \geq 3$ be an isometric immersion of a complete noncompact manifold M^m in unit sphere S^{m+l}. There exists a positive constant Λ depending only on m, such that if $||\Phi||_{L^m(M)} < \frac{m-2}{2C(m)\sqrt{2m(m-1)}(m-1)}$, then there admit no nontrivial $L^{2\beta}$ p-harmonic 1-forms on M, i.e. $H^{1,p}(L^{2\beta}(M)) = \{0\}$, where $p \geq 2$ and β satisfies the following inequality:*

$$m\left[2 - \sqrt{\frac{2}{m(m-1)}\left[(p-1)^2 + (2m-1)(m-1)\right]}\right]$$
$$< \beta < m\left[2 + \sqrt{\frac{2}{m(m-1)}\left[(p-1)^2 + (2m-1)(m-1)\right]}\right].$$

In particular, from $\beta = \frac{p}{2}$ and $H^{1,p}(L^p(M)) = \{0\}$, we know that M has only one p-nonparabolic end.

Remark 1.3 When $\beta = 1$ and $p = 2$, our result becomes the Theorem 1.2 in [25], so our result is a generalization of the result of Zhu and Fang [25].

2 Preliminaries

Let M^m be a complete submanifold immersed in a sphere S^{m+l}. Fix a point $x \in M$ and a local orthonormmal frame $\{e_1, \ldots, e_{m+l}\}$ of S^{m+l} such that $\{e_1, \ldots, e_m\}$ are tangent fields of M. For each α, $m+1 \leq \alpha \leq m+l$, define a line map $A_\alpha : T_xM \to T_xM$ by $\langle A_\alpha X, Y\rangle = \langle \overline{\nabla}_X Y, e_\alpha\rangle$, where X, Y are tangent fields and $\overline{\nabla}$ is the Riemannian connection of S^{m+n}. Denote by $h^\alpha_{ij} = \langle A_\alpha e_i, e_j\rangle$. The squared norm $|A|^2$ of the second fundamental form and the mean curvature vector H are defined by

$$|A|^2 = \sum_{ij\alpha}(h^\alpha_{ij})^2 \quad H = \sum_\alpha H^\alpha e_\alpha = \frac{1}{m}\sum_{i\alpha} h^\alpha_{ii} e_\alpha.$$

The traceless second fundamental form Φ is defined by

$$\Phi(X, Y) = A(X, Y) - \langle X, Y\rangle H,$$

for all vector fields X, Y on M. A simple computation shows that

$$|\Phi|^2 = |A|^2 - m|H|^2,$$

which measures how much the immersion deviates from being totally umbilical.

To prove our main result, we also need the following results. In [11], the author and H. Pan obtained the following Kato type inequality for p-harmonic 1-form.

Lemma 2.1 ([11]) *Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then, we have the following inequality:*

$$|\nabla(|\omega|^{p-2}\omega)|^2 \geq \left(1 + \frac{1}{(m-1)(p-1)^2}\right)|\nabla|\omega|^{p-1}|^2,$$

where $p \geq 2$.

In the following, we will refine the above inequality and obtain the following Kato type inequality for p-harmonic 1-form.

Lemma 2.2 *Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then, we have the following inequality:*

$$|\nabla(|\omega|^{p-2}\omega)|^2 \geq \left(1 + \frac{1}{m-1}\right)|\nabla|\omega|^{p-1}|^2, \tag{1}$$

where $p \geq 2$.

Proof When $p = 2$, ω is a 2-harmonic 1-form, i.e. harmonic 1-form, (1) is true. So we only need to the case for $p > 2$. We can choose a local orthonormal basis $e_1, \ldots, e_m$ with the dual basis $\theta_1, \ldots, \theta_m$ of M^m near a fixed point $q \in M$ such that $\nabla_{e_i} e_j(q) = 0$, $\omega_1(q) = \omega(e_1)(q) = |\omega|(q)$ and $\omega(e_i) = \omega_i = 0$ for $i \geq 2$. Writing

$$\omega = \sum_{i=1}^{m} \omega_i \theta_i.$$

We have

$$\mathrm{d}\omega = \sum_{i,j=1}^{m} \omega_{ij}\theta_j \wedge \theta_i$$

and

$$\delta(|\omega|^{p-2}\omega) = -|\omega|^{p-2}\sum_{i=1}^{m}[(p-2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]$$

Since ω is a p-harmonic 1-form, that is, $d\omega = 0$ and $\delta(|\omega|^{p-2}\omega) = 0$, therefore

$$\omega_{ij} = \omega_{ji}$$

for $i, j = 1, \cdots, m$ and

$$\sum_{i=1}^{m}[(p-2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}] = 0$$

and

$$\nabla_{e_i}|\omega| = \nabla_i|\omega| = \nabla_i\left(\sqrt{\sum_{j=1}^{m}\omega_j^2}\right) = \frac{\sum \omega_j\omega_{ij}}{|\omega|} = \omega_{1i}.$$

At the point q, we compute

$$\begin{aligned}
&|\nabla(|\omega|^{p-2}\omega)|^2 - |\nabla|\omega|^{p-1}|^2 \\
&= \sum_{i,j=1}^{m}|\omega|^{2(p-2)}[(p-2)\nabla_i(\ln|\omega|)\omega_j + \omega_{ij}]^2 \\
&\quad -\sum_{i=1}^{m}|\omega|^{2(p-2)}[(p-2)\nabla_i(\ln|\omega|)\omega_1 + \omega_{1i}]^2 \\
&\geq \sum_{i\neq 1}|\omega|^{2(p-2)}[(p-2)\nabla_i(\ln|\omega|)\omega_1 + \omega_{1i}]^2 \\
&\quad +\sum_{i\neq 1}|\omega|^{2(p-2)}[(p-2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]^2 \\
&= \sum_{i\neq 1}|\omega|^{2(p-2)}[(p-1)\omega_{1i}]^2 + \sum_{i\neq 1}|\omega|^{2(p-2)}[(p-2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii}]^2 \\
&\geq (p-1)^2\sum_{i\neq 1}|\omega|^{2(p-2)}[\omega_{1i}]^2 \\
&\quad +\frac{1}{m-1}|\omega|^{2(p-2)}[\sum_{i\neq 1}((p-2)\nabla_i(\ln|\omega|)\omega_i + \omega_{ii})]^2 \\
&= (p-1)^2\sum_{i\neq 1}|\omega|^{2(p-2)}[\omega_{1i}]^2 \\
&\quad +\frac{1}{m-1}|\omega|^{2(p-2)}[-(p-2)\nabla_1(\ln|\omega|)\omega_1 - \omega_{11}]^2 \\
&= (p-1)^2\sum_{i\neq 1}|\omega|^{2(p-2)}[\omega_{1i}]^2 + (p-1)^2\frac{1}{m-1}|\omega|^{2(p-2)}\omega_{11}^2 \\
&\geq \frac{(p-1)^2}{m-1}\sum_{i\neq 1}|\omega|^{2(p-2)}[\omega_{1i}]^2 + \frac{(p-1)^2}{m-1}|\omega|^{2(p-2)}\omega_{11}^2 \\
&\geq \frac{(p-1)^2}{m-1}|\omega|^{2(p-2)}\sum_{i=1}^{m}\omega_{1i}^2 = \frac{1}{(m-1)}|\nabla|\omega|^{p-1}|^2.
\end{aligned}$$

This proves the Lemma. □

Using Bochner's formula [2], we have the following results.

Lemma 2.3 *Let ω be a p-harmonic 1-form on Riemannian manifold M^m. Then, we have*

$$\frac{1}{2}\triangle|\omega|^{2(p-1)} = |\nabla(|\omega|^{p-2})\omega|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle + |\omega|^{2(p-2)} Ric^M(\omega,\omega). \tag{2}$$

From (1) and (2), we have

$$|\omega|^{p-1}\triangle|\omega|^{p-1} \geq \frac{1}{(m-1)}|\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle, + |\omega|^{2(p-2)} Ric^M(\omega,\omega), \tag{3}$$

where ω is a p-harmonic 1-form on Riemannian manifold M^m.

Lemma 2.4 ([23]) *Let M^m be an m-dimensional complete immersed submanifold in a Hadamard manifold N with the sectional curvature satisfying $0 < \delta \leq K_N$ for some constant δ. Then, the Ricci curvature of M satisfies*

$$Ric^M \geq (m-1)(|H|^2+\delta) - \frac{m-1}{m}|\Phi|^2 - \frac{(m-2)\sqrt{m(m-1)}}{m}|H||\Phi|. \tag{4}$$

Lemma 2.5 [13,25] *Let M^m be a complete noncompact oriented manifold isometrically immersed in a sphere S^{m+n}. Then, we have*

$$\left(\int_M |f|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leq C_0\left(\int_M |\nabla f|^2 + m^2\int_M (1+|H|^2) f^2\right) \tag{5}$$

for all $f \in C_0^1(M)$, where H is the mean curvature vector of M in S^{m+l} and C_0 is a constant given by the following

$$C_0 = C(m)^2\frac{8(m-1)^2}{(m-2)^2},$$

where $C(m)$ is Sobolev constant only depending on m.

In [11], the author and Pan proved the following result.

Lemma 2.6 ([11]) *Let $f : M^m \to R$ be a smooth function on Riemannian manifold M, and ω be a closed 1-form on M. Then, we have $|d(f\omega)| \leq |df||\omega|$.*

In the following, we recall the definition of the ends of Riemannian manifolds

Definition 2.7 Let $D \subset M$ be a compact subset of M. An end E of M with respect to D is a connected unbounded component of $M \backslash D$. When we say E is an end, it is implicitly assumed that E is an end with respect to some compact subset $D \subset M$.

As in usual harmonic function theory, we define the p-parabolicity and p-nonparabolicity of an end E as follows ([1,3,20]):

Definition 2.8 An end E of the Riemannian manifold M is called p-parabolic if for every compact subset $K \subset \overline{E}$

$$\mathrm{cap}_p(K, E) = \inf \int_E |\nabla u|^p = 0,$$

where the infimum is taken among all $u \in C_c^\infty(\overline{E})$ such that $u \geq 1$ on K. Otherwise, the end E is called p-nonparabolic.

Lemma 2.9 ([7,20]) *Let M be a Riemannian manifold with at least two p-nonparabolic ends. Then, there exists a nonconstant, bounded p-harmonic function $u \in C^{1,\alpha}(M)$ for some α such that $|\nabla u| \in L^p(M)$.*

3 Proof of the Main Results

In this section, we give the proof of our main result.

Proof of Theorem 1.2 Assume that ω is a p-harmonic 1-form on M^m. From (3) and (4), we have

$$\begin{aligned}
|\omega|^{p-1}\triangle|\omega|^{p-1} \geq\ & \frac{1}{m-1}|\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
& + (m-1)(|H|^2+1)|\omega|^{2(p-1)} - \frac{m-1}{m}|\Phi|^2|\omega|^{2(p-1)} \\
& - \frac{(m-2)\sqrt{m(m-1)}}{m}|H||\Phi||\omega|^{2(p-1)} \\
\geq\ & \frac{1}{m-1}|\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
& + (m-1)(|H|^2+1)|\omega|^{2(p-1)} - \frac{m-1}{m}|\Phi|^2|\omega|^{2(p-1)} \\
& - \frac{m-2}{2}|H|^2|\omega|^{2(p-1)} - \frac{(m-1)(m-2)}{2m}|\Phi|^2|\omega|^{2(p-1)} \\
=\ & \frac{1}{m-1}|\nabla|\omega|^{p-1}|^2 - \langle \delta d(|\omega|^{p-2}\omega), |\omega|^{p-2}\omega\rangle \\
& + \frac{m}{2}|H|^2|\omega|^{2(p-1)} + (m-1)|\omega|^{2(p-1)} - \frac{m-1}{2}|\Phi|^2|\omega|^{2(p-1)},
\end{aligned} \tag{6}$$

where we have used the Cauchy–Schwarz inequality in the second inequality.

For any $\alpha > 0$, we compute

$$\begin{aligned}|\omega|^{\alpha}\triangle|\omega|^{\alpha} &= |\omega|^{\alpha}\triangle\left[|\omega|^{(p-1)\frac{\alpha}{p-1}}\right]\\ &= |\omega|^{\alpha}\left[\frac{\alpha}{p-1}|\omega|^{\alpha-(p-1)}\triangle|\omega|^{p-1}\right.\\ &\quad\left.+\frac{\alpha}{p-1}\left(\frac{\alpha}{p-1}-1\right)|\omega|^{\alpha-2(p-1)}|\nabla|\omega|^{p-1}|^{2}\right]\\ &= \frac{\alpha}{p-1}|\omega|^{2\alpha-2(p-1)}|\omega|^{p-1}\triangle|\omega|^{p-1}\\ &\quad+\frac{\alpha}{p-1}\left(\frac{\alpha}{p-1}-1\right)|\omega|^{2\alpha-2(p-1)}|\nabla|\omega|^{p-1}|^{2}.\end{aligned} \tag{7}$$

From (6) and (7), we have

$$\begin{aligned}|\omega|^{\alpha}\triangle|\omega|^{\alpha} \geq{}& \frac{\alpha}{p-1}|\omega|^{2\alpha-2(p-1)}\left[\frac{1}{m-1}|\nabla|\omega|^{p-1}|^{2}-\langle\delta d(|\omega|^{p-2}\omega),|\omega|^{p-2}\omega\rangle\right.\\ &\left.+\ \frac{m}{2}|H|^{2}|\omega|^{2(p-1)}+(m-1)|\omega|^{2(p-1)}-\frac{m-1}{2}|\Phi|^{2}|\omega|^{2(p-1)}\right]\\ &+\ \frac{\alpha}{p-1}\left(\frac{\alpha}{p-1}-1\right)|\omega|^{2\alpha-2(p-1)}|\nabla|\omega|^{p-1}|^{2}\\ ={}& \frac{p-1}{\alpha}\left(\frac{1}{m-1}+\frac{\alpha}{p-1}-1\right)|\nabla|\omega|^{\alpha}|^{2}\\ &-\ \frac{\alpha}{p-1}\langle\delta d\left(|\omega|^{p-2}\omega\right),|\omega|^{2\alpha-p}\omega\rangle\\ &+\ \frac{\alpha}{p-1}\frac{m}{2}|H|^{2}|\omega|^{2\alpha}+\frac{\alpha}{p-1}(m-1)|\omega|^{2\alpha}-\frac{\alpha}{p-1}\frac{m-1}{2}|\Phi|^{2}|\omega|^{2\alpha}.\end{aligned} \tag{8}$$

Let $\phi \in C_0^{\infty}(M)$. Multiplying both sides of (8) by $|\phi|^2|\omega|^{2q\alpha}$, $q > 0$, and integrating over M, we have

$$\begin{aligned}&\frac{p-1}{\alpha}\left(\frac{1}{m-1}+\frac{\alpha}{p-1}-1\right)\int_M \phi^2|\omega|^{2q\alpha}|\nabla|\omega|^{\alpha}|^2\\ &\quad+\frac{\alpha}{p-1}\frac{m}{2}\int_M \phi^2|H|^2|\omega|^{2(q+1)\alpha}\\ &\quad+\frac{\alpha}{p-1}(m-1)\int_M \phi^2|\omega|^{2(q+1)\alpha} \leq \int_M \phi^2|\omega|^{(2q+1)\alpha}\triangle|\omega|^{\alpha}\\ &\quad+\frac{\alpha}{p-1}\int_M\langle\delta d\left(|\omega|^{p-2}\omega\right),\phi^2|\omega|^{2(q+1)\alpha-p}\omega\rangle\\ &\quad+\frac{\alpha}{p-1}\frac{m-1}{2}\int_M |\Phi|^2|\phi|^2|\omega|^{2(q+1)\alpha}\end{aligned}$$

$$
\begin{aligned}
&= -2\int_M \phi|\omega|^{(2q+1)\alpha}\langle\nabla\phi,\nabla|\omega|^\alpha\rangle-(2q+1)\int_M\phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\\
&\quad+\frac{\alpha}{p-1}\frac{m-1}{2}\int_M|\Phi|^2|\phi|^2|\omega|^{2(q+1)\alpha}\\
&\quad+\frac{\alpha}{p-1}\int_M\langle\delta d\left(|\omega|^{p-2}\omega\right),\phi^2|\omega|^{2(q+1)\alpha-p}\omega\rangle\\
&\le 2\int_M\phi|\omega|^{(2q+1)\alpha}|\nabla\phi||\nabla|\omega|^\alpha|-(2q+1)\int_M\phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\\
&\quad+\frac{\alpha}{p-1}\frac{m-1}{2}\int_M|\Phi|^2|\phi|^2|\omega|^{2(q+1)\alpha}\\
&\quad+\frac{\alpha}{p-1}\int_M\langle\delta d\left(|\omega|^{p-2}\omega\right),\phi^2|\omega|^{2(q+1)\alpha-p}\omega\rangle. \qquad (9)
\end{aligned}
$$

From Lemma 2.6, we have

$$
\begin{aligned}
&\left|\int_M\langle\delta d\left(|\omega|^{p-2}\omega\right),\phi^2|\omega|^{2(q+1)\alpha-p}\omega\rangle\right|\\
&\quad=\left|\int_M\left\langle d\left(|\omega|^{p-2}\omega\right),d\left(\phi^2|\omega|^{2(q+1)\alpha-p}\omega\right)\right\rangle\right|\\
&\quad\le\int_M\left|\omega|^2|\nabla|\omega|^{p-2}\right|\left|\nabla(\phi^2|\omega|^{2(q+1)\alpha-p})\right|\\
&\quad\le\int_M|\omega|^2|\nabla|\omega|^{p-2}||2\phi|\omega|^{2(q+1)\alpha-p}\nabla\phi+\phi^2\nabla|\omega|^{2(q+1)\alpha-p}|\\
&\quad\le\int_M\left[2\phi|\omega|^{2(q+1)\alpha-p+2}|\nabla|\omega|^{p-2}||\nabla\phi|+\phi^2|\omega|^2|\nabla|\omega|^{p-2}||\nabla|\omega|^{2(q+1)\alpha-p}|\right]\\
&\quad=\int_M\left[\frac{2(p-2)}{\alpha}\phi|\omega|^{(2q+1)\alpha}|\nabla|\omega|^\alpha||\nabla\phi|\right.\\
&\qquad\left.+\frac{(p-2)(2(q+1)\alpha-p)}{\alpha^2}\phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\right]. \qquad (10)
\end{aligned}
$$

From (9) and (10),

$$
\begin{aligned}
&C_1\int_M\phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2+\frac{\alpha}{p-1}\frac{m}{2}\int_M\phi^2|H|^2|\omega|^{2(q+1)\alpha}\\
&\quad+\frac{\alpha}{p-1}(m-1)\int_M\phi^2|\omega|^{2(q+1)\alpha}\\
&\quad\le\frac{\alpha}{p-1}\frac{m-1}{2}\int_M|\Phi|^2|\phi|^2|\omega|^{2(q+1)\alpha}\\
&\quad+\frac{2(2p-3)}{p-1}\int_M\phi|\omega|^{(2q+1)\alpha}|\nabla|\omega|^\alpha||\nabla\phi|, \qquad (11)
\end{aligned}
$$

where C_1 is a positive constant defined as follows.

$$C_1 = \frac{p-1}{\alpha}\left(\frac{1}{m-1} + \frac{\alpha}{p-1} - 1\right) + 2q + 1 - \frac{p-2}{p-1}\frac{2(q+1)\alpha - p}{\alpha}.$$

For any $\varepsilon_1 > 0$, by applying the Cauchy–Schwarz inequality, we have

$$\begin{aligned} & C_2 \int_M \phi^2 |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 + \frac{\alpha}{p-1}\frac{m}{2}\int_M \phi^2 |H|^2 |\omega|^{2(q+1)\alpha} \\ & \quad + \frac{\alpha}{p-1}(m-1)\int_M \phi^2 |\omega|^{2(q+1)\alpha} \\ & \le \frac{\alpha}{p-1}\frac{m-1}{2}\int_M |\Phi|^2 |\phi|^2 |\omega|^{2(q+1)\alpha} \\ & \quad + \frac{(2p-3)}{p-1}\frac{1}{\varepsilon_1}\int_M |\omega|^{(2q+2)\alpha} |\nabla\phi|^2, \end{aligned} \tag{12}$$

where C_2 is a positive constant defined as follows.

$$\begin{aligned} C_2 = & \frac{p-1}{\alpha}\left(\frac{1}{m-1} + \frac{\alpha}{p-1} - 1\right) + 2q + 1 \\ & - \frac{p-2}{p-1}\frac{2(q+1)\alpha - p}{\alpha} - \frac{(2p-3)}{p-1}\varepsilon_1. \end{aligned}$$

On the other hand, since $m \ge 3$, we use Hölder inequality, Sobolev inequality (5) and Cauchy–Schwarz inequality to obtain

$$\begin{aligned} & \int_M |\Phi|^2 |\phi|^2 |\omega|^{2(q+1)\alpha} \le \left(\int_{supp(\phi)} |\Phi|^m\right)^{\frac{2}{m}} \left(\int_M \left(\phi|\omega|^{(q+1)\alpha}\right)^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \\ & \le C_0 \left(\int_{supp(\phi)} |\Phi|^m\right)^{\frac{2}{m}} \int_M \left(|\nabla(\phi|\omega|^{(q+1)\alpha})|^2 + m^2(|H|^2+1)\phi^2|\omega|^{2(q+1)\alpha}\right) \\ & \le C_0 ||\Phi||^2_{L^m(M)} \int_M \Big[(1+\frac{1}{\varepsilon_2})|\omega|^{2(q+1)\alpha}|\nabla\phi|^2 \\ & \quad + (1+\varepsilon_2)(q+1)^2 |\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2\phi^2 \\ & \quad + m^2(|H|^2+1)\phi^2|\omega|^{2(q+1)\alpha}\Big], \end{aligned} \tag{13}$$

where $\varepsilon_2 > 0$ is a positive constant. From (12) and (13), we have

$$\begin{aligned} & C_3 \int_M \phi^2 |\omega|^{2q\alpha} |\nabla|\omega|^\alpha|^2 + C_4 \int_M \phi^2 |H|^2 |\omega|^{2(q+1)\alpha} \\ & \quad + C_5 \int_M \phi^2 |\omega|^{2(q+1)\alpha} \le C_6 \int_M |\omega|^{(2q+2)\alpha} |\nabla\phi|^2, \end{aligned} \tag{14}$$

where C_3, C_4, C_5 and C_6 are positive constants defined as follows.

$$
\begin{aligned}
C_3 &= C_2 - \frac{\alpha}{p-1}\frac{m-1}{2}C_0||\Phi||^2_{L^m(M)}(1+\varepsilon_2)(q+1)^2,\\
C_4 &= \frac{\alpha}{p-1}\frac{m}{2} - \frac{\alpha}{p-1}\frac{m-1}{2}m^2C_0||\Phi||^2_{L^m(M)},\\
C_5 &= \frac{\alpha}{p-1}(m-1) - \frac{\alpha}{p-1}\frac{m-1}{2}m^2C_0||\Phi||^2_{L^m(M)},\\
C_6 &= \frac{2p-3}{p-1}\frac{1}{\varepsilon_1} + \frac{\alpha}{p-1}\frac{m-1}{2}C_0||\Phi||^2_{L^m(M)}\left(1+\frac{1}{\varepsilon_2}\right) > 0.
\end{aligned}
$$

Since $||\Phi||_{L^m(M)} < \frac{m-2}{2C(m)\sqrt{2m(m-1)}(m-1)}$, it is easy to know that $C_4 > 0$ and $C_5 > 0$. Now taking $\beta = (1+q)\alpha > 0$, we consider the following constant:

$$
\begin{aligned}
\widetilde{C}_3 &= \frac{p-1}{\alpha}\left(\frac{1}{m-1} + \frac{\alpha}{p-1} - 1\right) + 2q + 1 - \frac{p-2}{p-1}\frac{2(q+1)\alpha - p}{\alpha}\\
&\quad - \frac{\alpha}{p-1}\frac{m-1}{2}C_0||\Phi||^2_{L^m(M)}(q+1)^2\\
&> \frac{p-1}{\alpha}\left(\frac{1}{m-1} + \frac{\alpha}{p-1} - 1\right) + 2q + 1 - \frac{p-2}{p-1}\frac{2(q+1)\alpha - p}{\alpha}\\
&\quad - \frac{\alpha}{p-1}\frac{m-1}{2}\frac{1}{m(m-1)}(q+1)^2\\
&= -\frac{1}{(p-1)\alpha}\left[\frac{1}{2m}\beta^2 - 2\beta - \frac{p^2-2p-m+2}{m-1}\right]. \qquad (15)
\end{aligned}
$$

By the assumption on β, we can obtain $\widetilde{C}_3 > 0$. Choosing ε_1 and ε_2 small enough, we have $C_3 > 0$ and

$$
\begin{aligned}
&\int_M \phi^2|\omega|^{2q\alpha}|\nabla|\omega|^\alpha|^2 + \frac{C_4}{C_3}\int_M \phi^2|H|^2|\omega|^{2\beta} + \frac{C_5}{C_3}\int_M \phi^2|\omega|^{2\beta}\\
&\quad \le \frac{C_6}{C_3}\int_M |\omega|^{2\beta}|\nabla\phi|^2. \qquad (16)
\end{aligned}
$$

Fix a point $x_0 \in M$. Let us choose a nonnegative smooth $\phi \in C_0^\infty(M)$ satisfying

$$
\phi = \begin{cases} 1 & x \in B_{x_0}(R),\\ 0 & x \in M\backslash B_{x_0}(2R) \end{cases} \qquad (17)
$$

and $|\nabla\phi| \le \frac{2}{R}$. From the definition of ϕ and (16), we have

$$\int_{B_{x_0}(R)} |\omega|^{2q\alpha}|\nabla|\omega|^{\alpha}|^2 + \frac{C_4}{C_3}\int_{B_{x_0}(R)} |H|^2|\omega|^{2\beta} + \frac{C_5}{C_3}\int_{B_{x_0}(R)} |\omega|^{2\beta}$$
$$\le \frac{C_6}{C_3 R^2}\int_M |\omega|^{2\beta}.$$

Since $|\omega| \in L^{2\beta}(M)$, letting $R \to \infty$, we have $\omega = 0$. By Lemma 2.9, M^m has only one p-parabolic end. This completes the proof of Theorem 1.2.

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