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ORIGINAL PAPER

Berge–Fulkerson Coloring for Infinite Families of Snarks

Ting Zheng[1] · **Rong-Xia Hao**[1]

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Abstract It is conjectured by Berge and Fulkerson that every bridgeless cubic graph has six perfect matchings such that each edge is contained in exactly two of them. Hägglund constructed Blowup(K_4, C) and Blowup(Prism, C_4). Based on these two graphs, Chen constructed infinite families of bridgeless cubic graphs $M_{0,1,2,\ldots,k-2,k-1}$ which are obtained from cyclically 4-edge-connected and admitting Fulkerson-cover cubic graphs $G_0, G_1, \ldots, G_{k-1}$ by recursive process. He obtained that every graph in $M_{0,1,2,3}$ has a Fulkerson-cover and gave the open problem that whether every graph in $M_{0,1,2,\ldots,k-2,k-1}$ has a Fulkerson-cover. In this paper, we solve this problem and prove that every graph in $M_{0,1,2,\ldots,k-2,k-1}$ has a Fulkerson-cover.

Keywords Berge–Fulkerson conjecture · Fulkerson-cover · Perfect matching

Mathematics Subject Classification Primary 05C70; Secondary 05C75 · 05C40 · 05C15

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✉ Rong-Xia Hao
rxhao@bjtu.edu.cn

Ting Zheng
15121553@bjtu.edu.cn

[1] Department of Mathematics, Beijing Jiaotong University, Beijing 100044, People's Republic of China

1 Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *circuit* of G is a 2-regular connected subgraph. An *even graph* is a graph with even degree at every vertex. A perfect matching of G is a 1-regular spanning subgraph of G. The excessive index of G, denoted by $\chi_e'(G)$, is the least integer k, such that G can be covered by k perfect matchings. A cubic graph is a *snark* if it is bridgeless and not 3-edge-colorable.

The following is a famous open problem called Berge–Fulkerson conjecture:

Conjecture 1.1 (Berge–Fulkerson Conjecture [6], or see [11]) *Every bridgeless cubic graph has six perfect matchings such that each edge belongs to exactly two of them.*

We call such six perfect matchings in the conjecture as *the Fulkerson-cover*. Let G be a cubic graph. The graph $2G$ is obtained from G by duplicating every edge to become a pair of parallel edges. A graph G is *Berge–Fulkerson colorable* if the graph $2G$ is 6-edge-colorable. It means that there exists a mapping from $E(2G)$ to $\{1, 2, \ldots, 6\}$ such that every vertex of $2G$ is incident with edges colored with all six colors. Clearly, the six perfect matchings in the conjecture correspond to the 6-edge-coloring of the graph $2G$. Thus Berge–Fulkerson colorable is an equivalent description of the Fulkerson-cover.

Although there are some results related with this conjecture, as examples, see [3,5,7,8,10,12], Berge–Fulkerson conjecture is still open for many bridgeless cubic graphs even for some simple snarks. Fan and Raspaud [5] in 1994 made a weaker conjecture that every bridgeless cubic graph contains three perfect matchings with empty intersection. There are some known partial results such as the verification [9] of Fan–Raspaud conjecture for oddness two graphs. However, this weaker conjecture remains also unsolved.

Hägglund [7] constructed Blowup(K_4, C) and Blowup(Prism, C_4). Based on Blowup(K_4, C), Esperet et al. [4] constructed infinite families of cyclically 4-edge-connected snarks with excessive index at least five. Based on these two graphs, Chen [2] constructed infinite families of cyclically 4-edge-connected snarks $E_{0,1,2,\ldots,(k-1)}$ obtained from cyclically 4-edge-connected snarks $G_0, G_1, \ldots, G_{k-1}$, in which $E_{0,1,2}$ is Esperet et al.'s construction. If only assume that each graph in $\{G_0, G_1, \ldots, G_{k-1}\}$ has a Fulkerson-cover, then these infinite families of bridgeless cubic graphs are denoted by $M_{0,1,2,\ldots,k-2,k-1}$. Chen [2] obtained that every graph in $M_{0,1}$ or in $M_{0,1,2,3}$ has a Fulkerson-cover and gave the following problem:

Problem 1.2 [2] If $H = \{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\} \in M_{0,1,2,\ldots,k-2,k-1}$, does H have a Fulkerson-cover?

In this paper, we solve Problem 1.2. The main result is Theorem 1.3.

Theorem 1.3 *Each graph in $M_{0,1,2,\ldots,k-2,k-1}$ for $k > 2$ has a Fulkerson-cover.*

2 Preliminaries

In this section, some necessary definitions, constructions and the lemma are given.

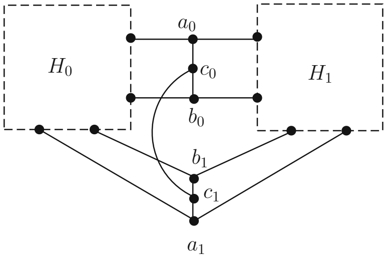

Fig. 1 $\{G; G_0, G_1\}$

Let $X \subseteq V(G)$ and $Y \subseteq E(G)$. We use $G\backslash X$ to denote the subgraph of G obtained from G by deleting all the vertices of X and all the edges incident with X. While $G\backslash Y$ to denote the subgraph of G obtained from G by deleting all the edges of Y. The edge-cut of G associated with X, denoted by $\partial_G(X)$, is the set of edges of G with exactly one end in X. The edge set $C = \partial_G(X)$ is called a k-edge-cut if $|\partial_G(X)| = k$. A cycle of G is a subgraph of G with each vertex of even degree. A circuit of G is a minimal 2-regular cycle of G. A graph G is called cyclically k-edge-connected if at least k edges must be removed to disconnect it into two components, each of which contains a circuit.

Let G_i be a cyclically 4-edge-connected snark with excessive index at least 5, for $i = 0, 1$. Let $x_i y_i$ be an edge of G_i and $x_i^0, x_i^1\,(y_i^0, y_i^1)$ be the neighbors of $x_i\ (y_i)$. Let H_i be the graph obtained from G_i by deleting the vertices x_i and y_i. Let $\{G; G_0, G_1\}$ be the graph obtained from the disjoint union of H_0, H_1 by adding six vertices $a_0, b_0, c_0, a_1, b_1, c_1$ and 13 edges $a_0 y_0^0, a_0 x_1^0, a_0 c_0, c_0 b_0, b_0 y_0^1, b_0 x_1^1, b_1 x_0^1, b_1 y_1^1, b_1 c_1, c_1 a_1, a_1 x_0^0, a_1 y_1^0, c_0 c_1$. The graphs of this type are denoted as $E_{0,1}$ (see Fig. 1).

The families of graphs $E_{0,1,\ldots,(k-1)}$ $(k \geq 2)$ and $M_{0,1,\ldots,(k-1)}$ $(k \geq 2)$ are constructed by Chen as follows:

1. $\{G; G_0, G_1\} \in E_{0,1}$ with $A_j = \{a_j, b_j, c_j\}$ for $j = 0, 1$.
2. For $3 \leq i \leq k$, $\{G; G_0, G_1, \ldots, G_{i-1}\}$ is obtained from $\{G; G_0, G_1, \ldots, G_{i-2}\} \in E_{0,1,\ldots,(i-2)}$ by adding H_{i-1} and $A_{i-1} = \{a_{i-1}, b_{i-1}, c_{i-1}\}$ and by inserting a vertex v_{i-3} into e_0, where e_0 is the edge incident with c_0 different from a_0c_0 and b_0c_0, such that the following conditions (i), (ii) and (iii) hold.

   (i) G_{i-1} is a cyclically 4-edge-connected snark with excessive index at least 5 (note that $x_{i-1}y_{i-1}$ is an edge of G_{i-1} and x_{i-1}^0, x_{i-1}^1 (resp. y_{i-1}^0, y_{i-1}^1) are the neighbors of x_{i-1} (resp. y_{i-1}));
   (ii) $H_{i-1} = G_{i-1}\backslash\{x_{i-1}, y_{i-1}\}$;
   (iii) a_{i-1} is adjacent to x_0^0 and y_{i-1}^0, b_{i-1} is adjacent to x_0^1 and y_{i-1}^1, a_{i-2} is adjacent to x_{i-1}^0 and y_{i-2}^0, b_{i-2} is adjacent to x_{i-1}^1 and y_{i-2}^1, c_{i-1} is adjacent to a_{i-1}, b_{i-1} and v_{i-3}, and the other edges of $\{G; G_0, G_1, \ldots, G_{i-2}\}$ remain the same:

3. $\{G; G_0, G_1, \ldots, G_{i-1}\} \in E_{0,1,\ldots,(i-1)}$.

For example, $\{G; G_0, G_1, G_2\}$ and $\{G; G_0, G_1, G_2, G_3\}$ are shown in Fig. 2.
The class of graphs constructed by Esperet et al. is a special case for $k = 3$ of $E_{0,1,\ldots,(k-1)}$. If the excessive index and non 3-edge-colorability of G_i $(i =$

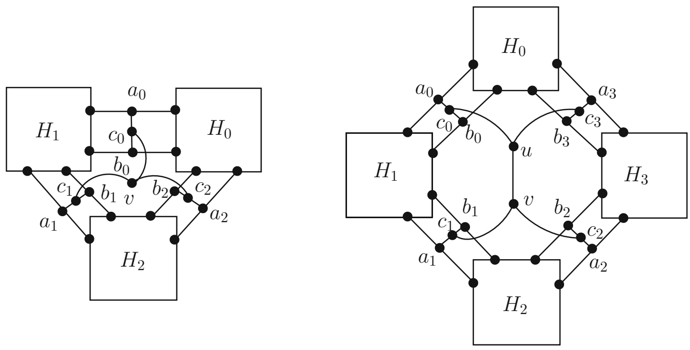

Fig. 2 $\{G; G_0, G_1, G_2\}$ and $\{G; G_0, G_1, G_2, G_3\}$

$0, 1, 2, \ldots, (k-1)$) are ignored and only assume that G_i has a Fulkerson-cover, then we obtain infinite families of bridgeless cubic graphs. We denote graphs of this type as $M_{0,1,2,\ldots,(k-1)}$ for $k \geq 2$.

The following Lemma is very import in our main proofs;

Lemma 2.1 (Hao et al. [8]) *A bridgeless cubic graph G has a Fulkerson-cover if and only if there are two disjoint matchings M_1 and M_2, such that $M_1 \cup M_2$ is a cycle and $\overline{G \backslash M_i}$ is 3-edge colorable, for each $i = 1, 2$, where $\overline{G \backslash M_i}$ is the graph obtained from $G \backslash M_i$ by suppressing all degree-2-vertices.*

3 Each Graph in $M_{0,1,2,\ldots,k-2,k-1}$ Has a Fulkerson Cover

We give the results according to the parity of k.

Theorem 3.1 *Let k be an even integer and $k \geq 4$. If $\Gamma \in M_{0,1,2,\ldots,k-2,k-1}$, then Γ has a Fulkerson-cover.*

Proof Since $\Gamma \in M_{0,1,2,\ldots,k-2,k-1}$, assume $\Gamma = \{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$.

Since G_i has a Fulkerson-cover, for each $i = 0, 1, \ldots, k-1$, suppose that $\{M_i^1, M_i^2, M_i^3\ , M_i^4, M_i^5, M_i^6\}$ is the Fulkerson-cover of G_i. Let B_i^2 be the set of edges in G_i covered twice by $\{M_i^1, M_i^2, M_i^3\}$ and B_i^0 be the set of edges in G_i which are not covered by $\{M_i^1, M_i^2, M_i^3\}$. Note that $B_i^2 \cup B_i^0$ is an even cycle, and $\overline{G_i \backslash B_i^2}$ and $\overline{G_i \backslash B_i^0}$ can be colored by three colors. Then B_i^2 and B_i^0 are the desired disjoint matchings of G_i as in Lemma 2.1. By choosing three perfect matchings of G_i, for each $i = 0, 1, \ldots, k-1$, we can obtain two desired disjoint matchings B_i^2 and B_i^0 such that $x_i y_i \in B_i^2 \cup B_i^0$ or $x_i, y_i \notin V(B_i^2 \cup B_i^0)$.

Three perfect matchings $\{M_i^1, M_i^2, M_i^3\}$ of G_i are chosen such that $x_i, y_i \notin V(B_i^2 \cup B_i^0)$ if i is even; And three perfect matchings $\{M_i^1, M_i^2, M_i^3\}$ of G_i are chosen such that $x_i y_i \in B_i^2 \cup B_i^0$ if i is odd. Without loss of generality, assume that $x_i y_i \in B_i^2$ and $x_i^0 x_i, y_i^0 y_i \in B_i^0$ for odd i.

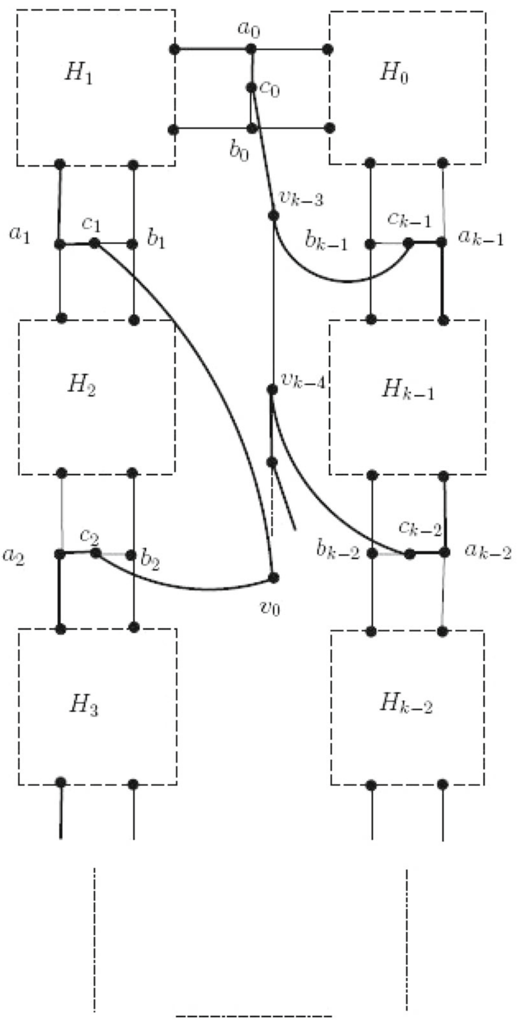

Fig. 3 $\{G; G_0, G_1, \ldots, G_{k-1}\}$ for even k

$$\text{Let } B_0 = \left(B_0^2 - x_0y_0\right) \cup \left(B_1^0 - \left\{x_1^0x_1, y_1^0y_1\right\}\right) \cup \bigcup_{i=2}^{k-1}\left(B_i^2 - x_iy_i\right) \cup \bigcup_{i=2}^{k-1} a_ic_i$$

$$\cup \bigcup_{j=1}^{\frac{k-4}{2}} v_{2j-1}v_{2j} \cup \left\{c_0v_{k-3}, c_1v_0, y_1^0a_1, x_1^0a_0\right\}, \text{ and}$$

$$B_2 = \left(B_0^0 - \left\{x_0x_0^0, y_0y_0^0\right\}\right) \cup \left(B_1^2 - x_1y_1\right) \cup \bigcup_{i=2}^{k-1}\left(B_i^0 - \left\{x_ix_i^0, y_iy_i^0\right\}\right)$$

$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left\{y_{2j+1}^0a_{2j+1}, x_{2j+1}^0a_{2j}\right\} \cup \bigcup_{i=2}^{k-1} v_{i-2}c_i \cup \{a_0c_0, a_1c_1\}.$$

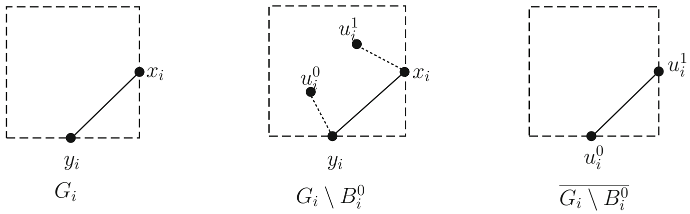

Fig. 4 $\overline{G_i \backslash B_i^0}$ with $x_i y_i \in B_i^2$

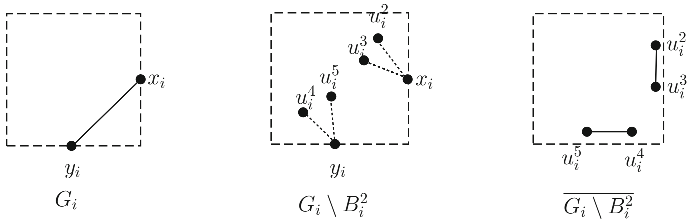

Fig. 5 $\overline{G_i \backslash B_i^2}$ with $x_i y_i \in B_i^2$

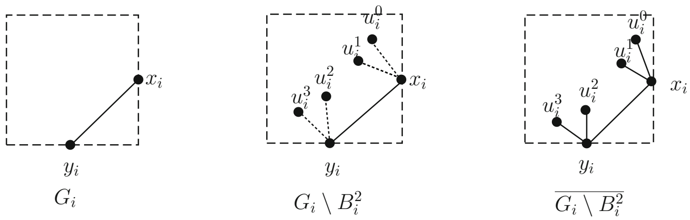

Fig. 6 $\overline{G_i \backslash B_i^2}$ with $x_i, y_i \notin V(B_i^2)$

Clearly, $B_0 \cup B_2$ is an even cycle C. See Fig. 3.

If i is odd, by $x_i y_i \in B_i^2$, there exists a maximal path containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^0$, say $u_i^0 \cdots y_i x_i \cdots u_i^1$, which corresponds to an edge $u_i^0 u_i^1$ in the graph $\overline{G_i \backslash B_i^0}$ (see Fig. 4). From $x_i^0 x_i$, $y_i^0 y_i \in B_i^0$, there exist two maximal paths containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^2$, say $u_i^2 \cdots x_i^0 x_i x_i^1 \cdots u_i^3$ and $u_i^5 \cdots y_i^0 y_i y_i^1 \cdots u_i^4$, which correspond to $u_i^2 u_i^3$ and $u_i^4 u_i^5$, respectively, in the graph $\overline{G_i \backslash B_i^2}$ (see Fig. 5).

If i is even, by $x_i, y_i \notin V(B_i^2)$, there exist four maximal paths containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^2$, say $u_i^0 \cdots x_i^1 x_i$ (maybe $u_i^0 = x_i^1$), $u_i^1 \cdots x_i^0 x_i$ (maybe $u_i^1 = x_i^0$), $u_i^2 \cdots y_i^1 y_i$ (maybe $u_i^2 = y_i^1$) and $u_i^3 \cdots y_i^0 y_i$, which correspond to four edges $u_i^0 x_i$, $u_i^1 x_i$, $u_i^2 y_i$ and $u_i^3 y_i$, respectively, in the graph $\overline{G_i \backslash B_i^2}$ (see Fig. 6).

Similarly, by $x_i, y_i \notin V(B_i^0)$, there exist four maximal paths containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^0$, say $u_i^4 \cdots x_i^1 x_i$ (maybe $u_i^4 = x_i^1$), $u_i^5 \cdots x_i^0 x_i$ (maybe $u_i^5 = x_i^0$), $u_i^6 \cdots y_i^1 y_i$ (maybe $u_i^6 = y_i^1$) and $u_i^7 \cdots y_i^0 y_i$ (maybe

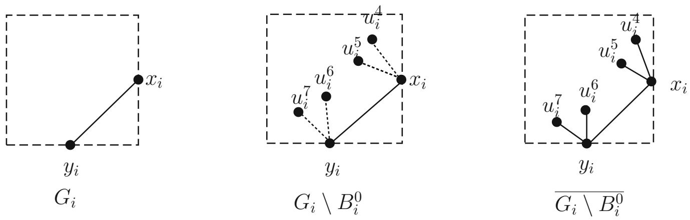

Fig. 7 $\overline{G_i\backslash B_i^0}$ with $x_i, y_i \notin V(B_i^2)$

$u_i^7 = y_i^0$), which correspond to four edges $u_i^4x_i$, $u_i^5x_i$, $u_i^6y_i$ and $u_i^7y_i$, respectively, in $\overline{G_i\backslash B_i^0}$ (see Fig. 7).

From the construction of Γ, we know that $\overline{\Gamma\backslash B_0}$ (see Fig. 8) is

$$\left(\overline{G_1\backslash B_1^0} - u_1^0u_1^1\right) \cup \bigcup_{j=0}^{\frac{k-2}{2}} \left(\overline{G_{2j}\backslash B_{2j}^2} - \{x_{2j}, y_{2j}\}\right) \cup \bigcup_{j=1}^{\frac{k-2}{2}} \left(\overline{G_{2j+1}\backslash B_{2j+1}^2}\right.$$
$$\left. - \left\{u_{2j+1}^2u_{2j+1}^3, u_{2j+1}^4u_{2j+1}^5\right\}\right) \cup \{u_1^0b_1, u_1^1b_0, b_1u_2^0, b_1u_2^1, u_0^2b_0, u_0^3b_0\}$$
$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left\{u_{2j}^3u_{2j+1}^2, u_{2j}^2b_{2j}, u_{2j+1}^3b_{2j}, u_{2j+1}^5u_{2j+2}^1, u_{2j+1}^4b_{2j+1}, u_{2j+2}^0b_{2j+1}, b_{2j}b_{2j+1}\right\}.$$

And $\overline{\Gamma\backslash B_2}$ (see Fig. 8) is

$$\left(\overline{G_1\backslash B_1^2} - \left\{u_1^2u_1^3, u_1^4u_1^5\right\}\right) \cup \bigcup_{j=0}^{\frac{k-2}{2}} \left(\overline{G_{2j}\backslash B_{2j}^0} - \{x_{2j}, y_{2j}\}\right)$$
$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left(\overline{G_{2j+1}\backslash B_{2j+1}^0} - u_{2j+1}^0u_{2j+1}^1\right) \cup \left\{u_1^4b_1, u_1^5u_2^5, b_1u_2^4, u_1^3b_0, u_0^7u_1^2, u_0^6b_0, b_0b_1\right\}$$
$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left\{u_{2j}^6b_{2j}, u_{2j}^7b_{2j}, b_{2j}u_{2j+1}^1, u_{2j+1}^0b_{2j+1}, b_{2j+1}u_{2j+2}^4, b_{2j+1}u_{2j+2}^5\right\}.$$

If i is odd, because B_i^2 and B_i^0 are the desired disjoint matchings of G_i as in Lemma 2.1, $\overline{G_i\backslash B_i^0}$ is 3-edge colorable. Thus there exists a 2-factor, say C_i^0, such that each component is an even circuit and $u_i^0u_i^1$ is not in the 2-factor C_i^0. Similarly, because $\overline{G_i\backslash B_i^2}$ is 3-edge colorable, there exists a 2-factor C_i^2 such that each component is an even circuit and $\{u_i^2u_i^3, u_i^4u_i^5\}$ is in the 2-factor C_i^2.

If i is even, because $\overline{G_i\backslash B_i^2}$ is 3-edge colorable, there exists a 2-factor C_i^2 such that each component is an even circuit and $u_i^0x_iu_i^1$ and $u_i^2y_iu_i^3$ are in the 2-factor

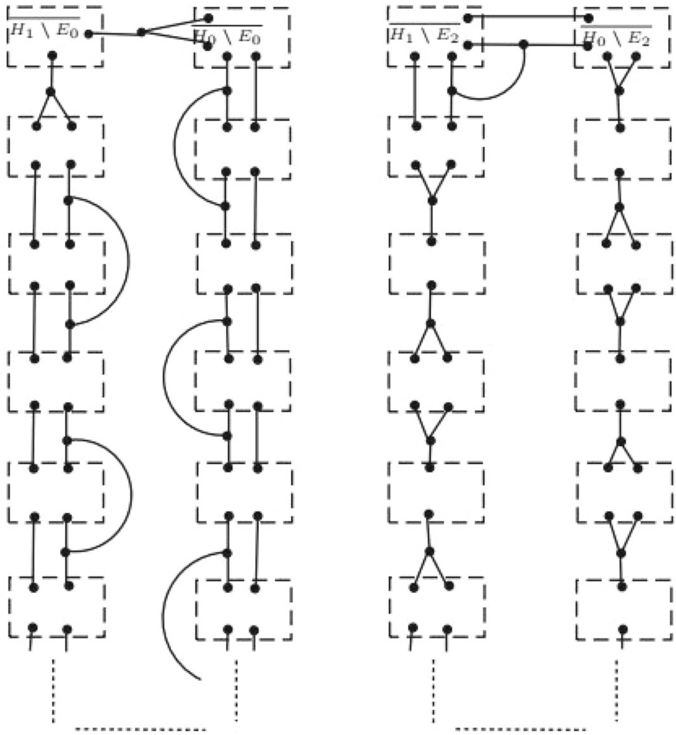

Fig. 8 $\overline{\Gamma \backslash B_0}$ and $\overline{\Gamma \backslash B_2}$ for even k

C_i^2. Because $\overline{G_i \backslash B_i^0}$ is 3-edge colorable, there exists a 2-factor C_i^0 such that each component is an even circuit and $\{u_i^4 x_i u_i^5, u_i^6 y_i u_i^7\}$ is in the 2-factor C_i^0.

Then $\overline{\Gamma \backslash B_0}$ has a 2-factor:

$$C_1^0 \cup \bigcup_{j=1}^{\frac{k-2}{2}} \left(C_{2j+1}^2 - \left\{u_{2j+1}^2 u_{2j+1}^3, u_{2j+1}^4 u_{2j+1}^5\right\}\right) \cup \bigcup_{j=0}^{\frac{k-2}{2}} \left(C_{2j}^2 - \{x_{2j}, y_{2j}\}\right)$$
$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left\{b_{2j} u_{2j+1}^3, b_{2j} u_{2j}^2, u_{2j}^3 u_{2j+1}^2, b_{2j+1} u_{2j+1}^4, b_{2j+1} u_{2j+2}^0, u_{2j+1}^5 u_{2j+2}^1\right\}$$
$$\cup \left\{b_0 u_0^3, b_0 u_0^2, b_1 u_2^0, b_1 u_2^1\right\}.$$

And each component is an even circuit.

Similarly $\overline{\Gamma \backslash B_2}$ has a 2-factor:

$$\bigcup_{j=1}^{\frac{k-2}{2}} C_{2j+1}^0 \cup \left(C_1^2 - \left\{u_1^2 u_1^3, u_1^4 u_1^5\right\}\right) \cup \bigcup_{j=0}^{\frac{k-2}{2}} \left(C_{2j}^0 - \{x_{2j}, y_{2j}\}\right)$$

$$\cup \left\{ b_0 u_0^6, b_0 u_1^3, u_0^7 u_1^2, b_1 u_1^4, u_1^5 u_2^5, b_1 u_2^4 \right\}$$
$$\cup \bigcup_{j=1}^{\frac{k-2}{2}} \left\{ u_{2j}^6 b_{2j}, u_{2j}^7 b_{2j}, u_{2j+2}^4 b_{2j+1}, u_{2j+2}^5 b_{2j+1} \right\}.$$

And each component is an even circuit.

So $\overline{\Gamma \backslash B_0}$ and $\overline{\Gamma \backslash B_2}$ are 3-edge colorable. Therefore, B_0 and B_2 are the desired matchings in Γ of Lemma 2.1. So $\Gamma = \{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$ has a Fulkerson-cover. □

Theorem 3.2 *Let k be an odd integer. If $H \in M_{0,1,2,\ldots,k-2,k-1}$, then H has a Fulkerson-cover.*

Proof By $H \in M_{0,1,2,\ldots,k-2,k-1}$, assume $H = \{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$. Since G_i has a Fulkerson-cover, for each $i = 0, 1, \ldots, k-1$, suppose that $\{M_i^1, M_i^2, M_i^3, M_i^4, M_i^5, M_i^6\}$ is the Fulkerson-cover of G_i. Let B_i^2 be the set of edges covered twice by $\{M_i^1, M_i^2, M_i^3\}$ and B_i^0 be the set of edges which are not covered by $\{M_i^1, M_i^2, M_i^3\}$. Now $B_i^2 \cup B_i^0$ is an even cycle, and $\overline{G_i \backslash B_i^2}$ and $\overline{G_i \backslash B_i^0}$ can be colored by three colors. Then B_i^2 and B_i^0 are the desired disjoint matchings of G_i as in Lemma 2.1. By choosing three perfect matchings of G_i, for each $i = 0, 1, \ldots, k-1$, we can obtain two desired disjoint matchings B_i^2 and B_i^0 such that $x_i y_i \in B_i^2 \cup B_i^0$ or $x_i, y_i \notin V(B_i^2 \cup B_i^0)$. If i is even and $i \neq 0$, three perfect matchings of G_i are chosen such that $x_i, y_i \notin V(B_i^2 \cup B_i^0)$. If i is odd or $i = 0$, three perfect matchings of G_i are chosen such that $x_i y_i \in B_i^2 \cup B_i^0$. Without loss of generality, assume that $x_i y_i \in B_i^2$ and $x_i^0 x_i, y_i^0 y_i \in B_i^0$ if $i = 0$ or i is odd.

Let

$$B_0 = \left(B_0^2 - x_0 y_0\right) \cup \left(B_1^0 - \left\{x_1^0 x_1, y_1^0 y_1\right\}\right) \cup \bigcup_{i=2}^{k-1} \left(B_i^2 - x_i y_i\right)$$
$$\cup \left\{y_1^0 a_1, x_1^0 a_0, c_1 v_0\right\} \cup \bigcup_{i=2}^{k-1} a_i c_i \cup \bigcup_{j=0}^{\frac{k-5}{2}} v_{2j+1} v_{2j+2},$$

If i is odd or $i = 0$, by $x_i y_i \in B_i^2$, there exists a maximal path containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^0$, say $u_i^0 \cdots y_i x_i \cdots u_i^1$, which corresponds to an edge $u_i^0 u_i^1$ in the graph $\overline{G_i \backslash B_i^0}$ (see Fig. 4); by $x_i^0 x_i, y_i^0 y_i \in B_i^0$, there exist two distinct maximal path containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^2$, say $u_i^2 \cdots x_i^0 x_i x_i^1 \cdots u_i^3$ and $u_i^5 \cdots y_i^0 y_i y_i^1 \cdots u_i^4$ which correspond to edges $u_i^2 u_i^3$ and $u_i^4 u_i^5$, respectively, in the graph $\overline{G_i \backslash B_i^2}$ (see Fig. 5).

$$B_2 = \left(B_1^2 - x_1 y_1\right) \cup \bigcup_{i=2}^{k-1} \left(B_i^0 - \left\{x_i x_i^0, y_i y_i^0\right\}\right) \cup \left(B_0^0 - \left\{x_0 x_0^0, y_0 y_0^0\right\}\right)$$

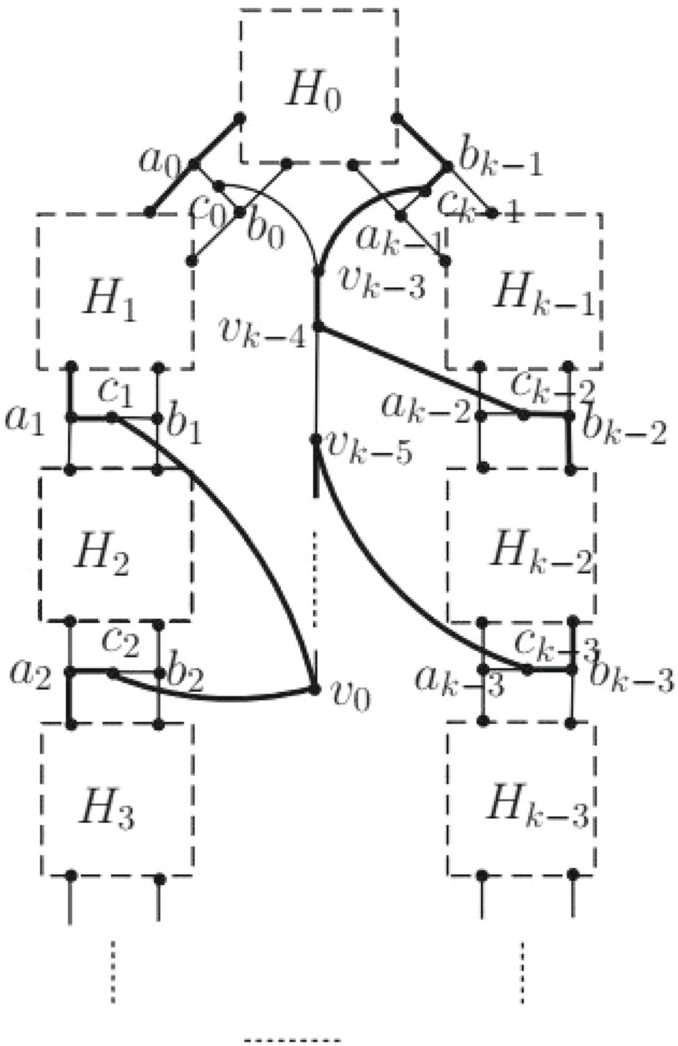

Fig. 9 $\{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$ for odd k

$$\cup \{a_1c_1\} \cup \bigcup_{i=1}^{\frac{k-1}{2}} \left\{a_{2i}x_{2i+1}^0, a_{2i+1}y_{2i+1}^0\right\} \cup \bigcup_{i=2}^{k-1} \{v_{i-2}c_i\}.$$

See Fig. 9. Clearly, $B_0 \cup B_2$ is an even cycle C.

If i is even and $i \neq 0$, since $x_i, y_i \notin V(B_i^2)$, there exist four maximal paths containing only 2-degree vertices as inter vertices in the graph $G_i \backslash B_i^2$, say $u_i^0 \cdots x_i^1 x_i$ (maybe $u_i^0 = x_i^1$), $u_i^1 \cdots x_i^0 x_i$ (maybe $u_i^1 = x_i^0$), $u_i^2 \cdots y_i^1 y_i$ (maybe $u_i^2 = y_i^1$) and $u_i^3 \cdots y_i^0 y_i$ (maybe $u_i^3 = y_i^0$), which correspond to edges $u_i^0 x_i$, $u_i^1 x_i$, $u_i^2 y_i$ and $u_i^3 y_i$, respectively, in the graph $\overline{G_i\backslash B_i^2}$ (See Fig. 6). Similarly, by $x_i, y_i \notin V(B_i^0)$, there exist four maximal paths containing only 2-degree vertices as inter vertices in the graph $G_i\backslash B_i^0$, say $u_i^4\cdots x_i^1 x_i$ (maybe $u_i^4 = x_i^1$), $u_i^5\cdots x_i^0 x_i$ (maybe $u_i^5 = x_i^0$), $u_i^6\cdots y_i^1 y_i$ (maybe $u_i^6 = y_i^1$) and $u_i^7\cdots y_i^0 y_i$ which correspond to edges $u_i^4 x_i$, $u_i^5 x_i$, $u_i^6 y_i$ and $u_i^7 y_i$,respectively, in the graph $\overline{G_i\backslash B_i^0}$ (see Fig. 7).

If $k = 1$, then $H = G_0$ which has a Fulkerson-cover.

If $k \geq 2$, we will prove $\overline{H\backslash B_0}$ and $\overline{H\backslash B_2}$ are 3-edge colorable in the following.

From the construction of H, one has that $\overline{H\backslash B_0}$ (see Fig. 10) is

$$\left(\overline{G_0\backslash B_0^2} - \left\{u_0^2u_0^3, u_0^4u_0^5\right\}\right) \cup \left(\overline{G_1\backslash B_1^0} - u_1^0u_1^1\right)$$
$$\cup \bigcup_{j=1}^{\frac{k-1}{2}} \left\{u_{2j}^2 b_{2j}, u_{2j+1}^3 b_{2j}, u_{2j+1}^2 u_{2j}^3\right\}$$
$$\cup \bigcup_{j=1}^{\frac{k-3}{2}} \left\{u_{2j+1}^4 b_{2j+1}, u_{2j+1}^5 u_{2j+2}^1, u_{2j+2}^0 b_{2j+1}, b_{2j} b_{2j+1}\right\}$$
$$\cup \left\{u_0^5 c_0, u_0^4 b_0, b_0 c_0, b_{k-1} c_0, u_1^1 b_0, u_1^0 b_1, u_2^0 b_1, u_2^1 b_1\right\} \cup Q_0 \cup Q_1,$$

where $Q_1 = \bigcup_{j=1}^{\frac{k-3}{2}}(\overline{G_{2j+1}\backslash B_{2j+1}^2} - \{u_{2j+1}^2 u_{2j+1}^3, u_{2j+1}^4 u_{2j+1}^5\})$ and $Q_0 = \bigcup_{j=1}^{\frac{k-1}{2}}(\overline{G_{2j}\backslash B_{2j}^2} - \{x_{2j}, y_{2j}\})$. And $\overline{H\backslash B_2}$ (see Fig. 10) is

$$\left(\overline{G_1\backslash B_1^2} - \left\{u_1^2u_1^3, u_1^4u_1^5\right\}\right) \cup \left(\overline{G_0\backslash B_0^0} - u_0^0u_0^1\right)$$
$$\cup \left\{u_1^5u_2^5, u_1^4 b_1, u_2^4 b_1, u_1^2 c_0, u_1^3 b_0, b_0 c_0, u_0^0 b_0, c_0 b_1\right\}$$
$$\cup \bigcup_{j=1}^{\frac{k-1}{2}} \left((\overline{G_{2j}\backslash B_{2j}^0} - \{x_{2j}, y_{2j}\})\right.$$
$$\left.\cup \left\{u_{2j}^6 b_{2j}, u_{2j+1}^1 b_{2j}, u_{2j}^7 b_{2j}\right\}\right)$$
$$\cup \bigcup_{j=1}^{\frac{k-3}{2}} \left(\left(\overline{G_{2j+1}\backslash B_{2j+1}^0} - \left\{u_{2j+1}^0 u_{2j+1}^1\right\}\right)\right.$$
$$\left.\cup \left\{u_{2j+1}^0 b_{2j+1}, u_{2j+2}^4 b_{2j+1}, u_{2j+2}^5 b_{2j+1}\right\}\right).$$

If i is odd or $i = 0$, because $\overline{G_i\backslash B_i^0}$ is 3-edge colorable, there exists a 2-factor C_i^0 such that each component is an even circuit and $u_i^0 u_i^1$ is not in the 2-factor C_i^0. Because $\overline{G_i\backslash B_i^2}$ is 3-edge colorable, there exists a 2-factor C_i^2 such each component is an even circuit and $u_i^2 u_i^3$, $u_i^4 u_i^5$ are in the 2-factor C_i^2.

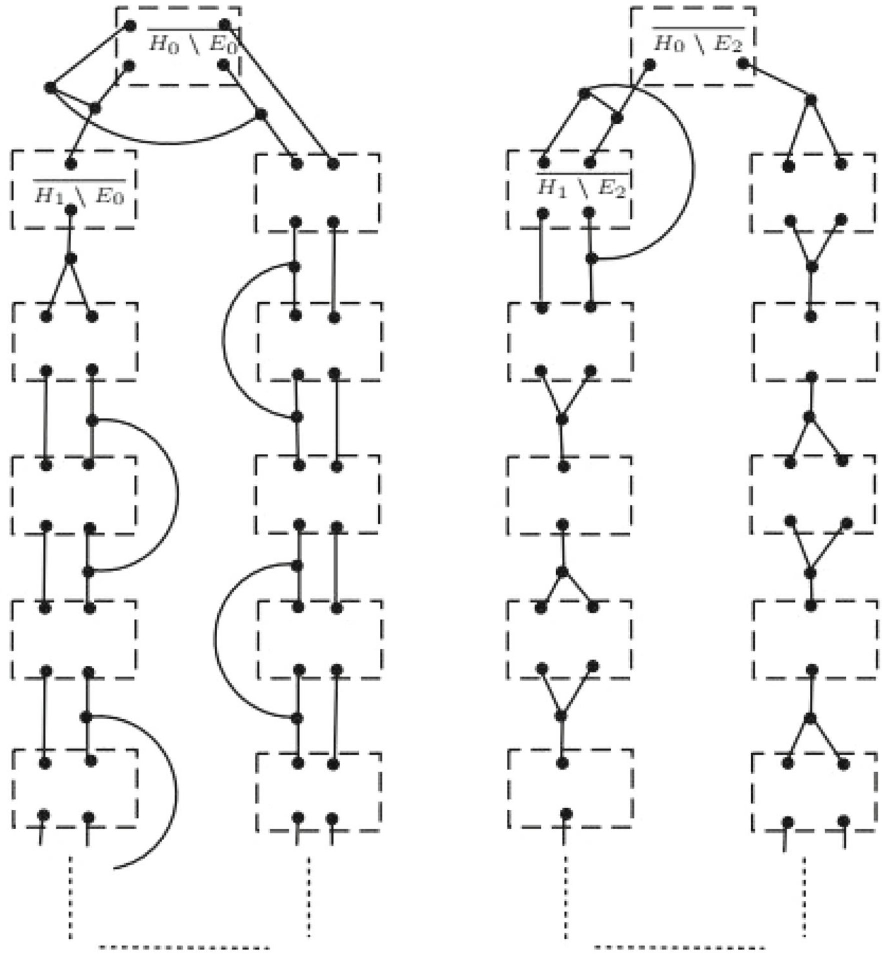

Fig. 10 $\overline{H \backslash B_0}$ and $\overline{H \backslash B_2}$ with odd k

If i is even and $i \neq 0$, because $\overline{G_i \backslash B_i^2}$ is 3-edge colorable, there exists a 2-factor C_i^2 such each component is an even circuit and two paths with length two $u_i^0 x_i u_i^1$ and $u_i^2 y_i u_i^3$ are in the 2-factor C_i^2. Similarly, because $\overline{G_i \backslash B_i^0}$ is 3-edge colorable, there exists a 2-factor C_i^0 such each component is an even circuit and $u_i^4 x_i u_i^5$, $u_i^6 y_i u_i^7$ are in the 2-factor C_i^0.

Then $\overline{H \backslash B_0}$ (see Fig. 10) has a 2-factor:

$$\left(C_0^2 - \left\{u_0^4 u_0^5, u_0^2 u_0^3\right\}\right) \cup C_1^0 \cup \bigcup_{j=1}^{\frac{k-3}{2}} \left(C_{2j+1}^2 - \left\{u_{2j+1}^2 u_{2j+1}^3, u_{2j+1}^4 u_{2j+1}^5\right\}\right)$$

$$\cup \bigcup_{j=1}^{\frac{k-1}{2}} \left(C_{2j}^2 - \{x_{2j}, y_{2j}\}\right) \cup \bigcup_{j=1}^{\frac{k-1}{2}} \left\{b_{2j}u_{2j+1}^3, b_{2j}u_{2j}^2, u_{2j}^3u_{2j+1}^2\right\}$$
$$\cup \bigcup_{j=1}^{\frac{k-3}{2}} \left\{b_{2j+1}u_{2j+1}^4, b_{2j+1}u_{2j+2}^0, u_{2j+1}^5u_{2j+2}^1\right\} \cup \left\{b_0u_0^4, b_1u_2^0, b_1u_2^1, b_0c_0, u_0^5c_0\right\}.$$

and each component is an even circuit. $\overline{H \backslash B_2}$ has a 2-factor:

$$C_0^0 \cup \left(C_1^2 - \left\{u_1^2u_1^3, u_1^4u_1^5\right\}\right) \cup \bigcup_{j=1}^{\frac{k-1}{2}} \left((C_{2j}^0 - \{x_{2j}, y_{2j}\}) \cup \left\{u_{2j}^6b_{2j}, u_{2j}^7b_{2j}\right\}\right)$$
$$\cup \bigcup_{j=1}^{\frac{k-3}{2}} C_{2j+1}^0 \cup \bigcup_{j=1}^{\frac{k-3}{2}} \left\{u_{2j+2}^4b_{2j+1}, u_{2j+2}^5b_{2j+1}\right\}$$
$$\cup \left\{c_0u_1^2, b_0c_0, b_0u_1^3, u_1^5u_2^5, u_1^4b_1, u_2^4b_1\right\}$$

And each component is an even circuit.

So $\overline{H \backslash B_0}$ and $\overline{H \backslash B_2}$ are 3-edge colorable. Therefore, B_0 and B_2 are the desired matchings of Lemma 2.1 and $H = \{G; G_0, G_1, \ldots, G_{k-2}, G_{k-1}\}$ has a Fulkerson-cover. □

From Theorems 3.1 and 3.2, we get the Theorem 1.3 that every graph in $M_{0,1,2,\ldots,(k-1)}$ has a Fulkerson-cover.

4 Remark on Treelike Snarks

The families of graphs $E_{0,1,\ldots,(k-1)}$ and $M_{0,1,\ldots,(k-1)}$ for $k \geq 2$, which are constructed by Chen [2], are set of graphs having all vertices not in $H_0, H_1, \ldots, H_{k-1}$ and not in a_i, b_i, c_i on a path. Abreu et al. [1] proposed Treelike snarks which generalize this idea by considering an arbitrary tree instead of a path, but the graphs H_i are all copies of the Petersen graph minus an edge. They proved that all such Treelike snarks have excessive index at least five.

In this paper, we prove that every graph in $M_{0,1,2,\ldots,k-2,k-1}$ has a Fulkerson-cover. Since the Petersen graph has a Fulkerson cover, as a directive corollary of this result, each graph in the subclass of treelike snarks obtained by considering a path instead of an arbitrary tree has a Fulkerson cover. So we give a conjecture as follows:

Conjecture 4.1 (Conjecture) *Every Treelike snarks proposed in* [1] *has a Fulkerson cover.*

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