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ORIGINAL PAPER

Finite Symmetric Graphs with 2-Arc-Transitive Quotients: General Affine Case

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Abstract Let G be a finite group and Γ a G-symmetric graph. Suppose that G is imprimitive on $V(\Gamma)$ with B a block of imprimitivity and $\mathcal{B} := \{B^g : g \in G\}$ is a system of imprimitivity of G on $V(\Gamma)$. Define $\Gamma_{\mathcal{B}}$ to be the graph with vertex set $\mathcal{B}$, such that two blocks $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least one edge of Γ joining a vertex in B and a vertex in C. Set $v = |B|$ and $k := |\Gamma(C) \cap B|$ where C is adjacent to B in $\Gamma_{\mathcal{B}}$ and $\Gamma(C)$ denotes the set of vertices of Γ adjacent to at least one vertex in C. Assume that $k = v - p \geq 1$, where p is an odd prime, and $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc-transitive. In this paper , we show that if the group induced on each block is an affine group then $v = 6$.

Keywords Symmetric graphs · Transitive groups and arc-transitive graphs

Mathematics Subject Classification 05C25

1 Introduction

Let G be a finite group. A graph Γ is called G-symmetric if Γ admits G as a group of automorphisms acting transitively on the set of vertices and the set of arcs of Γ, where an *arc* is an ordered pair of adjacent vertices. Suppose that G is imprimitive on $V(\Gamma)$ with B a block of imprimitivity

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$$\mathcal{B} := \{B^g : g \in G\}$$

is a system of imprimitivity of G on $V(\Gamma)$. Define $\Gamma_{\mathcal{B}}$ to be the graph with vertex set $\mathcal{B}$, such that two blocks $B, C \in \mathcal{B}$ are adjacent if and only if there exists at least one edge of Γ joining a vertex in B and a vertex in C. We call $\Gamma_{\mathcal{B}}$ the *quotient graph* of Γ with respect to $\mathcal{B}$. A graph Γ is called $(G, 2)$-arc-transitive if it admits G as a group of automorphisms acting transitively on the set of vertices and the set of 2-arcs of Γ, where a *2-arc* is an oriented path of length two. Denote by G_B the setwise stabilizer of B in G, and define $H := G_B^{\Gamma_{\mathcal{B}}(B)}$ to be the quotient group of G_B relative to the kernel of the induced action of G_B on the set of blocks of $\mathcal{B}$ adjacent to B in $\Gamma_{\mathcal{B}}$. We keep these notation in this paper. In [3], the following question was asked:

Question 1.1 *Under the assumption above, when is $\Gamma_{\mathcal{B}}$ a $(G, 2)$-arc-transitive graph?*

This question is studied in [3–8] and [9]. In this paper, we investigate the question above when H is an affine group, see [2] for definition of an affine group and other background in permutation groups. To state our main results, we need some definitions and notation.

Throughout this paper, $\Gamma_{\mathcal{B}}$ is a connected $(G, 2)$-arc-transitive graph with valency b. Let B and C be two vertices in $\Gamma_{\mathcal{B}}$ such that C is adjacent to B. Set $v = |B|$ and $k := |\Gamma(C) \cap B|$, where $\Gamma(C)$ denotes the set of vertices of Γ adjacent to at least one vertex in C. In this paper, we assume that $k = v - p \geq 1$, for some odd prime p, and H is the affine group $AGL_n(q)$. In [7], we have shown that if $q = 2$, then $v = 6$. In this paper, we consider the general case and shall prove theorem below which improves our results in [7]. In fact, in this paper, we show that H cannot be a 2-transitive subgroup of $AGL_n(q)$ for odd prime q.

Theorem 1.2 *Assume that $k = v - p \geq 1$, for some odd prime p, and H is an affine group. Then, $q = 2$ and $v = 6$.*

Theorem 1.2 shows that the graph which appears in [9, Theorem 3(e)] is the only graph satisfying the conditions of Theorem 1.2.

2 Proof of the Main Theorem

In this section, we prove Theorem 1.2. Therefore, we keep the assumptions in 1.2. We fix $B \in \mathcal{B}$. Let $\mathcal{U} := \Gamma_{\mathcal{B}}(B)$ be the set of blocks of $\mathcal{B}$ adjacent to B in $\Gamma_{\mathcal{B}}$. For $\alpha \in B$, let $\Gamma_{\mathcal{B}}(\alpha)$ be the set of blocks in $\mathcal{U}$ containing at least one neighbour of α in Γ, and let $r := |\Gamma_{\mathcal{B}}(\alpha)|$. Since Γ is G-symmetric and $\mathcal{B}$ is G-invariant, r, v and k are independent of the choice of α, B, and C, respectively, for each $C \in \mathcal{U}$. Denote by G_B the setwise stabilizer of B in G, and define $H := G_B^{\Gamma_{\mathcal{B}}(B)}$ to be the quotient group of G_B relative to the kernel of the induced action of G_B on $\mathcal{U}$.

Our strategy to prove Theorem 1.2 is to show that the minimal normal subgroup of H is 2-group. Then, Theorem 1.2 will follow from the main results in [7]. By our assumption, H is a 2-transitive affine group on $\mathcal{U}$. Therefore, we can control the structure of the minimal normal subgroup of H and the parameter b. This has been done in [9]. In fact, in [9] necessary conditions for $\Gamma_{\mathcal{B}}$ to be $(G, 2)$-arc-transitive were obtained in the case when $k = v - p \geq 1$, where p is an odd prime. In fact, Γ, $\Gamma_{\mathcal{B}}$, and H satisfy the conditions in the seventh row of Table 1 in [9]. Therefore, with our assumptions, the following theorem follows from the seventh row of Table 1 in [9].

Theorem 2.1 *We have $sp + 1 = b = q^n \geq 2$, where $s \geq 2$, p, and q are primes and $v = q^m p$, where $n - 1 \geq m \geq 1$.*

By Theorem 2.1, we get that $|\mathcal{U}| = sp+1 = q^n$ for some prime q and the minimal normal subgroup of H is an elementary abelian group of order q^n. We keep the notation in Theorem 2.1.

Now, set

$$\mathcal{U} := \{C, C_1, \ldots, C_{sp}\}, \quad W := \Gamma(C) \cap B, W_i = \Gamma(C_i) \cap B, \quad \text{for } i = 1, \ldots, sp.$$

Since in this paper, we investigate [9, Theorem 1.1(f)], our parameters are as in the last row [case (f)] of Table 1 in [9]. In fact, $a = q^m$, $v = ap$, $b = |\mathcal{U}| = sp + 1 = q^n$, $r = q^n(a-1)/a$ and $\lambda = |W \cap W_i| = p(a-2) + (b-a)/as$, for $i = 1, 2, \ldots, sp$. We have also that $2 \leq s \leq a - 1 \leq p - 2$. We have that $|B \backslash W| = p$, by our assumption. Let H_C be the stabilizer of C in H. Then, H_C leaves W and $B \backslash W$ invariant. Since $\Gamma_{\mathcal{B}}$ is $(G, 2)$-arc-transitive, H is 2-transitive on $\mathcal{U}$, and therefore, H_C is transitive on $\mathcal{U} \backslash \{C\}$. In fact, Γ, $\Gamma_{\mathcal{B}}$, and H satisfy the conditions in the seventh row of Table 1 in [9]. Therefore, we have that $H = N \rtimes H_C$ is an affine group (isomorphic to a subgroup of $\mathrm{AGL}(n, q)$), where $N \cong \mathbb{Z}_q^n$ is an elementary abelian group of order $sp + 1 = q^n$ and is the minimal normal subgroup of H acting regularly on $\mathcal{U}$ with $C_H(N) = N$, and H_C is isomorphic to a subgroup of $\mathrm{GL}_n(q)$ and acts transitively on subgroups of order q in N. We note that since the case $q = 2$ is considered in [7], so we may assume that q is an odd prime. Since $sp = q^n - 1$, we get that H_C is of even order. Let $P(N)$ be the set of subgroups of order q in N.

Since $\mathcal{U} \backslash \{C\}$ has exactly sp element and H_C is transitive on it, sp divides the order of H_C. Since p is a prime, H_C contains an element of order p, say, x and an involution say, z. Define

$$M := \langle x, z \rangle \leq H_C, \quad P := \langle N, x, z \rangle = N \rtimes M \leq H, \quad X := \langle x \rangle.$$

Lemma 2.2 *The following hold:*

(i) $C_N(x) = 1$ *and X has s orbit on $\mathcal{U} \backslash \{C\}$.*
(ii) *X fixes W and $B \backslash W$ setwise and is fixed-point-free on each of them.*
(iii) *X is regular on $B \backslash W$.*

Proof (i) By coprime action (see [1, Sect. 24]), we have $N = C_N(X) \oplus [N, x]$. Let I be the set of elements in $\mathcal{U}$ fixed by X. Then, for $y \in C_N(x)$, we have $C^y \in I$. If $C_i \in I$, then there is $y \in N$, such that $C_i = C^y$ which implies that $y \in C_N(x)$. Hence, $|I| = |C_N(x)|$.
Assume that $|I| \neq 1$. We note that for each $\{C_i, C_j\} \subset I$, $i \neq j$, we have $B \backslash W_i$ and $B \backslash W_j$ are sets with p element and x acts on each of them. This gives us that x acts trivially on $B \backslash (W_i \cap W_j)$. In fact, x acts trivially on $B \backslash (\bigcap_{C_i \in I} W_i)$.

Let $\alpha \in \bigcap_{C_i \in I} W_i$ and l be the number of orbits of X on $\mathcal{U}\backslash I$. Then, we have $r = |\Gamma_{\mathcal{B}}(\alpha)| = |I| + l$. On the other hand, we have $|I| + lp = |\mathcal{U}| = q^n$. These give us that $r - l = q^n - lp$ which implies that $q^n - q^{n-m} - l = q^n - lp$. Now, we have $l(p-1) = q^{n-m}$ which implies that $q = 2$ is even, a contradiction to our assumption. This contradiction shows that $\bigcap_{C_i \in I} W_i = \emptyset$, and then, x is trivial on B, a contradiction. Therefore, $|I| = 1 = |C_N(x)|$ and X has s orbits on $\mathcal{U}\backslash\{C\}$. Now i) holds.

(ii) Since $X \leq H_C$, it fixes W and $B\backslash W$ setwise. If a vertex $\alpha \in B\backslash W$ is fixed by a non-identity element of X, then it is fixed by every non-identity element of X. Since by (a), X has s orbits on $\mathcal{U}\backslash\{C\}$, we get that p divides r. However, $r = q^n - q^{n-m}$, and p does not divide r. Therefore, X is fixed-point-free on $B\backslash W$. A similar argument shows that X is fixed-point-free on W.

(iii) Since $|X| = |B\backslash W| = p$ is a prime and X acts fixed-point-freely on $B\backslash W$, X must be regular on $B\backslash W$ and the Lemma holds. □

Lemma 2.3 *No nonempty subset of W is N invariant.*

Proof Suppose to the contrary that $\emptyset \neq Y \subseteq W$ is N invariant. Since N is regular on $\mathcal{U}$, for each C_i, there exists a unique element $g_i \in N$ such that $C^{g_i} = C_i$. Hence, $W^{g_i} = \Gamma(C_i) \cap B$. Since Y is N invariant, we have $Y = Y^{g_i} \subseteq W^{g_i}$ for $i = 1, 2, \ldots, sp$, which implies that $q^n - 1 = sp = r - 1$, a contradiction. □

Lemma 2.4 *NX is transitive on B.*

Proof Let α^N be an N orbit on B, where $\alpha \in B$, and set $A = \cup_{g \in NX}(\alpha^N)^g$. Since $N \trianglelefteq NX \leq H$, $A \subseteq B$ and NX is transitive on A; thus, both A and $B\backslash A$ are NX invariant. In particular, both A and $B\backslash A$ are N invariant and X invariant. Since $A \neq \emptyset$, by Lemma 2.3, we have $A \cap (B\backslash W) \neq \emptyset$. On the other hand, by Lemma 2.2, X is transitive on $B\backslash W$. Since A is X invariant and $A \cap (B\backslash W) \neq \emptyset$, it follows that $B\backslash W \subseteq A$. Now that $B\backslash A \subseteq W$ and $B\backslash A$ is N invariant, by Lemma 2.3, $B\backslash A = \emptyset$, and hence, NX is transitive on $B = A$. □

Let $\alpha \in B$, and set

$$\mathcal{F} := \{(\alpha^N)^g : g \in X\}.$$

Since N is normal in P, $\mathcal{F}$ is a system of imprimitivity for P. Then, $|\alpha^N| = q^m = a$, $|\mathcal{F}| = p$, and $\mathcal{F}$ is the set of all N orbits on B.

Lemma 2.5 *We have $|\alpha^N \cap (B\backslash W)| = 1$. In fact, each element of $B\backslash W$ is in a unique element of $\mathcal{F}$ and each element of $\mathcal{F}$ contains a unique element of $B\backslash W$.*

Proof Let $h \in B\backslash W$. Since $\mathcal{F}$ is a system of imprimitivity for P, there exists $(\alpha^N)^g \in \mathcal{F}$, $g \in X$, such that $(\alpha^N)^g \cap (B\backslash W) \neq \emptyset$ and we may assume that $h \in \alpha^N$. Since X fixes $B\backslash W$ setwise and is transitive on $B\backslash W$, we have $h^X = B\backslash W$. By this and the fact $|\mathcal{F}| = p = |B\backslash W|$, we get that $\alpha^N \cap (B\backslash W) = \{h\}$. This shows that each element of $B\backslash W$ is in a unique element of $\mathcal{F}$ and each element of $\mathcal{F}$ contains a unique element of $B\backslash W$. □

For each $1 \neq g \in H$, we define B_g to be the set of all elements in B fixed by g.

Lemma 2.6 *For $1 \neq y \in N$, we have $|B_y| = a(q^{n-m} - 1)/s$. Furthermore, $|B_y \cap (B\backslash W)| = (q^{n-m} - 1)/s$*

Proof Since N has p orbits on B, by Burnside's Counting Theorem, we have $p = (ap + sp|B_y|)/q^n$ and then $|B_y| = a(q^{n-m} - 1)/s$. Since N acts on B_y and each orbit of N on B_y has a elements, by Lemma 2.4, we get that $|B_y \cap (B\backslash W)| = (q^{n-m} - 1)/s$ and the lemma is proved. □

Lemma 2.7 $[z, x] \neq 1$.

Proof Assume $[z, x] = 1$. Since $B\backslash W$ is of order p and M invariant, we get that z acts trivially on $B\backslash W$. Since $[z, N] \neq 1$, we get that there is $y \in N$, such that $y^z = y^{-1}$. Since N is regular on $\mathcal{U}$, we may assume that $C^y = C_1$, and then, z^y acts trivially on $B\backslash W_1$. This gives us that $\langle z, z^y\rangle$ acts trivially on $B\backslash(W \cup W_1)$. We note that $y^z = y^{-1}$ which implies that $z^y = y^{-2}z$. This implies that y acts trivially on $B\backslash(W \cup W_1)$. Therefore, $|B_y| \geq |B\backslash(W \cup W_1)|$. We have $|B\backslash(W \cup W_1)| = 2p + \lambda - sp$. Now, by this and Lemma 2.6, we get that $sp \geq ap$, which is a contradiction and the Lemma holds. □

Lemma 2.8 *If $\langle z, x\rangle \cong D_{2p}$, then $s = q - 1$.*

Proof Assume $s \neq q - 1$. Since $sp = q^n - 1$, we get that $s = (q - 1)f$ where $f \neq 1$ and f divides $(q^n - 1)/(q - 1)$. Set $N_1 = C_N(z)$ and let $|N_1| = q^{n_1}$. By Lemma 2.7, we get that $N_1 \neq 1$. Since all p involutions in $\langle z, x\rangle \cong D_{2p}$ are conjugate and by Lemma 2.2(i) $C_N(X) = 1$, we get that $N = N_1 \oplus (N_1)^x \oplus \cdots \oplus (N_1)^{x^{p-1}}$. This implies that $pn_1 = n$.

Let O be an orbit of X on B. Then $|B_z \cap O| = 1$. This tells us that $|B_z| = a = q^m$ and $|B_z \cap (B\backslash W)| = 1$. Now, by Lemma 2.5, we get that B_z is an N orbit. Furthermore, since z inverts each element in $[N, z]$, we get that $[z, N]$ is trivial on B_z. This shows that $n_1 = m$, and then, $pm = n$.

Let R be an orbit of X on $P(N)$. Then, $|R \cap P(N_1)| = 1$. By this and Lemma 2.2(i), we get that $(q^m - 1)/(q - 1) = |P(N_1)| \leq s/(q - 1)$. We note that by Lemma 2.2(i), we conclude that X has $s/(q - 1)$ orbits on $P(N)$ and by our assumption $s \leq a - 1 = q^m - 1$. Hence, $s = a - 1$, and then, $m \geq 2$. Now, we have that $p = (q^{pm} - 1)/(q^m - 1) = q^{(p-1)m} + \cdots + q + 1$ which implies that $q^{2(p-1)} \leq q^{(p-1)m} < p$. However, since p is odd, we have $p < 2(p - 1)$, a contradiction. Now, the Lemma is proved. □

Let y be an element of order q in N and $R_1 = C^{\langle y\rangle}$ be the orbit of $\langle y\rangle$ containing C. If $s = q - 1$, then N is transitive on $\mathcal{U}$ and X is transitive on $P(N)$. Therefore, $\bigcup_{l \in X} R_1^l = \mathcal{U}$ and $R_1^{x^i} \cap R_1^{x^j} = \emptyset$.

Lemma 2.9 $\langle z, x\rangle \cong S_p$.

Proof Assume that $\langle z, x\rangle \cong D_{2p}$. Then, by Lemma 2.8, we have $s = q - 1$ and $x^z = x^{-1}$. By Lemma 2.7, we get that there is $y \in N$ of order q, such that $[z, y] = 1$.

Let $R_1 = C^{\langle y\rangle}$ be the orbit of $\langle y\rangle$ contacting C. Then, z acts trivially on R_1. By coprime action, there is subgroup $Y_1 \leq N$ of order q, such that $Y_1^z = Y_1$ and $y \notin Y_1$. We may assume that $R_2 = R_1^x$ is the orbit of Y_1 on $\mathcal{U}$ containing C. We note that $(R_1^{x^{-1}}) \cap R_2 = \emptyset$. However, z acts on R_2, and since $x^z = x^{-1}$, we should have $R_2^z = R_1^{x^{-1}} = R_2$, a contradiction. This and Lemma 2.7 show that $\langle z, x\rangle \cong S_p$ and the Lemma holds. □

Lemma 2.10 *We have* $|B_y| = a$, $|B_y \cap (B\backslash W)| = 1$ *and* $m = n - 1$.

Proof Set $D = \langle z, x\rangle$. By Lemma 2.9, we have $D \cong S_p$. Therefore, the stabilizer of any element of $P(N)$ in D is isomorphic to S_{p-1}. We note that $D \leq H_C$. Now, let $Y = \langle y\rangle \in P(N)$ and D_1 be its stabilizer in D. Then, D_1 acts on $B_y \cap (B\backslash W)$. We note that D_1 acts also on W and $B\backslash W$. Since $D_1 \leq D \cong S_p$, we get that $|B_y \cap (B\backslash W)| = 1$ or $p - 1$. This gives us that $|B_y| = (p-1)a$ or a. By this and Lemma 2.6, we get that $|B_y| = a$, and then, $n - 1 = m$. Therefore, the lemma holds. □

Let $R_1 = \{C, C_1, \ldots, C_{q-1}\}$ and $W_i = \Gamma(C_i) \cap B$, $i = 1, 2, \ldots, q-1$. By Lemma 2.10, we may assume that $\{\alpha\} = (B\backslash W) \cap B_y$. Then, we have the following lemma.

Lemma 2.11 $\mathcal{U}\backslash R_1 = \Gamma_{\mathcal{B}}(\alpha)$.

Proof Since $\alpha \in B_y$ and $\alpha \notin W$, we get that $R_1 \cap \Gamma_{\mathcal{B}}(\alpha) = \emptyset$. We note that by Lemma 2.10, we have $m = n-1$ which implies that $a = q^{n-1}$. Now, for $i = 1, \ldots, q-1$, we have $|W\backslash(W \cap W_i)| = k - \lambda = ap - p - (p(a-2) + (q^n - a)/as) = p - (q-1)/s = p - 1$. By this and since $\alpha \notin W_i$, we get that $B\backslash(W \cup \{\alpha\}) \subset W_i$. This gives us that $\mathcal{U}\backslash R_1 = \Gamma_{\mathcal{B}}(\alpha)$ and the lemma holds. □

Now, we can proof Theorem 1.2.

Proof By Lemma 2.11, we have $r = q(p-1) = q^n - q^{n-m}$ which is impossible. Therefore, $q = 2$ and Theorem 1.2 follows from the main result in [7]. □

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