



Guaranteeing finite-time H_∞ performance for asynchronous discrete-time delayed switched systems with persistent dwell time structure

Haoliang Zhu¹

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Abstract

The simultaneous consideration of the disturbance attenuation problem and the asynchronous switching problem in the discrete-time linear switched systems with state delay and persistent dwell time structure is an important issue that has not been investigated yet to guarantee the finite-time H_∞ performance of these systems. To tackle this problem, this paper presents an asynchronous finite-time H_∞ control scheme for this class of systems through multiple Lyapunov–Krasovskii functional and persistent dwell time switching signals. To guarantee the finite-time H_∞ performance of the whole system, a series of non-convex conditions together with new conditions for the switching signal have been extracted. Also, the disturbance attenuation level is obtained. To be able to solve the non-convex problem with common software, the linear matrix inequality problem has been chosen. Finally, the proposed scheme has been validated through two numerical simulations and the results show the effectiveness of the proposed scheme.

Keywords Discrete-time delayed switched systems · Asynchronous switching · Finite-time H_∞ performance · Multiple Lyapunov–Krasovskii functional · Persistent dwell time switching signal · Linear matrix inequality

1 Introduction

Switch systems have recently received much attention as a class of hybrid systems because of their theoretical and practical significance. They consist of multiple modes with a sub-system in each mode and a switching signal that activates modes at different times. Switched systems can accurately describe the structural hybrid features of real-world application systems. They include a wide range of applications in power electronics, chemical processes, networked control, flight control, etc. Also, changing the parameters of a system, sudden changes in the environment, occurring faults, etc. provide the conditions for using the switched models for different systems (Sun and Ge 2011; Yang et al. 2014; Hespanha and Morse 1999). In addition to the dynamics assigned for subsystems, the features of the switching signals are effective in the closed-loop stability of the switched system (Lin and Antsaklis 2009). So, some of the important

Lyapunov functions including switched Lyapunov function, multiple Lyapunov functions, and common Lyapunov functions are usually used to ensure the stability of the system under the arbitrary, dwell time, average dwell time, and persistent dwell time switching signals (Lin and Antsaklis 2009; Shi et al. 2019).

In some past research studies, the stability of discrete-time delay switched systems has been analyzed without considering the uncertainties (Park et al. 2017). In some other works, conditions have been extracted for robust stability in the presence of uncertainties (Baleghi and Shafiei 2018a; Kermani and Sakly 2018). Moreover, considering the exogenous disturbances, H_∞ stability has been ensured in Wang et al. (2012), Tian et al. (2015) and Dong et al. 2019). The stabilization problem for discrete-time delayed switched systems has been investigated in many studies. Reference Zhang et al. (2007) presents a robust stabilization for the delayed switched systems in the presence of exogenous disturbances. Considering the polytopic uncertainties together with exogenous disturbances, H_∞ performance has been proven in Zhang et al. (2010). Using the switched Lyapunov function, the robust stabilization of the delayed switched systems

✉ Haoliang Zhu
zh115994338642@163.com

¹ Intelligent Manufacturing College, Nanning University, Nanning 541699, Guangxi, China

considering the norm-bounded uncertainties has been investigated in Du et al. (2008) and Zhang et al. (2008). Also, for the nonlinear delayed switched systems considering the parametric and norm-bounded uncertainties, robust stabilization has been proved in Xiang and Wang (2011) and Baleghi and Shafiei (2018b). The Lyapunov stability procedure considers the infinite-time response and does not focus on the transient response (Khalil 2014). However, for some cases with a large number of state variables, this fact does not work (Yan and Wu 2019; Zhang et al. 2015). In such systems, the stability analysis through the Lyapunov theory is not suitable (Amato et al. 2001). To tackle this problem, a definition of finite-time stability (FTS) (Weiss and Infante 1967) for nominal systems has been presented. Moreover, for systems considering exogenous disturbances, finite-time boundedness (FTB) stability has been presented in Amato et al. (2001) and Shen et al. (2015). In the FTB, we are ensured that the variables of the system do not exceed a prescribed bound considering the exogenous disturbances of finite duration. In many recent studies, delayed stochastic switched systems, singular switched systems, neutral switched systems, constrained switched systems, and other types of switched systems (Chen et al. 2016; Xiang et al. 2012a, b; Thanh et al. 2017; Lin et al. 2017; Gholami and Shafiei 2021; Wang et al. 2016, 2021; Zong et al. 2013; Cui et al. 2022; Mao et al. 2021; Mao et al. 2022) have been investigated for their responses in finite-time intervals.

In practical systems, not all switches are necessarily ideal and do not occur as expected. In other words, we might expect to have switched to a certain controller but still remain in the previous state. Therefore, we can encounter asynchronous switching, which may have consequences such as system instability. Also, the existence of exogenous disturbances reduces the control performance. To the best of our knowledge to date, the development of a finite-time control scheme to simultaneously encounter asynchronous switching and the effect of exogenous disturbances in delayed switched systems has not been investigated. Therefore, guaranteeing H_∞ finite-time performance in the switched systems with asynchronous and delay structures is considered the first motivation of this paper. In previous works extended by multiple Lyapunov–Krasovskii functional (MLKF), the stability boundary condition imposes a dwell-time limitation on the switching signal. Therefore, the second motivation is to simultaneously design the finite-time control scheme and the switching signal with the PDT structure to reduce the dwell-time limitation compared to other structures. According to these research motivations, for the first time, we intend to develop a simultaneous design of H_∞ finite-time scheme and PDT switching signal using MLKF for the delayed linear switched systems subjected to asynchronous switching and exogenous disturbances. The challenges and difficulties in

the proposed simultaneous design using MLKF for delayed switched systems are as follows:

- The Lyapunov–Krasovskii stability condition must be converted to an LMI Lyapunov–Krasovskii stability constraint. This conversion is complicated despite the state delay, exogenous disturbance and asynchronous switching terms. This problem has been solved using appropriate lemma and change of variables.
- The design of PDT switching signal guaranteeing finite-time bounded despite the exogenous disturbance and asynchronous switching terms is complicated. This problem is solved by specific design of PDT switching signal compared to other PDT designs.
- The computation of disturbance attenuation level guaranteeing non-weighted H_∞ finite-time performance despite the exogenous disturbance and asynchronous switching terms is complicated. This problem is solved by specific design of PDT switching signal and new assumptions on PDT structure.

The remainder of this essay is divided into the following categories. The topic is presented in Sect. 2, along with a number of lemmas and presumptions. Section 3 provides the results of the recommended scheme. The performance of the suggested scheme is supported by two numerical examples in Sect. 4, and the conclusion is offered in Sect. 5.

2 Problem formulation

Consider a discrete-time linear delayed switched system as follows:

$$x(d+1) = M_{\sigma(d)}x(d) + D_{\sigma(d)}x(d-h) + B_{\sigma(d)}u(d) + N_{\sigma(d)}\Delta(d) \quad (1)$$

$$x(r) = \theta(r), \quad r = d_0 - h, \dots, d_0 \quad (2)$$

$$z(d) = O_{\sigma(d)}x(d) + T_{\sigma(d)}\Delta(d) \quad (3)$$

where $x(d) \in R^{n_x}$ and $u(d) \in R^{n_u}$ are the state and the control input, respectively. The switching signal is revealed by $\sigma(d) : [0, \infty) \rightarrow M = \{1, 2, \dots, m\}$. h and $\theta(r)$ represents the state delay and function of initial conditions. d is an unknown constant state delay that has a known upper bound ($h \leq h_M$). $M_{\sigma(d)}$, $D_{\sigma(d)}$, $B_{\sigma(d)}$, $O_{\sigma(d)}$, $T_{\sigma(d)}$ and $N_{\sigma(d)}$ are known real matrices that have appropriate dimensions. It is noted that all subsystems are stabilizable and have a common equilibrium point. Also, there is a discrepancy d_{syn} between switching of controllers and the switching of subsystems. It is assumed that the upper bound of the mismatched intervals

$(d_{\text{syn}} \leq d_{\text{syn},m})$ is known. Hence, the state feedback control law is considered as follows:

$$u(d) = K_{\sigma(d-d_{\text{syn}})}x(d) \tag{4}$$

Therefore, the switching system (1) is recast for subsystem i in both the matched and mismatched periods:

$$x(d+1) = (M_i + B_i K_i)x(d) + D_i x(d-h) + N_i \Delta(d), \quad d \in T^\downarrow \tag{5}$$

$$x(d+1) = (M_j + B_j K_j)x(d) + D_j x(d-h) + N_j \Delta(d), \quad d \in T^\uparrow \tag{6}$$

where T^\downarrow and T^\uparrow denote matched intervals and mismatched intervals, respectively.

Assumption 1 For a given constant d_I , the exogenous disturbance $\Delta(d)$ is time-varying that satisfies the constraint:

$$\sum_{k=0}^{Nd_I} \Delta(d)^T \Delta(d) \leq ub, \quad ub > 0 \tag{7}$$

The calculation of the state feedback control law for the linear delayed switched system described in (1)–(3) with the aim of guaranteeing finite time H_∞ performance is the key subject of this paper. Hence, the following lemmas and definitions are needed.

Lemma 1 (Aminsafae and Shafiei 2019) The following inequalities are equivalent (Schur complement lemma):

$$\begin{pmatrix} V & W \\ W^T & X \end{pmatrix} > 0 \equiv \begin{cases} X > 0, & V - WX^{-1}W^T > 0 \\ V > 0, & X - W^T V^{-1}W > 0 \end{cases} \tag{8}$$

where $A = A^T$ and $C = C^T$ are nonsingular matrices.

Definition 1 (Finite time bounded) (Shi et al. 2018): if, for a given matrix $R > 0$, two positive constants k_1, k_2 , with $k_1 < k_2$, a positive integer d_I , and a switching signal $\sigma(d)$, we have: $x(0)^T R x(0) < k_1 \Rightarrow x(d)^T R x(d) < k_2, \forall r = 1, \dots, d_I$, then the discrete-time linear switched system (1) with $u(d) \equiv 0$ subject to an exogenous disturbance $\Delta(d)$ is

described as finite time bonded with regard to $(k_1, k_2, R, d_I, ub, \sigma)$.

Definition 2 (Finite time H_∞ performance) (Shi et al. 2018): if, for a given matrix $R > 0$, two positive constants k_1, k_2 , with $k_1 < k_2$, a positive integer d_I , and a switching signal $\sigma(d)$, we have: $x(0)^T R x(0) = 0 \Rightarrow x(d)^T R x(d) < k_2 \quad \forall r = 1, \dots, d_I$ and:

$$\sum_{d=0}^{d_I} z^T(d)z(d) \leq \gamma^2 \sum_{d=0}^{d_I} \Delta(d)^T \Delta(d) \tag{9}$$

then the discrete-time linear switched system (1) with $u(d) \equiv 0$ subject to an exogenous disturbance $\Delta(d)$ is said to have finite-time H_∞ performance with regard to $(0, k_2, R, d_I, ub, \sigma)$.

Definition 3 (PDT switching signal) (Fan et al. 2020): the PDT structure divides time interval into a number of stages. Each stage is broken down into a d_τ -portion and a d_T -portion. A designated sub-system is activated in the d_τ -portion with the running time at least d_τ . The total switching number in the d_T -portion satisfies the equation $N(d_{s_p+1}, d_{s_p+1}) \leq f d_T$, where f is the maximum switching rate, d_τ is persistent dwell-times, and d_T is persistence period.

3 Main results

For the closed-loop switched system described in (1)–(3), this section gives the necessary criteria to ensure the finite-time H_∞ performance using MLKF and PDT switching signals. The following theorem defines the critical findings of this study.

Theorem 1 Take into account the state feedback control rule and the discrete-time Lipschitz linear delayed switching system in (1)–(3) under Assumptions 1. In addition, constants for $\xi_s < 0, \xi_{as} > 0, \mu > 1$ and $\gamma > 0$ are provided. T, f , and $d_{\text{syn},m}$ are also chosen as the PDT structure’s parameters. If for each subsystem i , there exists matrices $S_i > 0, Q > 0$, and K_i such that the following inequalities be feasible:

$$\begin{pmatrix} -\bar{M}_i^T S_i \bar{M}_i + (1 + \xi_s) S_i - Q - O_i^T O_i & -\bar{M}_i^T S_i D_i & -\bar{M}_i^T S_i N_i - O_i^T T_i & 0 & \cdots & 0 \\ * & -D_i^T S_i D_i + Q & -D_i^T S_i N_i & 0 & \cdots & 0 \\ * & * & -N_i^T S_i N_i + \gamma^2 I - T_i^T T_i & 0 & \cdots & 0 \\ * & * & * & \xi_s Q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & \xi_s Q \end{pmatrix} \geq 0 \tag{10}$$

$$\begin{pmatrix} -\bar{M}_j^T S_i \bar{M}_j + (1 + \xi_{as}) S_i - Q - O_j^T O_j & -\bar{M}_j^T S_i M_{dj} & -\bar{M}_j^T S_i N_j - O_j^T T_j & 0 & \cdots & 0 \\ * & -D_j^T S_i D_j + Q & -D_j^T S_i E_j & 0 & \cdots & 0 \\ * & * & -N_j^T S_i N_j + \gamma^2 I - T_j^T T_j & 0 & \cdots & 0 \\ * & * & * & \xi_{as} Q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & \xi_{as} Q \end{pmatrix} \geq 0, \tag{11}$$

$i \neq j, \quad i, j \in M$

$$\mu S_j - S_i \geq 0 \tag{12}$$

mismatched intervals, and the stability boundary condition are considered as follows:

where

$$\bar{M}_i = M_i + B_i K_i, \quad \bar{M}_{i,j} = M_j + B_j K_i \tag{13}$$

and the PDT switching signal satisfies:

$$d_\tau > d_\tau^* = \max \left(\frac{1}{f}, -d_T + \frac{d_I((d_T f + 1) \ln \mu + (d_T f + 1) d_{syn,m} \ln(1 + \xi_{as}) - (d_T f + 1) d_{syn,m} \ln(1 + \xi_s))}{\ln k_2 \kappa_2 - \ln \gamma^2 ub - (d_T f + 1) \ln \mu - (d_T f + 1) d_{syn,m} \ln(1 + \xi_{as}) + (d_T f + 1) d_{syn,m} \ln(1 + \xi_s) - d_I \ln(1 + \xi_s)} \right) \tag{14}$$

then for the state feedback control law $u_i(d) = K_i x(d)$, the closed-loop switched system described in (1)–(3) has finite-time H_∞ performance with respect to $(0, k_2, R, d_I, ub, \tilde{\gamma}, \sigma)$ against asynchronous switching where $\tilde{\gamma} = \gamma \sqrt{\mu^{\frac{d_I}{d_T+d_\tau}+1} (d_T f + 1) (1 + \xi_s)^{d_I} (\frac{1+\xi_{as}}{1+\xi_s})^{\frac{d_I}{d_T+d_\tau}+1} (d_T f + 1) d_{syn,m}}$, and $\kappa_2 = \min_{i \in M} (\lambda_{\min}(\bar{S}_i))$.

Proof The proof of the theory is carried out in three steps.
Step 1: The candidate Lyapunov–Krasovskii functional, the Lyapunov stability condition in matched intervals and

$$\begin{aligned} v_{\sigma(d)}(d) &= \sum_{i=1}^2 v_i(d), \quad v_1(d) = x^T(d) S_{\sigma(d)} x(d), \\ v_2(d) &= \sum_{i=d-h}^{d-1} x^T(i) Q x(i) \tag{15} \\ v_{\sigma(d+1)}(d+1) - v_{\sigma(d)}(d) &\leq \xi_s v_{\sigma(d)}(d) - z^T(d) z(d) + \gamma^2 \Delta(d)^T \Delta(d), \quad d \in T^\downarrow \tag{16} \end{aligned}$$

$$\begin{aligned} v_{\sigma(d+1)}(d+1) - v_{\sigma(d)}(d) &\leq \xi_{as} v_{\sigma(d)}(d) - z^T(d) z(d) + \gamma^2 \Delta(d)^T \Delta(d), \quad d \in T^\uparrow \tag{17} \end{aligned}$$

$$v_{\sigma(d)}(x(d)) \leq \mu v_{\sigma(d-1)}(x(d)) \tag{18}$$

1. Proof of finite-time bounded respected to $(0, c_2, R, d_l, ub, \gamma, \sigma)$

Let $d_{s_{r+1}}$ and d_{s_r} stand for the switching instants that come following d_{s_r} in the r -th stage and the $(r + 1)$ th stage, respectively. Think about $\sigma(d_{s_{r+1}}) = n, \sigma(d_{s_r}) = m$, and $\xi = \begin{cases} \xi_s & d \in T^\downarrow \\ \xi_{as} & d \in T^\uparrow \end{cases}$. The fluctuation of the Lyapunov function between two successive samples in the r -th stage is derived as follows thanks to the PDT structure illustrated in Fig. 1 and combining (16)–(18):

$$\begin{aligned} v_n(x(d_{s_{r+1}})) &\leq \mu v_{\sigma(d_{s_{r+1}}-1)}(x(d_{s_{r+1}})) \\ &\leq \mu(1 + \xi) v_{\sigma(d_{s_{r+1}}-1)}(x(d_{s_{r+1}} - 1)) \\ &\quad + \mu\gamma^2 \Delta(d_{s_{r+1}} - 1)^T \Delta(d_{s_{r+1}} - 1) \end{aligned} \tag{19}$$

According to (19), the difference in the Lyapunov function between two successive switches at the r -th stage is calculated as:

$$\begin{aligned} v_n(x(d_{s_{r+1}})) &\leq \mu(1 + \xi)^{d_{s_{r+1}} - d_{s_{r+1}} - 1} v_{\sigma(d_{s_{r+1}}-1)}(x(d_{s_{r+1}} - 1)) \\ &\quad + \gamma^2 \Delta(d_{s_{r+1}} - 1)^T \Delta(d_{s_{r+1}} - 1) \\ &\quad + \mu(1 + \xi)\gamma^2 \Delta(d_{s_{r+1}} - 2)^T \Delta(d_{s_{r+1}} - 2) \\ &\quad + \dots + \mu(1 + \xi)^{d_{s_{r+1}} - d_{s_{r+1}} - 1} \gamma^2 \Delta(d_{s_{r+1}} - 1)^T \Delta(d_{s_{r+1}} - 1) \end{aligned} \tag{20}$$

Consider the total number of switching, the total running time of the matched intervals and the total running time of the mismatched intervals in a given interval as N, T_s and T_{as} , respectively. The fluctuation of the Lyapunov function in the whole r -th stage total interval is derived as per Definition 3 and utilizing (20):

$$\begin{aligned} v_n(x(d_{s_{r+1}})) &\leq \mu^{N(d_{s_r+1}, d_{s_{r+1}})}(1 + \xi)^{d_{s_{r+1}} - d_{s_r+1}} v_{\sigma(d_{s_{r+1}})}(x(d_{s_{r+1}})) \\ &\quad + \sum_{e=d_{s_p}}^{d_{s_{r+1}}} \mu^{N(e, d_{s_{r+1}})}(1 + \xi)^{d_{s_{r+1}} - e} \gamma^2 \Delta(e)^T \Delta(e) \\ &= \mu^{N(d_{s_r+1}, d_{s_{r+1}})}(1 + \xi_{as})^{T_{as}(d_{s_r+1}, d_{s_{r+1}})} \\ &\quad \times (1 + \xi_s)^{T_s(d_{s_p+1}, d_{s_{r+1}})} v_{\sigma(d_{s_p+1})}(x(d_{s_r+1})) \\ &\quad + \sum_{e=d_{s_p}}^{d_{s_{p+1}}} \mu^{N(e, d_{s_{r+1}})}(1 + \xi)^{d_{s_{r+1}} - e} \gamma^2 \Delta(e)^T \Delta(e) \\ &\leq \mu^{N(d_{s_r}, d_{s_{r+1}})}(1 + \xi_{as})^{T_{as}(d_{s_r}, d_{s_{r+1}})}(1 + \xi_s)^{T_s(d_{s_r}, d_{s_{r+1}})} \end{aligned}$$

$$\begin{aligned} &\times v_m(x(d_{s_r})) + \sum_{e=d_{s_p}}^{d_{s_{p+1}}} \mu^{N(e, d_{s_{r+1}})}(1 + \xi)^{d_{s_{r+1}} - e} \gamma^2 \Delta(e)^T \Delta(e) \\ &\Rightarrow v_n(x(d_{s_{p+1}})) \\ &\leq e^{N(d_{s_r}, d_{s_{r+1}}) \ln \mu + T_{as}(d_{s_r}, d_{s_{r+1}}) \ln(1 + \xi_{as}) + T_s(d_{s_r}, d_{s_{r+1}}) \ln(1 + \xi_s)} \\ &\quad \times v_m(x(d_{s_r})) \\ &\quad + \sum_{e=d_{s_p}}^{d_{s_{p+1}}} \mu^{N(e, d_{s_{r+1}})}(1 + \xi)^{d_{s_{r+1}} - e} \gamma^2 \Delta(e)^T \Delta(e) \end{aligned} \tag{21}$$

Based on (21), it is concluded that following inequality holds in interval $[d_0, d]$:

$$\begin{aligned} v_{\sigma(d)}(x(d)) &\leq e^{N(d_0, d) \ln \mu + T_{as}(d_0, d) \ln(1 + \xi_{as}) + T_s(d_0, d) \ln(1 + \xi_s)} \\ &\quad v_{\sigma(d_0)}(x(d_0)) \\ &\quad + \sum_{e=d_0}^d \mu^{N(e, d)}(1 + \xi)^{d - e} \gamma^2 \Delta(e)^T \Delta(e) \\ &\Rightarrow v_{\sigma(d)}(x(d)) \leq e^{N(d_0, d) \ln \mu + T_{as}(d_0, d) \ln(1 + \xi_{as}) + T_s(d_0, d) \ln(1 + \xi_s)} \\ &\quad v_{\sigma(d_0)}(x(d_0)) + \mu^{N(d_0, d)}(1 + \xi)^{d - d_0} \sum_{t=k_0}^k \gamma^2 \Delta(e)^T \Delta(e) \\ &\Rightarrow v_{\sigma(d)}(x(d)) \leq e^{N(d_0, d) \ln \mu + T_{as}(d_0, d) \ln(1 + \xi_{as}) + T_s(d_0, d) \ln(1 + \xi_s)} \\ &\quad \times (v_{\sigma(d_0)}(x(d_0)) + \sum_{e=d_0}^d \gamma^2 \Delta(e)^T \Delta(e)) \\ &\Rightarrow v_{\sigma(d)}(x(k)) \leq e^{N(d_0, d) \ln \mu + T_{as}(d_0, d) \ln(1 + \xi_{as}) + T_s(d_0, d) \ln(1 + \xi_s)} \\ &\quad (v_{\sigma(d_0)}(x(d_0)) + \gamma^2 ub) \end{aligned} \tag{22}$$

d_{T_r} and d_{τ_r} are the running times in the—portion and the T -portion, respectively, and together they make up the overall running time in the r -th stage. Since $d_{T_r} \leq d_T$ and $d_{\tau_r} \geq d_\tau$ are true, and supposing $d_\tau f > 1$, it is simple to arrive to $(d_\tau f - 1)(d_{T_r} - d_T) \leq 0$. The following results are obtained when the interval $[d_0, d]$ is in the r -th stage:

$$\left(\frac{d - d_0}{d_{\tau_r} + d_{T_r}} + 1\right)(d_{T_r} f + 1) \leq \left(\frac{d - d_0}{d_T + d_\tau} + 1\right)(d_T f + 1) \tag{23}$$

Additionally, (23) is valid for additional stages. One may write by adding up the steps between $[d_0, d]$:

$$\begin{aligned} N(d_0, d) &\leq \left(\frac{d - d_0}{d_T + d_\tau} + 1\right)(d_T f + 1), T_{as}(d_0, d) \\ &\leq \left(\frac{d - d_0}{d_T + d_\tau} + 1\right)(d_T f + 1) d_{syn, m} \end{aligned} \tag{24}$$

Substituting (24) into (22), we have:

$$\begin{aligned}
& v_{\sigma(d)}(x(d)) \\
& \leq e^{\left(\frac{d-d_0}{d_T+d_\tau}+1\right)(d_T f+1) \ln \mu + \left(\frac{d-d_0}{d_T+d_\tau}+1\right)(d_T f+1) d_{\text{syn}, m} \ln(1+\xi_{\text{as}}) + \left(d-d_0 - \left(\frac{d-d_0}{d_T+d_\tau}+1\right)(d_T f+1) d_{\text{syn}, m}\right) \ln(1+\xi_s)} (v_{\sigma(d_0)}(x(d_0)) + \gamma^2 ub) \\
& \Rightarrow v_{\sigma(d)}(x(d)) \\
& \leq e^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1) \ln \mu + \left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1) d_{\text{syn}, m} \ln(1+\xi_{\text{as}}) + \left(d_I - \left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1) d_{\text{syn}, m}\right) \ln(1+\xi_s)} (v_{\sigma(d_0)}(x(d_0)) + \gamma^2 ub) \\
& = e^{\left(\frac{d_I}{d_T+d_\tau}+1\right)((d_T f+1) \ln \mu + (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_{\text{as}}) - (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_s)) + d_I \ln(1+\xi_s)} (v_{\sigma(d_0)}(x(d_0)) + \gamma^2 ub) \tag{25}
\end{aligned}$$

Considering $\bar{S}_i = R^{-\frac{1}{2}} S_i R^{-\frac{1}{2}}$, we get:

$$\begin{aligned}
v_{\sigma(d_0)}(x(d_0)) &= x(d_0)^T S_{\sigma(d_0)} x(d_0) \\
&= x(d_0)^T R^{\frac{1}{2}} \bar{S}_{\sigma(d_0)} R^{\frac{1}{2}} x(d_0) \\
&\leq \max(\lambda_{\max}(\bar{S}_{\sigma(d_0)})) x(d_0)^T R x(d_0) \\
&\leq c_1 k_1 \tag{26}
\end{aligned}$$

$$\begin{aligned}
v_{\sigma(d)}(x(d)) &= x(d)^T S_{\sigma(d)} x(d) \\
&= x(d)^T R^{\frac{1}{2}} \bar{S}_{\sigma(d)} R^{\frac{1}{2}} x(d) \\
&\geq \min(\lambda_{\min}(\bar{S}_{\sigma(d)})) x(d)^T R x(d) \\
&\geq \kappa_2 x(d)^T R x(d) \tag{27}
\end{aligned}$$

Due to (26)–(27) and substituting (14) into (25), we get:

$$\begin{aligned}
& \kappa_2 x(d)^T R x(d) \\
& \leq v_{\sigma(d)}(x(d)) \Rightarrow x(d)^T R x(d) \\
& \leq \frac{1}{\kappa_2} v_{\sigma(d)}(x(d)) \\
& \leq \frac{1}{\kappa_2} e^{\left(\frac{d_I}{d_T+d_\tau}+1\right)((d_T f+1) \ln \mu + (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_{\text{as}}) - (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_s)) + d_I \ln(1+\xi_s)} \\
& \quad \times (v_{\sigma(d_0)}(x(d_0)) + \gamma^2 ub) \\
& = \frac{1}{\kappa_2} e^{\left(\frac{d_I}{d_T+d_\tau}+1\right)((d_T f+1) \ln \mu + (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_{\text{as}}) - (d_T f+1) d_{\text{syn}, m} \ln(1+\xi_s)) + d_I \ln(1+\xi_s)} \\
& \quad \times (k_1 \kappa_1 + \gamma^2 ub) \\
& \leq \frac{1}{\kappa_2} e^{\ln k_2 \kappa_2 - \ln \gamma^2 ub} (\gamma^2 ub) = k_2 \tag{28}
\end{aligned}$$

2. Proof of inequality (9)

Based on (22), the variation of Lyapunov function in interval $[k_0, k]$ is as follows:

$$\begin{aligned}
v_{\sigma(d)}(x(d)) &\leq e^{N(d_0, d) \ln \mu + T_{\text{as}}(d_0, d) \ln(1+\xi_{\text{as}}) + T_s(d_0, d) \ln(1+\xi_s)} v_{\sigma(d_0)}(x(d_0)) \\
&\quad + \sum_{e=d_0}^d \mu^{N(e, d)} (1+\xi)^{d-e} (-z(e)^T z(e) + \gamma^2 \Delta(e)^T \Delta(e)) \tag{29}
\end{aligned}$$

Due to $v_{\sigma(d_0)}(x(d_0)) = 0$ and $v_{\sigma(d)}(x(d)) \geq 0$, we have:

$$\begin{aligned} \Delta v_1 &= (\overline{M}_{\sigma(d)}x(d) + D_{\sigma(d)}x(d - h) + N_{\sigma(d)}\Delta(d))^T S_{\sigma(d)}(\overline{M}_{\sigma(d)}x(d) + D_{\sigma(d)}x(d - h) + N_{\sigma(d)}\Delta(d)) - x^T(d)S_{\sigma(k)}x(d) \\ &= x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}\overline{M}_{\sigma(d)}x(d) + x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h) + x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}N_{\sigma(d)}\Delta(d) \\ &\quad + x^T(d - h)D_{\sigma(d)}^T S_{\sigma(d)}\overline{M}_{\sigma(d)}x(d) + x^T(d - h)D_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h)S_{\sigma(d)}N_{\sigma(d)}\Delta(d) + \Delta(d)^T N_{\sigma(d)}^T \\ &\quad \times S_{\sigma(d)}\overline{M}_{\sigma(d)}x(d) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)}N_{\sigma(d)}\Delta(d) - x^T(d)S_{\sigma(d)}x(d) \end{aligned} \tag{33}$$

$$\begin{aligned} \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi)^{d-e}(-z(e)^T z(e) + \gamma^2 \Delta(e)^T \Delta(e)) &\geq 0 \\ \Rightarrow \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi)^{d-e}z(e)^T z(e) & \\ \leq \gamma^2 \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi)^{d-e} \Delta(e)^T \Delta(e) & \\ \Rightarrow \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi_s)^{T_s(d_0,d)}(1 + \xi_{as})^{T_{as}(d_0,d)}z(e)^T z(e) & \\ \leq \gamma^2 \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi_s)^{T_s(d_0,d)}(1 + \xi_{as})^{T_{as}(d_0,d)} \Delta(e)^T \Delta(e) & \\ \Rightarrow \sum_{e=d_0}^d z(e)^T z(e) & \\ \leq \gamma^2 \sum_{e=d_0}^d \mu^{N(e,d)}(1 + \xi_s)^{d-e} \left(\frac{1 + \xi_{as}}{1 + \xi_s} \right)^{T_{as}(d_0,d)} \Delta(e)^T \Delta(e) \end{aligned} \tag{30}$$

Substituting (24) in (30), we have:

$$\begin{aligned} \sum_{e=d_0}^d z(e)^T z(e) &\leq \gamma^2 \sum_{e=d_0}^d \mu^{\left(\frac{d-e}{d_T+d_\tau}+1\right)(d_T f+1)}(1 + \xi_s)^{d-e} \left(\frac{1 + \xi_{as}}{1 + \xi_s} \right)^{\left(\frac{d-e}{d_T+d_\tau}+1\right)(d_T f+1)d_{syn,m}} \Delta(e)^T \Delta(e) \\ &\Rightarrow \sum_{e=d_0}^d z(e)^T z(e) \leq \gamma^2 \sum_{e=d_0}^d \mu^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)}(1 + \xi_s)^{d_I} \left(\frac{1 + \xi_{as}}{1 + \xi_s} \right)^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)d_{syn,m}} \Delta(e)^T \Delta(e) \\ &\Rightarrow \sum_{e=d_0}^d z(e)^T z(e) \leq \gamma^2 \mu^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)}(1 + \xi_s)^{d_I} \left(\frac{1 + \xi_{as}}{1 + \xi_s} \right)^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)d_{syn,m}} \sum_{e=d_0}^d \Delta(e)^T \Delta(e) \\ &\Rightarrow \sum_{e=d_0}^d z(e)^T z(e) \leq \tilde{\gamma}^2 \sum_{t=d_0}^d \Delta(e)^T \Delta(e) \end{aligned} \tag{31}$$

where

$$\tilde{\gamma} = \gamma \sqrt{\mu^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)}(1 + \xi_s)^{d_I} \left(\frac{1 + \xi_{as}}{1 + \xi_s} \right)^{\left(\frac{d_I}{d_T+d_\tau}+1\right)(d_T f+1)d_{syn,m}}} \tag{32}$$

Step 2: The inequalities (16)–(17) are formulated as non-convex matrix inequality. Δv_1 is extended along the trajectories of the closed-loop system:

By extending Δv_2 as follows:

$$\Delta v_2 = x(d)^T Qx(d) - x(d - h)^T Qx(d - h) \tag{34}$$

Substituting (33) and (34) into (16):

$$\begin{aligned} &x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}\overline{M}_{\sigma(d)}x(d) + x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h) \\ &\quad + x^T(d)\overline{M}_{\sigma(d)}^T S_{\sigma(d)}N_{\sigma(d)}\Delta(d) + x^T(d - h)D_{\sigma(d)}^T S_{\sigma(d)}\overline{M}_{\sigma(d)}x(d) \\ &\quad + x^T(d - h)D_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h)S_{\sigma(d)}N_{\sigma(d)}\Delta(d) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)} \\ &\quad \times \overline{M}_{\sigma(d)}x(d) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)}D_{\sigma(d)}x(d - h) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)} \\ &\quad S_{\sigma(d)}N_{\sigma(d)}\Delta(d) - x^T(d)S_{\sigma(k)}x(d) + x(d)^T Q_{\sigma(d)}x(d) \\ &\quad - x(d - h)^T Q_{\sigma(d)}x(d - h) + x(d)^T O_{\sigma(d)}^T O_{\sigma(d)}x(d) \\ &\quad + \Delta(d)^T T_{\sigma(d)}^T T_{\sigma(d)}\Delta(d) + x(d)^T O_{\sigma(d)}^T T_{\sigma(d)}\Delta(d) \\ &\quad + \Delta(d)^T T_{\sigma(d)}^T O_{\sigma(d)}x(d) - \gamma^2 \Delta^T(d)\Delta(d) - \xi_s x(d)^T \\ &\quad S_{\sigma(d)}x(d) - \xi_s \sum_{i=d-h}^{d-1} x(i)Q_{\sigma(k)}x(i) \leq 0 \end{aligned} \tag{35}$$

According to the upper bound of the delay, if (36) holds, then (35) holds:

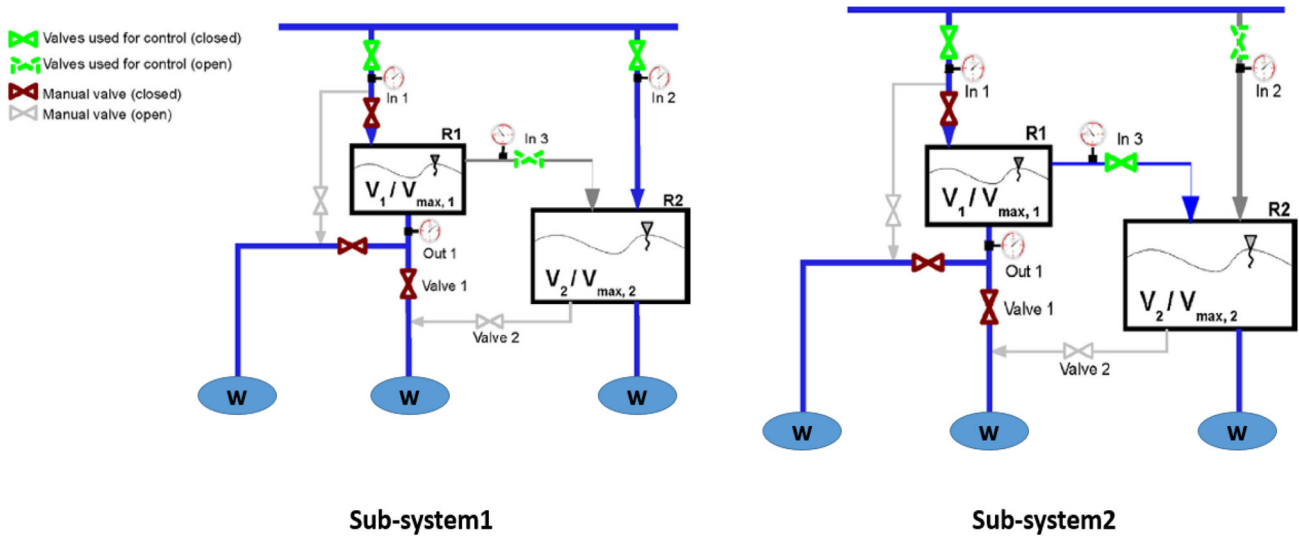


Fig. 1 Two sub-systems for activation of the drinking water supply network (DWSN)

$$\begin{aligned}
 & x^T(d) \bar{M}_{\sigma(d)}^T S_{\sigma(d)} \bar{M}_{\sigma(d)} x(d) + x^T(d) \bar{M}_{\sigma(d)}^T S_{\sigma(d)} D_{\sigma(d)} x(d-h) + x^T(d) \bar{M}_{\sigma(d)}^T \\
 & S_{\sigma(d)} N_{\sigma(d)} \Delta(d) + x^T(d-h) D_{\sigma(d)}^T S_{\sigma(d)} \bar{M}_{\sigma(d)} x(d) + x^T(d-h) D_{\sigma(d)}^T S_{\sigma(d)} D_{\sigma(d)} x(d-h) \\
 & + x^T(d-h) M_{d\sigma(d)}^T S_{\sigma(d)} N_{\sigma(d)} \Delta(d) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)} \bar{M}_{\sigma(d)} x(d) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)} M_{d\sigma(d)} \\
 & x(d-h) + \Delta(d)^T N_{\sigma(d)}^T S_{\sigma(d)} N_{\sigma(d)} \Delta(d) - x^T(d) S_{\sigma(k)} x(d) + x(d)^T Q_{\sigma(d)} x(d) \\
 & - x(d-h)^T Q_{\sigma(d)} x(d-h) + x(d)^T O_{\sigma(d)}^T O_{\sigma(d)} x(d) + \Delta(d)^T T_{\sigma(d)}^T T_{\sigma(d)} \Delta(d) \\
 & + x(d)^T O_{\sigma(d)}^T T_{\sigma(d)} \Delta(d) + \Delta(d)^T T_{\sigma(d)}^T O_{\sigma(d)} x(d) - \gamma^2 \Delta^T(d) \Delta(d) - \xi_s x(d)^T S_{\sigma(d)} x(d) \\
 & - \xi_s \sum_{i=d-h_M}^{d-1} x(i) Q_{\sigma(k)} x(i) \leq 0
 \end{aligned} \tag{36}$$

Considering $\sigma(d) = i$, (36) is rewritten as:

$$\begin{pmatrix} x(d) \\ x(d-h) \\ \Delta(d) \\ x(d-1) \\ \vdots \\ x(d-h_M) \end{pmatrix}^T \begin{pmatrix} -\bar{M}_i^T S_i \bar{M}_i + (1 + \xi_s) S_i - Q - O_i^T O_i & -\bar{M}_i^T S_i D_i & -\bar{M}_i^T S_i N_i - O_i^T T_i & 0 & \dots & 0 \\ * & -D_i^T S_i D_i + Q & -D_i^T S_i N_i & 0 & \dots & 0 \\ * & * & -N_i^T S_i N_i + \gamma^2 I - T_i^T T_i & 0 & \dots & 0 \\ * & * & * & \xi_s Q & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \dots & \xi_s Q \end{pmatrix} \begin{pmatrix} x(d) \\ x(d-h) \\ \Delta(d) \\ x(d-1) \\ \vdots \\ x(d-h_M) \end{pmatrix} \geq 0 \tag{37}$$

inequality (37) holds, if following inequality satisfy:

It is concluded that if (40) holds, (39) holds:

$$\begin{pmatrix} -\bar{M}_i^T S_i \bar{M}_i + (1 + \xi_s) S_i - Q - O_i^T O_i & -\bar{M}_i^T S_i D_i & -\bar{M}_i^T S_i N_i - O_i^T T_i & 0 & \cdots & 0 \\ * & -D_i^T S_i D_i + Q & -D_i^T S_i N_i & 0 & \cdots & 0 \\ * & * & -N_i^T S_i N_i + \gamma^2 I - T_i^T T_i & 0 & \cdots & 0 \\ * & * & * & \xi_s Q & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & \cdots & \xi_s Q \end{pmatrix} \geq 0 \tag{38}$$

Remark 1 The inequality (11) is also derived by (17), like the earlier step.

Step 3: The inequalities (18) are formulated as a non-convex matrix inequality. According to (18), we have

$$\begin{aligned} x(d)^T S_{\sigma(d)} x(d) + \sum_{i=d-h}^{d-1} x(i) Q x(i) \\ \leq \mu x(d)^T S_{\sigma(d-1)} x(d) + \mu \sum_{i=d-h}^{d-1} x(i) Q x(i) \end{aligned} \tag{39}$$

$$x(d)^T S_{\sigma(d)} x(d) \leq \mu x(d)^T S_{\sigma(d-1)} x(d) \Rightarrow S_{\sigma(d)} \leq \mu S_{\sigma(d-1)} \tag{40}$$

Considering $\sigma(d) = i$ and $\sigma(d-1) = j$, (12) is confirmed and Theorem 1 is proved.

Theorem 2 Through Theorem 1, if there exists matrices $u_i > 0$, $w > 0$, y_i , and x_i for each subsystem i such that the following inequalities be feasible:

$$\begin{pmatrix} (1 + \xi_s)(x_i^T + x_i - u_i) x_i^T O_i^T x_i^T & 0 & -x_i^T O_j^T T_j & 0 & 0 & 0 & x_i^T M_i^T + y_i^T B_i^T \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & * & w & 0 & 0 & 0 & 0 \\ * & * & * & x_i^T + x_i - w & 0 & 0 & x_i^T D_i^T \\ * & * & * & * & \gamma^2 I - T_i^T T_i & 0 & N_i^T \\ * & * & * & * & * & \xi_s(x_i^T + x_i - w) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \cdots & \xi_s(x_i^T + x_i - w) \\ * & * & * & * & * & * & * & u_i \end{pmatrix} \geq 0, \quad i \in M \tag{41}$$

$$\begin{pmatrix} (1 + \xi_{as})(x_i^T + x_i - u_i) x_i^T O_j^T x_i^T & 0 & -x_i^T O_j^T T_j & 0 & 0 & 0 & x_i^T M_j^T + y_i^T B_j^T \\ * & I & 0 & 0 & 0 & 0 & 0 \\ * & * & w & 0 & 0 & 0 & 0 \\ * & * & * & x_i^T + x_i - w & 0 & 0 & x_i^T D_j^T \\ * & * & * & * & \gamma^2 I - T_i^T T_i & 0 & N_j^T \\ * & * & * & * & * & \xi_{as}(x_i^T + x_i - w) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & * & * & * & \cdots & \xi_{as}(x_i^T + x_i - w) \\ * & * & * & * & * & * & * & u_i \end{pmatrix} \geq 0, \quad i \neq j, \quad i, j \in M \tag{42}$$

$$u_i - \mu^{-1}u_j \geq 0 \tag{43}$$

Then, for the state feedback control gain $K_i = y_i x_i^{-1}$ and the PDT switching signal satisfying (14), the closed-loop switched system described in (1)–(3) has finite-time $H\infty$ performance with respect to $(0, c_2, R, d_I, ub, \tilde{\gamma}, \sigma)$ against asynchronous switching.

Proof Firstly, the LMI constraint related to inequality (10) is proved. Inequality (10) can be rewritten as follows:

By applying Lemma 1 to the above inequality, we obtain:

$$\begin{pmatrix} (1 + \xi_s)S_i - Q - O_i^T O_i & 0 & -O_i^T T_i & 0 & \dots & 0 & \overline{M}_i^T \\ * & Q & 0 & 0 & 0 & 0 & D_i^T \\ * & * & \gamma^2 I - T_i^T T_i & 0 & 0 & 0 & N_i^T \\ * & * & * & \xi_s Q & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \dots & \xi_s Q & 0 \\ * & * & * & * & * & * & S_i^{-1} \end{pmatrix} \geq 0 \tag{45}$$

Consider $u_i = S_i^{-1} w$, $w = Q^{-1}$, and $y_i = K_i x_i$, we have:

$$\begin{pmatrix} (1 + \xi_s)u_i^{-1} - w^{-1} - O_i^T O_i & 0 & -O_i^T T_i & 0 & \dots & 0 & \overline{M}_i^T \\ * & w^{-1} & 0 & 0 & 0 & 0 & D_i^T \\ * & * & \gamma^2 I - T_i^T T_i & 0 & 0 & 0 & N_i^T \\ * & * & * & \xi_s w^{-1} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \dots & \xi_s w^{-1} & 0 \\ * & * & * & * & * & * & u_i \end{pmatrix} \geq 0 \tag{46}$$

$$\begin{pmatrix} (1 + \xi_s)S_i - Q - O_i^T O_i & 0 & -O_i^T T_i & 0 & \dots & 0 \\ * & Q & 0 & 0 & \dots & 0 \\ * & * & \gamma^2 I - T_i^T T_i & 0 & \dots & 0 \\ * & * & * & \xi_s Q & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ * & * & * & * & * & \xi_s Q \end{pmatrix} - \begin{pmatrix} \overline{M}_i^T \\ D_i^T \\ N_i^T \\ 0 \\ \dots \\ 0 \end{pmatrix} \tag{44}$$

$(S_i) \left(\overline{M}_i \ D_i \ N_i \ 0 \ \dots \ 0 \right) \geq 0$

Multiplying left and right sides of inequality (40) by $bdiag(x_i^T, x_i^T, I, x_i^T, \dots, x_i^T, I)$ and $bdiag(x_i, x_i, I, x_i, \dots, x_i, I)$, respectively:

$$\begin{pmatrix} (1 + \xi_s)x_i^T u_i^{-1} x_i - x_i^T w^{-1} x_i - x_i^T O_i^T O_i x_i & 0 & -x_i^T O_i^T T_i & 0 & \dots & 0 & x_i^T \overline{M}_i^T \\ * & x_i^T w^{-1} x_i & 0 & 0 & 0 & 0 & x_i^T D_i^T \\ * & * & \gamma^2 I - T_i^T T_i & 0 & 0 & 0 & N_i^T \\ * & * & * & \xi_s x_i^T w^{-1} x_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & \dots & \xi_s x_i^T w^{-1} x_i & 0 \\ * & * & * & * & * & * & u_i \end{pmatrix} \geq 0 \tag{47}$$

Utilizing Lemma 1 several times, we have:

$$\begin{pmatrix} (1 + \xi_s)x_i^T u_i^{-1} x_i & x_i^T O_i^T & x_i^T & 0 & -x_i^T O_i^T T_i & 0 & 0 & 0 & x_i^T M_i^T + y_i^T B_i^T \\ * & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & w & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & x_i^T w^{-1} x_i & 0 & 0 & 0 & 0 & x_i^T D_i^T \\ * & * & * & * & \gamma^2 I - T_i^T T_i & 0 & 0 & 0 & N_i^T \\ * & * & * & * & * & \xi_s x_i^T w^{-1} x_i & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & * & * & * & * & \dots & \xi_s x_i^T w^{-1} x_i & 0 \\ * & * & * & * & * & * & * & * & u_i \end{pmatrix} \geq 0 \tag{48}$$

From the feasibility of (49), it can be concluded that $x_i^T u_i^{-1} x_i > 0$ and $x_i^T w^{-1} x_i > 0$. Therefore, x_i and is full rank. Developing $(u_i - x_i)^T u_i^{-1} (u_i - x_i) \geq 0$ and $(w - x_i)^T w^{-1} (w - x_i) \geq 0$:

$$x_i^T u_i^{-1} x_i \geq x_i + x_i^T - u_i \tag{49}$$

$$x_i^T w^{-1} x_i \geq x_i + x_i^T - w \tag{50}$$

According to (49)–(50), it can be inferred that if the inequality (41) holds, the above inequality also.

Remark 2 The inequality (42) can also be derived by converting the inequality (11) as described above.

Based on inequality (12), mentioned change of variable and using Lemma 1 twice, we have

$$S_i \leq \mu S_j \Rightarrow u_i^{-1} \leq \mu u_j^{-1} \Rightarrow u_i - \mu^{-1} u_j \geq 0 \tag{51}$$

Therefore, inequality (51) confirms (43) and Theorem 2 is proved.

Remark 3 It is worth pointing out that we must firstly solve LMI problem (41)–(43) to gain P_i , which can let us find κ_1 and κ_2 , then switching signal is designed by substituting κ_1 and κ_2 into (14).

Remark 4 To achieve a feasible solution for the LMI problem described in (41)–(43), decision parameters (μ, ξ_s, ξ_{as}) play important roles.

4 Numerical simulations

Through two numerical examples, we examine the performance of the suggested approach in this section. MATLAB simulations have been carried out using the YALMIP toolset.

Example 1 This example examines a drinking water supply network (DWSN) (Benallouch et al. 2014). Pipes, valves, reservoirs, water storage, pumps, water collection, water purification, and consumers are typical components of the DWSN. A linear discrete-time switching system with two subsystems that has reservoir volumes as its state vector and all controlled reservoir inflows as its input vector can be used to describe the behavior of the DWSN (Fig. 1). Two subsystems are thought of as:

Sub-system 1:

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_1 = \begin{pmatrix} 0.1501 & 0.0001 \\ -0.1001 & -0.1002 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

$$N_1 = [0.1 \ 0.1]^T, \quad O_1 = I_2, \quad T_1 = I_3, \quad T_s = 0.5 \text{ min},$$

$$h_M = 2 \text{ min}, \quad h = 1 \text{ min} \tag{52}$$

Sub-system 2:

$$M_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad D_2 = \begin{pmatrix} 0.1401 & 0.0001 \\ -0.0401 & -0.0501 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix},$$

$$N_2 = [0.1 \ 0.1]^T, \quad O_2 = I_2, \quad T_2 = I_3, \quad T_s = 0.5 \text{ min},$$

$$h_M = 2 \text{ min}, \quad h = 1 \text{ min} \tag{53}$$

The goal is for closed-loop system to have the finite time H_∞ performance with respect to $(0, 30, I, 30, 3, \sigma)$. The parameters of the controller and switching signal are selected as $\xi_s = 0.01, \xi_{as} = 0.04, \mu = 1.08, d_{syn,m} = 1 \text{ min}, d_T = 10 \text{ min}$ and $f = 0.5 \frac{1}{\text{min}}$. By Solving the LMI problem (41)–(43), we obtain:

$$S_1 = \begin{pmatrix} 4.7269 & -0.0370 \\ -0.0370 & 4.7269 \end{pmatrix}, \quad K_1 = \begin{pmatrix} -0.6646 & -0.0103 \\ -0.0103 & -0.6646 \\ 0 & 0 \end{pmatrix},$$

$$S_2 = \begin{pmatrix} 4.6351 & 0.0213 \\ 0.0213 & 4.6351 \end{pmatrix},$$

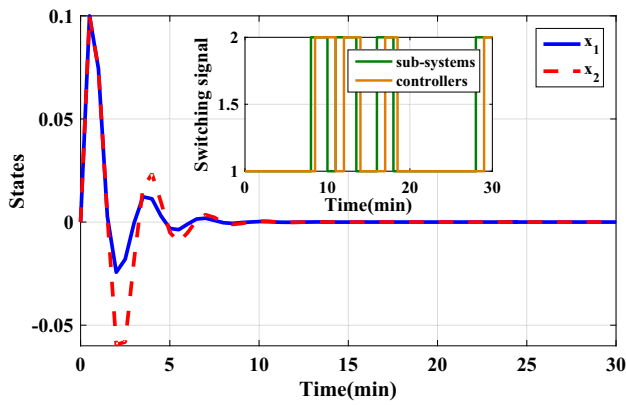


Fig. 2 The system states of the drinking water supply network (DWSN)

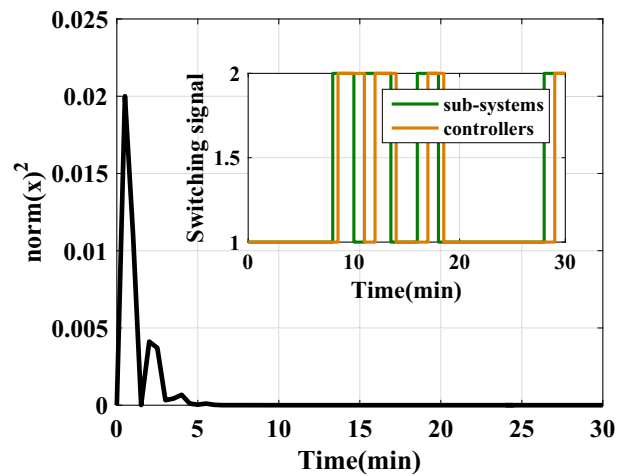


Fig. 4 The time response of $\|x\|^2$

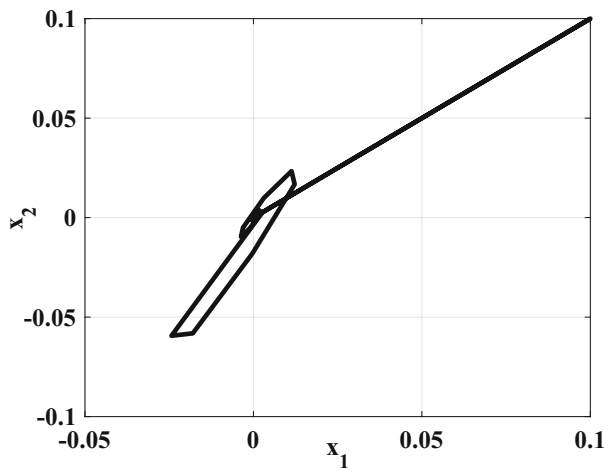


Fig. 3 The states trajectory of the drinking water supply network (DWSN)

$$K_2 = \begin{pmatrix} -0.4793 & -0.2113 \\ -0.2122 & -0.4746 \\ 0.2880 & -0.2820 \end{pmatrix},$$

$$Q = 10^{-5} \begin{pmatrix} 0.4356 & -0.0008 \\ -0.0008 & 0.4354 \end{pmatrix} \quad (54)$$

By having S_1, S_2 , and $R = I$, we obtain $\kappa_2 = \min_{i \in M}(\lambda_{\min}(\bar{S}_i)) = 4.6138$. Finally, considering $\gamma = 1$ and using (14), we obtain $d_\tau^* = 6.60$ min. An example of a switching signal satisfying the design parameters is shown in Fig. 2. Figure 2 also shows the system states that ultimately converge to zero and consequently, the specified bound is satisfied. Figure 3 shows the system state trajectory that gives a better view of the variations in states.

In Fig. 4, the time response of $x^T x$ is depicted. From this graph, it can be deduced that the closed-loop switched system satisfies finite time H_∞ criteria with specification $(0, 30, I, 30, 3, \sigma)$ against asynchronous switching. In fact, the proposed scheme controls the rate of energy variation in

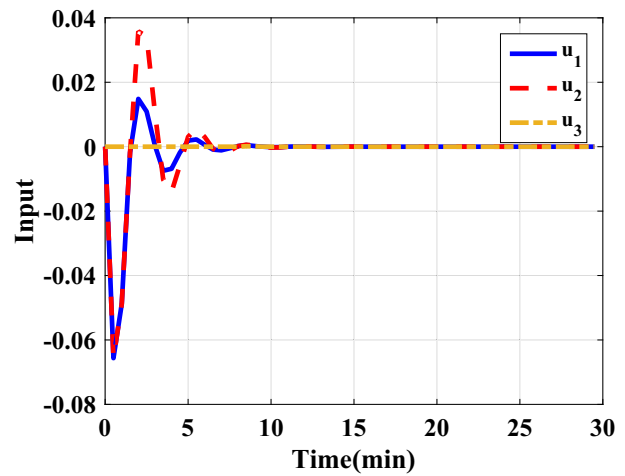


Fig. 5 The control input of the drinking water supply network (DWSN)

both matched and mismatched time intervals with the presence of asynchronous switching, allowing $x^T x$ to satisfy the specified bound in the time interval. The control input for the switched system is shown in Fig. 5.

The comparison between the proposed scheme and control scheme extended by Gholami and Shafiei (2021) is shown in Table 1. This table shows that proposed scheme has less convergence time and less average of $\|x\|$. In fact, the control scheme extended by Gholami and Shafiei (2021) does not take into account the asynchronous switching phenomena and considers the switching signal based on the average dwell time structure that is robust to a smaller class of switching signal compared to a persistent dwell time structure. These issues reduce control performance.

Example 2 Think about a model of discretized water pollution. The intended model depicts, respectively, the concentrations of dissolved oxygen and biological oxygen demand

Table 1 Comparison of proposed scheme and control scheme extended by Gholami and Shafiei (2021)

Methods	Average of $\ x\ $	Convergence time of states ($ x_i < 0.02, i = 1, 2$) (min)
Control scheme extended by Gholami and Shafiei (2021)	0.02	5
Proposed scheme	0.008	3

per unit volume at time d . The purpose of this example is to illustrate how the requirements in Theorem 2 may be used to create the suggested scheme for this model. There is a switch between the two subsystems listed below:

Subsystem 1:

$$\begin{aligned}
 M_1 &= \begin{pmatrix} 0.8794 & 0 \\ -0.2773 & 0.8208 \end{pmatrix}, & D_1 &= \begin{pmatrix} 0.0926 & 0 \\ 0.02294 & 0.0659 \end{pmatrix}, \\
 B_1 &= \begin{pmatrix} 0.1126 & 0 \\ -0.0176 & 0.1270 \end{pmatrix}, \\
 N_1 &= [0.1 \ 0.1]^T, & O_1 &= I_2, & T_1 &= I_2, & h_M &= 4s, \\
 h &= 1s, & T_s &= 1s
 \end{aligned} \tag{55}$$

Subsystem 2:

$$\begin{aligned}
 M_2 &= \begin{pmatrix} 0.8620 & 0 \\ -0.1848 & 0.7730 \end{pmatrix}, & D_2 &= \begin{pmatrix} 0.1257 & 0 \\ 0.0288 & 0.1606 \end{pmatrix}, \\
 B_2 &= \begin{pmatrix} 0.1394 & 0 \\ -0.0148 & 0.1058 \end{pmatrix}, \\
 N_2 &= [0.1 \ 0.1]^T, & O_2 &= I_2, & T_2 &= I_2, & h_M &= 4s, \\
 h &= 1s, & T_s &= 1s
 \end{aligned} \tag{56}$$

We would make the closed-loop water pollution system finite time stable with respect to $(0, 20, I, 30, 3, \sigma)$. The parameters of the controller and switching signal are selected as $\xi_s = 0.02, \xi_{as} = 0.05, \mu = 1.05, d_{syn,m} = 2s, d_T = 10s$ and $f = 0.25Hz$. Solving the LMI problem (41)–(43), we obtain:

$$\begin{aligned}
 S_1 &= \begin{pmatrix} 13.4200 & -0.2164 \\ -0.2164 & 13.7033 \end{pmatrix}, & K_1 &= \begin{pmatrix} -4.5824 & 1.0925 \\ 1.0582 & -4.1406 \end{pmatrix}, \\
 S_2 &= \begin{pmatrix} 13.3287 & 0.2231 \\ 0.2231 & 13.6123 \end{pmatrix}, & K_2 &= \begin{pmatrix} -4.1138 & 0.4719 \\ 0.4525 & -4.7902 \end{pmatrix}, \\
 Q &= \begin{pmatrix} 0.3719 & 0.0842 \\ 0.0842 & 0.4257 \end{pmatrix}
 \end{aligned} \tag{57}$$

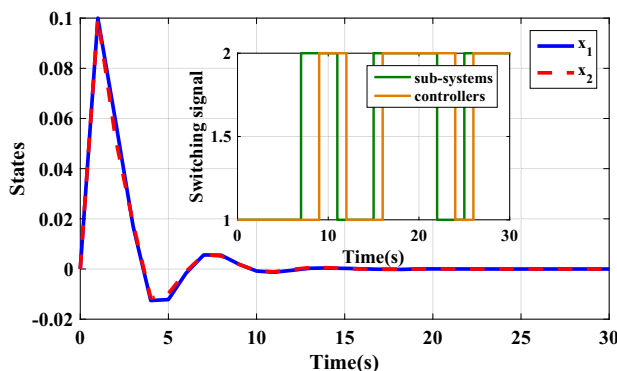


Fig. 6 The system states of the water pollution system

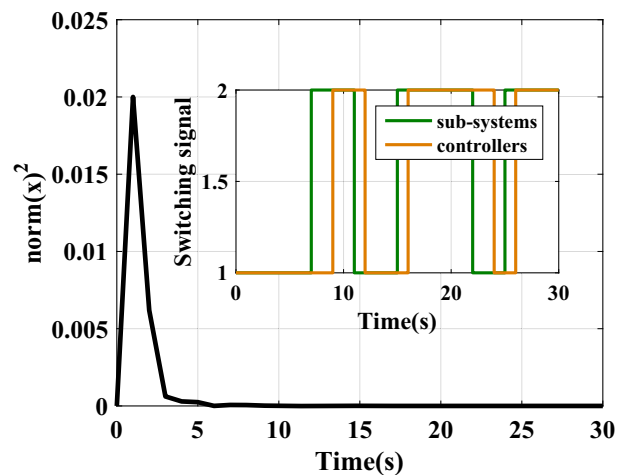


Fig. 7 The time response of $\|x\|^2$

Based on S_1, S_2 , and $R = I$, we obtain $\kappa_1 = \max_{i \in M}(\lambda_{\max}(\bar{S}_i)) = 13.82$ and $\kappa_2 = \min_{i \in M}(\lambda_{\min}(\bar{S}_i)) = 13.20$. Finally, considering $\gamma = 1$ and using (12), we obtain $d_\tau^* = 6.27$. Figures 6, 7 and 8 show the system states, the variation of Lyapunov function, the time response of $\|x\|^2$ and control input of water pollution system. Analysis of these figures confirms that closed-loop system is finite time stable with respect to $(1, 20, I, 30, 3, \sigma)$, and the required performance is satisfied.

5 Conclusion

In this study, using multiple Lyapunov–Krasovskii functionals, a finite-time H_∞ state feedback strategy against asynchronous switching was developed in the linear delayed switched systems with the PDT switching signal. A number of non-convex constraints as well as additional switching signal conditions were extracted to ensure the system’s finite-time H_∞ criteria. The level of disturbance attenuation is also measured. The linear matrix inequality problem was selected

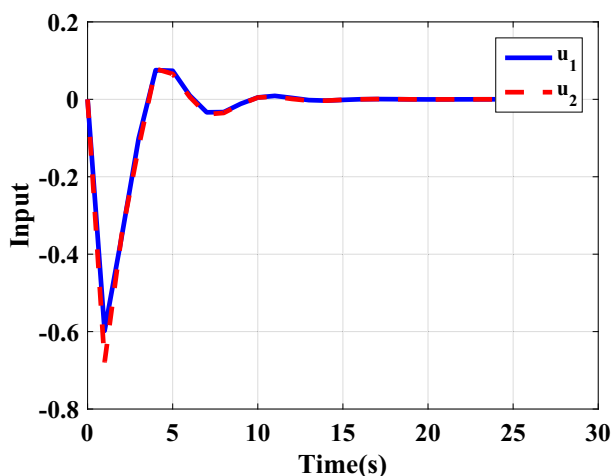


Fig. 8 The input of the water pollution system

so that the non-convex issue could be solved using standard tools. Finally, numerical simulations verified the recommended strategy, and the results showed it to be effective. Future studies will find it difficult to construct a finite-time H_∞ asynchronous output feedback strategy in linear switched systems.

Author contributions HZ: writing-original draft preparation, conceptualization, supervision, project administration.

Data availability The authors do not have permissions to share data.

Declarations

Conflict of interest The author declares that there are no conflict of interest.

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