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PERSPECTIVE

Toward differential geometry of statistical submanifolds

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Abstract
A brief introduction of doubly minimal submanifolds of statistical manifolds is given. A complex submanifold of a holomorphic statistical manifold is doubly minimal. Similar properties are obtained in the case where the ambient space is a Sasakian statistical manifold.

Keywords Statistical submanifolds · Doubly minimal · Doubly totally umbilical

1 Introduction

A differential geometric study of submanifolds of statistical manifolds is developing as an interesting research field. Although many research papers have been published, it still has room for improvement on the basic part. The elementary differential geometry of surfaces in the Euclidean 3-space is a hometown of the submanifold theory. In this theory, we first see a totally geodesic surface (a plane, a part of a plane) and a totally umbilical surface (a sphere, a part of a sphere, in addition) as fundamental objects, which are characterized in terms of the second fundamental forms and should be studied deeply. Moreover, a minimal surface has appealed to many mathematicians, which is a surface with zero mean curvature vector field. In fact, beautiful and exciting examples of such surfaces have been explicitly founded. In the statistical submanifold theory, what are the counterparts of such submanifolds?

The author hopes that this small article will be useful in attracting interest in these issues, though it does not have enough results. We here introduce doubly minimal submanifolds of statistical manifolds, and will indicate that such submanifolds arise from a special class of minimal submanifolds of Riemannian manifolds with other additional structures like Kähler structures. We have that a complex submanifold of

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a holomorphic statistical manifold is doubly minimal (Theorem 3.1). We also study doubly minimal submanifolds of a Sasakian statistical manifold (Theorem 3.3 and Proposition 3.4).

2 Doubly totally-umbilical submanifolds

Throughout this paper, M denotes a smooth manifold of dimension $m \geq 2$, and all the objects are assumed to be smooth. $\Gamma(E)$ denotes the set of sections of a vector bundle $E \to M$. For example, $\Gamma(TM^{(p,q)})$ means the set of all the tensor fields on M of type (p,q), and $\Gamma(TM) = \Gamma(TM^{(1,0)})$ means the set of all the vector fields on M.

Let ∇ be an affine connection on M, and $g \in \Gamma(TM^{(0,2)})$ a Riemannian metric. We denote by ∇^g the Levi-Civita connection of g. A pair (∇, g) is called a *statistical structure* on M if ∇ is of torsion free, and the Codazzi equation

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z)$$

holds for any $X, Y, Z \in \Gamma(TM)$. A manifold equipped with a statistical structure is called a *statistical manifold*.

For an affine connection ∇ on a Riemannian manifold (M, g), define ∇^* by the formula

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla^*_X Z)$$

for any $X, Y, Z \in \Gamma(TM)$. Then ∇^* is an affine connection on M which is called the *dual* connection of ∇ with respect to g.

For a statistical structure (∇, g), we set

$$K^{(\nabla,g)}_X Y = \nabla_X Y - \nabla^g_X Y$$

for any $X, Y \in \Gamma(TM)$. Then $K = K^{(\nabla,g)} \in \Gamma(TM^{(1,2)})$ satisfies

$$K_X Y = K_Y X, \quad g(K_X Y, Z) = g(Y, K_X Z). \tag{2.1}$$

Remark 2.1 (1) For a Riemannian metric g and a $(1,2)$-tensor field K satisfying (2.1), a pair $(\nabla = \nabla^g + K, g)$ is a statistical structure.

(2) For a statistical structure (∇, g) and a real number $\alpha \in \mathbb{R}$, set

$$\nabla^{(\alpha)} = \nabla^g + \alpha K^{(\nabla,g)}.$$

Then $(\nabla^{(\alpha)}, g)$ is a statistical structure with $\nabla^{(1)} = \nabla$, $\nabla^{(0)} = \nabla^g$, and $\nabla^{(-1)} = \nabla^*$. Moreover, $(\nabla^{(\alpha)})^* = \nabla^{(-\alpha)}$ holds.

We will now fix the notation in the statistical submanifold theory. Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a statistical manifold of dimension $n = m + p$, and M a manifold of dimension m as before.

Let $\iota : M \to \widetilde{M}$ be an immersion of M into $\widetilde{M}$. The readers not familiar with the submanifold theory can consider ι as the inclusion map of $M \subset \widetilde{M}$, that is, M is a subset of $\widetilde{M}$ and $\iota : M \ni x \mapsto x \in \widetilde{M}$, and may omit ι, ι_* and ι^* in the following.

We then define g and ∇ on M by

$$g = \iota^* \widetilde{g}, \quad g(\nabla_X Y, Z) = \widetilde{g}(\widetilde{\nabla}_X \iota_* Y, \iota_* Z)$$

for $X, Y, Z \in \Gamma(TM)$. We show that (∇, g) is a statistical structure on M, and call it the statistical structure *induced* by ι from $(\widetilde{\nabla}, \widetilde{g})$. Then (M, ∇, g) is called a *statistical submanifold* of $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.

In another setting, for two statistical manifolds (M, ∇, g), $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ and an immersion $\iota : M \to \widetilde{M}$, ι is called a *statistical immersion* if the statistical structure induced by ι from $(\widetilde{\nabla}, \widetilde{g})$ coincides with (∇, g). In both settings, $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ is often called the ambient space.

We denote the orthogonal decomposition of the induced bundle $\iota^* T\widetilde{M} \to M$ with respect to $\widetilde{g}$ by

$$\iota^* T\widetilde{M} = TM \oplus TM^{\perp} \tag{2.2}$$

and the orthogonal projection by

$$(\ )^{\top} : \Gamma(\iota^* T\widetilde{M}) \to \Gamma(TM), \quad (\ )^{\perp} : \Gamma(\iota^* T\widetilde{M}) \to \Gamma(TM^{\perp}).$$

For the simplicity, the induced connection $\iota^* \widetilde{\nabla}$ is written as $\widetilde{\nabla}$. By using the decomposition (2.2), we define B, A, $\nabla^{\perp}$ by

$$\begin{aligned} \widetilde{\nabla}_X \iota_* Y &= \iota_* \nabla_X Y + B(X, Y), \quad X, Y \in \Gamma(TM), \\ \widetilde{\nabla}_X \xi &= -\iota_* A_{\xi} X + \nabla^{\perp}_X \xi, \quad \xi \in \Gamma(TM^{\perp}),\ X \in \Gamma(TM). \end{aligned}$$

Then we call $B \in \Gamma(T^{\perp}M \otimes TM^{(0,2)})$ the *second fundamental form* for ι with respect to $\widetilde{\nabla}$. We call $A \in \Gamma((TM^{\perp})^* \otimes TM^{(1,1)})$ the *shape operator*, $\nabla^{\perp} : \Gamma(TM^{\perp}) \times \Gamma(TM) \to \Gamma(TM^{\perp})$ the *normal connection*.

Taking $\widetilde{\nabla}^{(\alpha)}$ in Remark 2.1 as the connection of the ambient space, we define $B^{(\alpha)}$, $A^{(\alpha)}$, $\nabla^{\perp(\alpha)}$ in the same fashion:

$$\widetilde{\nabla}^{(\alpha)}_X \iota_* Y = \iota_* \nabla^{(\alpha)}_X Y + B^{(\alpha)}(X, Y), \quad \widetilde{\nabla}^{(\alpha)}_X \xi = -\iota_* A^{(\alpha)}_{\xi} X + \nabla^{\perp}{}^{(\alpha)}_X \xi.$$

Moreover, we write $B^* = B^{(-1)}$, $\widehat{B} = B^{(0)}$, $A^* = A^{(-1)}$, $\widehat{A} = A^{(0)}$, $\nabla^{\perp *} = \nabla^{\perp(-1)}$ and $\widehat{\nabla}^{\perp} = \nabla^{\perp(0)}$.

For a statistical submanifold (M, ∇, g) of $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$, the following hold for each $\alpha \in \mathbb{R}$:

$$\begin{aligned} &\widetilde{g}(B^{(\alpha)}(X, Y), \xi) = g(A^{(-\alpha)}_{\xi} X, Y), \\ &X\widetilde{g}(\xi, \eta) = \widetilde{g}(\nabla^{\perp}{}^{(\alpha)}_X \xi, \eta) + \widetilde{g}(\xi, \nabla^{\perp}{}^{(-\alpha)}_X \eta), \end{aligned}$$

$$\widehat{B} = \frac{1}{2}(B^{(\alpha)} + B^{(-\alpha)}), \quad \widehat{A} = \frac{1}{2}(A^{(\alpha)} + A^{(-\alpha)}),$$
$$\widehat{\nabla}^{\perp} = \frac{1}{2}(\nabla^{\perp(\alpha)} + \nabla^{\perp(-\alpha)})$$

for $X, Y \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(TM^{\perp})$.

Moreover, we define the *mean curvature vector field* with respect to $\widetilde{\nabla}^{(\alpha)}$ as

$$H^{(\alpha)} = \frac{1}{m}\mathrm{tr}_g B^{(\alpha)},$$

and set $H = H^{(1)}$, $\widehat{H} = H^{(0)}$ and $H^* = H^{(-1)}$. We remark that

$$\widehat{H} = \frac{1}{2}(H^{(\alpha)} + H^{(-\alpha)}), \quad H^{(\alpha)} = \frac{1+\alpha}{2}H + \frac{1-\alpha}{2}H^*.$$

Definition 2.2 Let (M, ∇, g) be a statistical submanifold of $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$. (1) M is said to be *doubly totally-geodesic* if $B = B^* = 0$. (2) M is said to be *doubly totally-umbilical* if $B = g \otimes H$ and $B^* = g \otimes H^*$. (3) M is said to be *doubly minimal* if $H = H^* = 0$.

Remark 2.3 The following conditions are equivalent: (1) A statistical submanifold (M, ∇, g) is doubly totally-umbilical. (2) $B^{(\alpha)} = g \otimes H^{(\alpha)}$ for all $\alpha \in \mathbb{R}$. (3) $B^{(\alpha_j)} = g \otimes H^{(\alpha_j)}$ for some $\alpha_1 \neq \alpha_2 \in \mathbb{R}$. (4) $B = g \otimes H$ and $A_\xi = \widetilde{g}(H^*, \xi)\mathrm{id}$.

The reader will be able to list the similar properties for the other notions in Definition 2.2.

In our setting, the term *auto-parallel* and the term totally-geodesic coincide with each other. See [8] for example, in which related Information Geometric objects are studied. The second fundamental form is sometimes called the *embedding curvature*.

We now denote by $R^{\nabla} \in \Gamma(TM^{(1,3)})$ the curvature tensor field for a connection ∇:

$$R^{\nabla}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \Gamma(TM)$.

Example 2.4 The triplet $((\mathbb{R}^+)^m, g_0, \nabla)$ defined below is a statistical manifold with $R^{\nabla^{g_0}} = R^{\nabla} = 0$.

$$(\mathbb{R}^+)^m = \{p \in \mathbb{R}^m \mid y^1(p) > 0, \ldots, y^m(p) > 0\},$$
$$g_0 = \sum_{j=1}^{m}(dy^j)^2, \text{ that is, the restriction of the Euclidean metric,}$$
$$\nabla_{(\partial_i)_p}\partial_j = -\delta_{ij}\{y^j(p)\}^{-1}(\partial_j)_p, \text{ where } \partial_j = \partial/\partial y^j.$$

Theorem 2.5 *A round hypersphere of center the origin is the only doubly totally-umbilical hypersurface of* $((\mathbb{R}^+)^m, g_0, \nabla)$ *which is not doubly totally-geodesic.*

See [5] for example.

Example 2.6 The triplet $(\widetilde{M}, \widetilde{\nabla} = \nabla^{\widetilde{g}} + \widetilde{K}, \widetilde{g})$ defined below is a statistical manifold with $R^{\widetilde{\nabla}} = 0$.

$$\begin{aligned}
&\widetilde{M} = \mathbb{H}^n = \{y = (y^1, \dots, y^{n-1}, y^n) \in \mathbb{R}^n \mid y^n > 0\},\\
&\widetilde{g} = (y^n)^{-2} \sum_{A=1}^{n} (dy^A)^2,\\
&\widetilde{K}(\widetilde{\partial}_i, \widetilde{\partial}_j) = \delta_{ij}(y^n)^{-1}\widetilde{\partial}_n, \qquad i, j = 1, \dots, n-1,\\
&\widetilde{K}(\widetilde{\partial}_i, \widetilde{\partial}_n) = \widetilde{K}(\widetilde{\partial}_n, \widetilde{\partial}_i) = (y^n)^{-1}\widetilde{\partial}_i,\\
&\widetilde{K}(\widetilde{\partial}_n, \widetilde{\partial}_n) = 2(y^n)^{-1}\widetilde{\partial}_n,
\end{aligned}$$

where $\widetilde{\partial}_A = \partial/\partial y^A$, $A = 1, \dots, n$.
(1) For $(a^1, \dots, a^p) \in \mathbb{R}^p$, $m + p = n$, the inclusion map

$$\iota_1 : \mathbb{H}^m \ni (x^1, \dots, x^{m-1}, x^m) \mapsto (a^1, \dots, a^p, x^1, \dots, x^{m-1}, x^m) \in \mathbb{H}^n \tag{2.3}$$

is doubly totally-geodesic; $B = B^* = 0$.
(2) For $(a^1, \dots, a^{p-1}, a^p) \in \mathbb{R}^{p-1} \times \mathbb{R}^+$, the inclusion map

$$\iota_2 : \mathbb{R}^m \ni (x^1, \dots, x^m) \mapsto (x^1, \dots, x^m, a^1, \dots, a^{p-1}, a^p) \in \mathbb{H}^n \tag{2.4}$$

is doubly totally-umbilical. In fact, we have

$$\begin{aligned}
g &= (a^p)^{-2} \sum_{j=1}^{m} (dx^j)^2, \qquad \nabla_{\partial/\partial x^j} \frac{\partial}{\partial x^i} = \nabla^g_{\partial/\partial x^j} \frac{\partial}{\partial x^i} = 0,\\
B &= 2a^p g \otimes \frac{\partial}{\partial y^n} = g \otimes H, \quad B^* = 0.
\end{aligned}$$

A hypersurface of the form in (2) is studied in [4].

3 Doubly minimal submanifolds

In this section, we introduce typical examples of doubly minimal statistical immersions.

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{J})$ be a *holomorphic* statistical manifold. By definition, $(\widetilde{M}, \widetilde{\nabla} = \nabla^{\widetilde{g}} + \widetilde{K}, \widetilde{g})$ is a statistical manifold with an almost complex structure $\widetilde{J} \in \Gamma(T\widetilde{M}^{(1,1)})$ such that $(\widetilde{g}, \widetilde{J})$ is a Kähler structure on $\widetilde{M}$ and

$$\widetilde{K}_X \widetilde{J} Y + \widetilde{J} \widetilde{K}_X Y = 0$$

holds for $X, Y \in \Gamma(T\widetilde{M})$. It is easy to show that

$$\widetilde{\nabla}_X(\widetilde{J}Y) = \widetilde{J}\widetilde{\nabla}^*_X Y$$

for $X, Y \in \Gamma(T\widetilde{M})$. See [4] for example.

Let (M, g, J) be a Kähler manifold of dimension $m = 2l$. Let $\iota : M \to \widetilde{M}$ be a holomorphic isometric immersion, that is, $\iota^*\widetilde{g} = g$ and

$$\iota_* JX = \widetilde{J}\iota_* X$$

holds for $X \in \Gamma(TM)$. We often call such an M a complex submanifold of $\widetilde{M}$. It is known that M is a minimal submanifold of $\widetilde{M}$ as well.

Let (∇, g) be the statistical structure on M induced by ι from $(\widetilde{\nabla}, \widetilde{g})$. Then (∇, g, J) is a holomorphic statistical structure on M. Besides, we have the following:

Theorem 3.1 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{J})$ be a holomorphic statistical manifold and $\iota : (M, g, J) \to (\widetilde{M}, \widetilde{g}, \widetilde{J})$ a holomorphic isometric immersion. Then ι is doubly minimal. In fact,*

$$B(X, JY) = \widetilde{J}B^*(X, Y) \tag{3.1}$$

holds for $X, Y \in \Gamma(TM)$.

Proof We have for $X, Y \in \Gamma(TM)$,

$$\begin{aligned}
\widetilde{\nabla}_X(\widetilde{J}\iota_* Y) &= \widetilde{J}\widetilde{\nabla}^*_X \iota_* Y = \widetilde{J}\{\iota_* \nabla^*_X Y + B^*(X, Y)\} \\
&= \iota_* J\nabla^*_X Y + \widetilde{J}B^*(X, Y), \\
\widetilde{\nabla}_X \iota_* JY &= \iota_* \nabla_X(JY) + B(X, JY),
\end{aligned}$$

which imply that $\nabla_X(JY) = J\nabla^*_X Y$ and Equation (3.1). In the same way, we have

$$B^*(X, JY) = \widetilde{J}B(X, Y).$$

Using them and the symmetricity of the second fundamental forms, we have $B(X, X) + B(JX, JX) = B(X, X) + \widetilde{J}B^*(JX, X) = B(X, X) + \widetilde{J}^2 B(X, X) = 0$. Since (g, J) is a Hermitian structure, taking orthonormal frame fields of the form $\{e_{2j-1}, Je_{2j-1}\}_{j=1,\ldots,l}$, we calculate the mean curvature vector field by

$$\begin{aligned}
2lH &= \sum_{j=1}^{l}\{B(e_{2j-1}, e_{2j-1}) + B(e_{2j}, e_{2j})\} \\
&= \sum_{j=1}^{l}\{B(e_{2j-1}, e_{2j-1}) + B(Je_{2j-1}, Je_{2j-1})\} = 0.
\end{aligned}$$

We have $H^* = 0$ in the similar fashion. Therefore, ι is doubly minimal. □

Remark 3.2 Although Theorem 3.1 itself seems trivial, it is an interesting problem to determine when a doubly minimal submanifold is holomorphic conversely.

Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a *Sasakian* statistical manifold. By definition, $(\widetilde{M}, \widetilde{\nabla} = \nabla^{\widetilde{g}} + \widetilde{K}, \widetilde{g})$ is a statistical manifold with a Sasakian structure $(\widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ such that

$$\widetilde{K}_X \widetilde{\phi} Y + \widetilde{\phi} \widetilde{K}_X Y = 0 \tag{3.2}$$

holds for $X, Y \in \Gamma(T\widetilde{M})$. We briefly review the notion of Sasakian structures; $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold, $\widetilde{\phi} \in \Gamma(T\widetilde{M}^{(1,1)})$, $\widetilde{\xi} \in \Gamma(T\widetilde{M})$, and the triplet $(\widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ satisfies (1) $\widetilde{\phi}\widetilde{\xi} = 0$, (2) $\widetilde{g}(\widetilde{\xi}, \widetilde{\xi}) = 1$, (3) $\widetilde{\phi}^2 X = -X + \widetilde{g}(X, \widetilde{\xi})\widetilde{\xi}$, (4) $\widetilde{g}(\widetilde{\phi}X, Y) + \widetilde{g}(X, \widetilde{\phi}Y) = 0$, (5) $(\nabla^{\widetilde{g}}_X \widetilde{\phi})Y = \widetilde{g}(Y, \widetilde{\xi})X - \widetilde{g}(Y, X)\widetilde{\xi}$ for $X, Y \in \Gamma(T\widetilde{M})$. We also remark that the dimension of such a manifold is odd.

Let $(\widetilde{M}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a Sasakian manifold. We set $\widetilde{K} \in \Gamma(T\widetilde{M}^{(1,2)})$ as

$$\widetilde{K}_X Y = \widetilde{g}(X, \widetilde{\xi})\widetilde{g}(Y, \widetilde{\xi})\widetilde{\xi} \tag{3.3}$$

for $X, Y \in \Gamma(T\widetilde{M})$. Then, the quadruplet $(\widetilde{g}, \widetilde{\nabla} = \nabla^{\widetilde{g}} + \widetilde{K}, \widetilde{\phi}, \widetilde{\xi})$ is a Sasakian statistical structure, because $\widetilde{K}$ satisfies (2.1) and (3.2).

Let (M, g, ϕ, ξ) be a Sasakian manifold, and $\iota : M \to \widetilde{M}$ an *invariant* immersion, that is,

$$\iota^* \widetilde{g} = g, \quad \iota_* \circ \phi = \widetilde{\phi} \circ \iota_*, \quad \text{and} \quad \widetilde{\xi} \circ \iota = \xi$$

hold. Let (∇, g) be the statistical structure on M induced by ι from $(\widetilde{\nabla}, \widetilde{g})$. Then (∇, g, ϕ, ξ) is a Sasakian statistical structure on M.

Theorem 3.3 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a Sasakian statistical manifold and $\iota : (M, g, \phi, \xi) \to (\widetilde{M}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ an invariant immersion. Then ι is doubly minimal. In fact,*

$$B(X, \phi Y) = \widetilde{\phi} B^*(X, Y)$$

holds for $X, Y \in \Gamma(TM)$.

See [6] for example.

An immersion ι of M into a Sasakian manifold $(\widetilde{M}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ is said to be *C-totally real* if $\widetilde{g}(\iota_* X, \widetilde{\xi}) = 0$ for all $X \in \Gamma(TM)$. In particular, a C-totally real submanifold M is said to be *Legendrian* if $\dim \widetilde{M} = 2 \dim M + 1$.

Proposition 3.4 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ be a Sasakian statistical manifold with $\widetilde{K}$ in (3.3). If $\iota : M \to (\widetilde{M}, \widetilde{g}, \widetilde{\phi}, \widetilde{\xi})$ is a C-totally real immersion, then $B = B^* = \widehat{B}$. In particular, if ι is a minimal C-totally real immersion, then it is doubly minimal.*

Proof Since $\widetilde{K}(\iota_* X, \iota_* Y) = 0$ for any $X, Y \in \Gamma(TM)$, we have that $\widetilde{\nabla}_X \iota_* Y = \nabla^{\widetilde{g}}_X \iota_* Y$, and that $B = \widehat{B}$. □

Example 3.5 (1) Let S^{2n-1} be a unit hypersphere in the Euclidean space $\mathbb{R}^{2n}$. Let J be a standard almost complex structure on $\mathbb{R}^{2n}$ considered as $\mathbb{C}^n$. Set $\xi = -JN$, where N is a unit normal vector filed of S^{2n-1}. Define $\phi \in \Gamma(T(S^{2n-1})^{(1,1)})$ by $\phi X = JX - g(JX, N)N$. Denote by g the standard metric of the hypersphere. Then such a (g, ϕ, ξ) is known as a standard Sasakian structure of S^{2n-1}. We set $K \in \Gamma(T(S^{2n-1})^{(1,2)})$ as in (3.3). Then, the quadruplet $(g, \nabla = \nabla^g + K, \phi, \xi)$ is a Sasakian statistical structure on S^{2n-1}.

(2) The natural inclusion of S^3 into S^5 is an invariant immersion between two Sasakian manifolds defined above. By Theorem 3.3, it is a doubly minimal immersion, in fact, a doubly totally-geodesic immersion between two Sasakian statistical manifolds.

(3) The immersion from a torus into the above Sasakian manifold (S^5, g, ϕ, ξ) defined by

$$S^1 \times S^1 \ni (u, v) \mapsto \frac{1}{\sqrt{3}}(\cos u, \sin u, \cos v, \sin v, \cos(u+v), -\sin(u+v)) \in S^5$$

is a minimal C-totally real immersion. By Proposition 3.4, this torus is a doubly minimal submanifold of the above statistical manifold (S^5, ∇, g).

In the end, we will state a non-existence theorem of doubly minimal immersions. For a statistical manifold (M, ∇, g), we define

$$U^{(\nabla,g)} = 2R^{\nabla^g} - \frac{1}{2}(R^{\nabla} + R^{\nabla^*}).$$

Then $U = U^{(\nabla,g)} \in \Gamma(TM^{(1,3)})$ has similar properties to those of the Riemannian curvature tensor field $R^{\nabla^g} \in \Gamma(TM^{(1,3)})$ for g, from which we can define $\rho^U = \mathrm{tr}_g \mathrm{Ric}^U$ like the scalar curvature. The tensor field U vanishes for a Hessian manifold of constant Hessian curvature zero, for example. See [7] for details.

Theorem 3.6 *Let $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$ be a Hessian manifold of constant Hessian curvature zero. Suppose that a statistical manifold (M, ∇, g) has a point x such that $\rho^U(x) > 0$. Then there exists no doubly minimal statistical immersion of (M, ∇, g) into $(\widetilde{M}, \widetilde{\nabla}, \widetilde{g})$.*

Proof We prove in [7] the following inequality at each point in M:

$$\frac{1}{2}(\|H\|^2 + \|H^*\|^2) \geq \frac{1}{m(m-1)}\rho^U \tag{3.4}$$

The theorem is a direct consequence. □

It is a classical result that a 2-dimensional Riemannian manifold with a positive Gaussian curvature point admits no minimal isometric immersion into the Euclidean space. In the case where the ambient space is the Euclidean space, Theorem 3.6 reduces to this fact. See [7] again for details and a generalization of Theorem 3.6.

4 Appendix

We will present several related problems in this section, which is added by following the suggestion of the editorial committee.

1. Characterize doubly minimal statistical immersions from a variation problem. In the Riemannian submanifold theory, minimal immersions are characterized as critical points of the volume functional determined by the induced metric. What is the counterpart of this fact in the statistical submanifold theory? Is it related to the stability in the Riemannian minimal submanifold theory?

2. Characterize statistical structures which admit doubly minimal statistical immersions into *standard* statistical manifolds. In other words, construct the statistical submanifold version of the following theorem (See [3]).

Theorem 4.1 *Let (M, g) be a simply connected Riemannian manifold of dimension 2. Denote by K_g the Gaussian curvature of g. Let $M(c)$ be the 3-dimensional space form of section curvature $c \in \mathbb{R}$. Suppose that $K_g < c$ everywhere, and set a Riemannian metric on M by $\widehat{g} = (c - K_g)g$. The Riemannian manifold (M, g) admits an isometric minimal immersion into $M(c)$ if and only if the Gaussian curvature of $\widehat{g}$ satisfies $K_{\widehat{g}} = 1 + \dfrac{c}{K_g - c}$.*

This theme also includes an important problem which is to determine the counterpart for $M(c)$. Is our tensor field U useful for this problem? Is the property to admit *many* doubly totally-geodesic submanifolds useful for this problem as well?

The elementary contents of this article should have been written much earlier than many detailed works on inequalities for statistical submanifolds got published. We refer the readers to the surveys [1, 2] due to B.-Y. Chen.

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Declarations

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