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ORIGINAL RESEARCH

Asymptotic and bootstrap confidence intervals of generalized process capability index C_{py} for exponentially distributed quality characteristic

Mahendra Saha[1] · Sumit Kumar[1] · Sudhansu S. Maiti[2] · Abhimanyu Singh Yadav[1]

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Abstract

The process capability index (PCI) is a useful tool for assessing the capability of a manufacturing process. It plays an important role in monitoring and analyzing process quality and productivity. Since PCI is based on sample observations, it is a point estimate of the true PCI. It is well known that confidence interval (CI) provides much more information about the population characteristic of interest than does a point estimate. In this article, asymptotic confidence interval (ACI) and three bootstrap confidence intervals (BCIs) namely, standard bootstrap (*s*-boot), percentile bootstrap (*p*-boot), and Student's *t* bootstrap (*t*-boot) of the generalized process capability index (GPCI) C_{py}, defined as the ratio of proportion of specification conformance to proportion of desired conformance, are studied through simulation when the underlying distribution follows exponential distribution. Method of maximum likelihood is used to estimate the parameter of the model. Monte Carlo simulation has been carried out to investigate the estimated average widths and coverage probabilities of the ACI and BCIs of C_{py}. Finally, two data sets have been analyzed for illustrative purpose.

Keywords Process capability index · Generalized process capability index · Maximum-likelihood estimate · Asymptotic confidence interval · Bootstrap confidence intervals

1 Introduction

From statistical point of view, we can employ several techniques to measure the capability of a manufacturing process, namely, graphical methods, design of experiments, and process capability indices (PCIs). PCI is a indicator function which establishes the relationship between the actual process performance and manufacturing specifications, designed by the designers or customers. The first PCI C_p, introduced by Juran (1974), has taken into account the process standard deviation (σ) only. However, C_p did not detect departure of the process mean (μ) from the specification center. Then, second-generation PCI C_{pk} came in the trend, which was developed by Kane (1986) by taking into account both μ and σ together, while C_{pk} did not consider the target value (T). Then, Hsiang and Taguchi (1985) introduced a new index C_{pm}, independently suggested by Chan et al. (1988), which took into account the target value (T) together with μ and σ. The third-generation PCI was developed by Pearn et al. (1992), abbreviated as 'PKJ', applicable in both the conditions when the process distribution is off-centered as well as off-target. All PCIs were designed to measure the process capability when the studied characteristic of the process is normal. Vannman (1995) defined a class of PCIs, depending on two non-negative parameters, given as

$$C_p(u, v) = \frac{d - u|\mu - m|}{3\sqrt{\sigma^2 + v(\mu - T)^2}},$$

where $m = (U + L)/2$, the specification center; U and L are the upper and the lower specification limits, respectively. The four basic indices, C_p, C_{pk}, C_{pm}, and C_{pmk}, are special cases of $C_p(u, v)$ by letting $u = 0$ or 1 and $v = 0$ or 1. More specially, $C_p(0, 0) = C_p$, $C_p(1, 0) = C_{pk}$, $C_p(0, 1) = C_{pm}$, and $C_p(1, 1) = C_{pmk}$, respectively.

In many cases, if the assumption of normality is violated, then PCIs calculated using the conventional methods could often lead to erroneous and misleading interpretation of the capability of manufacturing processes. Clements (1989)

✉ Sumit Kumar
stats.sumitbhal@gmail.com

[1] Department of Statistics, Central University of Rajasthan, Ajmer, Rajasthan, India

[2] Department of Statistics, Visva-Bharati University, Santiniketan, India

introduced a method by taking into account non-normal PCIs by considering the Pearsonian system and using their quantiles. Gilchrist (1993) introduced a quantile transformation which was based on standardized distribution instead of Pearson distribution. Johnson et al. (1994) applied the Clement's method and also obtained the estimators of PCIs. Mukherjee (1995) introduced a method of obtaining PCI for non-normal manufacturing process based on tolerance interval/prediction interval. An alteration of the Clement's method has been derived by Pearn and Chen (1995). Chen and Pearn (1997) proposed generalizations of $C_p(u, v)$ for any underlying distribution as

$$C_{Np}(u, v) = \frac{d - u|M - m|}{3\sqrt{\left(\frac{F_{0.99865} - F_{0.00135}}{6}\right)^2 + v(M - T)^2}},$$

where F_α is the αth percentile and M is the median of the process distribution, respectively. We observed that the generalizations were developed by replacing μ by M and σ by $\left(F_{0.99865} - F_{0.00135}\right)/6$ in $C_p(u, v)$, given by Vannman (1995). By setting (u, v) as (0, 0), (1, 0), (0, 1), and (1, 1) leads to the four basic indices for any underlying distributions, namely, C_{Np}, C_{Npk}, C_{Npm}, and C_{Npmk} (see Zwick 1995; Tong and Chen 1998), respectively.

Maiti et al. (2010) suggested a generalized process capability index (GPCI), defined as the ratio of proportion of specification conformance to proportion of desired conformance. PCI C_{py} can be measured under unilateral as well as bilateral specification limits and normal as well as non-normal process distributions, given as

$$C_{py} = \frac{F(U) - F(L)}{F(\mathrm{UDL}) - F(\mathrm{LDL})} = \frac{p}{p_0},$$

where $F(t) = P(X \leq t)$ is the cumulative distribution function of the quality characteristic X, p is the process yield, and p_0 is the desirable yield. LDL and UDL are the abbreviation of lower desirable limit and upper desirable limit, respectively. Sometimes, the practitioners may realized the LDL as lower tolerance limit (LTL) and UDL as upper tolerance limit (UTL), respectively. The beauty of the GPCI is that most of the PCIs defined in the literature are directly or indirectly related with this index.

In recent time, competitiveness in quality is not only central to profitability, but crucial to business survival, and industry is viable only if they provide satisfaction to their consumers. Use of PCIs in industry motivated the statistician and quality control engineers to focus on the point estimation and construction of confidence intervals (CIs) of these PCIs (see, Chan et al. 1988). The construction of confidence limits for PCIs was introduced by Hsiang and Taguchi (1985). Then, many other researchers have developed numerous techniques and tables to construct confidence limits for these PCIs. Chen and Hsu (1995) proposed the asymptotic distribution of C_{pmk}. Several researchers have developed numerous techniques and tables to construct confidence limits for these PCIs. Initially, the CIs were constructed for a normally distributed process, and nowadays, a frequently used approach is a non-parametric statistical method called bootstrap technique, introduced by Efron (1982), which did not require the assumption of process distribution for obtaining the confidence limits. Owing to cost and/or time constraints, data of large sample sizes are seldom realized in industry, and therefore, limited or insufficient data are used by the engineers/practitioners for statistical inference. When utilizing the bootstrap method to construct confidence limits for the PCIs, Wasserman and Franklin (1991, 1992) and Price and Price (1993) still used the conventional PCIs for non-normal process distribution. Later on, efforts were also made by several authors to develop the construction of CIs for non-normally distributed manufacturing processes (see, Peng 2010a, b; Rao et al. 2016; Dey et al. 2017; Saha et al. 2018).

In this article, our objective is to find out the asymptotic confidence interval (ACI) and bootstrap confidence intervals (BCIs), namely, standard bootstrap (*s*-boot), percentile bootstrap (*p*-boot), and Student's *t* bootstrap (*t*-boot) of C_{py} for exponentially distributed quality characteristic. To the best of our knowledge thus so far, no attempt has been made to study these ACI and BCIs based on C_{py} for exponentially distributed quality characteristic. Our aim is to fill up this gap through this present work.

The rest of the article is organised as follows: In Sect. 2, we have discussed some earlier work regarding C_{py} under the exponential distribution. ACI and BCIs of C_{py} based on exponential distribution have been discussed in Sect. 3. In addition, Monte Carlo simulation study is carried out to see the performance of the proposed ACI and BCIs of C_{py} under exponential distribution in terms of average width and coverage probabilities in Sect. 4. In Sect. 5, real-life examples are presented for illustrative purposes of the proposed study. The article ends with a brief conclusion given in Sect. 6.

2 Review of the earlier works

The assumption of exponentially distributed quality characteristics is generally valid for data that have a natural one-sided boundary with a large probability mass concentrated near this boundary. Gunter (198) pointed out some cases where this distributional assumption seems to be reasonable. Yeh and Bhattacharya (1998) pointed out that the exponential distribution arises frequently in industrial process. Perakis and Xekalaki

(2002) investigated the properties of the PCI in the case where the process distribution is exponential. In addition, the inferential aspects of the index C_{py} have been pointed out in case of exponentially distributed quality characteristic by Maiti et al. (2010), Maiti and Saha (2011). A process whose distribution can be regarded to be the exponential with parameter λ, the GPCI is given by (see, Maiti et al. 2010)

$$C_{py} = \frac{1}{p_0}\left(e^{-\lambda L} - e^{-\lambda U}\right).$$

In practice, the true value of λ is unknown. Using the invariance properties of the MLE, Maiti et al. (2010) obtained the MLE of C_{py}, given as follows:

$$\hat{C}_{py} = \frac{1}{p_0}\left(\mathrm{e}^{-\frac{L}{\bar{X}}} - \mathrm{e}^{-\frac{U}{\bar{X}}}\right).$$

Since $Z = \sum_{i=1}^{n} X_i$ is complete sufficient statistic for λ, using Lehmann–Scheffe theorem, the MVUE of C_{py} is given as

$$\begin{aligned}\tilde{C}_{py} &= \frac{1}{p_0}\left(\int_L^U f(x|\bar{X})\mathrm{d}x\right)\\ &= \frac{1}{p_0}\left\{\sum_{r=1}^{n-1}(-1)^r\binom{n-1}{r}\left(\frac{1}{n\bar{X}}\right)^r(L^r - U^r)\right\},\end{aligned}$$

where $\bar{X}$ is the sample mean and

$$f(x|\bar{X}) = \frac{n-1}{n\bar{X}}\left[1 - \frac{x}{n\bar{X}}\right]^{n-2}; \quad 0 < x < n\bar{X}.$$

By carrying out Monte Carlo simulation study with $\lambda = 0.2, 0.5, 0.7\ 1.0$ and with $n = 25, 50, 100, 150$, Maiti et al. (2010) showed that for $\lambda < 0.5$, $\hat{C}_{py}$ (MLE of C_{py}) gives better result than $\tilde{C}_{py}$ (MVUE of C_{py}) in MSE sense, but, for $\lambda > 0.5$, it is reverse. Maiti and Saha (2011) have looked into the inferential aspects of the GPCI C_{py} for exponential process distribution. Where only one specification limit has been set, this may be either U (e.g., surface roughness), or L (e.g., time until failure) is set, they have obtained the exact lower confidence limit of the index C_{py} in case of exponential process distribution. Maiti and Saha (2012) has also obtained the Bayesian estimation of the index C_{py} in case of exponential process distribution under squared error loss function. The Bayes' estimate of the index C_{py} has been compared with the frequentist counterpart.

3 Asymptotic confidence interval of C_{py}

In this section, we have derived the asymptotic distribution and $100(1-\alpha)\%$ ACI of C_{py} under the assumption of exponential distribution. In Sect. 1, some recent work related to the ACI has been discussed. It is to be noted that, to derive the ACI of C_{py}, we have to derive at first the asymptotic distribution of θ. It is to be noted that

$$\sqrt{n}(\hat{\theta} - \theta) \sim N\left(0,\ \theta^2\right)$$

and the $100(1-\alpha)\%$ ACI for θ is given by

$$\left\{\hat{\theta} \pm Z_{(\alpha/2)} \cdot \sqrt{\frac{\hat{\theta}^2}{n}}\right\},$$

where $Z_{(\alpha/2)}$ is the upper $(\alpha/2)$th point of the standard normal distribution. Here, we have derived the ACI of C_{py} using the MLE of the parameter θ. It can be easily shown that for large n:

$$\frac{\hat{C}_{py} - C_{py}}{\sqrt{Var(C_{py})}} \sim N(0,\ 1),$$

where

$$\mathrm{Var}(C_{py}) = \left(\frac{\partial C_{py}}{\partial\theta}\right)^2 \times \left(\frac{1}{E\left(-\frac{\partial^2 l}{\partial\theta^2}\right)}\right),$$

the log-likelihood function (l) of the parameter,

$$l = n\log\theta - \theta\sum_{i=1}^{n} x_i,$$

$$E\left(-\frac{\partial^2 l}{\partial\theta^2}\right) = \frac{n}{\theta^2}$$

and

$$\frac{\partial C_{py}}{\partial\theta} = \frac{1}{p_0}\left\{Ue^{-U\theta} - Le^{-L\theta}\right\}.$$

Hence, the $100(1-\alpha)\%$ ACI C_{py} can be easily obtained as

$$\left\{\hat{C}_{py} \pm Z_{(\alpha/2)}.\sqrt{\mathrm{Var}(\hat{C}_{py})}\right\}.$$

4 Bootstrap confidence intervals of C_{py}

In this section, we obtained the CIs of C_{py} using bootstrap method. Franklin and Wasserman (1991) and Choi and Bai (1996) have used bootstrap methods for estimating various PCIs. Here, we have obtained three BCIs, namely, Standard bootstrap (*s*-boot), Percentile bootstrap (*p*-boot), and Student's *t* bootstrap (*t*-boot) for calculating CIs of the GPCI C_{py}.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n drawn from exponential distribution with parameter θ.

ALGORITHM

- **Step 1:** From the given random sample of size n, we compute MLE $\hat{\theta}$ of θ. A bootstrap sample of size n is obtained from the original sample by putting 1 / n as mass at each point, denoted by $X_1^*, X_2^*, \ldots, X_n^*$.
- **Step 2:** We compute the MLE $\hat{\theta}^*$ of θ as well as $\hat{C}_{py}^*$ of C_{py}. The mth bootstrap estimator of C_{py} is computed as $\hat{C}_{py}^{*(m)} = \hat{C}_{py}(X_1^*, X_2^*, \ldots, X_n^*)$.
- **Step 3:** There are total number of n^n re-samples and we calculate B values of $\hat{C}_{py}^*$ from these re-samples. Each of these $\hat{C}_{py}^*$ would be estimator of $\hat{C}_{py}$. The arrangement of the entire collection in ascending would constitute an empirical bootstrap distribution $\left\{ \hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B \right\}$, will be denoted as $\hat{C}_{py}^{*(1)} \le \hat{C}_{py}^{*(2)} \le \cdots \le \hat{C}_{py}^{*(B)}$.

Here, in this study, we considered $B = 1000$ bootstrap samples.

4.1 Standard bootstrap (*s*-boot) confidence interval

Let $\bar{\hat{C}}_{py}^*$ and Se^* be the sample mean and sample standard deviation of $\left\{ \hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B \right\}$, that is

$$\bar{\hat{C}}_{py}^* = \frac{1}{B} \sum_{j=1}^{B} \hat{C}_{py}^{*(j)}$$

and

$$Se^* = \sqrt{\frac{1}{(B-1)} \sum_{j=1}^{B} \left(\hat{C}_{py}^{*(j)} - \bar{\hat{C}}_{py}^* \right)^2},$$

respectively. A $100(1-\alpha)\%$ ACI of the index C_{py} is given by

$$\left\{ \hat{C}_{py}^* - z_{(\alpha/2)} \cdot Se^*,\ \hat{C}_{py}^* + z_{(\alpha/2)} \cdot Se^* \right\}.$$

4.2 Percentile bootstrap (*p*-boot) confidence interval

Let $\hat{C}_{py}^{*(\tau)}$ be the τ percentile of $\left\{ \hat{C}_{py}^{*(j)}; j = 1, 2, \ldots, B \right\}$, i.e., $\hat{C}_{py}^{*(\tau)}$ is such that

$$\frac{1}{B} \sum_{j=1}^{B} I\left(\hat{C}_{py}^{*(j)} \le \hat{C}_{py}^{*(\tau)} \right) = \tau; \quad 0 < \tau < 1,$$

where $I(.)$ is the indicator function. A $100(1-\alpha)\%$ ACI of the index C_{py} is given by

$$\left\{ \hat{C}_{py}^{*(B\cdot(\alpha/2))},\ \hat{C}_{py}^{*(B\cdot(1-\alpha/2))} \right\},$$

where $\hat{C}_{py}^{*(r)}$ is the rth ordered value on the list of the B bootstrap estimators of C_{py}.

4.3 Student's *t* bootstrap (*t*-boot) confidence interval

Let S^* be the sample standard deviation of $\left\{ \hat{C}_{py}^{*(j)};\ j = 1, 2, \ldots, B \right\}$, that is

$$S^* = \sqrt{\frac{1}{B} \sum_{j=1}^{B} \left(\hat{C}_{py}^{*(j)} - \bar{\hat{C}}_{py}^* \right)^2},$$

where

$$\bar{\hat{C}}_{py}^* = \frac{1}{B} \sum_{j=1}^{B} \hat{C}_{py}^{*(j)}.$$

In addition, let $\hat{t}^{*(\tau)}$ be the τ percentile of $\left\{ \frac{\hat{C}_{py}^{*(j)} - \hat{C}_{py}}{S^*} \right\}; j = 1, 2, \ldots, B$, i.e., $\hat{t}^{*(\tau)}$ is such that

$$\frac{1}{B} \sum_{j=1}^{B} I\left(\frac{\hat{C}_{py}^{*(j)} - \hat{C}_{py}}{S^*} \le \hat{t}^{*(\tau)} \right) = \tau; \quad 0 < \tau < 1,$$

where $I(.)$ is the indicator function. A $100(1-\alpha)\%$ ACI of the index C_{py} is given by

$$\left\{ \bar{\hat{C}}_{py}^* - \hat{t}^{*(\alpha/2)} \cdot S^*,\ \bar{\hat{C}}_{py}^* + \hat{t}^{*(\alpha/2)} \cdot S^* \right\}.$$

To study the different CIs, we consider their estimated average widths. For each of the methods considered, the average width of the BCI is calculated based on the B different trials. The average width are given by

$$\text{Average width} = \frac{\sum_{i=1}^{B} \left(U_{P_i} - L_{W_i} \right)}{B},$$

and

$$\text{Coverage probability} = \frac{\text{Number} \left(L_W \le C_{py} \le U_P \right)}{B},$$

where L_W and U_P are the $100(1-\alpha)\%$ CI based on B replicates.

5 Simulation and discussion

Maiti et al. (2010) has shown that the MLE performed better than the MVUE of the GPCI C_{py} in case of exponentially distributed quality characteristics for their considered setup in the simulation study. In this section, a simulation study has been carried out to see the performance of ACI and BCIs of the GPCI C_{py} for exponentially distributed quality characteristics using MLE of the parameter θ. Here, we have considered the sample sizes $n = 10, 20,\ 30,\ 50$, and 100, and set the lower and the upper specification limits as 0 and 8.5, respectively. For each design, $B = 1000$ bootstrap samples with each of size n are drawn from the original sample and replicated 5000 times. The 95% ACI and BCIs are constructed for each methods. Simulations are performed using programs written in the open source statistical package *R* (see, Ihaka and Gentleman 1996).

The average widths, the difference between the upper and the lower specification limits and corresponding coverage probabilities, the true values of C_{py}, covered by $100(1-\alpha)\%$ CI, are calculated for comparing the performance of ACI and BCIs of C_{py}. Tables 1, 2 represent the estimated average widths and coverage probabilities of 95% ACI and BCIs of the index C_{py}. The results show that p-boot provide smaller average widths than ACI and BCIs (s-boot and t-boot) for $\theta \leq 0.10$ and t-boot provide smaller average widths than ACI and BCIs (s-boot and p-boot) for $\theta > 0.10$, respectively. In addition, it has been observed that, as the sample sizes increases, the average widths decrease in almost all cases.

6 Data analysis

In this section, real data sets are considered to illustrate ACI and BCIs (viz., s-boot, p-boot, and t-boot) of the GPCI C_{py} for exponential distribution using the MLE of the parameter. At first, we have checked whether the considered data sets come from the exponential distribution by the goodness of fit test, which are based on five statistics using the log-likelihood function evaluated at the MLE ($l(\hat{\theta})$). Let c be the total number of parameters to be fitted, n, the sample size, then the criteria are: Akaike information criteria

Table 1 C_{py} and its estimated average widths and coverage probabilities of ACI for exponential distribution

n	θ	C_{py}	Confidence limits		Average width	Coverage probability
			L	U		
10	0.10	0.602721	0.390397	0.843577	0.453180	0.910
20	0.10	0.602721	0.449501	0.778317	0.328816	0.924
30	0.10	0.602721	0.474479	0.744704	0.270225	0.931
50	0.10	0.602721	0.499644	0.709978	0.210334	0.945
100	0.10	0.602721	0.531367	0.680808	0.149440	0.951
10	0.25	0.926912	0.770140	1.075994	0.305854	0.860
20	0.25	0.926912	0.815899	1.038788	0.222889	0.906
30	0.25	0.926912	0.835535	1.019794	0.184259	0.914
50	0.25	0.926912	0.851559	0.998029	0.146469	0.941
100	0.25	0.926912	0.875389	0.978891	0.103501	0.933
10	0.50	1.037616	0.984680	1.076701	0.092020	0.817
20	0.50	1.037616	1.002437	1.064307	0.061869	0.843
30	0.50	1.037616	1.011034	1.059300	0.048265	0.859
50	0.50	1.037616	1.018043	1.054466	0.036423	0.902
100	0.50	1.037616	1.023407	1.049406	0.025998	0.928
10	0.75	1.050838	1.035132	1.060944	0.025812	0.747
20	0.75	1.050838	1.042590	1.056593	0.014003	0.816
30	0.75	1.050838	1.045121	1.055082	0.009961	0.804
50	0.75	1.050838	1.046898	1.053985	0.007086	0.840
100	0.75	1.050838	1.048263	1.053013	0.004749	0.883
10	1.00	1.052417	1.047848	1.055285	0.007437	0.698
20	1.00	1.052417	1.050582	1.053633	0.003051	0.763
30	1.00	1.052417	1.051247	1.053246	0.001999	0.805
50	1.00	1.052417	1.051579	1.052997	0.001417	0.811
100	1.00	1.052417	1.051943	1.052783	0.000839	0.879
10	1.25	1.052605	1.051439	1.053368	0.001920	0.666
20	1.25	1.052605	1.052171	1.052886	0.000715	0.732
30	1.25	1.052605	1.052312	1.052792	0.000480	0.788
50	1.25	1.052605	1.052435	1.052716	0.000280	0.813
100	1.25	1.052605	1.052523	1.052665	0.000142	0.838

Table 2 C_{py} and its estimated average widths and coverage probabilities of BCIs for exponential distribution

n	θ	C_{py}	Average width			Coverage probability		
			s-boot	p-boot	t-boot	s-boot	p-boot	t-boot
10	0.10	0.602721	0.448249	0.440395	0.505157	0.916	0.927	0.915
20	0.10	0.602721	0.326798	0.324286	0.363725	0.939	0.940	0.947
30	0.10	0.602721	0.270090	0.268997	0.296787	0.939	0.938	0.951
50	0.10	0.602721	0.209855	0.209023	0.226023	0.952	0.953	0.956
100	0.10	0.602721	0.149234	0.148908	0.157721	0.943	0.940	0.953
10	0.25	0.926912	0.282853	0.271228	0.215613	0.876	0.910	0.784
20	0.25	0.926912	0.215431	0.210767	0.180329	0.913	0.951	0.848
30	0.25	0.926912	0.179311	0.176866	0.155780	0.904	0.927	0.859
50	0.25	0.926912	0.142110	0.141100	0.128030	0.936	0.950	0.892
100	0.25	0.926912	0.102809	0.102401	0.095589	0.952	0.953	0.927
10	0.50	1.037616	0.099858	0.093496	0.042178	0.879	0.919	0.708
20	0.50	1.037616	0.062315	0.059550	0.031209	0.902	0.935	0.721
30	0.50	1.037616	0.051284	0.049577	0.029181	0.904	0.939	0.769
50	0.50	1.037616	0.037994	0.037152	0.024435	0.924	0.936	0.796
100	0.50	1.037616	0.025961	0.025632	0.019016	0.942	0.946	0.840
10	0.75	1.050838	0.036302	0.033167	0.008902	0.906	0.910	0.142
20	0.75	1.050838	0.018277	0.016955	0.005539	0.919	0.940	0.433
30	0.75	1.050838	0.014148	0.013309	0.005235	0.931	0.942	0.607
50	0.75	1.050838	0.008933	0.008545	0.003962	0.935	0.944	0.741
100	0.75	1.050838	0.005442	0.005306	0.003077	0.928	0.941	0.796
10	1.0	1.052417	0.013821	0.012030	0.001940	0.942	0.932	0.031
20	1.0	1.052417	0.005699	0.005057	0.001034	0.941	0.942	0.149
30	1.0	1.052417	0.003733	0.003373	0.000855	0.933	0.945	0.257
50	1.0	1.052417	0.002065	0.001911	0.000622	0.929	0.940	0.484
100	1.0	1.052417	0.001053	0.001004	0.000443	0.921	0.938	0.681
10	1.25	1.052605	0.005933	0.004863	0.000505	0.955	0.930	0.005
20	1.25	1.052605	0.002027	0.001696	0.000236	0.959	0.930	0.040
30	1.25	1.052605	0.001049	0.000891	0.000150	0.953	0.942	0.079
50	1.25	1.052605	0.000486	0.000428	0.000095	0.953	0.958	0.259
100	1.25	1.052605	0.000222	0.000206	0.000069	0.939	0.941	0.225

[$AIC = -2l(\hat{\theta}) + 2c$], Bayesian information criteria [$BIC = -2l(\hat{\theta}) + 2\ln(n)$], Hannan–Quinn information criterion [$HQIC = -2l(\hat{\theta}) + 2c\ln(\ln(n))$], Consistent Akaike information criterion [$CAIC = -2l(\hat{\theta}) + 2c * n/(n-c-1)$], and Kolmogorov–Smirnov statistic [$D = \max_{1\le i\le N}\left(F(X_i) - \frac{i-1}{N}, \frac{i}{N} - F(X_i)\right)$], and the results are reported in Table 3.

- **Data Set I:** The manufacturer of integrated circuits comprises the initial process of wafer and the final process of packaging. In an integrated circuit packaging factory, the manufacturing process generally includes the several steps, details are given in Leiva et al. (2014). In the wire bonding process, one of the important factors is the ball size which directly related the quality level. Thus, the quality characteristic to be monitored is the ball size (1 mil = 1 / 1000 in = 0.00254 mm). The data set is given below: 2.891, 4.035, 4.495, 2.890, 2.312, 3.158, 5.22 8, 3.334, 5.896, 5.639, 3.842, 1.590, 1.954, 1.842, 0. 680, 2.752, 1.301, 2.260, 0.889, 2.381, 0.619, 2.788, 1.050, 3.750, 3.508, 6.123, 6.549, 5.954, 2.207, 4.4 17, 4.805, 1.516, 2.227, 2.797, 1.636, 1.066, 0.940, 4.101, 4.542, 1.295, 1.770, 3.492, 5.706, 3.722, 6.6 44, 2.472, 1.383, 4.494, 1.694, 2.892, 2.111, 3.591, 2.093, 3.222, 2.891, 2.582, 0.665, 3.234, 1.102, 1.0 83, 1.508, 1.811, 2.803, 6.659, 0.923, 6.229, 3.177, 2.333, 1.311, 4.419, 2.495, 0.921, 4.061, 9.725, 1.6 00, 4.281, 3.360, 1.131, 1.618, 4.489, 3.696, 1.982, 2.413, 5.480, 1.992, 2.573, 1.845, 4.620, 6.221, 1.6 94, 4.882, 1.380, 3.982, 2.260, 2.366, 2.899, 3.782, 2 .336, 1.175, 3.055 Figure 1 displays the density, Q–Q plot, and fitted density of exponential distribution for the given data set, and Table 4 represents the 95% ACI and BCIs (s-boot, p-boot, and t-boot) of the index C_{py}. Here, the quality characteristic to be monitored is the ball size with the upper and lower specification limits $L = 0.50$ mil and $U = 8.00$ mil (1 mil=1 / 1000 in=0.00254 mm), respectively.

- **Data Set II:** This data set is used by Bhaumik et al. (2009), which is vinyl chloride data obtained from clean up gradient monitoring wells in mg/l. The data set is given below: 5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8, 0.8, 0.4, 0.6, 0.9, 0.4, 2, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1, 0.2, 0.1, 0.1, 1.8, 0.9, 2, 4, 6.8, 1.2, 0.4, 0.2 Figure 2 displays the density, *Q*–*Q* plot, and fitted density of exponential distribution for the given data set, and Table 4 represents the 95% ACI and BCIs (*s*-boot, *p*-boot, and *t*-boot) of the index C_{py}. Here, for the data set, we hypothetically choose $L = 0.5$ and $U = 7.5$, respectively.

From these given data sets, estimated widths of ACI and BCIs at 95% level of significance of the index C_{py} for the exponential distribution are represented in Table 4. For these data sets, it has been observed that *t*-boot provide smaller widths than ACI and BCIs (*s*-boot and *p*-boot). These results adequately support the simulation output also.

7 Conclusions

In this article, methodology has been proposed to analyze the capability of any manufacturing process using GPCI, C_{py}, where the quality characteristic follows exponential process distribution. As Maiti et al. (2010) has shown that the MLE performed better than the MVUE of C_{py} through simulation study under exponentially distributed quality characteristics, maximum-likelihood method of estimation of the parameter has been used for further analysis. In simulation study, ACI and BCIs are compared through average widths and corresponding coverage probabilities for exponentially distributed quality characteristics. It is to be noted that, in simulation study, *t*-boot performed better than ACI and other two BCIs for larger values of parameter, whereas *p*-boot performed better than ACI and other two BCIs for smaller values of the parameter, respectively. The real-life examples are also

Table 3 Model fitting summary for the considered data sets

Data set	Model	MLE	Log-likelihood	AIC	BIC	HQIC	CAIC	K–S statistic	*p* value
I	Exponential	0.329392	− 211.0505	424.1009	426.7061	425.1553	424.1417	0.9300	0.3587
II	Exponential	0.532081	− 55.4526	112.9052	114.4316	113.4257	113.0302	0.5588	0.9220

Table 4 Widths of ACI & BCI (*s*-boot, *p* boot, & *t*-boot) of C_{py}

Data	Distributions	$\hat{\theta}$	$\hat{C}_{py}$	ACI	BCI		
					s-boot	*p*-boot	*t*-boot
I	Exponential	0.329392	0.817307	0.020332	0.021810	0.019053	0.005972
II	Exponential	0.532081	0.787283	0.092077	0.102937	0.092175	0.044442

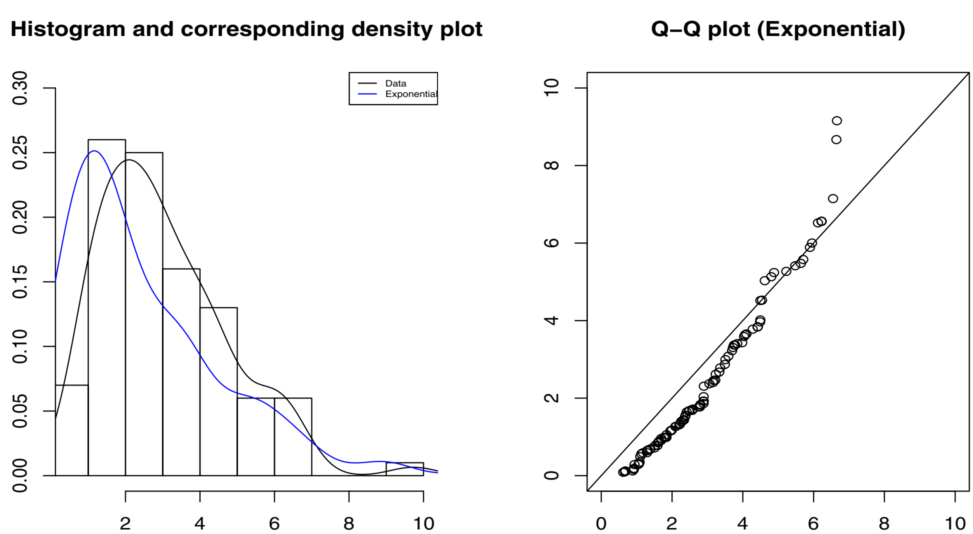

Fig. 1 Histogram, density, and Q–Q plot

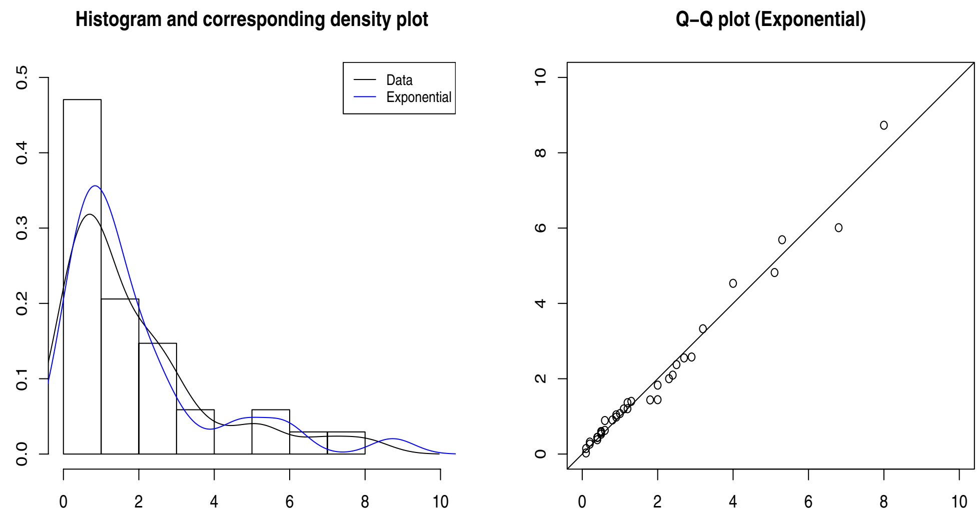

Fig. 2 Histogram, density, and Q–Q plot

adequately in favour to the simulation results. Even in case of limited data, engineers/practitioners can easily study the capability of any manufacturing process in case of exponentially distributed quality characteristics and can utilize the bootstrap technique to study CI of the PCIs. Furthermore, the results can be applied to any manufacturing industries for analyzing the capabilities for production process.

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