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Existence and uniqueness results for an elliptic equation with blowing-up coefficient and singular lower order term

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Abstract

In this paper, we study a class of nonlinear elliptic problems whose model is the following

$$\begin{cases} -\operatorname{div}\Big(b(u)|\nabla u|^{p-2}\nabla u\Big) = f\Big(1+\dfrac{1}{|u|^{\gamma}}\Big) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N (N \geq 2)$, $\gamma > 0$, b is a positive continuous function which blows up for a finite value of the unknown u. We will prove existence and uniqueness of a renormalized nonnegative solution in the case where the nonnegative source f belongs to $L^1(\Omega)$.

Keywords Nonlinear elliptic equations · Blowing-up coefficients · Singular lower order term · Renormalized solutions · Existence · Uniqueness

Mathematics Subject Classification 35J60 · 35J75

1 Introduction

In this paper we are interested in the existence and uniqueness of a renormalized solution for a classe of nonlinear elliptic equations of the type

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$$\begin{cases} -\operatorname{div}\Big(a(x,u,\nabla u)\Big) = f\Big(1+\dfrac{1}{u^{\gamma}}\Big) \text{ in } \Omega, \\ u \geq 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded open subset of $\mathbb{R}^N (N \geq 2)$, $\gamma > 0$, f is a nonnegative function which belongs to $L^1(\Omega)$ and $a(x,u,\nabla u)$ is a Carathéodory function which blows up at a finite value of the unknown u. More precisley, let $m > 0$ and assume that there exists a function $b \in C^0((-\infty, m), (0, +\infty))$ which satisfies $b(s) > 0$, $\forall s \in (-\infty, m)$, $\lim_{s\to m^-} b(s) = +\infty$ and such that $a(x,s,\xi)\xi \geq \alpha b(s)|\xi|^p$ for almost every $x \in \Omega$, for any $s \in (-\infty, m)$ and for any $\xi \in \mathbb{R}^N$, with $\alpha > 0$.

When the function b blows up at a finite value $m > 0$, $b(s) \geq \alpha_0 > 0$ for any $s \in (-\infty, m)$ and $\gamma = 0$, problems similar to (1.1) have been considered in the literature under various assumptions and in different context on the equations, for more details, we refer to [1, 2, 16, 19, 21, 22, 27, 33, 35]. In these papers, it is natural to look for solutions to (1.1) that are less or equal than m depending on the nature of the integral $\int_0^m b(s)\,ds$. Indeed, if $\int_0^m b(s)\,ds = +\infty$, then the solutions do not reach m almost everywhere in Ω, so that one can give a sense to the field $a(x,u,\nabla u)\nabla u$ at $\{u = m\}$ which insures that u is a weak solution. Otherwise, if $\int_0^m b(s)\,ds < +\infty$, then the solutions may attain the value m almost everywhere in Ω (i.e. $meas(\{u = m\}) > 0$) and the energy term $a(x,u,\nabla u)\nabla u$ is well defined at $\{u = m\}$ provided the hypothesis of the smallness on the Lebesgue norm of the data f. In order to avoid this assumption on f, the framework of renormalized or entropy solutions is then employed and allows us to get the existence result and then to give a sense to the energy term $a(x,u,\nabla u)\nabla u$ on the set $\{u < m\}$ even if the datum f is merely integrable.

Now if $b \equiv 1$, problem (1.1) has been studied by many authors in the past. In the linear case and if $f(x)\Big(1+\dfrac{1}{s^{\gamma}}\Big) = g(x,s)$, we refer in particular to the classical papers: [38] by Stuart, [12] by Crandall et al. and [24] by Lazer and McKenna. In [12, 38] the authors proved the existence and regularity results of classical nonnegative solutions (i.e a $C^2(\Omega) \cap C_0(\overline{\Omega})$ solutions) if $g(x,s)$ and the boundary $\partial\Omega$ are smooth enough. In [3], existence and regularity of solutions has been studied by Boccardo and Orsina when the datum f belongs to $L^m(\Omega)$, $m \geq 1$. They have proved the existence and regularity of solutions by discussing the cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$. In the nonlinear case, the authors in [10, 11, 31, 32, 36] proved the existence of weak solutions when the main operator satisfies the Leray–Lions assumptions and the datum f belongs to $L^m(\Omega)$, $m \geq 1$ or belongs to the space of the Radon mesure. For a review of more results about problems having singular lower order term, we refer to [5–11, 13, 15, 17, 20, 28, 31, 34, 39] and the references therein.

In the present paper, motivated by the works [1, 18, 23], we focus on the existence and uniqueness of a renormalized solution of problem (1.1). Here, in the left hand side of (1.1), we asume that the main operator has a singularity at $u = m$, the tem in right hand side is singular at $u = 0$ and f nonnegative and belongs to L^1. So, to give a sense to our problem, we have to manage both the sets $\{u = m\}$ and $\{u = 0\}$. For this purpose, we will use the framework of renormalized solutions introduced in [14, 29, 30] for L^1 or measure data in order to handle the singularity of the coefficient b near m. In the spirit of [1], we give a definition of renormalized solutions by considering the formulation of the problem (1.1) in $\{u < m\}$ and to precise carfully the behavior of the energy term near to set $\{u = m\}$. On the other hand, to deal with the singular term in the right hand of (1.1), it is not convenient, since the principal operator is singular at $\{u = m\}$ and does not satisfy any growth assumption with respect to u to apply the strong maximum principle. To bypass this difficulty, we will use suitable test function as in [15, 18, 23] to handle the set where the solution is near to zero.

As far as the uniqueness of a renormalized solution for (1.1) is concerned, in [18] the authors proved the uniqueness of an entropy solution to (1.1) in the case where the principle operator degenerates at infinity by using an additional assumption on the Carathéodory function a and the fact the singular term is nonincreasing. In this work, we show that the singular term in the right hand side of (1.1) will help us to extend and improve the uniqueness result proved in [1]. Indeed, in [1] it was given a partial uniqueness result by assuming that $\{u_1 = m\} = \{u_2 = m\}$, where u_1 and u_2 are two nonnegative renormalized solutions. Our aim is then to establish the uniqueness result by avoiding the use of this assumption.

The paper is organized as follows. In Sect. 2 we precise the assumptions on the data and we state the definition of solutions and the main results. In Sect. 3 we prove our existence result by means of an approximation procedure. Section 4 is devoted to the study the case of strong singularity $\gamma > 1$. In Sect. 5 we will establish the uniqueness result for the renormalized solution of problem (1.1).

2 Assumptions on the data and statements of main results

Let us specify the assumptions of the problem (1.1) that we will study. Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$, $\gamma > 0$. Let $1 < p < N$, $m > 0$ and $a(x, s, \xi) : \Omega \times (-\infty, m) \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function such that

$$a(x, s, \xi) = b(s)\overline{a}(x, s, \xi) \quad \text{a.e. } x \in \Omega,\ \forall s \in (-\infty, m),\ \forall \xi \in \mathbb{R}^N,$$

where b is a continuous function of $C^0((-\infty, m), \mathbb{R}^+)$ satisfying

$$b(s) > 0,\ \forall s \in (-\infty, m),\ \lim_{s \to m^-} b(s) = +\infty,\ \text{and} \int_0^m b^{\frac{1}{p-1}}(s)\, ds < +\infty. \tag{2.1}$$

The Carathéodory function $\overline{a}(x, s, \xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following assumptions

$$\overline{a}(x, s, \xi).\xi \geq \alpha|\xi|^p \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,\ \text{with } \alpha > 0. \tag{2.2}$$

$$\overline{a}(x, s, t\xi) = t^{p-1}\overline{a}(x, s, \xi), \quad \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,\ \forall t \geq 0. \tag{2.3}$$

For any $k > 0$, there exist a constant $C_k > 0$ and a positive function $L \in L^{p'}(\Omega)$ with $p' = \frac{p}{p-1}$ such that

$$|\overline{a}(x,s,\xi)| \leq C_k(L(x) + |\xi|^{p-1}) \quad \text{a.e. } x \in \Omega,\ \forall s \in (-k,k),\ \forall \xi \in \mathbb{R}^N, \tag{2.4}$$

$$[\overline{a}(x,s,\xi) - \overline{a}(x,s,\xi')][\xi - \xi'] > 0,\ \text{a.e. } x \in \Omega,\ \forall s \in \mathbb{R},\ \forall \xi, \xi' \in \mathbb{R}^N, \xi \neq \xi'. \tag{2.5}$$

The nonnegative function f is measurable such that

$$f \in L^1(\Omega). \tag{2.6}$$

Throughout the paper, we will make use of the following functions: for every $k, l > 0$ and $r \in \mathbb{R}$, the functions T_k, T_l^k and S_k are defined by

$$T_k(r) = \begin{cases} r & \text{if } |r| \leq k, \\ k\frac{r}{|r|} & \text{if } |r| > k, \end{cases}$$

$$T_l^k(r) = \begin{cases} -k & \text{if } r \leq -k, \\ r & \text{if } -k \leq r \leq l, \\ l & \text{if } r \geq l, \end{cases}$$

and

$$S_k(r) = \begin{cases} 1 & \text{if } |r| \leq k, \\ \frac{2k-|r|}{k} & \text{if } k \leq |r| \leq 2k, \\ 0 & \text{if } |r| > 2k. \end{cases}$$

For $j \geq 1$ fixed, we define the functions

$$h_j(r) = \begin{cases} 1 & \text{if } r \leq m - \frac{2}{j}, \\ j(m - \frac{1}{j} - r) & \text{if } m - \frac{2}{j} \leq r \leq m - \frac{1}{j}, \\ 0 & \text{if } r \geq m - \frac{1}{j}. \end{cases}$$

For the sake of simplicity, we will use when referring to the integrals the following notation

$$\int_\Omega f = \int_\Omega f(x)\,\mathrm{d}x.$$

Finally, throughout this paper, the symbol $\omega(n, \sigma, j)$ will denote any quantity that vanishes as the argument goes to its natural limit (that is $n \to +\infty$, $\sigma \to 0$ and $j \to +\infty$).

Now we specify the definition of renormalized solutions for problem (1.1) which can be seen as an adaptation of the one introduced in [1] (see also [19]).

Definition 2.1 (The case $\gamma \le 1$) A positive function u in $W_0^{1,1}(\Omega)$ is a renormalized solution of problem (1.1) if

$$T_k(u) \in W_0^{1,p}(\Omega), \text{ for every } k > 0, \tag{2.7}$$

$$0 \le u \le m, \quad \text{a.e. in } \Omega, \tag{2.8}$$

$$a(x, u, \nabla u)\chi_{\{0\le u<m\}} \in (L^{p'}(\Omega))^N. \tag{2.9}$$

For any function $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$, such that $\nabla\varphi = 0$ a.e. on $\{x \in \Omega, u(x) = m\}$ one has

$$\lim_{\sigma\to 0} \frac{1}{\sigma}\int_{\{m-2\sigma\le u\le m-\sigma\}} a(x, u, \nabla u)\nabla u\varphi = \int_{\{u=m\}} f\Big(1 + \frac{1}{u^\gamma}\Big)\varphi, \tag{2.10}$$

and if, for every function $S \in W^{1,\infty}(\mathbb{R})$ such that the support of S is compact and $S(m) = 0$, the solution u satisfies

$$\frac{f}{u^\gamma} S(u)\varphi \in L^1(\Omega), \tag{2.11}$$

$$\int_\Omega S(u)\, a(x, u, \nabla u)\chi_{\{u<m\}}\nabla\varphi + \int_\Omega S'(u)\, a(x, u, \nabla u)\nabla u\chi_{\{u<m\}}\varphi$$
$$= \int_\Omega f\Big(1 + \frac{1}{u^\gamma}\Big)S(u)\varphi, \tag{2.12}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Remark 2.2 Due to (2.7) and (2.8), we deduce that u belongs to $W_0^{1,p}(\Omega)$. Indeed, Indeed, let $\varepsilon > 0$, by (2.7) with $k = m + \varepsilon$, one has $T_{m+\varepsilon}(u) \in W_0^{1,p}(\Omega)$, so using (2.8) and since we can write

$$\int_\Omega |\nabla u|^p = \int_{\{0<u\le m\}} |\nabla u|^p + \int_{\{m<u\ m+\varepsilon\}} |\nabla u|^p,$$

from where we deduce that $u \in W_0^{1,p}(\Omega)$.

We want also to point out that since we deal with nonnegative solutions, only the behavior near the set $\{u = m\}$ appears in the above definition. Finally, it is easy to see, according to the conditions (2.7), (2.9), (2.11) and the assumptions (2.1)–(2.6), that each term in the formulation (2.12) is well defined.

Now we state the first main result of this paper.

Theorem 2.3 *Assume that* (2.1)–(2.6) *hold true. If* $\gamma \le 1$*, then, there exists at least a renormalized solution* u *of problem* (1.1).

The second main result deals with the uniqueness of a renormalized solution of problem (1.1) under additional assumption on the Carathéodory function $\overline{a}$. We have the following result

Theorem 2.4 *Assume that* (2.1)–(2.6) *hold true. Moreover, assume that for every* $k > 0$*, there exist* $\gamma_k \geq 0$ *and* E_k *in* $L^{p'}(\Omega)$ *such that*

$$\left|\overline{a}(x,s,\xi) - \overline{a}(x,s',\xi)\right| \leq |s-s'|\Big[E_k(x) + \gamma_k|\xi|^{p-1}\Big], \tag{2.13}$$

for almost every $x \in \Omega$*, for every* s *and* s' *such that* $|s| \leq k$ *and* $|s'| \leq k$*, and for every* $\xi \in \mathbb{R}^N$*. If* $\gamma \leq 1$*, then, there exists a unique renormalized solution* u *to problem* (1.1).

3 A priori estimates and existence result

In order to prove our existence result, we need to consider the following approximate problem of (1.1).

$$\begin{cases} -\operatorname{div}\Big(a_n(x,u_n,\nabla u_n)\Big) = f_n\Big(1 + \dfrac{1}{(|u_n| + \frac{1}{n})^{\gamma}}\Big) \text{ in } \Omega, \\ u_n = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.1}$$

where for any $n \in \mathbb{N}^*$, for almost every $x \in \Omega$, for every $s \in \mathbb{R}$ and for every $\forall\xi \in \mathbb{R}^N$, we have set $a_n(x,s,\xi) = a(x, T^n_{m-\frac{1}{n}}(s), \xi)$, $b_n(s) = b(T^n_{m-\frac{1}{n}}(s))$, and $f_n = T_n(f)$. By the classical results in [25, 26] and by means of the Schauder's fixed theorem, there exists at least a weak solution $u_n \in W_0^{1,p}(\Omega)$ of problem (3.1) such that

$$\int_\Omega a_n(x,u_n,\nabla u_n)\nabla v = \int_\Omega f_n\Big(1 + \frac{1}{(|u_n| + \frac{1}{n})^{\gamma}}\Big)v, \ \forall v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega). \tag{3.2}$$

Moreover, as the right hand side belongs to $L^\infty(\Omega)$, thanks to [37], we deduce that u_n belongs to $L^\infty(\Omega)$.
Now if we take $v = u_n^-$ in (3.2), where $s^- = \min(s,0)$, the assumption (2.1) and the positivity of the right hand side of (3.1) lead to

$$\begin{aligned} \inf_{\{-n\leq s\leq 0\}} b(s) \int_\Omega |\nabla u_n^-|^p &\leq \int_\Omega a_n(x,u_n,\nabla u_n)\nabla u_n^- \\ &\leq \int_\Omega f_n\Big(1 + \frac{1}{(|u_n| + \frac{1}{n})^{\gamma}}\Big)u_n^- \leq 0, \end{aligned}$$

from where we deduce that $u_n^- = 0$, so that $u_n \geq 0$.
In order to achieve our existence results stated in Theorem 2.3, the proof needs to be split into 5 steps.
⋆ **Step 1**. We give some a priori estimates and pointwise convergence results related to the approximate solutions u_n. To this end, let $\varepsilon > 0$, with $\varepsilon < \frac{1}{n}$ and taking

$(T_k(u_n)+\varepsilon)^\gamma - \varepsilon^\gamma$ as a test function in (3.1), by (2.1), we obtain

$$\gamma \int_\Omega b_n(u_n)\frac{|\nabla T_k(u_n)|^p}{(T_k(u_n)+\varepsilon)^{1-\gamma}} \le \|f\|_{L^1(\Omega)}(k+\varepsilon)^\gamma + \int_\Omega f_n \frac{(T_k(u_n)+\varepsilon)^\gamma}{(u_n+\frac{1}{n})^\gamma}$$
$$\le \|f\|_{L^1(\Omega)}(k+\varepsilon)^\gamma + \|f\|_{L^1(\Omega)}. \tag{3.3}$$

On the other hand, using the continuity of the function b and the definition of $T^n_{m-\frac{1}{n}}$, one gets

$$\int_\Omega b_n(u_n)\frac{|\nabla T_k(u_n)|^p}{(T_k(u_n)+\varepsilon)^{1-\gamma}} \ge \inf_{\{0\le s\le m-\frac{1}{n}\}} b(s)\int_\Omega \frac{|\nabla T_k(u_n)|^p}{(T_k(u_n)+\varepsilon)^{1-\gamma}},$$

so, from (3.3) one obtains

$$\int_\Omega |\nabla T_k(u_n)|^p \le C\Big((k+\varepsilon)+(k+\varepsilon)^{1-\gamma}\Big). \tag{3.4}$$

Thus, letting ε goes to zero in (3.4), it follows that

$$\int_\Omega |\nabla T_k(u_n)|^p \le C(k+k^{1-\gamma}), \tag{3.5}$$

where C is a constant which does not depend on the index n of the sequence. Moreover, by (3.5), we deduce from a classical argument (see, e.g. [29]) that, up to a subsequence still indexed by n,

$$u_n \to u \ \text{ a.e. in } \ \Omega. \tag{3.6}$$
$$T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ W_0^{1,p}(\Omega), \tag{3.7}$$

where u is a measurable function which is finite almost everywhere in Ω. Indeed, using (3.5) and Poincaré inequality, we get

$$meas\{u_n \ge k\} \le C\Big(\frac{1}{k^{p-1}} + \frac{1}{k^{p-1+\gamma}}\Big),$$

then, letting k goes to infinity leads to

$$\lim_{k\to+\infty} \sup_n meas\{u_n \ge k\} = 0.$$

Hence, by (3.6) and Fatou's lemma, we deduce that u is finite almost everywhere in Ω.

Next, we use $T_k(v_n)$ as test function in (3.1), where $v_n = \displaystyle\int_0^{u_n} b_n(s)^{\frac{1}{p-1}}\,ds$, by the assumption (2.1), we obtain

$$\int_\Omega |\nabla T_k(v_n)|^p \le k\|f\|_{L^1(\Omega)} + \int_\Omega f_n \frac{T_k(v_n)}{(u_n+\frac{1}{n})^\gamma}. \tag{3.8}$$

As regards the second term in the right-hand side of (3.8), we have

$$\int_\Omega f_n \frac{T_k(v_n)}{(u_n+\frac{1}{n})^\gamma} = \int_{\{u_n\le m-\frac{1}{n}\}} f_n \frac{T_k(v_n)}{(u_n+\frac{1}{n})^\gamma} + \int_{\{u_n\ge m-\frac{1}{n}\}} f_n \frac{T_k(v_n)}{(u_n+\frac{1}{n})^\gamma}.$$

Thanks to Hôpital rule, it is easy to see, since $\gamma \le 1$ that $\dfrac{1}{s^\gamma}\displaystyle\int_0^s b_n^{\frac{1}{p-1}}(r)\,dr$ is bounded near to zero, then from (3.8), we deduce that

$$\int_\Omega |\nabla T_k(v_n)|^p \le C\Big(1+k\Big), \tag{3.9}$$

where C is a constant independent of n. So, by virtue of the classical arguments, we deduce that for a subsequence still indexed by n

$$v_n \to v = \int_0^u b(s)^{\frac{1}{p-1}}\,ds \ \text{ a.e. in } \ \Omega. \tag{3.10}$$

$$T_k(v_n) \rightharpoonup T_k(v) \ \text{ weakly in } \ W_0^{1,p}(\Omega). \tag{3.11}$$

In view of (3.9) and by means of Poincaré inequality, we get

$$meas\{v_n \ge k\} \le C\Big(\frac{1}{k^p}+\frac{1}{k^{p-1}}\Big),$$

then, letting k goes to infinity leads to

$$\lim_{k\to+\infty} \sup_n meas\{v_n \ge k\} = 0,$$

so, by using (3.10) and Fatou's lemma, it follows that v is almost everywhere finite. In the following, we are going to prove that $\dfrac{f_n}{(u_n+\frac{1}{n})^\gamma}$ is bounded in $L^1_{loc}(\Omega)$ independently of n. Let $0 \le \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and for some $k < \frac{m}{2}$ we take $S_k(u_n)\varphi$ as test function in (3.1), by dropping the positive terms, we obtain

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\le k\}}\varphi \le \int_\Omega S_k(u_n)a_n(x,u_n,\nabla u_n)\nabla\varphi,$$

then by (3.5) and the assumptions (2.4), it yields that

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n\le k\}}\varphi \le \int_\Omega |a(x,T_{2k}(u_n),\nabla T_{2k}(u_n))||\nabla\varphi| \le C_k, \tag{3.12}$$

where C_k is a constant which depends on k and not on the index n of the sequence. Then, by choosing for example $k=\frac{m}{3}$ and observing that

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi = \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi\chi_{\{u_n\le\frac{m}{3}\}} + \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi\chi_{\{u_n>\frac{m}{3}\}},$$

using (3.12), it follows that

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi \le C, \tag{3.13}$$

for every $0\le\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ and where the constant C is independent of n. To verify that (2.8) of the Definition 2.1 holds, we will argue as in [1, 2]. Indeed, by taking $T_{2m}(u_n)-T_m(u_n)$ as test function in (3.1) and in view of the approximation of b, $\overline{a}$ and (2.2), we obtain

$$\begin{aligned}\alpha b\left(m-\frac{1}{n}\right)&\int_\Omega |\nabla(T_{2m}(u_n)-T_m(u_n))|^p\\ &\le \int_{\{u_n>m\}} f_n\left(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\right)(T_{2m}(u_n)-T_m(u_n)),\end{aligned}$$

since $|T_{2m}(s)-T_m(s)|\le m$ for every $s\in\mathbb{R}$, we obtain

$$\begin{aligned}\alpha b\left(m-\frac{1}{n}\right)&\int_\Omega |\nabla(T_{2m}(u_n)-T_m(u_n))|^p\\ &\le \int_{\{u_n>m\}} f\left(1+\frac{1}{m^\gamma}\right)m \le (m+m^{1-\gamma})\|f\|_{L^1(\Omega)},\end{aligned}$$

from where with the help of Poincaré inequality, we get

$$\int_\Omega |T_{2m}(u_n)-T_m(u_n)|^p \le \frac{C}{b(m-\frac{1}{n})}\|f\|_{L^1(\Omega)}.$$

So, in view of (3.6), Fatou's lemma together with the fact that $b(m-\dfrac{1}{n})$ goes to infinity as $n\to+\infty$, we deduce that

$$T_{2m}(u)-T_m(u)=0 \text{ a.e. in } \Omega,$$

As a consequence, (2.8) of the Definition 2.1 holds.

Now for $j > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$, we choose $1 - S_j(v_n)$ as test function in (3.1), which gives

$$\begin{aligned}\frac{1}{j}\int_{\{j\le v_n\le 2j\}} & b_n^{\frac{1}{p-1}}(u_n)a_n(x,u_n,\nabla u_n)\nabla u_n \\ &\le \int_{\{v_n\ge j, u_n\le m-\frac{1}{n}\}} f\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)(1-S_j(v_n)) \\ &\quad + \int_{\{v_n\ge j, u_n\ge m-\frac{1}{n}\}} f\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)(1-S_j(v_n)), \end{aligned} \tag{3.14}$$

since $j > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$ implies that the first term in the right hand side of (3.14) is equal to zero, one has

$$\frac{1}{j}\int_{\{j\le v_n\le 2j\}} b_n^{\frac{1}{p-1}}(u_n)a_n(x,u_n,\nabla u_n)\nabla u_n \le \Big(1+\frac{1}{m^\gamma}\Big)\int_{\{v_n\ge j\}} f \tag{3.15}$$

so, letting n tends to $+\infty$ and then j tends to $+\infty$ in (3.15), using (3.6) and the equi-integrability of f, it follows that

$$\frac{1}{j}\int_{\{j\le v_n\le 2j\}} b_n^{\frac{1}{p-1}}(u_n)a_n(x,u_n,\nabla u_n)\nabla u_n = \omega(n,j). \tag{3.16}$$

⋆ **Step 2**. We have now all the ingredients to show that the sequence $T_k(v_n)$ converges to $T_k(v)$ strongly in $W_0^{1,p}(\Omega)$, for all $k > 0$. For any given $j \ge 1$ and $k > 0$, we choose $S_j(v_n)(T_k(v_n) - T_k(v))$ as test function in (3.1), it results

$$\begin{aligned}\int_\Omega & S_j(v_n)a_n(x,u_n,\nabla u_n)\nabla(T_k(v_n)-T_k(v)) \\ &= \frac{1}{j}\int_{\{j\le v_n\le 2j\}} a_n(x,u_n,\nabla u_n)\nabla v_n(T_k(v_n)-T_k(v)) \\ &\quad + \int_\Omega f_n\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)S_j(v_n)(T_k(v_n)-T_k(v)). \end{aligned} \tag{3.17}$$

For the first term in the left hand side of (3.17), let us remark, that for $j, k > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$, one has $0 \le v_n \le j$ is equivalent to $0 \le u_n \le \overline{j}$ and $0 \le v_n \le k$ is equivalent to $0 \le u_n \le \overline{k}$ respectively. So, choosing $j > k$, $n > \overline{k}$ and by the assumption (2.3), one can write

$$\begin{aligned}\int_\Omega & S_j(v_n)a_n(x,u_n,\nabla u_n)\nabla(T_k(v_n)-T_k(v)) \\ &= \int_{\{0\le v_n\le k\}} S_j(v_n)a_n(x,u_n,\nabla u_n)\nabla(T_k(v_n)-T_k(v)) \end{aligned}$$

$$
\begin{aligned}
&+\int_{\{k\le v_n\le 2j\}} S_j(v_n)a_n(x,u_n,\nabla u_n)\nabla(T_k(v_n)-T_k(v))\\
&=\int_\Omega (\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v_n))-\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v)))\nabla(T_k(v_n)-T_k(v))\\
&+\int_{\{k\le v_n\le 2j\}} S_j(v_n)\overline{a}(x,T_{\overline{j}}(u_n),\nabla v_n)\nabla(T_k(v_n)-T_k(v))\\
&+\int_\Omega \overline{a}(x,T_{\overline{j}}(u_n),\nabla T_k(v))\nabla(T_k(v_n)-T_k(v)).
\end{aligned}
$$

Then, we can rewrite (3.17) as follows

$$
\begin{aligned}
&\int_\Omega (\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v_n))-\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v)))\nabla(T_k(v_n)-T_k(v))\\
&=-\int_{\{k\le v_n\le 2j\}} S_j(v_n)\overline{a}(x,T_{\overline{j}}(u_n),\nabla v_n)\nabla(T_k(v_n)-T_k(v))\\
&-\int_\Omega \overline{a}(x,T_{\overline{j}}(u_n),\nabla T_k(v))\nabla(T_k(v_n)-T_k(v))\\
&+\frac{1}{j}\int_{\{j\le v_n\le 2j\}} a(x,u_n,\nabla u_n)\nabla v_n(T_k(v_n)-T_k(v))\\
&+\int_\Omega f_nS_j(v_n)(T_k(v_n)-T_k(v))+\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}S_j(v_n)(T_k(v_n)-T_k(v)).
\end{aligned}
\tag{3.18}
$$

Let us now analysis each terms on the right hand side of (3.18), for the first term, in view of (3.9) and the assumption (2.4), one has $\overline{a}(x,T_{\overline{j}}(u_n),\nabla T_j(v_n))$ is bounded in $(L^{p'}(\Omega))^N$ uniformly in n, and then

$$
\overline{a}(x,T_{\overline{j}}(u_n),\nabla T_j(v_n))\rightharpoonup\sigma_j \text{ weakly in } (L^{p'}(\Omega))^N. \tag{3.19}
$$

So, by (3.11), letting $n\to+\infty$, it yields

$$
\begin{aligned}
&\int_{\{k\le v_n\le 2j\}} S_j(v_n)\overline{a}(x,T_{\overline{j}}(u_n),\nabla v_n)\nabla(T_k(v_n)-T_k(v))\\
&\quad=-\int_{\{k\le v_n\le 2j\}} S_j(v_n)\overline{a}(x,T_{\overline{j}}(u_n),\nabla v_n)\nabla T_k(v)\\
&\quad=-\int_{\{k\le v\le 2j\}} S_j(v)\sigma_j\nabla T_k(v)+\omega(n)=\omega(n).
\end{aligned}
$$

For the second term in the right hand side of (3.18), due to (2.4), (3.6), (3.11) and Lebesgue's convergence theorem, we obtain

$$
\overline{a}(x,T_{\overline{j}}(u_n),\nabla T_k(v))\to\overline{a}(x,T_{\overline{j}}(u),\nabla T_k(v)) \text{ strongly in } (L^{p'}(\Omega))^N,
$$

and by (3.11), it results

$$\int_\Omega \overline{a}(x, T_{\overline{j}}(u_n), \nabla T_k(v))\nabla(T_k(v_n) - T_k(v)) = \omega(n).$$

As regards the third term, thanks to (3.16), it follows that

$$\begin{aligned}&\left|\frac{1}{j}\int_{\{j\le v_n\le 2j\}} a(x, u_n, \nabla u_n)\nabla v_n(T_k(v_n) - T_k(v))\right|\\ &\quad\le \frac{2k}{j}\int_{\{j\le v_n\le 2j\}} b_n(u_n)a(x, u_n, \nabla u_n)\nabla u_n = \omega(n, j).\end{aligned}$$

By Lebesgue's convergence theorem, it is easy to check that

$$\int_\Omega f_n S_j(v_n)(T_k(v_n) - T_k(v)) = \omega(n).$$

Now, let $\delta \in (0, m)$ such that $\delta \notin \{\eta > 0 : meas(\{u = \eta\}) > 0\}$, we split the last term in the right hand side of (3.18) on the sets $\{u_n \le \delta\}$ and $\{u_n > \delta\}$, we have

$$\begin{aligned}&\int_\Omega (\overline{a}(x, T_{\overline{k}}(u_n), \nabla T_k(v_n) - \overline{a}(x, T_{\overline{k}}(u_n), \nabla T_k(v)))\nabla(T_k(v_n) - T_k(v))\\ &\quad+\int_{\{u_n\le\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)T_k(v)\\ &=\int_{\{u_n\le\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)T_k(v_n)\\ &\quad+\int_{\{u_n>\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)(T_k(v_n) - T_k(v)) + \omega(n, j)\\ &= (A) + (B) + \omega(n, j).\end{aligned} \tag{3.20}$$

By dropping the second term in the left side of (3.20) since it is positive, we will focus only on the terms (A) and (B). For the term (A), due to (3.6), letting n goes to infinity and then δ goes to zero in $\chi_{\{u_n\le\delta\}}$, it gives

$$\chi_{\{u_n\le\delta\}} \to \chi_{\{u\le\delta\}} \text{ a.e. in } \Omega,$$

and

$$\chi_{\{u\le\delta\}} \to 1 \text{ a.e. in } \{u = 0\}.$$

Using the boundedness of the term $\dfrac{1}{s^\gamma}\displaystyle\int_0^s b_n^{\frac{1}{p-1}}(r)\,dr$ near to zero, (3.6) and (3.10) it follows that

$$\int_{\{u_n\le\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)T_k(v_n) = \int_{\{u\le\delta\}} f\frac{T_k(v)}{u^\gamma}S_j(v)+\omega(n)$$
$$= \int_{\{u=0\}} f\frac{T_k(v)}{u^\gamma} + \omega(n,\delta).$$

Note that if $\gamma < 1$ one has

$$\int_{\{u=0\}} f\frac{T_k(v)}{u^\gamma} = 0,$$

and so $A = \omega(n,\delta)$.
If $\gamma = 1$, by means of (3.6) and Fatou's Lemma in (3.13) we obtain that $\frac{f}{u^\gamma} \in L^1_{loc}(\Omega)$, so that $\{u = 0\} \subset \{f = 0\}$ up to a set of zero Lebesgue measure, then, we deduce that $A = \omega(n,\delta)$.
As regards the term (B), by virtue of the Lebesgue's convergence theorem, we obtain

$$\int_{\{u_n>\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)(T_k(v_n)-T_k(v))$$
$$= \int_{\{u>\delta\}} \frac{f}{u^\gamma}S_j(v)(T_k(v)-T_k(v))+\omega(n)=\omega(n).$$

By collecting all the previous convergence results, we arrive at

$$\int_\Omega (\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v_n)-\overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v)))\nabla(T_k(v_n)-T_k(v)) = \omega(n,\delta,j).$$

Hence, thanks to Lemma 5 in [4], we conclude that

$$T_k(v_n) \to T_k(v) \ \text{ strongly in } \ W_0^{1,p}(\Omega). \tag{3.21}$$

In particular, there exists a subsequence such that ∇v_n converge to ∇v almost everywhere in Ω. On the other hand, in view of (2.1) and (3.6), one has

$$\frac{1}{b_n(u_n)} \to \frac{1}{b(u)} \ \text{ a.e. in } \ \Omega,$$

then, it results

$$\nabla u_n = \frac{1}{b_n^{\frac{1}{p-1}}(u_n)}\nabla v_n \to \nabla u \text{ a.e. in } \ \Omega, \tag{3.22}$$

which, in turn, implies that

$$\nabla v = b^{\frac{1}{p-1}}(u)\nabla u \text{ a.e. in } \ \{0\le u<m\}.$$

Since a is a Carathéodory function, one gets

$$a_n(x, u_n, \nabla u_n) \to \overline{a}(x, u, \nabla v) \text{ a.e. in } \Omega, \tag{3.23}$$

$$\sigma_j = \overline{a}(x, T_{\overline{j}}(u), \nabla T_j(v)) \text{ a.e. in } \Omega. \tag{3.24}$$

Moreover, for $\ell < m$, using (3.22) and (2.4), it follows that

$$a(x, T_\ell(u_n), \nabla T_\ell(u_n)) \rightharpoonup a(x, T_\ell(u), \nabla T_\ell(u)) \text{ weakly in } (L^{p'}(\Omega))^N.$$

On the other hand, by (2.8), we can write

$$a(x, T_\ell(u), \nabla T_\ell(u)) = a(x, T_\ell(u), \nabla T_\ell(u))\chi_{\{0\le u<\ell\}} + a(x, T_\ell(u), \nabla T_\ell(u))\chi_{\{\ell\le u\le m\}},$$

so, using assumption (2.3), we obtain

$$a(x, T_\ell(u), \nabla T_\ell(u))\chi_{\{\ell\le u\le m\}} = a(x, \ell, 0) = 0.$$

Hence,

$$a(x, T_\ell(u), \nabla T_\ell(u)) = a(x, u, \nabla u)\chi_{\{0\le u<\ell\}} \in (L^{p'}(\Omega))^N,$$

for every $\ell < m$. Since $\ell < m$ is arbitrary, this allows us to deduce, by letting $\ell \to m^-$ that $a(x, u, \nabla u)\chi_{\{0\le u<m\}}$ belongs to $(L^{p'}(\Omega))^N$, so that (2.9) of Definition 2.1 holds. Moreover, using (2.2) and Hölder inequality, we obtain

$$\int_\Omega |\nabla v|^p = \int_\Omega b^{\frac{p}{p-1}}(u)|\nabla u|^p \chi_{\{0\le u<m\}} \le \frac{1}{\alpha}\int_\Omega a(x, u, \nabla u)\chi_{\{0\le u<m\}}\nabla v$$
$$\le \frac{1}{\alpha}\Big(\int_\Omega |a(x, u, \nabla u)|^{p'}\chi_{\{0\le u<m\}}\Big)^{\frac{1}{p'}}\Big(\int_\Omega |\nabla T_M(v)|^p\Big)^{\frac{1}{p}},$$

where $M = \displaystyle\int_0^m b^{\frac{1}{p-1}}(s)\,ds$. This means that v belongs to $W_0^{1,p}(\Omega)$.

⋆ **Step 3**. We will prove that (2.10) of Definition 2.1 holds. Let $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ such that $\nabla\varphi = 0$ almost everywhere in $\{x \in \Omega, u(x) = m\}$ with $\varphi \ge 0$. Let us take $\dfrac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))S_j(v_n)\varphi$ as test function in (3.1) which gives

$$\int_\Omega S_j(v_n)a_n(x, u_n, \nabla u_n)\nabla\varphi\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))$$
$$+\int_\Omega S_j'(v_n)a_n(x, u_n, \nabla u_n)\nabla v_n\varphi\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))$$
$$+\frac{1}{\sigma}\int_{\{m-2\sigma\le u_n\le m-\sigma\}} S_j(v_n)a_n(x, u_n, \nabla u_n)\nabla u_n\varphi$$
$$= \int_\Omega f_n\Big(1 + \frac{1}{(u_n + \frac{1}{n})^\gamma}\Big)\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\varphi S_j(v_n). \tag{3.25}$$

To study each term of (3.25). Let $j > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$, so that $0 \le v_n \le j$ implies that $0 \le u_n \le \overline{j}$. Then, by taking $n > \overline{j}$ and in view of (3.6), (3.10), (3.19) and (3.24), we obtain

$$\begin{aligned}
&\int_\Omega S_j(v_n)a_n(x, u_n, \nabla u_n)\nabla\varphi\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\\
&= \int_\Omega S_j(v_n)\overline{a}(x, T_{\overline{j}}(u_n), \nabla T_{2j}(v_n))\nabla\varphi\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\\
&= \int_\Omega S_j(v)\overline{a}(x, T_{\overline{j}}(u), \nabla T_{2j}(v))\nabla\varphi\frac{1}{\sigma}(T_{m-\sigma}(u) - T_{m-2\sigma}(u)) + \omega(n)\\
&= \int_{\{u=m\}} S_j(v)\overline{a}(x, T_{\overline{j}}(u), \nabla T_{2j}(v))\nabla\varphi + \omega(n, \sigma) = \omega(n, \sigma).
\end{aligned}$$

Due to, (3.6), (3.10), (3.21), (3.24) and the assumptions on b, it yields that

$$\begin{aligned}
&\frac{1}{\sigma}\int_{\{m-2\sigma\le u_n\le m-\sigma\}} S_j(v_n)\frac{1}{b_n^{\frac{1}{p-1}}(u_n)}\overline{a}(x, T_{\overline{j}}(u_n), \nabla T_{2j}(v_n))\nabla T_{2j}(v_n)\varphi\\
&= \frac{1}{\sigma}\int_{\{m-2\sigma\le u\le m-\sigma\}} a(x, u, \nabla u)\nabla u\varphi + \omega(n, j).
\end{aligned}$$

By (3.16), it is easy to check that

$$\begin{aligned}
&\left|\int_\Omega S_j'(v_n)a_n(x, u_n, \nabla v_n)\nabla v_n\varphi\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\right|\\
&\le \|\varphi\|_{L^\infty(\Omega)}\frac{1}{j}\int_{\{j\le v_n\le 2j\}} a_n(x, u_n, \nabla u_n)\nabla v_n = \omega(n, j).
\end{aligned}$$

For the last term, with the help of (3.6), (3.10) and the Lebesgue's convergence theorem, we obtain

$$\begin{aligned}
&\int_\Omega f_n\Big(1 + \frac{1}{(u_n + \frac{1}{n})^\gamma}\Big)\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\varphi S_j(v_n)\\
&= \int_{\{u_n\ge m-\sigma\}} f_n\Big(1 + \frac{1}{(u_n + \frac{1}{n})^\gamma}\Big)\frac{1}{\sigma}(T_{m-\sigma}(u_n) - T_{m-2\sigma}(u_n))\varphi S_j(v_n)\\
&= \int_{\{u\ge m-\sigma\}} f_n\Big(1 + \frac{1}{u^\gamma}\Big)\frac{1}{\sigma}(T_{m-\sigma}(u) - T_{m-2\sigma}(u))\varphi S_j(v) + \omega(n)\\
&= \int_{\{u=m\}} f\Big(1 + \frac{1}{u^\gamma}\Big)\varphi + \omega(n, \sigma).
\end{aligned}$$

Where we have used the fact that $\frac{1}{\sigma}(T_{m-\sigma}(u) - T_{m-2\sigma}(u)) \to \chi_{\{u=m\}}$. Therefore, we deduce that (2.10) holds true for every $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $\nabla\varphi = 0$ almost everywhere on $\{x \in \Omega, u(x) = m\}$ with $\varphi \ge 0$.

⋆ **Step 4**. Now we will show that

$$\lim_{n\to+\infty}\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi=\int_\Omega \frac{f}{u^\gamma}\varphi,$$

for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \geq 0$. Having in mind (3.13) and using Fatou's Lemma, we deduce that $\dfrac{f}{u^\gamma}\varphi$ belongs to $L^1(\Omega)$, for every $\varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi \geq 0$.

Now as in Step 2, Let $\delta < \frac{m}{2} < n$ such that $\delta \notin \{\eta > 0 : meas(\{u=\eta\}) > 0\}$ which is at most countable, we split the term $\displaystyle\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi$ $(0\leq \varphi \in W_0^{1,p}(\Omega)\cap L^\infty(\Omega))$ as follows

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi=\int_{\{u_n>\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi+\int_{\{u_n\leq\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi. \tag{3.26}$$

We want to pass to the limit as n tends to infinity and δ goes to zero in (3.26). For the first term in the right hand side of (3.26), since we can check that

$$\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi\chi_{\{u_n>\delta\}}\leq \frac{1}{\delta^\gamma}f\varphi \in L^1(\Omega),$$

one can apply Lebesgue convergence theorem to obtain (as $n\to\infty$)

$$\lim_{n\to+\infty}\int_{\{u_n>\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi=\int_{\{u>\delta\}} \frac{f}{u^\gamma}\varphi.$$

Moreover, since $\dfrac{f}{u^\gamma}$ belongs to $L^1(\Omega)$, we pass to the limit as δ goes to zero to deduce that

$$\lim_{\delta\to 0}\lim_{n\to+\infty}\int_{\{u_n>\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi=\int_{\{u>0\}} \frac{f}{u^\gamma}\varphi.$$

On the other hand, since $\dfrac{f}{u^\gamma}\varphi \in L^1(\Omega)$ implies that $\{u=0\}\subset\{f=0\}$ up to a set of zero Lebesgue measure, we deduce that

$$\int_{\{u>0\}} \frac{f}{u^\gamma}\varphi=\int_\Omega \frac{f}{u^\gamma}\varphi.$$

Next, we deal with the second term in the right hand side of (3.26) as n tends to infinity and δ goes to zero. We choose $S_\delta(u_n)\varphi$ as test function in (3.1), dropping the positive terms we have

$$\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_\delta(u_n)\varphi \le \int_\Omega S_\delta(u_n)a_n(x,u_n,\nabla u_n)\nabla\varphi$$
$$\le \int_\Omega S_\delta(u_n)a(x,T_{2\delta}(u_n),\nabla T_{2\delta}(u_n))\nabla\varphi. \tag{3.27}$$

Since $S_\delta(u_n)b(T_{2\delta}(u_n))$ converges to $S_\delta(u)b(T_{2\delta}(u))$ *-weakly in $L^\infty(\Omega)$ as $n\to+\infty$, using (3.19) and (3.24), we obtain

$$\int_\Omega S_\delta(u_n)a(x,T_{2\delta}(u_n),\nabla T_{2\delta}(u_n))\nabla\varphi = \int_\Omega S_\delta(u)a(x,u,\nabla u)\nabla\varphi + \omega(n).$$

Then, letting n tends to infinity in (3.27), we obtain

$$\limsup_{n\to+\infty}\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_\delta(u_n)\varphi \le \int_\Omega S_\delta(u)a(x,u,\nabla u)\nabla\varphi,$$

Moreover, since δ is chosen smaller enough, for $\ell\in(2\delta,m)$, one has

$$|S_\delta(u)a(x,u,\nabla u)| \le |a(x,T_\ell(u),\nabla T_\ell(u))| \in L^{p'}(\Omega),$$

so, by means of Lebesgue's convergence theorem, letting δ goes to zero and since $\nabla u=0$ almost everywhere in $\{u=0\}$ (thanks to Stampacchia's result because u belongs to $W_0^{1,p}(\Omega)$) and since $a(x,s,0)=0$ a.e. $x\in\Omega$, for every $s\in\mathbb{R}$), we obtain

$$\lim_{\delta\to 0}\limsup_{n\to+\infty}\int_{\{u_n\le\delta\}}\Big(\frac{f_n}{(u_n+\frac{1}{n})^\gamma}\Big)\varphi \le \int_{\{u=0\}} a(x,u,\nabla u)\nabla\varphi = 0.$$

Hence, we conclude that

$$\lim_{n\to+\infty}\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\varphi = \int_\Omega \frac{f}{u^\gamma}\varphi, \tag{3.28}$$

for every $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\ge 0$. In particular, u satisfies (2.11) of Definition 2.1.

⋆ **Step 5**. End of the proof. In this step, we are in position to show that u satisfies (2.12) of the Definition 2.1. Let $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ with $\varphi\ge 0$ and let $S\in W^{1,\infty}(\mathbb{R})$ such that the support of S is compact with $S(m)=0$. By choosing $S_j(v_n)S(u_n)\varphi$ as test function in (3.1), it results

$$\int_\Omega S_j(v_n)\,a_n(x,u_n,\nabla u_n)\nabla(S(u_n)\varphi) = \int_\Omega f_n S_j(v_n)S(u_n)\varphi$$
$$+\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)S(u_n)\varphi \tag{3.29}$$

Now we pass to the limit in each term of (3.29) as n goes to infinity and then as j tends to infinity. Since for $j > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$, $0 \le v_n \le j$ implies that $0 \le u_n \le \overline{j}$. For $n > \overline{j}$, using (3.6), (3.10), (3.19), (3.21) and (3.24), we obtain

$$\begin{aligned}
&\int_\Omega S_j(v_n)\, a_n(x, u_n, \nabla u_n)\nabla(S(u)\varphi) \\
&\quad = \int_\Omega S_j(v)\, \overline{a}(x, T_{\overline{j}}(u), \nabla T_{2j}(v))\nabla(S(u)\varphi) + \omega(n) \\
&\quad = \int_\Omega a(x, u, \nabla u)\chi_{\{0\le u<m\}}\nabla(S(u)\varphi) + \omega(n, j). \qquad (3.30)
\end{aligned}$$

Now using (3.16), one has

$$\begin{aligned}
&\left|\frac{1}{j}\int_{\{j\le v_n\le 2j\}} S(u_n)\, a_n(x, u_n, \nabla u_n)\nabla v_n\varphi\right| \\
&\quad \le \|S\|_{L^\infty(\mathbb{R})}\|\varphi\|_{L^\infty(\Omega)}\frac{1}{j}\int_{\{j\le v_n\le 2j\}} a_n(x, u_n, \nabla u_n)\nabla v_n = \omega(n, j). \qquad (3.31)
\end{aligned}$$

By means of Lebesgue's convergence theorem, one can check that

$$\int_\Omega f_n S_j(v_n)S(u_n)\varphi = \int_\Omega f S(u)\varphi + \omega(n). \qquad (3.32)$$

To deal with the second term in the right hand side of (3.29). Let us split it in two terms

$$\begin{aligned}
&\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)S(u_n)\varphi \\
&\quad = \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)S^+(u_n)\varphi - \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)S^-(u_n)\varphi. \qquad (3.33)
\end{aligned}$$

Now we follow the approach of the Step 4. Let $\delta \in (0, \frac{m}{2})$, we split the first term on the right hand side of (3.33) as

$$\begin{aligned}
\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S_j(v_n)S^+(u_n)\varphi &= \int_{\{u_n>\delta\}} \frac{f}{u^\gamma} S_j(v_n)S^+(u_n)\varphi \\
&\quad + \int_{\{u_n\le\delta\}} \frac{f}{u^\gamma} S_j(v_n)S^+(u_n)\varphi. \qquad (3.34)
\end{aligned}$$

To deal with the second term on the right hand side of (3.34), we take $S_\delta(u_n)S^+(u_n)\varphi$ as test function in (3.1), dropping the positive terms, we obtain

$$\begin{aligned}
&\int_{\{u_n\le\delta\}} \frac{f_n}{(u_n+\frac{1}{n})^\gamma} S^+(u_n)\, S_\delta(u_n)\varphi \le \int_\Omega S_\delta(u_n)a_n(x, u_n, \nabla u_n)\nabla(S^+(u_n)\varphi) \\
&\quad = \int_\Omega S_\delta(u_n)a(x, T_{2\delta}(u_n), \nabla T_{2\delta}(u_n))\nabla(S^+(u_n)\varphi),
\end{aligned}$$

by raisoning as in step 4 above, one can pass to the limit as n goes to $+\infty$ in the above inequality to deduce that

$$\limsup_{n\to+\infty}\int_{\{u_n\le\delta\}}\frac{f_n}{(u_n+\frac{1}{n})^\gamma}S_\delta(u_n)\varphi\le\int_\Omega S_\delta(u)a(x,T_{2\delta}(u),\nabla T_{2\delta}(u))\nabla(S^+(u)\varphi),$$

For the first term on the right hand side of (3.34), we follow again the proof of the Step 4 to deduce that

$$\lim_{n\to+\infty}\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}S^+(u_n)\varphi=\int_\Omega\frac{f}{u^\gamma}S^+(u)\varphi.$$

Similarly, one has also

$$\lim_{n\to+\infty}\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}S^-(u_n)\varphi=\int_\Omega\frac{f}{u^\gamma}S^-(u)\varphi.$$

This allows us to conclude that (recall that $S_j(s)\to 1$ as $j\to+\infty$)

$$\int_\Omega f_n\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)S_j(v_n)S(u_n)\varphi=\int_\Omega f\Big(1+\frac{1}{u^\gamma}\Big)S(u)\varphi+\omega(n,j). \tag{3.35}$$

Therefore, putting together (3.30), (3.31), (3.32) and (3.35), it results that u satisfies (2.12) of Definition 2.1 for every $\varphi\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$, $\varphi\ge 0$. Since we can write $\varphi=\varphi^+-\varphi^-$, we conclude the proof of Theorem 2.3.

4 The strongly singular case: $\gamma > 1$

In this section we deal with the strongly singular case $\gamma>1$. In this case, the a priori estimates on u_n derived from the approximation (3.1) hold only locally in $W^{1,p}(\Omega)$. However, one can show that $T_k^{\frac{\gamma-1+p}{p}}(u)$ belongs to $W_0^{1,p}(\Omega)$, which gives a sense to the solution u on the boundary $\partial\Omega$. To this end, we choose $T_k(u_n)^\gamma$ as test function in (3.1), using assumptions (2.1) and (2.2), we obtain

$$\begin{aligned}\alpha\gamma\inf_{\{0\le s\le m-\frac{1}{n}\}}b(s)\int_\Omega|\nabla T_k(u_n)^{\frac{\gamma-1+p}{p}}|^p&\le\alpha\gamma\int_\Omega b_n(u_n)|\nabla T_k(u_n)|^pT_k(u_n)^{\gamma-1}\\&\le\|f\|_{L^1(\Omega)}(k^\gamma+1).\end{aligned}$$

Then, we conclude that

$$\int_\Omega|\nabla T_k(u_n)^{\frac{\gamma-1+p}{p}}|^p\le C(1+k^\gamma).$$

By reasoning as in Step 1, we obtain

$$meas\{u_n \geq k\} \leq C\Big(\frac{1+k^{\gamma}}{k^{\gamma-1+p}}\Big),$$

and passing to the limit as k goes to infinity, leads to

$$\lim_{k\to+\infty} \sup_n meas\{u_n \geq k\} = 0.$$

In the following, we prove that $T_k(u_n)$ is bounded in $W^{1,p}_{loc}(\Omega)$. Let $\varphi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ such that $\varphi \geq 0$ and taking $(k-v_n)^+\varphi^p$ as test function in (3.1) to obtain

$$\begin{aligned} p\int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}(k-v_n)^+ - \int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla T_k(v_n)\varphi^p \\ = \int_{\Omega} f_n\Big(1+\frac{1}{(u_n+\frac{1}{n})^{\gamma}}\Big)(k-v_n)^+\varphi^p. \end{aligned} \tag{4.1}$$

Note that for $k > \int_0^m b^{\frac{1}{p-1}}(r)\,dr$, $0 \leq v_n \leq k$ implies that $0 \leq u_n \leq \overline{k}$ with $\overline{k}$ is independente of n. For $n > \overline{k}$, since the right hand side of (4.1) is positive, we have

$$\int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla T_k(v_n)\varphi^p \leq p\int_{\Omega} \overline{a}(x,T_{\overline{k}}(u_n),\nabla T_k(v_n))\nabla\varphi\varphi^{p-1}(k-v_n)^+,$$

by using assumptions (2.3), (2.4) and Young inequality, we obtain

$$\begin{aligned} \int_{\Omega} a_n(x,u_n,\nabla u_n)\nabla T_k(v_n)\varphi^p &\leq 2kp\,C_{\overline{k}}\int_{\Omega}(L(x)+|\nabla T_k(v_n)|^{p-1})|\nabla\varphi| \\ &\leq C_{\overline{k}}\|\varphi\|^{p-1}_{L^{\infty}(\Omega)}\Big(\|L\|^{p'}_{L^{p'}(\Omega)}+\|\nabla\varphi\|^{p}_{L^{p}(\Omega)}\Big)+\frac{1}{p'}\int_{\Omega}|\nabla T_k(v_n)|^p\varphi^p. \end{aligned}$$

So, by to assumption (2.2), we derive

$$\int_{\Omega} |\nabla T_k(v_n)|^p\varphi^p \leq C_k, \tag{4.2}$$

for every $\varphi \in W_0^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ such that $\varphi \geq 0$ and where C_k is a constant which depends on k and not the index n of the sequence. As a consequence, we conclude that

$$T_k(v_n) \rightharpoonup T_k(v) \ \text{ weakly in } \ W^{1,p}_{loc}(\Omega), \tag{4.3}$$

$$v_n \to v \ \text{ a.e. in } \ \Omega. \tag{4.4}$$

Next, we choose $T_k(v_n)^\gamma$ as test function in (3.1) to obtain

$$\gamma\int_\Omega T_k(v_n)^{\gamma-1}a_n(x,u_n,\nabla u_n)\nabla T_k(v_n)=\int_\Omega f_n\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)T_k(v_n)^\gamma, \tag{4.5}$$

by splitting the right hand side of (4.5) on the sets $\{u_n\le m-\frac{1}{n}\}$ and $\{u_n>m-\frac{1}{n}\}$, recalling that $\frac{1}{s}\int_0^s b_n^{\frac{1}{p-1}}(r)\,dr$ is bounded near to zero, we obtain

$$\begin{aligned}
&\gamma\int_\Omega T_k(v_n)^{\gamma-1}a_n(x,u_n,\nabla u_n)\nabla T_k(v_n)\\
&\le\int_{\{u_n\le m-\frac{1}{n}\}} f_n\Big(\frac{T_k(v_n)}{u_n+\frac{1}{n}}\Big)^\gamma+\int_{\{u_n>m-\frac{1}{n}\}} f_n\Big(\frac{T_k(v_n)}{u_n+\frac{1}{n}}\Big)^\gamma+\int_\Omega f_nT_k(v_n)^\gamma,\\
&\le\Big(C+\Big(\frac{k}{m}\Big)^\gamma+1\Big)\|f\|_{L^1(\Omega)}.
\end{aligned}$$

Then, by assumption (2.1), it follows that

$$\int_\Omega|\nabla(T_k(v_n))^{\frac{\gamma-1+p}{p}}|^p\le C(k^\gamma+1)\|f\|_{L^1(\Omega)}.$$

Moreover, using Poincaré inequality, we get

$$meas\{v_n\ge k\}\le\frac{C(k^\gamma+1)}{k^{\gamma-1+p}},$$

then, letting k goes to infinity leads to

$$\lim_{k\to+\infty}\sup_n meas\{v_n\ge k\}=0.$$

Therefore, by Fatou's lemma, we conclude that v is almost everywhere finite in Ω. Now we are in position to prove that $T_k(u_n)$ is bounded in $W^{1,p}_{loc}(\Omega)$. Let us start by splitting the integral $\int_\Omega|\nabla T_k(u_n)|^p\varphi^p$ on the sets $\{u_n\le m-\frac{1}{n}\}$ and $\{u_n>m-\frac{1}{n}\}$, we have

$$\int_\Omega|\nabla T_k(u_n)|^p\varphi^p=\int_{\{u_n\le m-\frac{1}{n}\}}|\nabla T_k(u_n)|^p\varphi^p+\int_{\{u_n>m-\frac{1}{n}\}}|\nabla T_k(u_n)|^p\varphi^p. \tag{4.6}$$

Setting $L = \int_0^m b^{\frac{1}{p-1}}(s)\,ds$. For the first term in the right hand side of (4.6), by (2.1) and (4.2), we obtain

$$\int_{\{u_n \le m-\frac{1}{n}\}} |\nabla T_k(u_n)|^p \varphi^p \le \sup_{s\in[0,m-\frac{1}{n}]} \frac{1}{b(s)^{p'}} \int_\Omega |\nabla T_L(v_n)|^p \varphi^p \le C.$$

As regards the second term in the right hand side of (4.6). Let $k > m$ and define for $s \ge 0$ the function $\psi_{k,m}(s) = k - m + \frac{1}{n} - (T_k(s) - T_{m-\frac{1}{n}}(s))$. By taking $\psi_{k,m}(u_n)\varphi^p$ as test function in (3.1) we obtain

$$\begin{aligned}
&-\int_{\{m-\frac{1}{n}\le u_n<k\}} a_n(x,u_n,\nabla u_n)\nabla u_n \varphi^p + p\int_\Omega a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}\psi_{k,m}(u_n)\\
&= \int_\Omega f_n\Big(1+\frac{1}{(u_n+\frac{1}{n})^\gamma}\Big)\psi_{k,m}(u_n)\varphi^p \ge 0.
\end{aligned}$$

By dropping the positive term and using assumptions (2.1) and (2.2), we get

$$\begin{aligned}
&\alpha b(m-\frac{1}{n})\int_{\{m-\frac{1}{n}\le u_n<k\}} |\nabla u_n|^p\varphi^p\\
&\le p\int_{\{0\le u_n\le m-\frac{1}{n}\}} a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}(k-m+\frac{1}{n})\\
&\quad + p\int_{\{m-\frac{1}{n}\le u_n<k\}} a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}\psi_{k,m}(u_n).
\end{aligned} \tag{4.7}$$

For the first term in the right of (4.7), by (2.4), Young's inequality and (4.2), we thus have

$$\begin{aligned}
&p\int_{\{0\le u_n\le m-\frac{1}{n}\}} a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}(k-m+\frac{1}{n})\\
&= p\int_{\{0\le u_n\le m-\frac{1}{n}\}} a(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}(k-m+\frac{1}{n})\\
&= p\int_{\{0\le u_n\le m-\frac{1}{n}\}} \overline{a}_n(x,u_n,\nabla v_n)\nabla\varphi\varphi^{p-1}(k-m+\frac{1}{n})\\
&\le pC(k-m+1)\Big(\int_\Omega L(x)|\nabla\varphi|\|\varphi\|^{p-1}_{L^\infty(\Omega)} + \int_\Omega |\nabla T_L(v_n)|^{p-1}\varphi^{p-1}|\nabla\varphi|\Big)\\
&\le C_k\Big(\int_\Omega |\nabla T_L(v_n)|^p\varphi^p + |\nabla\varphi|^p + L(x)^{p'}\Big) \le C_k,
\end{aligned}$$

where C_k is a constant does not depend on n. For the second term in the right hand side of (4.7), using (2.4) and Young's inequality, we obtain

$$p\int_{\{m-\frac{1}{n}\le u_n<k\}} a_n(x,u_n,\nabla u_n)\nabla\varphi\varphi^{p-1}\psi_{k,m}(u_n)$$
$$\le \frac{\alpha}{p}b(m-\frac{1}{n})\int_{\{m-\frac{1}{n}\le u_n<k\}}|\nabla u_n|^p\varphi^p + C_k b(m-\frac{1}{n})\Big(\int_\Omega |\nabla\varphi|^p + L(x)^{p'}\varphi^p\Big).$$

So, from (4.7), it follows that

$$\int_{\{m-\frac{1}{n}\le u_n<k\}}|\nabla u_n|^p\varphi^p \le C_k\Big(\int_\Omega |\nabla\varphi|^p + L(x)^{p'}\Big).$$

Hence, $T_k(u_n)$ is bounded in $W^{1,p}_{loc}(\Omega)$.

Let us mention that the same approach used to establish the existence result stated in Theorem 2.3 in the case $\gamma \le 1$ can be adapted to the strongly singular case by localizing the proof. We have then the following result

Theorem 4.1 *Assume that* (2.1)–(2.6) *hold true. If* $\gamma > 1$*, then, there exists at least a renormalized solution* u *of problem* (1.1) *in the sense that* $T_k^{\frac{\gamma-1+p}{p}}(u)$ *belongs to* $W_0^{1,p}(\Omega)$ *for any* $k>0$ *and*

$$\begin{aligned}
&T_k(u)\in W^{1,p}_{loc}(\Omega),\\
&\quad 0\le u\le m, \quad a.e.\ in\ \Omega,\\
&\quad a(x,u,\nabla u)\chi_{\{0\le u<m\}}\in (L^{p'}_{loc}(\Omega))^N,\\
&\quad \lim_{\sigma\to 0}\frac{1}{\sigma}\int_{\{m-2\sigma\le u\le m-\sigma\}} a(x,u,\nabla u)\nabla u\varphi = \int_{\{u=m\}} f\Big(1+\frac{1}{u^\gamma}\Big)\varphi,
\end{aligned}$$

for every $\varphi\in C_c^1(\Omega)$*. Moreover, for every function* $S\in W^{1,\infty}(\mathbb{R})$ *such that the support of* S *is compact and* $S(m)=0$*, the solution* u *satisfies*

$$\frac{f}{u^\gamma}S(u)\varphi\in L^1(\Omega),$$

and

$$\int_\Omega S(u)\,a(x,u,\nabla u)\chi_{\{u<m\}}\nabla\varphi + \int_\Omega S'(u)\,a(x,u,\nabla u)\nabla u\chi_{\{u<m\}}\varphi$$
$$=\int_\Omega f\Big(1+\frac{1}{u^\gamma}\Big)S(u)\varphi,$$

for every $\varphi\in C_c^1(\Omega)$.

5 Uniqueness result of the renormalized solution

In this section, we are going to establish the uniqueness of a renormalized to problem (1.1) stated in Theorem 2.4.

Proof of Theorem 2.4 Let us consider two renormalized solutions u_1 and u_2 to (1.1) in the sense of Definition 2.1. We choose for any $\sigma > 0$ and $j \geq 1$, $S = h_j$ and $\varphi = \frac{1}{\sigma} T_\sigma(v_1 - v_2)$ in the formulation (2.12) with $v_i = \int_0^{u_i} b^{\frac{1}{p-1}}(s)\, ds$, $i = 1, 2$. Note that the function $\frac{1}{\sigma} T_\sigma(v_1 - v_2)$ belongs to $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ since $v_i \in W_0^{1,p}(\Omega)$. Then, by taking the difference of the two formulations (2.12) for u_1 and u_2 and by setting $h(s) = 1 + \frac{1}{s^\gamma}$, $s \in [0, +\infty[$, one gets

$$\begin{aligned} &\frac{1}{\sigma} \int_\Omega (h_j(u_1)\overline{a}(x, u_1, \nabla v_1) - h_j(u_2)\overline{a}(x, u_2, \nabla v_2))\nabla T_\sigma(v_1 - v_2) \\ &\quad + \frac{1}{\sigma} \int_\Omega (h'_j(u_1)a(x, u_1, \nabla u_1)\nabla u_1 - h'_j(u_2)a(x, u_2, \nabla u_2)\nabla u_2)T_\sigma(v_1 - v_2) \\ &= \frac{1}{\sigma} \int_\Omega f(h(u_1)h_j(u_1) - h(u_2)h_j(u_2))T_\sigma(v_1 - v_2). \end{aligned} \tag{5.1}$$

Now we investigate the behaviors of each term in (5.1) when σ goes to 0 and then as j goes to $+\infty$. Let us start by studying the first term in left hand side of (5.1) that can be rewritten as

$$\begin{aligned} &\frac{1}{\sigma} \int_\Omega \Big(h_j(u_1)\overline{a}(x, u_1, \nabla v_1) - h_j(u_2)\overline{a}(x, u_2, \nabla v_2)\Big)\nabla T_\sigma(v_1 - v_2) \\ &= \frac{1}{\sigma} \int_\Omega h_j(u_1)\Big(\overline{a}(x, u_1, \nabla v_1) - \overline{a}(x, u_1, \nabla v_2)\Big)\nabla T_\sigma(v_1 - v_2) \\ &\quad + \frac{1}{\sigma} \int_\Omega h_j(u_2)\Big(\overline{a}(x, u_1, \nabla v_2) - \overline{a}(x, u_2, \nabla v_2)\Big)\nabla T_\sigma(v_1 - v_2) \\ &\quad + \frac{1}{\sigma} \int_\Omega (h_j(u_1) - h_j(u_2))\overline{a}(x, u_1, \nabla v_2)\nabla T_\sigma(v_1 - v_2) \\ &= A_{j,\sigma} + B_{j,\sigma} + C_{j,\sigma}, \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} A_{j,\sigma} &= \frac{1}{\sigma} \int_\Omega h_j(u_1)(\overline{a}(x, u_1, \nabla v_1) - \overline{a}(x, u_1, \nabla v_2))\nabla T_\sigma(v_1 - v_2), \\ B_{j,\sigma} &= \frac{1}{\sigma} \int_\Omega h_j(u_2)(\overline{a}(x, u_1, \nabla v_2) - \overline{a}(x, u_2, \nabla v_2))\nabla T_\sigma(v_1 - v_2), \end{aligned}$$

and

$$C_{j,\sigma} = \frac{1}{\sigma} \int_\Omega (h_j(u_1) - h_j(u_2))\overline{a}(x, u_1, \nabla v_2)\nabla T_\sigma(v_1 - v_2).$$

Let us observe that, by assumption (2.5), one has

$$A_{j,\sigma} \geq 0. \tag{5.3}$$

For the term $B_{j,\sigma}$, since the two solutions u_1 and u_2 belong to $[0, m]$, by the assumption (2.13), we obtain

$$|B_{j,\sigma}| \leq \frac{1}{\sigma} \int_{\{0<|v_1-v_2|<\sigma\}} |u_1 - u_2| \Big[E_m(x) + \gamma_m |\nabla v_2|^{p-1} \Big] |\nabla(v_1 - v_2)|.$$

On the other hand, using the assumption (2.1) on b and the fact that u_1 and u_2 belong to $[0, m]$, there exists a constant $C > 0$ such that

$$|u_1 - u_2| \leq C|v_1 - v_2|, \tag{5.4}$$

and since v_1 and v_2 belong to $W_0^{1,p}(\Omega)$, it follows that

$$\Big[E_m(x) + \gamma_m |\nabla v_2|^{p-1} \Big] |\nabla(v_1 - v_2)| \in L^1(\Omega).$$

Then, letting σ goes to zero in $B_{j,\sigma}$ yielding to

$$|B_{j,\sigma}| \leq \int_{\{0<|v_1-v_2|<\sigma\}} \Big[E_m(x) + \gamma_m |\nabla v_2|^{p-1} \Big] |\nabla(v_1 - v_2)| = \omega(\sigma). \tag{5.5}$$

As regards the term $C_{j,\sigma}$, using the lipschitz regularity of h_j, the inequality (5.4), since v_1 and v_2 belongs to $W_0^{1,p}(\Omega)$ and $\overline{a}(x, T_m(u_1), \nabla v_2)$ belongs to $(L^{p'}(\Omega))^N$ we obtain

$$|C_{j,\sigma}| \leq C_j \int_{\{0<|v_1-v_2|<\sigma\}} |\overline{a}(x, T_m(u_1), \nabla v_2)| |\nabla(v_1 - v_2)| = \omega(\sigma), \tag{5.6}$$

where $C_j > 0$ is a constant which does not depend on σ. Therefore, from (5.1), (5.2), (5.3), (5.5) and (5.6) we deduce that

$$\begin{aligned} &\frac{1}{\sigma} \int_\Omega f(h(u_2)h_j(u_2) - h(u_1)h_j(u_1)) T_\sigma(v_1 - v_2) \\ &\quad - \frac{j}{\sigma} \int_{\{m-\frac{2}{j} \leq u_1 \leq m-\frac{1}{j}\}} a(x, u_1, \nabla u_1) \nabla u_1 T_\sigma(v_1 - v_2) \\ &\quad + \frac{j}{\sigma} \int_{\{m-\frac{2}{j} \leq u_2 \leq m-\frac{1}{j}\}} a(x, u_2, \nabla u_2) \nabla u_2 T_\sigma(v_1 - v_2) + \omega(\sigma) \leq 0. \end{aligned} \tag{5.7}$$

Moreover, since the function $s \in \mathbb{R}^+ \mapsto h(s)h_j(s)$ is nonincreasing, one can pass to the limit using Fatou's Lemma as σ goes to zero (recalling that $\frac{1}{\sigma} T_\sigma(s) \to sign(s)$ as $\sigma \to 0$) to obtain

$$\int_{\Omega} f(h(u_2)h_j(u_2) - h(u_1)h_j(u_1))sign(v_1 - v_2)$$
$$-j\int_{\{m-\frac{2}{j}\le u_2\le m-\frac{1}{j}\}} a(x,u_2,\nabla u_2)\nabla u_2 sign(v_1-v_2)$$
$$\le j\int_{\{m-\frac{2}{j}\le u_1\le m-\frac{1}{j}\}} a(x,u_1,\nabla u_1)\nabla u_1 sign(v_1-v_2) + \omega(\sigma),$$

in view of (2.2) and since $-1 \le sign(s) \le 1$ for every $s \in \mathbb{R}$, it follows that

$$\int_{\Omega} f(h(u_2)h_j(u_2) - h(u_1)h_j(u_1))sign(v_1 - v_2)$$
$$-j\int_{\{m-\frac{2}{j}\le u_2\le m-\frac{1}{j}\}} a(x,u_2,\nabla u_2)\nabla u_2$$
$$\le j\int_{\{m-\frac{2}{j}\le u_1\le m-\frac{1}{j}\}} a(x,u_1,\nabla u_1)\nabla u_1 + \omega(\sigma). \tag{5.8}$$

Applying again Fatou's Lemma, letting j tends to $+\infty$ in (5.8) (recall that $h_j(s) \to \chi_{\{0\le s<m\}}$ as $j \to +\infty$) and using (2.10) with $\varphi = 1$, we get

$$\int_{\Omega} f(h(u_2)\chi_{\{0\le u_2<m\}} - h(u_1)\chi_{\{0\le u_1<m\}})sign(v_1 - v_2)$$
$$+h(m)\int_{\{u_1=m\}} f sign(v_1 - v_2)$$
$$\le h(m)\int_{\{u_1=m\}} f + h(m)\int_{\{u_2=m\}} f + \omega(\sigma, j).$$

Moroever, since we can write

$$\int_{\Omega} f(h(u_2)\chi_{\{0\le u_2<m\}} - h(u_1)\chi_{\{0\le u_1<m\}})sign(v_1 - v_2)$$
$$= \int_{\Omega} f|h(u_1) - h(u_2)| - h(m)\int_{\{u_2=m\}} f sign(v_1 - v_2)$$
$$+h(m)\int_{\{u_1=m\}} f sign(v_1 - v_2),$$

we easily obtain

$$\int_{\Omega} f|h(u_1) - h(u_2)| - h(m)\int_{\{u_2=m\}} f sign(v_1 - v_2)$$
$$+h(m)\int_{\{u_1=m\}} f sign(v_1 - v_2)$$
$$\le h(m)\int_{\{u_1=m\}} f + h(m)\int_{\{u_2=m\}} f + \omega(\sigma, j). \tag{5.9}$$

On the other hand, let us observe that $v_1 \leq v_2$ almost everywhere in $\{u_2 = m\}$, this implies that

$$\int_{\{u_2=m\}} f sign(v_1 - v_2) = -\int_{\{u_2=m\}} f.$$

Similarly, one has $v_1 \geq v_2$ almost everywhere in $\{u_1 = m\}$, which leads to

$$\int_{\{u_1=m\}} f sign(v_1 - v_2) = \int_{\{u_1=m\}} f.$$

So, by cancelling the equal term in (5.9), we deduce that

$$\int_\Omega f|h(u_1) - h(u_2)| \leq \omega(\sigma, j).$$

Since $f > 0$ almost everywhere in Ω, the previous inequality leads to $u_1 = u_2$ almost everywhere in Ω. Therefore, the Theorem 2.4 is then established. □

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