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Exact convergence rate of solutions for a semilinear heat equation with a critical and a supercritical exponent revisited

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Abstract

We study the behavior of solutions of the Cauchy problem for a semilinear heat equation with supercritical and critical nonlinearity in the sense of Joseph and Lundgren. It is known that if two solutions are initially close enough near the spatial infinity, then these solutions approach each other in the above cases. In this paper, for the supercritical case, we give a lower bound of a convergence rate that leads to the exact convergence rate together with our previous result. Also for the critical case, we give the exact convergence rate of solutions depending on two approaching initial data near spatial infinity again by using a different function than the previous results. For the critical case, this rate contains a logarithmic factor which is not contained in the supercritical nonlinearity case. Proofs are given by a comparison method based on matched asymptotic expansion.

Keywords Cauchy problem · Semilinear heat equation · Stationary solution · Convergence · Critical exponent

Mathematics Subject Classification 35K15 · 35B35 · 35B40 · 35B33

1 Introduction and results

In this paper, we investigate the behavior of solutions of the Cauchy problem

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, & x \in \mathbb{R}^N, \quad t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where $u = u(x,t)$, Δ is the Laplace operator with respect to x, $p > 1$, and $u_0 \not\equiv 0$ is a given continuous function on $\mathbb{R}^N$ that decays to zero as $|x| \to \infty$. The problem (1.1) has been studied in many papers, since Fujita studied the blow-up problem [10]. Among them, the stability problem of stationary solutions is one of the most important problems and we study the problem (1.1) along this line.

It is known that there exist critical exponents p that govern the structure of solutions. The exponent

$$p_S = \begin{cases} \frac{N+2}{N-2} & N > 2, \\ \infty & N \leq 2, \end{cases}$$

is well known as the Sobolev exponent that is critical for the existence of positive stationary solution of (1.1). Namely, there exists a classical positive radial solution φ of

$$\Delta\varphi + \varphi^p = 0, \qquad x \in \mathbb{R}^N,$$

if and only if $p \geq p_S$ [1, 2, 12]. We denote the solution by $\varphi = \varphi_\alpha(r)$, $r = |x|$, $\alpha > 0$, where $\varphi_\alpha(0) = \alpha$. Then $\varphi_\alpha(r)$ satisfies the initial value problem

$$\begin{cases} \varphi_{\alpha,rr} + \frac{N-1}{r}\varphi_{\alpha,r} + \varphi_\alpha^p = 0, \\ \varphi_\alpha(0) = \alpha, \quad \varphi_{\alpha,r}(0) = 0. \end{cases} \tag{1.2}$$

For each $\alpha > 0$, the solution φ_α is strictly decreasing in $|x|$ and satisfies

$$\varphi_\alpha \to 0 \quad \text{as} \quad |x| \to \infty.$$

We extend the solution by setting $\varphi_\alpha = -\varphi_{-\alpha}$ for $\alpha < 0$ and $\varphi_0 = 0$. Then the set $\{\varphi_\alpha; \alpha \in \mathbb{R}\}$ forms a one-parameter family of radial stationary solutions.

The exponent

$$p_c = \begin{cases} \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} & N > 10, \\ \infty & N \leq 10, \end{cases}$$

is another important exponent which appeared first in [15]. It is known that for $p_S \leq p < p_c$, any pair of positive stationary solutions intersects each other. For $p \geq p_c$, Wang [20] showed that the family of stationary solutions for (1.2) forms a simply ordered set, that is, φ_α is strictly increasing in α for each x. We call it the ordering property of $\{\varphi_\alpha\}$. Moreover, φ_α satisfies

$$\lim_{\alpha\to 0} \varphi_\alpha(|x|) = 0, \quad \lim_{\alpha\to\infty} \varphi_\alpha(|x|) = \varphi_\infty(|x|),$$

for each x, where $\varphi_\infty(|x|)$ is a singular stationary solution given by

$$\varphi_\infty(|x|) = L|x|^{-m}, \quad x \in \mathbb{R}^N \setminus \{0\},$$

with

$$m = \frac{2}{p-1}, \quad L = \{m(N-2-m)\}^{1/(p-1)}. \tag{1.3}$$

It was also shown in [13] that each positive stationary solution has the expansion

$$\varphi_\alpha(|x|) = \begin{cases} L|x|^{-m} - a_\alpha|x|^{-m-\lambda_1} + \text{h.o.t.} & p > p_c, \\ L|x|^{-m} - a_\alpha|x|^{-m-\lambda}\log|x| + \text{h.o.t.} & p = p_c, \end{cases} \tag{1.4}$$

as $|x| \to \infty$, where for $p \geq p_c$, λ_1, λ is a positive constant. λ_1 is given by

$$\lambda_1 = \lambda_1(N, p) := \frac{N-2-2m-\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

λ is given later and $a_\alpha = a(\alpha)$ is a positive number that is monotone decreasing in α. Note that λ_1 is a smaller root of the quadratic equation

$$h(\lambda) := \lambda^2 - (N-2-2m)\lambda + 2(N-2-m) = 0. \tag{1.5}$$

We define for $p > p_c$ by

$$\lambda_2 = \lambda_2(N, p) := \frac{N-2-2m+\sqrt{(N-2-2m)^2 - 8(N-2-m)}}{2},$$

a larger root of the quadratic equation (1.5).

For the stability problem, Gui et al. [13, 14] proved that any regular positive radial stationary solution is unstable in any reasonable sense if $p_S < p < p_c$ and "weakly asymptotically stable" in a weighted L^∞ norm if $p \geq p_c$. For $p > p_c$, Poláčik and Yanagida [17, 18] improved the above results and proved that the solutions approach a set of stationary solutions for a wide class of the initial data. As a by-product, they also showed the existence of global unbounded solutions. We note that the study of global unbounded solutions of (1.1) [3, 4] is closely related to our problem on bounded solutions mentioned later.

Later, Fila et al. [5] studied the convergence of solutions of (1.1). They considered the following more general problem: Let u and $\tilde{u}$ denote solutions of (1.1) with initial data u_0, $\tilde{u}_0$ respectively. Where, u_0 and $\tilde{u}_0$ are continuous functions and we always assume this assumption In the following. They studied how fast these two solutions approach each other as $t \to \infty$. In particular, in the case of $\tilde{u}_0 = \varphi_\alpha(|x|)$, then the rate of approach corresponds to the convergence rate to the stationary solution. More precisely, they showed that if $p > p_c$, $m + \lambda_1 < l < m + \lambda_2$ and initial functions are under some stationary solution and approaches the decay rate of t^{-l} near spatial infinity then the difference between the values of the two solutions decays in time the exact rate $t^{-(l-m-\lambda_1)/2}$.

The above result is no longer valid for large l and in fact they found a universal lower bound for the rate of approach which applies to any initial data. More precisely, they showed that if $p \geq p_c$ and $0 \leq \tilde{u}_0(x) < u_0(x) \leq \varphi_\infty(|x|)$ then difference between the values of the two solutions decays more slowly in time than the rate $t^{-(N-m-\lambda_1)/2}$. We note that there exists a gap of the convergence rate between the rate $t^{-(\lambda_2-\lambda_1)/2}$ which is obtained for the case $l = m + \lambda_2$ and a universal lower bound of the rate $t^{-(N-m-\lambda_1)/2}$.

On the other hand, for the grow-up problem which can be regarded as a stability problem of singular stationary solution, a sharp universal upper bound of the grow-up rate was found by Mizoguchi [16], and optimal lower bound of the grow-up rate was found by Fila et al. [4]. The results on the grow-up problem strongly suggest that the above result of the convergence rate is not optimal.

For $p > p_c$, We obtain a sharp bound of the convergence rate in the case of $m + \lambda_1 < l < m + \lambda_2 + 2$ which leads to its optimal convergence rate in [7]. In fact, we improve the results in [5]. More precisely, we had already proved following Theorems in [7].

Theorem A *Let $p > p_c$. Suppose that $|u_0|, |\tilde{u}_0| \leq \varphi_\alpha(|x|)$ with some α. If $m + \lambda_1 < l < m + \lambda_2 + 2$, and satisfy*

$$|u_0(x) - \tilde{u}_0(x)| \leq m_1(1 + |x|)^{-l}$$

with some $m_1 > 0$. Then there exists constant $M_1 > 0$ such that

$$\|u(\cdot, t) - \tilde{u}(\cdot, t)\|_{L^\infty} \leq M_1(t + 3)^{(l-m-\lambda_1)/2}$$

for all $t > 0$.

After, Stinner studied a similar problem in [19] with the critical exponent $p = p_c$ in the case of $m + \lambda < l < m + \lambda + 2$ and shows the exact convergence rate of the solution that approaches a stationary solution $\varphi_\alpha(|x|)$. In this case, the equation (1.5) has the double root

$$\lambda := \frac{N - 2 - 2m}{2}.$$

Actually, the sharp convergence rate $t^{-(l-m-\lambda)/2}(\log t)^{-1}$ is obtained in [19]. The characteristic point is that the convergence rate contains a logarithmic factor. We remark that our result in [7] Theorem 1.4 and [9] Theorem 1.1 shows that the convergence rate can not be extended in the case of $l > m + \lambda_2 + 2$ for $p > p_c$ and $l > m + \lambda + 2$ for $p = p_c$.

Our purpose of this paper is to show there exists the exact estimate of the convergence rate with the approaching solutions which applies an approaching two initial data for $p > p_c$ and $p = p_c$ by using partially different function in [19] mentioned later. In fact, with the supercritical exponent $p > p_c$, we prove a lower estimate of the convergence rate of the solutions and with the critical exponent $p = p_c$, we can prove again the same results in [19] as follows. Our results also show that a logarithmic factor appears for the critical case.

Theorem 1.1 *Let $p > p_c$. Suppose that $|u_0|, |\tilde{u}_0| \le \varphi_\alpha(|x|)$ with some α. If $m+\lambda_1 < l < m+\lambda_2+2$, and satisfy*

$$|u_0(x) - \tilde{u}_0(x)| \ge c_1(1+|x|)^{-l}$$

with some $c_1 > 0$. Then there exists constant $C_1 > 0$ such that

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \ge C_1(t+3)^{-(l-m-\lambda_1)/2}$$

for all $t > 0$.

Theorem 1.2 *Let $p = p_c$. Suppose that $|u_0|, |\tilde{u}_0| \le \varphi_\alpha(|x|)$ with some α. If $m+\lambda < l < m+\lambda+2$, and satisfy*

$$|u_0(x) - \tilde{u}_0(x)| \le c_2(1+|x|)^{-l}$$

with some $c_2 > 0$. Then there exists constant $C_2 > 0$ such that

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \le C_2(t+3)^{-(l-m-\lambda)/2}(\log(t+3)^{1/2})^{-1}$$

for all $t > 0$.

Theorem 1.3 *Let $p = p_c$. Suppose that $|u_0|, |\tilde{u}_0| \le \varphi_\alpha(|x|)$ with some α If $m+\lambda < l < m+\lambda+2$, and satisfy*

$$|u_0(x) - \tilde{u}_0(x)| \ge c_3(1+|x|)^{-l}$$

with some $c_3 > 0$. Then there exists constant $C_3 > 0$ such that

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \ge C_3(t+3)^{-(l-m-\lambda)/2}(\log(t+3)^{1/2})^{-1}$$

for all $t > 0$.

Proofs of the above theorems are obtained by a comparison technique that is based on matched asymptotic expansion. For the critical case, the inner expansion is the same as in [19] and the difference of our proof is the careful description of the outer expansion by differential equation. More precisely with the critical exponent, for the upper bound Stiner uses Kummer's function as in [6] and for the lower bound, the same technical function as used in [5], Although we do not use these methods. In fact, we will use the solutions of a differential equations which behaves in a self-similar way near the spatial infinity and make super and sub-solutions by using these solutions in the outer region. Then we construct super and sub-solutions by matching these inner and outer solutions.

This paper is organized as follows. In Sect. 2, we recall preliminary results in [4] and [5]. We note that the result of this section imply the reason why logarithmic factor appear for the critical case. The formal analysis in this section will give the idea of constructing super and sub-solutions, and a matching condition of these expansion implies the convergence rate. In Sect. 3, we prove Theorem 1.1 and note that the result together with our result Theorem A shows that the exact convergence rate are obtained. In Sect. 4, we prove Theorem 1.2, and in Sect. 5, we prove Theorem 1.3.

2 Preliminary results on the linearized equation

In this section, we summarize previous results on the linear equation that are needed in subsequent sections. For proofs of the results, see [4, 5, 8].

We consider radial solutions $u = U(r,t)$, $r = |x|$, of the linearized equation of (1.1) at φ_α. Namely, let $\mathcal{P}_\alpha$ be the linearized operator defined by

$$\mathcal{P}_\alpha U := U_{rr} + \frac{N-1}{r}U_r + p{\varphi_\alpha}^{p-1}U$$

and let $U(r,t)$ be a solution of

$$\begin{cases} U_t = \mathcal{P}_\alpha U, & r > 0, \quad t > 0, \\ U_r(0,t) = 0, & t > 0, \\ U(r,0) = U_0(r), & r \geq 0, \end{cases} \tag{2.1}$$

where U_0 is a continuous function that decays to zero as $r \to \infty$. From the maximum principle, we see that $U(\cdot,t) > 0$ for all $t > 0$ if $U_0 \geq 0$ and $U_0 \not\equiv 0$. We will describe some fundamental properties for the solution of (2.1).

2.1 Comparison principle

Let u and $\tilde{u}$ be solutions of (1.1) with initial data u_0 and $\tilde{u}_0$ respectively. We recall some comparison results for $u - \tilde{u}$ and the solution U of (2.1), which comes from the ordering property and the convexity of nonlinearity.

Lemma 2.1 ([5] Lemma 2.1) *Let $p \geq p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy* (H1). *If*

$$|u_0(x) - \tilde{u}_0(x)| \leq U_0(|x|), \quad x \in \mathbb{R}^N,$$

then

$$|u(x,t) - \tilde{u}(x,t)| \leq U(|x|,t), \quad x \in \mathbb{R}^N$$

for all $t > 0$.

Lemma 2.2 ([5] Lemma 2.2) *Let $p \geq p_c$. Suppose that u_0 and $\tilde{u}_0$ satisfy*

$$\varphi_\alpha(|x|) \leq \tilde{u}_0(x) \leq u_0(x) \leq \varphi_\infty(|x|), \quad x \in \mathbb{R}^N \setminus \{0\}$$

with some $\alpha > 0$. If

$$0 \leq U_0(|x|) \leq u_0(x) - \tilde{u}_0(x), \quad x \in \mathbb{R}^N,$$

then

$$0 \le U(|x|,t) \le u(x,t) - \tilde{u}(x,t), \quad x \in \mathbb{R}^N$$

for all $t > 0$.

2.2 Formal matched asymptotics

By the above comparison results, we may only consider the convergence of radial solution of the linearized equation (2.1). In the following, we recall the asymptotic analysis, which is only formal but will be useful in the rigorous analysis in subsequent sections.

First, following Fila et al. [5], the formal expansion of a solution of (2.1) near the origin is given by

$$U(r,t) = \sigma(t)\psi(r) + \sigma_t(t)\Psi(r) + \text{h.o.t.}, \tag{2.2}$$

where, $\sigma(t) = U(0,t)$, ψ and Ψ satisfy

$$\begin{cases} \mathcal{P}_\alpha \psi = 0, & r > 0, \\ \psi(0) = 1, & \psi_r(0) = 0 \end{cases} \tag{2.3}$$

and

$$\begin{cases} \mathcal{P}_\alpha \Psi = \psi, & r > 0, \\ \Psi(0) = 0, & \Psi_r(0) = 0, \end{cases} \tag{2.4}$$

respectively (see also [5] and [11] for details). We recall some results in [5] on the above linear differential equations (2.3) and (2.4) in the following.

Lemma 2.3 ([5] Lemma 2.3, [8] Lemma 2.3) *For all* $\alpha > 0$ *and* $r \ge 0$, $\alpha \mapsto \varphi_\alpha(r)$ *is differentiable and*

$$\psi(r) := \frac{\partial}{\partial\alpha}\varphi_\alpha$$

satisfies (2.3)*. Moreover, if* $p = p_c$*, then* $\psi(r)$ *is positive and satisfies*

$$\psi(r) = c_\alpha r^{-m-\lambda}\log r + o(r^{-m-\lambda}\log r) \quad \textit{as} \quad r \to \infty,$$

and if if $p > p_c$*, then* $\psi(r)$ *is positive and satisfies*

$$\psi(r) = c_\alpha r^{-m-\lambda} + o(r^{-m-\lambda}) \quad \textit{as} \quad r \to \infty,$$

where c_α *is a constant given by* $c_\alpha = \frac{a_1\lambda}{m}\alpha^{-\frac{m+\lambda}{m}}$ *and* $a_1 = a(1)$ *is a constant independent of* α.

Remark 2.1 The function ψ defined in Lemma 2.3 satisfies $\psi_r < 0$ for all $r > 0$. Indeed, we see from (2.3) that ψ does not attain a positive local minimum by the positivity of φ_α and ψ.

Lemma 2.4 ([5] Lemma 2.4, [8] Lemma 2.5) *If $p \geq p_c$, then the solution Ψ of (2.4) has the following properties*:

(i) *Ψ/ψ is strictly increasing in $r > 0$. In particular, Ψ is positive for all $r > 0$.*
(ii) *If $p = p_c$, then Ψ satisfies*

$$\Psi(r) = C_\alpha r^{-m-\lambda+2} \log r + o(r^{-m-\lambda+2} \log r) \quad as \quad r \to \infty,$$

and if $p > p_c$, then Ψ satisfies

$$\Psi(r) = C_\alpha r^{-m-\lambda+2} + o(r^{-m-\lambda+2}) \quad as \quad r \to \infty,$$

where

$$C_\alpha = \frac{c_\alpha}{g(m+\lambda-2)} > 0, \quad g(\mu) := h(\mu - m).$$

Next, let us consider the expansion of a solution of (2.1) near $r = \infty$. By the expansion of $\varphi_\alpha(r)$ near $r = \infty$, $U(r,t)$ satisfies approximately

$$U_t = U_{rr} + \frac{N-1}{r} U_r + \frac{pL^{p-1}}{r^2} U, \quad r \simeq \infty. \tag{2.5}$$

Following [3, 4], we assume that U is of a self-similar form for $r \gg 1$

$$U(r,t) = t^{-l/2} F(\eta), \quad \eta = t^{-1/2} r. \tag{2.6}$$

so that the specific scaling for $r \gg 1$ corresponding to the outer region is in fact $r = O(t^{1/2})$ as $t \to \infty$. Substituting this in (2.5), we see that F satisfies

$$F_{\eta\eta} + \frac{N-1}{\eta} F_\eta + \frac{\eta}{2} F_\eta + \frac{l}{2} F + \frac{pL^{p-1}}{\eta^2} F = 0. \tag{2.7}$$

In order that the outer expansion matches with the inner solution (2.2), $F(\eta)$ must satisfy

$$\lim_{\eta \to 0} \eta^{m+\lambda} F(\eta) = a_0 > 0 \tag{2.8}$$

in view of the spatial order of Lemma 2.3, where a_0 is a constant depending on initial data.

We will know in the next section that (2.7) has a positive solution satisfying (2.8).

For $p = p_c$, we show a formal analysis in [8] here again for the reader's convenience. By matching the inner expansion (2.2) by using Lemmas 2.3, 2.4,

$$\begin{aligned} U(r,t) &= \sigma(t)\psi(r) + \sigma_t(t)\Psi(r) + \text{h.o.t.} \\ &\simeq \sigma(t)r^{-m-\lambda}\log r + \sigma_t(t)r^{-m-\lambda+2}\log r + \text{h.o.t.} \\ &\simeq \sigma(t)t^{-(m+\lambda)/2}\eta^{-m-\lambda}(\log \eta t^{1/2}) \\ &\quad + \sigma_t(t)t^{-(m+\lambda)/2}\eta^{-m-\lambda+2}(\log \eta t^{1/2}) + \text{h.o.t.} \end{aligned}$$

and the outer expansion (2.6),

$$U(r,t) = t^{-l/2}F(\eta),$$

we obtain

$$\sigma(t)t^{(-m-\lambda)/2}\log t^{l/2} \simeq t^{-(l/2)}.$$

This implies the convergence rate

$$\sigma(t) \simeq t^{-(l-m-\lambda)/2}(\log t^{1/2})^{-1}$$

which is the same convergence rate given in Theorems 1.2 and 1.3. We use theses results, and also obtain

$$\sigma_t \simeq -\frac{l-m-\lambda}{2}t^{-(l-m-\lambda)/2-1}(\log t^{1/2})^{-1} - \frac{1}{2}t^{-(l-m-\lambda)/2-1}(\log t^{1/2})^{-2}.$$

We substitute above results in (2.2), then we obtain a formal expansion near the origin as follows.

$$\begin{aligned} U(r,t) &= \sigma(t)\psi(r) + \sigma_t(t)\Psi(r) + \text{h.o.t.} \\ &\simeq t^{-q}(\log t^{1/2})^{-1}\psi(r) - (qt^{-q-1}(\log t^{1/2})^{-1} \\ &\quad - \frac{1}{2}t^{-q-1}(\log t^{1/2})^{-2})\Psi(r). \end{aligned} \tag{2.9}$$

For $p > p_c$, we also obtain

$$\begin{aligned} U(r,t) &= \sigma(t)\psi(r) + \sigma_t(t)\Psi(r) + \text{h.o.t.} \\ &\simeq t^{-q}\psi(r) - qt^{-q-1}\Psi(r), \end{aligned}$$

by a similar argument where $q = (l-m-\lambda)/2$ (See [7]).

The above expansions suggest the constructions of inner super and sub-solutions.

2.3 Properties of self-similar solutions

In this subsection, we recall the behavior of solutions of (2.7) satisfying

$$\lim_{\eta\to 0} \eta^{m+\mu} F(\eta) = a_0 > 0,$$

where $a_0 > 0$ is a constant and $\mu = \lambda_1$ or λ. To this end, we set

$$f(\eta) = \eta^{m+\mu} F(\eta).$$

Substituting this in (2.7), we see that $f(\eta)$ satisfies

$$\begin{cases} f_{\eta\eta} + \frac{N-1-2(m+\mu)}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{l-m-\mu}{2} f = 0, & \eta > 0, \\ f(0) = a_0 > 0, \quad f_\eta(0) = 0. \end{cases} \tag{2.10}$$

The following lemma characterizes the behavior of f as $\eta \to \infty$, and explains why $l = m + \mu + 2$ is critical.

Lemma 2.5 ([4] Lemma 3.1) *For $p > p_c$, let f be the solution of* (2.10).

(i) *If $l \in (m + \lambda_1, m + \lambda_2 + 2)$, then $f > 0$ and $f_\eta < 0$ for all $\eta > 0$. Moreover, for each $\eta_0 > 0$, there exist $d^-(\eta_0) > 0$ such that*

$$f(\eta) \geq d^-(\eta_0)\eta^{-(l-m-\lambda_1)} \quad \textit{for} \quad \eta \geq \eta_0,$$

and $d^+ > 0$ such that

$$f(\eta) \leq d^+\eta^{-(l-m-\lambda_1)} \quad \textit{for all} \quad \eta > 0.$$

(ii) *If $l = m + \lambda_2 + 2$, then $f(\eta)$ is given explicitly by $f(\eta) = a_0 \exp(-\eta^2/4)$.*
(iii) *If $l > m + \lambda_2 + 2$, then $f(\eta)$ vanishes at some finite η.*

Lemma 2.6 *For $p = p_c$, let f be the solution of* (2.10).

(i) *If $l \in (m + \lambda, m + \lambda + 2)$, then $f > 0$ and $f_\eta < 0$ for all $\eta > 0$. Moreover, for each $\eta_0 > 0$, there exist $d^-(\eta_0) > 0$ such that*

$$f(\eta) \geq d^-(\eta_0)\eta^{-(l-m-\lambda)} \quad \textit{for} \quad \eta \geq \eta_0,$$

and $d^+ > 0$ such that

$$f(\eta) \leq d^+\eta^{-(l-m-\lambda)} \quad \textit{for all} \quad \eta > 0.$$

(ii) *If $l = m + \lambda + 2$, then $f(\eta)$ is given explicitly by $f(\eta) = a_0 \exp(-\eta^2/4)$.*
(iii) *If $l > m + \lambda + 2$, then $f(\eta)$ vanishes at some finite η.*

The proof of Lemma 2.6 is the same as the proof of Lemma 3.1 in [4]. So, we omit the proof here.

3 Lower bound for the supercritical exponent

In this section, we prove that a lower bound of the convergence rate exists, which applies to initial data close at most of the negative polynomial order from above or below to a stationary solution in the case $\tilde{u} = \varphi_\alpha$.

3.1 Outer sub-solution

In this subsection, we construct a suitable outer sub-solution of (2.1).

First, we recall that f satisfies

$$f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, \tag{3.1}$$

where $n = N - 2(m + \lambda_1)$, $\beta = l - m - \lambda_1$ and satisfies $0 < \beta < 2 + \lambda_2 - \lambda_1$. Although this solution was already used for the construction of super-solution to (2.1) used in the previous result in [7], we need to modify this solution to construct a sub-solution of (2.1) in an outer region as follows.

We take δ satisfies $0 < \delta < \min\{2 + \lambda_2 - \lambda_1 - \beta, 1\}$, put $\tilde{\beta} = \beta + \delta$ and define $\tilde{f}$ that satisfies

$$\begin{cases} \tilde{f}_{\eta\eta} + \frac{n-1}{\eta} \tilde{f}_\eta + \frac{\eta}{2} \tilde{f}_\eta + \frac{\tilde{\beta}}{2} \tilde{f} = 0, & \eta > 0, \\ \tilde{f}(0) = a_0 > 0, \quad \tilde{f}_\eta(0) = 0. \end{cases}$$

Lemma 3.1 *For $p > p_c$, define $F^-(\eta; b_1) := \eta^{-m-\lambda_1} f^-(\eta; b_1) = \eta^{-m-\lambda_1}(f(\eta) - b_1 \tilde{f}(\eta))$ and*

$$U^-_{\text{out}}(r, t) := \begin{cases} 0 & \text{for } 0 \le \eta < \eta_1, \\ (t+\tau)^{-\frac{l}{2}} F^-(\eta; b_1) = (t+\tau)^{-\frac{l}{2}} \eta^{-m-\lambda_1} f^-(\eta; b_1) & \text{for } \eta \ge \eta_1, \end{cases}$$

where $\eta = (t+\tau)^{-1/2} r$, $b_1, \tau, \eta_1 > 0$ are sufficient large constant determined later. Then U^-_{out} is a sub-solution of (2.1).

Proof It is trivial that $U \equiv 0$ is a sub-solution. Then we only check the case where $\eta \ge \eta_1$.

First, we fix any $\eta_0 > 1$. We can take positive constant a^-_α satisfies

$$\varphi_\alpha \ge L r^{-m} - a^-_\alpha r^{-m-\lambda_1} \quad \text{for} \quad r \ge 3, \tag{3.2}$$

from (1.4) and $d^+, \tilde{d}^-(\eta_0) > 0$ satisfy

$$f(\eta) \le d^+ \eta^{-\beta} \text{ for all } \eta > 0, \tag{3.3}$$

$$\tilde{f}(\eta) \ge \tilde{d}^-(\eta_0) \eta^{-\tilde{\beta}} \text{ for } \eta > \eta_0 \tag{3.4}$$

from Lemma 2.5 respectively. We take any $\varepsilon > 0$ and sufficiently large $\tau > 0$ satisfies

$$mp(N-2-m)\left(1-\left(1-\frac{a_\alpha^-}{L}(\eta_0\tau)^{-1/2\lambda_1}\right)^{p-1}\right) < \varepsilon \quad \text{for} \quad \eta \geq \eta_0 \tag{3.5}$$

and

$$\eta_0 \tau^{1/2} > 3.$$

We take b_1 satisfies

$$\frac{b_1\delta}{2}\tilde{d}^-(\eta_0)\eta_0^{2-\delta} - \varepsilon d^+ > 0 \text{ and } f(\eta_0) - b_1\tilde{f}(\eta_0) \leq 0. \tag{3.6}$$

We define

$$\eta_1 = \inf\{\eta > 1 | f^-(\rho; b_1) = f(\rho) - b_1\tilde{f}(\rho) > 0, \ \text{ for } \ \rho > \eta\}. \tag{3.7}$$

Then we find $\eta_1 \geq \eta_0$ is well defined from Lemma 2.5 and the definition. We note that F^- is positive for $\eta > \eta_1$ from (3.7) and $\eta_1 \geq \eta_0$ from (3.6).

Next, in general setting of our problem, the following differential inequality is computed the same as in [8].

$$\begin{aligned}
&U^-_{\mathrm{out},t} - \mathcal{P}_\alpha U^-_{\mathrm{out}} \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1}\left(\frac{l}{2}F^- + \frac{\eta}{2}F^-_\eta + F^-_{\eta\eta} + \frac{N-1}{\eta}F^-_\eta + p(t+\tau)\varphi_\alpha^{p-1}F^-\right) \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Bigg(f^-_{\eta\eta} + \frac{N-1-2(m+\lambda_1)}{\eta}f^-_\eta + \frac{\eta}{2}f^-_\eta + \frac{l-m-\lambda_1}{2}f^- \\
&\qquad + p(t+\tau)\varphi_\alpha^{p-1}f^- + \eta^{-2}\left((m+\lambda_1+1)(m+\lambda_1) - (N-1)(m+\lambda_1)\right)f^-\Bigg) \\
&\quad \leq -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Bigg(f^-_{\eta\eta} + \frac{N-1-2(m+\lambda_1)}{\eta}f^-_\eta + \frac{\eta}{2}f^-_\eta \\
&\qquad + \frac{l-m-\lambda_1}{2}f^- + p(t+\tau)\left(Lr^{-m} - a_\alpha^- r^{-m-\lambda_1}\right)^{p-1}f^- \\
&\qquad + \eta^{-2}\left((m+\lambda_1+1)(m+\lambda_1) - (N-1)(m+\lambda_1)\right)f^-\Bigg) \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Bigg(f^-_{\eta\eta} + \frac{N-1-2(m+\lambda_1)}{\eta}f^-_\eta + \frac{\eta}{2}f^-_\eta \\
&\qquad + \frac{l-m-\lambda_1}{2}f^- + p(t+\tau)L^{p-1}r^{-m(p-1)}\left(1-\frac{a_\alpha^-}{L}r^{-\lambda_1}\right)^{p-1}f^- \\
&\qquad + \eta^{-2}\left((m+\lambda_1+1)(m+\lambda_1) - (N-1)(m+\lambda_1)\right)f^-\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Big(f^-_{\eta\eta} + \frac{n-1}{\eta} f^-_\eta + \frac{\eta}{2} f^-_\eta + \frac{\beta}{2} f^- \\
&\quad + \eta^{-2}\Big((m+\lambda_1+1)(m+\lambda_1) - (N-1)(m+\lambda_1) \\
&\quad + pL^{p-1}\left(1 - \frac{a^-_\alpha}{L} r^{-\lambda_1}\right)^{p-1}\Big) f^-\Big) \\
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Big(f^-_{\eta\eta} + \frac{n-1}{\eta} f^-_\eta + \frac{\eta}{2} f^-_\eta + \frac{\beta}{2} f^- \\
&\quad + \eta^{-2}\Big((m+\lambda_1)^2 + 2(m+\lambda_1) - N(m+\lambda_1) \\
&\quad + (m+2)(N-2-m)\left(1 - 1 + \left(1 - \frac{a^-_\alpha}{L} r^{-\lambda_1}\right)^{p-1}\right)\Big) f^-\Big) \\
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Big(f^-_{\eta\eta} + \frac{n-1}{\eta} f^-_\eta + \frac{\eta}{2} f^-_\eta + \frac{\beta}{2} f^- \\
&\quad + \eta^{-2}\Big(\lambda_1^2 - (N-2-m)\lambda_1 + m^2 + 2m - Nm \\
&\quad + (m+2)(N-2-m)\left(1 - 1 + \left(1 - \frac{a^-_\alpha}{L} r^{-\lambda_1}\right)^{p-1}\right)\Big) f^-\Big) \\
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda_1}\Big(f^-_{\eta\eta} + \frac{n-1}{\eta} f^-_\eta + \frac{\eta}{2} f^-_\eta + \frac{\beta}{2} f^- \\
&\quad + \eta^{-2}\Big(\lambda_1^2 - (N-2-2m)\lambda_1 + 2(N-2-m) \\
&\quad - (m+2)(N-2-m)\left(1 - \left(1 - \frac{a^-_\alpha}{L} r^{-\lambda_1}\right)^{p-1}\right)\Big) f^-\Big).
\end{aligned}
$$

Here we use (1.3), (3.2).

Finally, we substitute $f^-(\eta; b_1) = f(\eta) - b_1\tilde{f}(\eta)$ with $r = (t+\tau)^{1/2}\eta$. Then, we use (1.5), (3.1), (3.2), (3.3), (3.4), (3.5) and there by we can simplify the above inequality as follows.

$$
\begin{aligned}
&U^-_{\mathrm{out},t} - \mathcal{P}_\alpha U^-_{\mathrm{out}} \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda} \\
&\qquad \left(f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f - b_1\left(\tilde{f}_{\eta\eta} + \frac{n-1}{\eta}\tilde{f}_\eta + \frac{\eta}{2}\tilde{f}_\eta + \frac{\tilde{\beta}}{2}\tilde{f}\right) + \frac{b_1\delta}{2}\tilde{f} \right. \\
&\qquad \left. - \eta^{-2}(m+2)(N-2-m)\left(1 - \left(1 - \frac{a^-_\alpha}{L}(\eta_0\tau)^{-\lambda}\right)^{p-1}\right)(f - b_1\tilde{f})\right) \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_1\delta}{2}\tilde{f} - \varepsilon\eta^{-2} f\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_1\delta}{2}\tilde{d}^-(\eta_0)\eta^{-\tilde{\beta}} - \varepsilon\eta^{-2}d^+\eta^{-\beta}\right)\\
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_1\delta}{2}\tilde{d}^-(\eta_0)\eta^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta}\\
&\le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_1\delta}{2}\tilde{d}^-(\eta_0)\eta_1^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta}\\
&\le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_1\delta}{2}\tilde{d}^-(\eta_0)\eta_0^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta}\\
&< 0 \quad \text{for} \quad \eta \ge \eta_1
\end{aligned}
$$

from (3.5) and (3.6).We complete the proof. □

3.2 Inner sub-solution and matching

We use the same inner sub-solution as in [5] Lemma 4.1.

Lemma 3.2 *For any $q > 0$, we define*

$$U_{\text{in}}^-(r,t) := (t+\tau)^{-q}\psi(r).$$

If τ is sufficiently large, then there exists a constant $B_1 > 0$ satisfies $B_1\tau^{1/2} > 3$ and $c > 0$ such that the following inequalities hold :

(i) $U_{\text{in},t}^- < \mathcal{P}_\alpha U_{\text{in}}^-$ *for all* $r > 0$ *and* $t > 0$.

(ii) $U_{\text{in}}^-(r,t) > 0$ *for all* $t > 0$ *and* $r \in [0, B_1(t+\tau)^{\frac{1}{2}}]$.

(iii) $cU_{\text{in}}^-(r,t) < U_{\text{out}}^-(r,t)$ *at* $r = B_1(t+\tau)^{\frac{1}{2}}$ *for all* $t > 0$.

Proof We compute the same as in [5, 7] and see

$$
\begin{aligned}
U_{\text{in},t}^- - \mathcal{P}_\alpha U_{\text{in}}^- &= -q(t+\tau)^{-q-1}\psi(r) - (t+\tau)^{-q}\mathcal{P}_\alpha\psi(r)\\
&= -q(t+\tau)^{-q-1}\psi(r)\\
&< 0, \quad \text{for} \quad r > 0, t > 0,
\end{aligned}
$$

from Lemma 2.3. Hence U_{in}^- is a sub-solution of (2.1) which proves(i). Next, let us shows (ii), (iii). We set

$$q := \frac{l-m-\lambda_1}{2},$$

then by Lemma 2.3, we can choose positive constants c_α^+ such that

$$\psi(r) \le c_\alpha^+ r^{-m-\lambda_1} \quad \text{for} \quad r \ge 3. \tag{3.8}$$

First, we fix η_0 in Lemma 3.1 and take $\tau, b_1 > 0$ as sufficiently large such that the condition in Lemma 3.1 holds. We found there exists the maximum point of $f^-(\eta)$

denote η_M from the construction of f^- and Lemma 2.5 then we fix $\eta_1 < B_1 < \eta_M$. finally, we take $c > 0$ satisfies

$$f^-(B_1) - cc_\alpha^+ > 0. \tag{3.9}$$

It is clear that

$$U_{\text{in}}^-(r,t) > 0, \quad t > 0, r \in [0, B_1(t+\tau)^{1/2}].$$

Indeed, we recall

$$U_{\text{in}}^-(r,t) := (t+\tau)^{-q}\psi(r) > 0 \text{ for } t \geq 0, r \geq 0,$$

which prove (ii) by using positivity of $\psi(r)$.

Next, let us show $U_{\text{out}}^-(r_1,t) - cU_{\text{in}}^-(r_1,t) > 0$ at $r_1(t) := B_1(t+\tau)^{1/2}$ and $\eta_1 := (t+\tau)^{-1/2}r_1(t)$, namely at $\eta_1 = B_1$. We obtain

$$\begin{aligned}
&U_{\text{out}}^-(r_1(t),t) - cU_{\text{in}}^-(r_1(t),t) \\
&\quad = (t+\tau)^{-\frac{l}{2}}F^-(\eta_1) - c(t+\tau)^{-q}\psi(r_1) \\
&\quad \geq (t+\tau)^{-\frac{l}{2}}B_1^{-(m+\lambda)}f^-(B_1) \\
&\qquad - c(t+\tau)^{-\frac{l-m-\lambda}{2}}c_\alpha^+ r_1^{-(m+\lambda)} \\
&\quad = (t+\tau)^{-\frac{l}{2}}\left(B_1^{-(m+\lambda)}f^-(B_1) - cc_\alpha^+ B_1^{-(m+\lambda)}\right) \\
&\quad > (t+\tau)^{-\frac{l}{2}}\left(f^-(B_1) - cc_\alpha^+\right)B_1^{-(m+\lambda)} \\
&\quad > 0, \quad \text{for all } t > 0,
\end{aligned}$$

by (3.8), (3.9) and $B_1\tau^{1/2} > 3$, thus (iii) is proved. Then we complete the proof. □

Proposition 3.3 *Suppose that $m + \lambda_1 < l < m + \lambda_2 + 2$ and*

$$U_0(r) \geq c_1(1+r)^{-l}, \quad r \geq 0$$

with some $c_1 > 0$. Then there exists a constant $C_1', \tau > 0$ such that the solution of (2.1) *satisfies*

$$\|U(\cdot,t)\| \geq C_1'(t+\tau)^{-(l-m-\lambda_1)/2} \quad \textit{for all } t > 0.$$

Proof Recall $c > 0$ satisfies $f^-(B_1) - cc_\alpha^+ > 0$. Let $U_{\text{out}}^-(r,t)$ and $U_{\text{in}}^-(r,t)$ be as in Lemmas 3.1 and 3.2 respectively, and define

$$U^-(r,t) := \begin{cases} cU_{\text{in}}^-(r,t) & \text{for } r < r^*(t), \\ U_{\text{out}}^-(r,t) & \text{for } r \geq r^*(t), \end{cases}$$

where $r^*(t)$ is defined

$$r^*(t) := \sup\{r > 0 | cU_{\text{in}}^-(\rho, t) > U_{\text{out}}^-(\rho, t) \text{ for } \rho \in [0, r)\}.$$

From Lemma 3.2 (iii), we obtain

$$0 < r^*(t) < r_1(t) < \infty \quad \text{for all } t > 0.$$

We note that $r^*(t) \in (0, \infty]$ is well defined since $0 < r^*(t) < \infty$.

From the construction of U^-, it attains the exact decay rate at the origin. Thus it is shown that $U^-(r, t)$ is a sub-solution of (2.1) which satisfies

$$\begin{aligned} U^-(0, t) &= cU_{\text{in}}^-(0, t) \\ &= c(t + \tau)^{-(l-m-\lambda_1)/2} \psi(0) \\ &= c(t + \tau)^{-(l-m-\lambda_1)/2} \end{aligned}$$

for all $t > 0$.

We will show that the $C^- U^-(r, 0)$ lies below the initial data $U_0(r)$ if we take a constant $C^- > 0$ sufficiently small. In fact, we can take a constant $C^- > 0$ small enough to hold that

$$C^- U^-(r, 0) \le U_0(r), \quad r \ge 0.$$

Indeed, if we take $C^- > 0$ so small that

$$C^- c\tau^{-(l-m-\lambda_1)/2} \le c_1 \tag{3.10}$$

and

$$c_1 - C^- d^+ > 0 \tag{3.11}$$

Then we find

$$\begin{aligned} C^- cU_{\text{in}}^-(r, 0) &\le C^- cU_{\text{in}}^-(0, 0) \\ &\le C^- c\tau^{-(l-m-\lambda_1)/2} \psi(0) \\ &= C^- c\tau^{-(l-m-\lambda_1)/2} \\ &\le c_1 \le U_0(r) \quad \text{for } 0 \le r \le r^*(0) \end{aligned}$$

from (3.10) and

$$\begin{aligned} &U_0(r) - C^- U_{\text{out}}^-(r, 0) \\ &\quad \ge c_1(1 + r)^{-l} - C^- U_{\text{out}}^-(r, 0) \\ &\quad = c_1(1 + r)^{-l} - C^- \tau^{-\frac{l}{2}} F^-(\eta) \end{aligned}$$

$$
\begin{aligned}
&= c_1(1+r)^{-l} - C^- \tau^{-\frac{l}{2}} \eta^{-m-\lambda_1} (f(\eta) - b\tilde{f}(\eta)) \\
&> c_1(1+r)^{-l} - C^- \tau^{-\frac{l}{2}} \eta^{-m-\lambda_1} f(\eta) \\
&\geq c_1(1+r)^{-l} - C^- \tau^{-\frac{l}{2}} \eta^{-m-\lambda_1} d^+ \eta^{-(l-m-\lambda_1)} \\
&= c_1(1+r)^{-l} - C^- d^+ \tau^{-\frac{l}{2}} (\tau^{-1/2} r)^{-l} \\
&= c_1(1+r)^{-l} - C^- d^+ r^{-l} \\
&> (c_1 - C^- d^+) r^{-l} \\
&\geq 0 \text{ for } r \geq r^*(0)
\end{aligned}
$$

by using (3.3) and (3.11). Then the initial condition is satisfied by the above argument, and by the comparison principle, we obtain

$$
C^- U^-(r,t) \leq U(r,t) \quad \text{for} \quad r > 0, \ t > 0.
$$

We obtain

$$
\|U(\cdot,t)\|_{L^\infty} \geq \|C^- U^-(\cdot,t)\|_{L^\infty} \geq C^- c U^-(0,t) = C^- c (t+\tau)^{-(l-m-\lambda_1)/2}
$$

Then, we replace $C^- c$ with C_1', we finish the proof. □

Proof of Theorem 1.1 We take

$$
U_0(r) := \min_{|x|=r} (u_0(x) - \tilde{u}_0(x)), \quad r \geq 0.
$$

Then by Lemma 2.2, Proposition 3.3, we have

$$
\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \geq C_1 (t+3)^{-(l-m-\lambda_1)/2}
$$

for all $t > 0$ with some constant $C_1 > 0$. □

4 Upper bound for the critical nonlinearity

In the following sections, we always assume the critical case $p = p_c$.

In this section, we prove that there exists a upper bound of the convergence rate which applies to an initial data close from above or below to a stationary solution in the case $\tilde{u} = \varphi_\alpha$. First, we recall the initial value problem (2.10):

$$
\begin{cases}
f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, \quad \eta > 0, \\
f(0) = a_0 > 0, \quad f_\eta(0) = 0,
\end{cases}
$$

where $n = N - 2(m+\lambda)$, $\beta = l - m - \lambda$, $m + \lambda < l < m + \lambda + 2$.

4.1 Outer super-solution

In this subsection, we will construct a suitable super-solution of (2.1) in the same way as that in [7, 8].

Lemma 4.1 *We define*

$$U_{\text{out}}^{+}(r,t) := (t+\tau)^{-\frac{l}{2}} F^{+}(\eta) = (t+\tau)^{-\frac{l}{2}} \eta^{-m-\lambda} f(\eta),$$

where τ *is a positive constant determined later.Then* U_{out}^{+} *is a super-solution of* (2.1).

Proof Although the proof proceeds the same as in [7, 8] Lemma 3.1, we show it here for the reader's convenience. We note that $F^{+}(\eta)$ satisfies

$$F_{\eta\eta}^{+} + \frac{N-1}{\eta} F_{\eta}^{+} + \frac{\eta}{2} F_{\eta}^{+} + \frac{l}{2} F^{+} + \frac{pL^{p-1}}{\eta^2} F^{+} = 0.$$

Then we have

$$\begin{aligned}
&U_{\text{out},t}^{+} - \mathcal{P}_{\alpha} U_{\text{out}}^{+} \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1} \left(\frac{l}{2} F + \frac{\eta}{2} F_{\eta}^{+} + F_{\eta\eta}^{+} + \frac{N-1}{\eta} F_{\eta}^{+} + p(t+\tau)\varphi_{\alpha}^{p-1} F^{+} \right) \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1} \left(-\frac{pL^{p-1}}{\eta^2} F^{+}(\eta) + p(t+\tau)\varphi_{\alpha}^{p-1} F^{+}(\eta) \right) \\
&\quad = (t+\tau)^{-\frac{l}{2}-1} \left(p(t+\tau)(\varphi_{\infty}^{p-1}(r) - \varphi_{\alpha}^{p-1}(r)) \right) F^{+}(\eta) \\
&\quad = (t+\tau)^{-\frac{l}{2}-1} \left(p(t+\tau)(\varphi_{\infty}^{p-1}(r) - \varphi_{\alpha}^{p-1}(r)) \right) \eta^{-m-\lambda} f(\eta).
\end{aligned}$$

Then the ordering property of $\{\varphi_{\alpha}\}$ and the positivity of $f(\eta)$, we have

$$U_{\text{out},t}^{+} - \mathcal{P}_{\alpha} U_{\text{out}}^{+} > 0$$

for all $r, t > 0$. □

4.2 Inner super-solution and matching

We use the same inner super-solution in [19] Lemma 3.2 which is appeared in the formal analysis from (2.9).

Lemma 4.2 *For* $q > 0$. *We define*

$$\begin{aligned}
U_{\text{in}}^{+}(r,t) &:= (t+\tau)^{-q} (\log(B_3 (t+\tau)^{1/2}))^{-1} \psi(r) \\
&\qquad - (t+\tau)^{-q-1} \left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2} (\log(B_3(t+\tau)^{1/2}))^{-2} \right) \Psi(r),
\end{aligned}$$

where $q = (l - m - \lambda)/2$. If $\tau > 0$ is sufficiently large, then there exist constants $B_3 > 0$ satisfies $B_3\tau^{1/2} > 3$ and $c > 0$ such that the following inequalities hold.

(i) $U^+_{\mathrm{in},t} \geq \mathcal{P}_\alpha U^+_{\mathrm{in}}$ *for all* $r > 0$ *and* $t > 0$.
(ii) $U^+_{\mathrm{in}}(r,t) > 0$ *for all* $t > 0$ *and* $r \in [0, B_3(t+\tau)^{\frac{1}{2}}]$.
(iii) $U^+_{\mathrm{in}}(r,t) > cU^+_{\mathrm{out}}(r,t)$ *at* $r = B_3(t+\tau)^{\frac{1}{2}}$ *for all* $t > 0$.

Proof Although proof is similar manner in [19] Lemma 3.2, we prove the Lemma here for the reader's convenience. First, we prove (i) for any $B_3 > 0$ determined later.

$$\begin{aligned}
U^+_{\mathrm{in},t} - \mathcal{P}_\alpha U^+_{\mathrm{in}} &= -q(t+\tau)^{-q-1}(\log B_3(t+\tau)^{1/2})^{-1}\psi(r)\\
&\quad + (q+1)(t+\tau)^{-q-2}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r)\\
&\quad - (t+\tau)^{-q}\left((\log(B_3(t+\tau)^{1/2}))^{-2}B_3^{-1}(t+\tau)^{-1/2}B_3/2(t+\tau)^{-1/2}\right)\psi(r)\\
&\quad + (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-2}B_3^{-1}(t+\tau)^{-1/2}B_3/2(t+\tau)^{-1/2}\right)\Psi(r)\\
&\quad + (t+\tau)^{-q-1}\left((\log(B_3(t+\tau)^{1/2}))^{-3}B_3^{-1}(t+\tau)^{-1/2}B_3/2(t+\tau)^{-1/2}\right)\Psi(r)\\
&\quad - (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\mathcal{P}_\alpha\psi(r)\\
&\quad + (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\mathcal{P}_\alpha\Psi(r)\\
&= -q(t+\tau)^{-q-1}(\log B_3(t+\tau)^{1/2})^{-1}\psi(r)\\
&\quad + (q+1)(t+\tau)^{-q-2}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r)\\
&\quad - \frac{1}{2}(t+\tau)^{-q-1}\left((\log(B_3(t+\tau)^{1/2}))^{-2}\right)\psi(r)\\
&\quad + \frac{1}{2}(t+\tau)^{-q-2}\left(q(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r)\\
&\quad + \frac{1}{2}(t+\tau)^{-q-2}\left((\log(B_3(t+\tau)^{1/2}))^{-3}\right)\Psi(r)\\
&\quad + (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\psi(r)\\
&= (q+1)(t+\tau)^{-q-2}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r)\\
&\quad + \frac{1}{2}(t+\tau)^{-q-2}\left(q(\log(B_3(t+\tau)^{1/2}))^{-2} + (\log(B_3(t+\tau)^{1/2}))^{-3}\right)\Psi(r)\\
&\geq 0, \quad \text{for all } r \geq 0, t > 0,
\end{aligned}$$

by Lemma 2.3 and $B_3\tau^{1/2} > 3$. Hence U^+_{in} is a super-solution of (2.1).

Next, let us show (ii) and (iii). By Lemmas 2.3 and 2.4, we can choose positive constant c^-_α and C^+_α such that

$$\psi(r) \geq c^-_\alpha r^{-m-\lambda}\log r \quad \text{for} \quad r \geq 3, \tag{4.1}$$

and

$$\Psi(r) \le C_\alpha^+ r^{-m-\lambda+2} \log r \quad \text{for} \quad r \ge 3, \tag{4.2}$$

respectively. Then we fix $B_3 > 0$ such that

$$c_\alpha^- - C_\alpha^+ \left(q + \frac{1}{2}\right) B_3^2 > 0. \tag{4.3}$$

Next, we take $\tau > 1$ so large that

$$B_3 \tau^{\frac{1}{2}} > 3 \tag{4.4}$$

and satisfies

$$\tau - \left(q + \frac{1}{2}\right) \frac{\Psi(3)}{\psi(3)} > 0. \tag{4.5}$$

Finally $c > 0$ so small such that

$$c_\alpha^- - C_\alpha^+ \left(q + \frac{1}{2}\right) B_3^2 > cf(B_3). \tag{4.6}$$

Let us now verify (ii) and (iii). For $r \in [0, 3]$, it follows from due to the monotonicity of Ψ/ψ, positivity of $\psi(r)$ (see Lemma 2.4 and Remark 2.1), (4.4) and (4.5) that

$$\begin{aligned}
U_{\text{in}}^+(r,t) &= (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r) \\
&\quad - (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r) \\
&= (t+\tau)^{-q-1}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r) \\
&\quad \left((t+\tau) - \left(q + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-1}\right)\frac{\Psi(r)}{\psi(r)}\right) \\
&\ge (t+\tau)^{-q-1}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r)\left(\tau - \left(q+\frac{1}{2}\right)\frac{\Psi(r)}{\psi(r)}\right) \\
&\ge (t+\tau)^{-q-1}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r)\left(\tau - \left(q+\frac{1}{2}\right)\frac{\Psi(3)}{\psi(3)}\right) \\
&> 0 \quad \text{for all} \quad t > 0.
\end{aligned}$$

For $r \in [3, B_3(t+\tau)^{\frac{1}{2}}]$, (4.1), (4.2), (4.3) and (4.4) yield

$$\begin{aligned}
U_{\text{in}}^+(r,t) &= (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r) \\
&\quad - (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r)
\end{aligned}$$

$$
\begin{aligned}
&= (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1} \\
&\quad \left(\psi(r) - \left(q + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-1}\right)(t+\tau)^{-1}\Psi(r)\right) \\
&\geq (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1} \\
&\quad \left(c_\alpha^- - C_\alpha^+\left(q + \frac{1}{2}(\log B_3\tau^{1/2})^{-1}\right) r^2(t+\tau)^{-1}\right) r^{-(m+\lambda)}\log r \\
&\geq (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\left(c_\alpha^- - C_\alpha^+\left(q+\frac{1}{2}\right)B_3^2\right) r^{-(m+\lambda)}\log r \\
&> 0 \quad \text{for all} \quad t > 0,
\end{aligned}
$$

which proves (ii). This also shows in view of (4.1), (4.2), (4.6) and the definition of q that

$$
\begin{aligned}
U_{\text{in}}^+(r,t) &= (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\psi(r) \\
&\quad - (t+\tau)^{-q-1}\left(q(\log(B_3(t+\tau)^{1/2}))^{-1} + \frac{1}{2}(\log(B_3(t+\tau)^{1/2}))^{-2}\right)\Psi(r) \\
&\geq (t+\tau)^{-q}(\log(B_3(t+\tau)^{1/2}))^{-1}\left(c_\alpha^- - C_\alpha^+\left(q+\frac{1}{2}\right)B_3^2\right) r^{-(m+\lambda)}\log r \\
&= (t+\tau)^{-q}(\log r)^{-1}\left(c_\alpha^- - C_\alpha^+\left(q+\frac{1}{2}\right)B_3^2\right) r^{-(m+\lambda)}\log r \\
&= (t+\tau)^{-(l-m-\lambda)/2}\left(c_\alpha^- - C_\alpha^+\left(q+\frac{1}{2}\right)B_3^2\right) B_3^{-(m+\lambda)}(t+\tau)^{-(m+\lambda)/2} \\
&= (t+\tau)^{-l/2}\left(c_\alpha^- - C_\alpha^+\left(q+\frac{1}{2}\right)B_3^2\right) B_3^{-(m+\lambda)} \\
&> c(t+\tau)^{-l/2}B_3^{-(m+\lambda)}f(B_3)
\end{aligned}
$$

at $r = B_3(t+\tau)^{\frac{1}{2}}$. On the other hand, we have at $r = B_3(t+\tau)^{\frac{1}{2}}$ that is to say at $\eta = B_3$,

$$
\begin{aligned}
cU_{\text{out}}^+(r,t) &= c(t+\tau)^{-l/2}F^+(\eta) \\
&= c(t+\tau)^{-l/2}B_3^{-(m+\lambda)}f(B_3).
\end{aligned}
$$

Hence we obtain by (4.6)

$$
cU_{\text{out}}^+(r,t) < U_{\text{in}}^+(r,t) \quad \text{at} \quad r = B_3(t+\tau)^{\frac{1}{2}}, \quad t > 0.
$$

Thus (iii) is proved. □

Since the super-solution $U_{\text{in}}^+(r,t)$ decays too slowly as $r \to \infty$, we shall only use it in the inner region $r \leq r^*(t)$ with suitable positive function $r^*(t)$ noted in the above. In the outer region, we shall work with a different class of super-solutions defined in Lemma 4.1 is already mentioned.

Proposition 4.3 *Suppose that $m+\lambda<l<m+\lambda+2$ and*

$$0<U_0(r)\le c_2(1+r)^{-l},\quad r\ge 0$$

with some $c_2>0$. Then there exist constant $C_2',\tau>0$ such that the solution of (2.1) satisfies

$$\|U(\cdot,t)\|_{L^\infty}\le C_2'U^+(0,t)=C_2'(t+\tau)^{(l-m-\lambda)/2}(\log(t+\tau)^{1/2})^{-1}\quad \textit{for all}\quad t>0.$$

Proof Let $U^+_{\rm out}$ and $U^+_{\rm in}$ be as given in Lemmas 4.1 and 4.2 respectively, and define

$$U^+(r,t):=\begin{cases}U^+_{\rm in}(r,t) & \text{for } r<r^*(t),\\ cU^+_{\rm out}(r,t) & \text{for } r\ge r^*(t),\end{cases}$$

where $c>0$ is given in Lemma 4.2. Put

$$r^*(t):=\sup\{r>0\,|\,U^+_{\rm in}(\rho,t)<cU^+_{\rm out}(\rho,t)\text{ for }\rho\in[0,r]\}.$$

We note that $r^*(t)\in(0,\infty]$ is well defined for each $c>0$, in view of Lemma 4.2 (iii). It is clear that

$$U^+_r(0,t)=U^+_{{\rm in},r}(0,t)=0,\quad t>0.$$

We will show that the initial data $U_0(r)$ lies below $C^+U^+(r,0)$ if we take a constant $C^+>0$ sufficiently large. In fact, we see from Lemma 2.6 that for $r\ge r^*(0)$,

$$\begin{aligned}U^+(r,0)&=cU^+_{\rm out}(r,0)\\&=c\tau^{-\frac{l}{2}}F^+(\eta)\\&=c\tau^{-\frac{l}{2}}\eta^{-(m+\lambda)}f(\eta)\\&=c\tau^{-\frac{l}{2}}\tau^{-(m+\lambda)/2}r^{-(m+\lambda)}f(\tau^{-1/2}r)\end{aligned}$$

Then, we show there exists $C^+>0$ so large that

$$U_0=c_1(1+r)^{-l}\le C^+U^+(r,0)\quad\text{for}\quad r\ge r^*(0).$$

Indeed, we can take $C^+>0$ satisfies

$$C^+d^-(\eta_2)ca_0\tau^{\frac{m+\lambda}{2}}-c_1>0,$$

where $\eta_2=\tau^{-1/2}r^*(0)$ then we obtain

$$\begin{aligned}
&C^+U^+(r,0)-U_0(r)\\
&\quad = C^+ca_0\tau^{-\frac{l}{2}}\tau^{(m+\lambda)/2}r^{-(m+\lambda)}f(\eta)-c_1(1+r)^{-l}\\
&\quad \ge C^+ca_0\tau^{-\frac{l-m-\lambda}{2}}r^{-(m+\lambda)}d^-(\eta_2)\eta^{-(l-m-\lambda)}-c_1(1+r)^{-l}\\
&\quad = C^+d^-(\eta_2)ca_0\tau^{-\frac{l-m-\lambda}{2}}(\tau^{-1/2}r)^{-l}-c_1(1+r)^{-l}\\
&\quad \ge Cd^-(\eta_2)^-ca_0\tau^{\frac{m+\lambda}{2}}(1+r)^{-l}-c_1(1+r)^{-l}\\
&\quad \ge \left(C^+d^-(\eta_2)ca_0\tau^{\frac{m+\lambda}{2}}-c_1\right)(1+r)^{-l}\\
&\quad \ge 0 \quad \text{for} \quad r\ge r^*(0),
\end{aligned}$$

On the other hand, for $0\le r\le r^*(0)$, we have

$$U^+(r,0)=U_{\text{in}}^+(r,0)=\tau^{-q-1}(\log B_3\tau^{1/2})^{-1}\psi(r)\left(\tau-\left(q+\frac{1}{2}(\log B_3\tau)^{1/2})^{-1}\right)\frac{\Psi(r)}{\psi(r)}\right).$$

This shows that $U^+(r,0)$ is monotone decreasing in $r\in[0,r^*(0)]$, and U_{in}^+ attains its minimum at $r=r^*(0)$ (see Lemma 2.4 and Remark 2.1). Hence it is sufficient to choose C^+ so large that

$$C^+U_{\text{in}}^+(r^*(0),0)\ge c_1.$$

By taking larger C^+ that satisfies the above conditions, we see that U_0 satisfies

$$0<U_0(r)\le C^+U^+(r,0),\quad r\ge 0.$$

Then by the comparison principle, we obtain

$$0<U(r,t)\le C^+U^+(r,t),\quad r\ge 0,\quad t>0.$$

Since U^+ attains the exact decay rate at the origin we retake C^+ with C_2'. we finish the proof. □

Proof of Theorem 1.2 We take

$$U_0(r):=\max_{|x|=r}|u_0(x)-\tilde{u}_0(x)|>0,\quad r\ge 0.$$

Then by Lemma 2.1, Proposition 4.3, then U satisfies

$$\|U(\cdot,t)\|_{L^\infty}\le C^+U^+(0,t)\le C(t+3)^{-\frac{l-m-\lambda}{2}}(\log(t+3)^{1/2})^{-1}\quad \text{for all } t>0,$$

with some constant $C>0$. The proof is now complete. □

5 Lower bound for the critical exponent

In this section, we prove that there exists a lower bound of the convergence rate for the critical case. To this end, we proceed to construct a sub-solution as proven in the preceding section.

5.1 Outer sub-solution

In this subsection, we construct a suitable outer sub-solution of (2.1). First, we recall that f satisfies

$$\begin{cases} f_{\eta\eta} + \frac{n-1}{\eta} f_\eta + \frac{\eta}{2} f_\eta + \frac{\beta}{2} f = 0, \quad \eta > 0, \\ f(0) = a_0 > 0, \quad f_\eta(0) = 0. \end{cases} \tag{5.1}$$

where $n = N - 2(m + \lambda)$, $\beta = l - m - \lambda$ and satisfies $0 < \beta < 2$. Although this solution is used to make a super-solution of (2.1) used in the previous section. Then we need to modify this solution to construct a sub-solution of (2.1) in an outer region as follows.

We take $\delta > 0$ satisfies $\delta < 2 - \beta$ and put $\tilde{\beta} = \beta + \delta$ define $\tilde{f}$ satisfies

$$\begin{cases} \tilde{f}_{\eta\eta} + \frac{n-1}{\eta} \tilde{f}_\eta + \frac{\eta}{2} \tilde{f}_\eta + \frac{\tilde{\beta}}{2} \tilde{f} = 0, \quad \eta > 0, \\ \tilde{f}(0) = a_0 > 0, \quad \tilde{f}_\eta(0) = 0. \end{cases} \tag{5.2}$$

Lemma 5.1 *We define*

$$F^-(\eta) := \eta^{-m-\lambda} f^-(\eta), \quad f^-(\eta) := f(\eta) - b_2 \tilde{f}(\eta)$$

and

$$U^-_{\text{out}}(r,t) := \begin{cases} 0 & \text{for } 0 \le \eta < \eta_2, \\ (t+\tau)^{-\frac{l}{2}} F^-(\eta) = (t+\tau)^{-\frac{l}{2}} \eta^{-m-\lambda} f^-(\eta) & \text{for } \eta \ge \eta_2, \end{cases}$$

with $\eta = (t+\tau)^{-1/2} r$, where constants $\tau, b_2, \eta_2 > 0$ are determined later. Then U^-_{out} is a sub-solution of (2.1).

Proof It is trivial that 0 is a sub-solution. Then we only check the case where $\eta \ge \eta_2$. We fix any $\eta_0 > 1$ and take any $\varepsilon > 0$ and sufficiently large $\tau > 0$ satisfies

$$(m+2)(N-2-m)\left(1 - \left(1 - \frac{a^-_\alpha}{L}(\eta_0\tau)^{-1/2\lambda} \log(\eta_0\tau)\right)^{p-1}\right) < \varepsilon \tag{5.3}$$

and

$$\eta_0 \tau^{1/2} > 3.$$

We define

$$F^-(\eta) := \eta^{-m-\lambda} f^-(\eta), \quad f^-(\eta) := f(\eta) - b_2\tilde{f}(\eta).$$

where b_2 is a constant determined later. There exist constants $d_-(\eta_0) > 0$ such that

$$f(\eta) \geq d_-(\eta_0)\eta^{-(l-m-\lambda)} \quad \text{for} \quad \eta \geq \eta_0. \tag{5.4}$$

and $d_+ > 0$ such that

$$\tilde{f}(\eta) \leq d^+\eta^{-(l-m-\lambda)} \quad \text{for all} \quad \eta > 0 \tag{5.5}$$

from Lemma 2.6. We take $b_2 > 0$ satisfies

$$f(\eta_0) - b_2\tilde{f}(\eta_0) \leq 0 \tag{5.6}$$

and

$$\frac{b_2\delta}{2}d^-(\eta_0)\eta_0^{2-\delta} - \varepsilon d^+ > 0. \tag{5.7}$$

Then we can define $\eta_2 \geq \eta_0$ as

$$\eta_2 := \inf\{\rho | f(\eta) - b_2\tilde{f}(\eta) > 0 \text{ for } \eta > \rho > \eta_0\} \tag{5.8}$$

and find η_2 is well defined from Lemma 2.6, (5.2) and (5.6).

First, for our problem's general setting, the following results are obtained in the same way as in the computation of Lemma 3.1.

$$\begin{aligned}
&U^-_{\mathrm{out},t} - \mathcal{P}_\alpha U^-_{\mathrm{out}} \\
&= -(t+\tau)^{-\frac{l}{2}-1}\left(\frac{l}{2}F^- + \frac{\eta}{2}F^-_\eta + F^-_{\eta\eta} + \frac{N-1}{\eta}F^-_\eta + p(t+\tau)\varphi_\alpha^{p-1}F^-\right) \\
&\leq -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\Bigg(f^-_{\eta\eta} + \frac{N-1-2(m+\lambda)}{\eta}f^-_\eta + \frac{\eta}{2}f^-_\eta + \frac{l-m-\lambda}{2}f^- \\
&\quad + p(t+\tau)\left(Lr^{-m} - a^-_\alpha r^{-m-\lambda}\log r\right)^{p-1} f^- \\
&\quad + \eta^{-2}\left((m+\lambda+1)(m+\lambda) - (N-1)(m+\lambda)\right) f^-\Bigg) \\
&= -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\Bigg(f^-_{\eta\eta} + \frac{n-1}{\eta}f^-_\eta + \frac{\eta}{2}f^-_\eta + \frac{\beta}{2}f^- \\
&\quad + \eta^{-2}\Bigg(\lambda^2 - (N-2-2m)\lambda + 2(N-2-m) \\
&\quad - (m+2)(N-2-m)\left(1 - \left(1 - \frac{a^-_\alpha}{L}r^{-\lambda}\log r\right)^{p-1}\right)\Bigg) f^-\Bigg).
\end{aligned}$$

Here we use (1.3) and (3.2). Next, we substitute $f^-(\eta) = f(\eta) - b_2\tilde{f}(\eta)$ with $r = (t+\tau)^{1/2}\eta$, we use $m+\lambda = (N-2)/2$, (1.5), and find we only need to consider the range $\eta > \eta_2$ from (5.8). Then we can use (5.1), (5.2), (5.3), (5.4), (5.5) and deduce the above inequality as follows.

$$
\begin{aligned}
&U^-_{\mathrm{out},t} - \mathcal{P}_\alpha U^-_{\mathrm{out}} \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\Bigg(f_{\eta\eta} + \frac{n-1}{\eta}f_\eta + \frac{\eta}{2}f_\eta + \frac{\beta}{2}f \\
&\qquad - b_2\left(\tilde{f}_{\eta\eta} + \frac{n-1}{\eta}\tilde{f}_\eta + \frac{\eta}{2}\tilde{f}_\eta + \frac{\tilde{\beta}}{2}\tilde{f}\right) + \frac{b_2\delta}{2}\tilde{f} \\
&\qquad - \eta^{-2}(m+2)(N-2-m)\left(1 - \left(1 - \frac{a^-_\alpha}{L}(\eta_0\tau)^{-\lambda}\log(\eta_0\tau)\right)^{p-1}\right)(f - b_2\tilde{f})\Bigg) \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_2\delta}{2}\tilde{f} - \varepsilon\eta^{-2}f\right) \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_2\delta}{2}d^-(\eta_0)\eta^{-\tilde{\beta}} - \varepsilon\eta^{-2}d^+\eta^{-\beta}\right) \\
&\quad = -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_2\delta}{2}d^-(\eta_0)\eta^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta} \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_2\delta}{2}d^-(\eta_0)\eta_2^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta} \\
&\quad \le -(t+\tau)^{-\frac{l}{2}-1}\eta^{-m-\lambda}\left(\frac{b_2\delta}{2}d^-(\eta_0)\eta_0^{2-\delta} - \varepsilon d^+\right)\eta^{-2-\beta} \\
&\quad \le 0 \quad \text{for} \quad \eta \ge \eta_2
\end{aligned}
$$

by (5.7). We complete the proof. □

5.2 Inner sub-solution and matching

We use a similar inner sub-solution as in [19] Lemma 4.1.

Lemma 5.2 *For any $q > 0$, we define*

$$U^-_{\mathrm{in}}(r,t) := (t+\tau)^{-q}(\log(t+\tau)^{1/2})^{-1}\psi(r).$$

If τ is sufficiently large, then there exist constants $B_2 > 0$ satisfies $B_2\tau^{1/2} > 3$ and $c > 0$ such that the following inequalities hold :

(i) $U^-_{\mathrm{in},t} < \mathcal{P}_\alpha U^-_{\mathrm{in}}$ *for all* $r > 0$ *and* $t > 0$.
(ii) $U^-_{\mathrm{in}}(r,t) > 0$ *for all* $t > 0$ *and* $r \in [0, B_2(t+\tau)^{\frac{1}{2}}]$.
(iii) $cU^-_{\mathrm{in}}(r,t) < U^-_{\mathrm{out}}(r,t)$ *at* $r = B_2(t+\tau)^{\frac{1}{2}}$ *for all* $t > 0$.

Proof The proof of (i) is computed in the same way as in [19] Lemma 4.1, Although for the reader's convenience, we will show it here.

$$\begin{aligned} U^-_{\mathrm{in},t} - \mathcal{P}_\alpha U^-_{\mathrm{in}} &= -q(t+\tau)^{-q-1}(\log(t+\tau)^{1/2})^{-1}\psi(r) \\ &\quad - \frac{1}{2}(t+\tau)^{-q-1/2}((t+\tau)^{1/2})^{-1}(\log(t+\tau)^{1/2})^{-2}\psi(r) \\ &\quad - (t+\tau)^{-q}(\log(t+\tau)^{1/2})^{-1}\mathcal{P}_\alpha\psi(r) \\ &= -q(t+\tau)^{-q-1}(\log(t+\tau)^{1/2})^{-1}\psi(r) \\ &\quad - \frac{1}{2}(t+\tau)^{-q-1/2}((t+\tau)^{1/2})^{-1}(\log(t+\tau)^{1/2})^{-2}\psi(r) \\ &< 0, \quad \text{for} \quad r>0, t>0, \end{aligned}$$

by Lemma 2.3. Hence U^-_{in} is a sub-solution of (2.1) which proves(i).

Next, let us shows (ii), (iii). We set

$$q := \frac{l-m-\lambda}{2}$$

in Lemma 5.2. By Lemma 2.3, We can choose positive constants c^+_α such that

$$\psi(r) \le c^+_\alpha r^{-m-\lambda}\log r \quad \text{for} \quad r \ge 3, \tag{5.9}$$

We found there exists the maximum point of $F^-(\eta)$ denote η_M from the construction of F^- and Lemma 2.6. First, we fix $\eta_2 < B_2 < \eta_M$. Next, we take $c > 0$ satisfies

$$f^-(B_2) - cc^+_\alpha(\log B_2 + 1) > 0 \tag{5.10}$$

Finally, we take $\tau > 0$ is sufficient large such that the condition in Lemma 5.1 holds and

$$\tau^{1/2} > 3 \tag{5.11}$$

It is clear that

$$U^-_{\mathrm{in}}(r,t) > 0, \quad t>0, r\in[0, B_2(t+\tau)^{1/2}].$$

Indeed, we recall

$$U^-_{\mathrm{in}}(r,t) := (t+\tau)^{-q}(\log(t+\tau)^{1/2})^{-1}\psi(r) > 0 \text{ for } t\ge 0, r\ge 0,$$

which prove (ii) by using positivity of $\psi(r)$ and (5.11).

Next, let us show $U^-_{\mathrm{out}}(r_2,t) - cU^-_{\mathrm{in}}(r_2,t) > 0$ at $r_2(t) := B_2(t+\tau)^{1/2}$ and $\eta_3 := (t+\tau)^{-1/2}r_2(t)$, namely at $\eta_3 = B_2$. Noting $B_2 > \eta_2 > 1$, we obtain

$$
\begin{aligned}
&U^-_{\mathrm{out}}(r_2(t),t) - cU^-_{\mathrm{in}}(r_2(t),t)\\
&\quad = (t+\tau)^{-\frac{l}{2}}F^-(\eta_3) - c(t+\tau)^{-q}(\log(t+\tau)^{1/2})^{-1}\psi(r_2)\\
&\quad \geq (t+\tau)^{-\frac{l}{2}}B_2^{-(m+\lambda)}f^-(B_2)\\
&\qquad - c(t+\tau)^{-\frac{l-m-\lambda}{2}}(\log(t+\tau)^{1/2})^{-1}c_\alpha^+ r_2^{-(m+\lambda)}\log r_2\\
&\quad = (t+\tau)^{-\frac{l}{2}}\left(B_2^{-(m+\lambda)}f^-(B_2) - cc_\alpha^+ B_2^{-(m+\lambda)}\frac{\log B_2(t+\tau)^{1/2}}{\log(t+\tau)^{1/2}}\right)\\
&\quad = (t+\tau)^{-\frac{l}{2}}\left(f^-(B_2) - cc_\alpha^+\frac{\log B_2 + \log(t+\tau)^{1/2}}{\log(t+\tau)^{1/2}}\right)B_2^{-(m+\lambda)}\\
&\quad = (t+\tau)^{-\frac{l}{2}}\left(f^-(B_2) - cc_\alpha^+\left(\frac{\log B_2}{\log(t+\tau)^{1/2}}+1\right)\right)B_2^{-(m+\lambda)}\\
&\quad > (t+\tau)^{-\frac{l}{2}}\left(f^-(B_2) - cc_\alpha^+\left(\frac{\log B_2}{\log\tau^{1/2}}+1\right)\right)B_2^{-(m+\lambda)}\\
&\quad > (t+\tau)^{-\frac{l}{2}}(f^-(B_2) - cc_\alpha^+(\log B_2+1))B_2^{-(m+\lambda)}\\
&\quad > 0, \quad \text{for all } t>0,
\end{aligned}
$$

by (5.9), (5.10) and (5.11), thus (iii) is proved. Then we complete the proof. □

Proposition 5.3 *Suppose that* $m+\lambda < l < m+\lambda+2$ *and*

$$U_0(r) \geq c_3(1+r)^{-l}, \quad r \geq 0$$

with some $c_3 > 0$. *Then there exist constant* $C_3', \tau > 0$ *such that the solution of* (2.1) *satisfies*

$$\|U(\cdot,t)\|_{L^\infty} \geq C_3'(t+\tau)^{-\frac{l-m-\lambda}{2}}(\log(t+\tau)^{1/2})^{-1} \quad \textit{for all } t>0.$$

Proof Let $U^-_{\mathrm{out}}(r,t)$ and $U^-_{\mathrm{in}}(r,t)$ be as in Lemmas 5.1 and 5.2 respectively, and define

$$U^-(r,t) := \begin{cases} cU^-_{\mathrm{in}}(r,t) & \text{for } r < r^*(t),\\ U^-_{\mathrm{out}}(r,t) & \text{for } r \geq r^*(t),\end{cases}$$

where $r^*(t)$ is defined

$$r^*(t) := \sup\{r>0 \mid cU^-_{\mathrm{in}}(\rho,t) > U^-_{\mathrm{out}}(\rho,t) \text{ for } \rho\in[0,r)\}.$$

From Lemma 5.2 (iii), we obtain

$$0 < r^*(t) < r_2(t) < \infty \quad \text{for all } t>0.$$

We note that $r^*(t) \in (0,\infty]$ is well defined since $0 < r^*(t) < \infty$. From the construction of U^-, it attains the exact decay rate at the origin. Thus it is shown that $U^-(r,t)$ is a sub-solution of (2.1) which satisfies

$$
\begin{aligned}
U^-(0,t) &= cU_{\text{in}}^-(0,t)\\
&= c(t+\tau)^{-(l-m-\lambda)/2}(\log(t+\tau)^{1/2})^{-1}\psi(0)\\
&= c(t+\tau)^{-(l-m-\lambda)/2}(\log(t+\tau)^{1/2})^{-1}
\end{aligned}
$$

for all $t > 0$.

We will show that the $C^-U^-(r,0)$ lies below the initial data $U_0(r)$ if we take a constant $C^- > 0$ sufficiently small. In fact, we can take a constant $C^- > 0$ small enough to hold that

$$C^-U^-(r,0) \le U_0(r), \quad r \ge 0.$$

Indeed, if we take $C^- > 0$ so small that

$$C^-c\tau^{-1}(\log \tau^{1/2})^{-1} \le c_3$$

and

$$C^-d^+r^{-l} \le c_3(1+r)^{-l} \text{ for } r \ge r^*(0).$$

Then we see that

$$
\begin{aligned}
C^-U^-(r,0) &= C^-cU_{\text{in}}^-(r,0)\\
&= C^-c\tau^{-(l-m-\lambda)/2}(\log\tau^{1/2})^{-1}\psi(0)\\
&= C^-c\tau^{-1}(\log\tau^{1/2})^{-1}\\
&\le c_3 \le U_0(r) \quad \text{for } 0 \le r \ \le r^*(0)
\end{aligned}
$$

and

$$
\begin{aligned}
&U_0(r) - C^-U^-(r,0)\\
&\quad \ge c_3(1+r)^{-l} - C^-U_{\text{out}}^-(r,0)\\
&\quad = c_3(1+r)^{-l} - C^-\tau^{-\frac{l}{2}}F^-(\eta)\\
&\quad = c_3(1+r)^{-l} - C^-\tau^{-\frac{l}{2}}\eta^{-m-\lambda}(f(\eta) - b_2\tilde{f}(\eta))\\
&\quad > c_3(1+r)^{-l} - C^-\tau^{-\frac{l}{2}}\eta^{-m-\lambda}f(\eta)\\
&\quad \ge c_3(1+r)^{-l} - C^-\tau^{-\frac{l}{2}}\eta^{-m-\lambda}d^+\eta^{-(l-m-\lambda)}\\
&\quad = c_3(1+r)^{-l} - C^-d^+\tau^{-\frac{l}{2}}(\tau^{-1/2}r)^{-l}\\
&\quad = c_3(1+r)^{-l} - C^-d^+r^{-l}\\
&\quad \ge 0 \text{ for } r \ge r^*(0)
\end{aligned}
$$

by using (5.4). Then the initial condition is satisfied by the above argument, and by the comparison principle, we obtain

$$C^{-}U^{-}(r,t) \le U(r,t) \quad \text{for} \quad r>0,\ t>0.$$

We replace $C^{-}c$ with C_3'. Since U^{-} attains the exact decay rate at the origin, we finish the proof. □

Proof of Theorem 1.3 We take

$$U_0(r) := \min_{|x|=r} |u_0(x) - \tilde{u}_0(x)| > 0, \quad r \ge 0.$$

Then by Lemma 2.2, Proposition 5.3, we have

$$\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_{L^\infty} \ge C_3' U^{-}(0,t) \ge C_3 (t+3)^{-(l-m-\lambda)} (\log(t+3)^{1/2})^{-1}$$

for all $t>0$ with some constant $C_3>0$. □

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Declarations

Conflict of interest The author declares that he has no conflict of interest.

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