Journal of Elliptic and Parabolic Equations (2024) 10:415–474
https://doi.org/10.1007/s41808-024-00264-w

Stochastic homogenization of nonlocal reaction–diffusion problems of gradient flow type

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Received: 10 July 2023 / Accepted: 22 January 2024 / Published online: 14 March 2024
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Abstract

In this paper we study the stochastic homogenization of reaction–diffusion problems whose the diffusion terms are gradients of random nonlocal convex and Fréchet-differentiable functionals and the reaction terms are random CP-structured reaction functionals as introduced in Anza Hafsa et al. (Asymptot Anal 115(3–4):169–221, 2019). We provide an application to spatial population dynamics.

Keywords Nonlocal reaction–diffusion equations · Mosco-convergence · Γ-convergence · Convergence of nonlocal reaction–diffusion equations · Stochastic homogenization

Mathematics Subject Classification 35K57 · 35B27 · 35R60 · 45K05 · 49K45

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1 Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space, let $T > 0$ and let $O \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary. In this paper we study the stochastic homogenization of reaction–diffusion problems of the form:

$$(\mathscr{P}_{\varepsilon,\omega}) \begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t) + \nabla \mathscr{E}_\varepsilon(\omega, u_\varepsilon^\omega(t)) = F_\varepsilon(\omega, t, u_\varepsilon^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u_\varepsilon^\omega(0) = u_{0,\varepsilon}^\omega \in L^2(O), \end{cases} \tag{1.1}$$

where, for each $\varepsilon > 0$, the diffusion term is the gradient of a random nonlocal functional $\mathscr{E}_\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ of type:

$$\mathscr{E}_\varepsilon(\omega, u) = \frac{1}{4\varepsilon^d} \int_O \int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{u(x) - u(y)}{\varepsilon}\right)^2 dxdy + \mathscr{D}_\varepsilon(\omega, u) \tag{1.2}$$

with $J : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty[$ and $\mathscr{D}_\varepsilon : \Omega \times L^2(O) \to \mathbb{R}$ a nonlocal functional characterizing the fact that $(\mathscr{P}_{\varepsilon,\omega})$ is of Neumann–Cauchy nonhomogenous or Dirichlet–Cauchy type, and the reaction term is a random CP-structured reaction functional $F_\varepsilon : \Omega \times [0, T] \times L^2(O) \to L^2(O)$, see Definition 2.9.

Roughly, our main result (see Theorem 3.19) is to prove that as $\varepsilon \to 0$, $(\mathscr{P}_{\varepsilon,\omega})$ converges almost surely, in a variational sense, to

$$(\mathscr{P}_{\mathrm{hom},\omega}) \begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla \mathscr{E}_{\mathrm{hom}}(\omega, u^\omega(t)) = G^\omega(t, u^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u^\omega(0) = u_0^\omega \in \mathrm{dom}(\mathscr{E}_{\mathrm{hom}}(\omega, \cdot)), \end{cases} \tag{1.3}$$

where $u_{0,\varepsilon}^\omega \rightharpoonup u_0^\omega$ in $L^2(O)$, $F_\varepsilon(\omega, \cdot, u_\varepsilon^\omega) \rightharpoonup G^\omega(\cdot, u^\omega)$ in $L^2([0, T]; L^2(O))$ with $G^\omega \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ (see the definition in Sect. 2.1). The functional $\mathscr{E}_{\mathrm{hom}} : \Omega \times L^2(O) \to [0, \infty]$

is the almost sure Mosco-limit of $\mathscr{E}_\varepsilon$ (see Theorem 4.8) and is given in its domain by

$$\mathscr{E}_{\text{hom}}(\omega, u) = \int_O f_{\text{hom}}(\omega, \nabla u(x))dx$$

with $f_{\text{hom}} : \Omega \times \mathbb{R}^d \to [0, \infty[$ a quadratic function defined as the limit of a suitable subadditive process (see Propositions 3.14 and 3.17).

To our knowledge, in a deterministic framework, the convergence of problems of type (1.1) without reaction term and with J depending only on the third variable has been firstly addressed by Andreu, Mazón, Rossi and Toledo in [6, 7] (see also [8]) using semi-group theory and the convergence of their resolvents. They prove the convergence to a local Cauchy problem. In the scope of homogenization, the convergence of nonlocal energies of type (1.2) has been recently studied by Braides and Piatnitski in [13] in the periodic case (see also [22]), and in [12] in a stochastic case (see also [23]). For a general Γ-convergence approach to non-local to local limits, we refer to the book [1] (see, in particular, [1, Chapter 9] which is devoted to non-local to local parabolic problems).

In our work, under a stationarity hypothesis on J but without ergodicity assumption, we establish the almost sure Mosco convergence of such nonlocal functionals (see Theorem 4.8) yielding, as a consequence, the almost sure convergence of $(\mathscr{P}_{\varepsilon,\omega})$ to $(\mathscr{P}_{\text{hom},\omega})$ with Neumann–Cauchy homogeneous or Dirichlet–Cauchy boundary conditions (see Theorem 3.19).

Nonlocal problems of type (1.1) are well adapted for spatial population dynamics where the density J in (1.2) accounts for the number of individuals at time t in O which jump from y to x. The nonlocal diffusion term can be explained for example by the dispersion of population of species (seeds, larvae) by wind or water, the population can be transported over long distances which increases their survival and reproduction (see [20, 21, 24]). In Sect. 5 we consider such a population dynamics model with a reaction term of the form:

$$F_\varepsilon(\omega, t, u)(x) = r\left(\omega, t, \frac{x}{\varepsilon}\right) u(x) \left(1 - \frac{u(x)}{K\left(\omega, t, \frac{x}{\varepsilon}\right)}\right) - hu(x),$$

with $h \geq 0$ and $r, K \in L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$ such that $r > 0$ and $K \geq \gamma > 0$, where r is the growth rate, K is the carrying capacity and h the percentage of harversting. By applying our convergence result, we show (see Corollary 5.8) that as $\varepsilon \to 0$, the nonlocal reaction–diffusion problems $(\mathscr{P}_{\varepsilon,\omega})$ almost surely converge to a local reaction–diffusion problem of type (1.3) with $G^\omega = F_{\text{hom}}(\omega, \cdot, \cdot)$ where

$$F_{\text{hom}}(\omega, t, u)(x) := r_{\text{hom}}(\omega, t)u(x) \left(1 - \frac{u(x)}{K_{\text{hom}}(\omega, t)}\right) - hu(x),$$

where $r_{\mathrm{hom}}(\omega,\cdot):[0,T]\to[0,\infty[$ and $K_{\mathrm{hom}}(\omega,\cdot):[0,T]\to[0,\infty[$ are given by

$$\begin{cases} r_{\mathrm{hom}}(\omega,t)=\mathbb{E}^{\mathscr{I}}\left(\int_{]0,1[^d} r(\cdot,t,y)dy\right)(\omega) \\ K_{\mathrm{hom}}(\omega,t)=\dfrac{\mathbb{E}^{\mathscr{I}}\left(\int_{]0,1[^d} r(\cdot,t,y)dy\right)(\omega)}{\mathbb{E}^{\mathscr{I}}\left(\int_{]0,1[^d} \frac{r(\cdot,t,y)}{K(\cdot,t,y)}dy\right)(\omega)} \end{cases}$$

with $\mathbb{E}^{\mathscr{I}}$ being the conditional mathematical expectation with respect to σ-algebra $\mathscr{I}$ of invariant sets with respect to the dynamical system $(\Omega,\mathscr{F},\mathbb{P},\{T_z\}_{z\in\mathbb{Z}^d})$ (see Sect. 3.1 for more details). The distinguishing feature here is that in the formula of the homogenized reaction functional, the homogenized carrying capacity K_{hom} is given by a mixture between carrying capacity and growth rate.

1.1 Plan of the paper

Section 2 is devoted to existence, uniqueness for nonlocal reaction diffusion problems of gradient flow type, and boundedness of the solutions when the reaction term is a CP-structured reaction functional (see Definition 2.9 and Corollary 2.11). For this, we develop the nonlocal framework for dealing with Neumann–Cauchy homogeneous (see Sect. 2.1.1), nonhomogeneous (see Sect. 2.1.2) and Dirichlet–Cauchy (see 2.1.3) nonlocal reaction–diffusion problems. In addition, in Sect. 2.2 we treat the invasion property for for nonlocal problems with CP-structured autonomous reaction functionals.

Section 3 is devoted to the statement of the main result. In Sect. 3.1 we precise the probability setting and recall some tools from ergodic theory (see Definitions 3.1–3.2 and 3.5–3.6 and Theorem 3.7). By applying Corollary 2.11 we obtain existence and uniqueness of bounded solutions for random Neumann–Cauchy homogeneous and Dirichlet–Cauchy nonlocal reaction–diffusion problems: this is discribed in Sect. 3.2. The main result of the paper is stated in Sect. 3.3 (see Theorem 3.19). To identify the homogenized diffusion term we need a suitable subadditive theorem that we state and prove in Sect. 3.3 (see Proposition 3.17). Note that we do not deal with the convergence of Neumann–Cauchy nonhomogeneous nonlocal reaction–diffusion problems. Indeed, the mathematical analysis seems technically more tricky but we hope to cover this case in the future.

Section 4 is devoted to the proof of Theorem 3.19. Its proof, which is given in Sect. 4.4, follows from two theorems. The first one (see Theorem 4.1) is an abstract convergence result for passing from nonlocal to local: it is stated and proved in Sect. 4.1. The second one (see Theorem 4.8) establishes the almost sure Mosco-convergence of the energies corresponding to the diffusion term: it is stated and proved in Sect. 4.3. The proof of Theorem 4.8 uses Proposition 3.17 together with some lemmas. These lemmas are stated and proved in Sect. 4.2.

Section 5 is devoted to the application of the results to spatial population dynamics. In Sect. 5.1 we begin by giving a heuristic derivation of the model. Then, in Sect. 5.1, we precise the mathematical description of the model in showing that it can studied in the general framework developed in Sects. 2–3. Finally, by applying Theorem 3.19, in Sect. 5.3 we obtain the homogenized model (see Corollary 5.8). Besides population dynamics, another example of field of application (not adressed in our paper) is peridynamics, for which we refer to [11, 18].

For convenience of the reader, in the appendix we recall some classical definitions and results that we use in the paper.

Notation. Throughout the paper we will use the following notation.

- Given $x_0 \in \mathbb{R}^d$ we denote the open (resp. closed) ball of radius $r > 0$ centered at x_0 by $B_r(x_0)$ (resp. $\overline{B}_r(x_0)$).
- The closure (resp. interior) of a set $A \subset \mathbb{R}^d$ is denoted by $\overline{A}$ (resp. $\mathrm{int}(A)$).
- The Lebesgue measure on $\mathbb{R}^d$ with $d \in \mathbb{N}^*$ is denoted by $\mathscr{L}^d$ and for each Borel set $A \subset \mathbb{R}^d$, the measure of A with respect to $\mathscr{L}^d$ is denoted by $\mathscr{L}^d(A)$.
- The class of bounded Borel subsets of $\mathbb{R}^d$ is denoted by $\mathscr{B}_\mathrm{b}(\mathbb{R}^d)$.
- The space of continuous piecewise affine functions from O to $\mathbb{R}$ is denoted by $\mathrm{Aff}(O)$.
- Given $(a, b) \in \mathbb{R}^2$ with $a \leq b$, the space of $u \in L^2(O)$ such that $a \leq u \leq b$ is denoted by $L^2(O; [a, b])$.
- The space of continuous functions from $[0, T]$ to $L^2(O)$ is denoted by $C([0, T]; L^2(O))$.
- The space of absolutely continuous functions from $[0, T]$ to $L^2(O)$ is denoted by $AC([0, T]; L^2(O))$.
- The class of reaction functionals $F : [0, T] \times L^2(O) \to L^2(O)$ satisfying R_1–R_1 is denoted by $\mathscr{F}_{(R_1)\text{-}(R_2)}$.
- The class of CP-structured reaction functionals $F : [0, T] \times L^2(O) \to L^2(O)$ is denoted by $\mathscr{F}_{\mathrm{CP}}$.
- Given $\{u_n\}_n \subset C([0, T]; L^2(O))$, by $u_n \to u$ in $C([0, T]; L^2(O))$ we mean that $\lim\limits_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^2(O)} = 0$. By $\frac{du_n}{dt} \rightharpoonup \frac{du}{dt}$ in $L^2([0, T]; L^2(O))$ we mean that for every $v \in L^2([0, T]; L^2(O))$, $\int_0^T \langle \frac{du_n}{dt}(t), v(t)\rangle dt \to \int_0^T \langle \frac{du}{dt}(t), v(t)\rangle dt$ as $n \to \infty$, where $\langle\cdot, \cdot\rangle$ denotes the scalar product in $L^2(O)$.

2 Nonlocal reaction–diffusion problems of gradient flow type

2.1 Existence and uniqueness of bounded solutions for nonlocal problems with CP-structured reaction functionals

Given $T > 0$ and $\mathscr{E} : L^2(O) \to [0, \infty[$ a convex and Fréchet-differentiable function, we consider the following reaction–diffusion problem of gradient flow type:

$$(\mathscr{P}_{\mathscr{E}}^{u_0,F}) \begin{cases} \dfrac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(t, u(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in L^2(O), \end{cases}$$

where the reaction term $F : [0, T] \times L^2(O) \to L^2(O)$ is a Borel measurable map satisfying the following two conditions:

(R_1) there exists $L \in L^2([0, T])$ such that for every $(u, v) \in L^2(O) \times L^2(O)$ and every $t \in [0, T]$,

$$\|F(t, u) - F(t, v)\|_{L^2(O)} \le L(t)\|u - v\|_{L^2(O)};$$

(R_2) $\|F(\cdot, 0)\|_{L^2(O)} \in L^2([0, T])$.

From now on, the class of Borel measurable maps $F : [0, T] \times L^2(O) \to L^2(O)$ verifying (R_1)–(R_2) is denoted by $\mathscr{F}_{(R_1)\text{-}(R_2)}$. The following result is a straightforward consequence of [4, Theorem 2.2, p. 16].

Theorem 2.1 *If $u_0 \in L^2(O)$ and $F \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ then $(\mathscr{P}_{\mathscr{E}}^{u_0,F})$ admits a unique solution $u \in AC([0, T]; L^2(O))$. Moreover, if $F(\cdot, u(\cdot)) \in AC([0, T]; L^2(O))$ then u admits a right derivative $\frac{d^+u}{dt}(t)$ at every $t \in]0, T[$ which satisfies $\frac{d^+u}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(t, u(t))$.*

In this paper we consider reaction–diffusion problems with nonlocal diffusion terms, i.e. when $\mathscr{E} : L^2(O) \to [0, \infty[$ is a nonlocal functional.

2.1.1 Neumann–Cauchy homogeneous nonlocal problems

Let $J : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to [0, \infty[$ be a Borel measurable function satisfying the following conditions:

(NL_1) J is symmetric, i.e. for every $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$J(x, y, x - y) = J(y, x, y - x);$$

(NL_2) there exists a $(\mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable function $\overline{J} : \mathbb{R}^d \to [0, \infty[$ with $\text{supp}(\overline{J}) = \overline{B}_{R_J}(0)$ for some $R_J > 0$ and $\int_{\mathbb{R}^d} \overline{J}(\xi)d\xi = 1$ such that for every $(x, y, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d$,

$$0 \le J(x, y, \xi) \le \overline{J}(\xi).$$

Remark 2.2 The function $\overline{J}$ is assumed to be compactly supported for simplifying certain calculations. Without major difficulties, by using a truncation argument, we could take $\overline{J}$ growing as $\frac{1}{1+|\xi|^{d+2+\kappa}}$ with $\kappa > 0$.

Let $O \subset \mathbb{R}^d$ be a bounded open set and let $\mathscr{J} : L^2(O) \to [0, \infty[$ be defined by

$$\mathscr{J}(u) := \frac{1}{4}\int_O\int_O J(x, y, x - y)(u(x) - u(y))^2 dxdy.$$

It is easy to see that $\mathscr{J}$ is convex and Fréchet-differentiable, and by the Riesz representation theorem, for each $u \in L^2(O)$, the gradient of $\mathscr{J}$ at u, denoted by $\nabla \mathscr{J}(u)$, is such that

$$\mathscr{J}'(u)(v) = \langle \nabla \mathscr{J}(u), v \rangle = \int_O \nabla \mathscr{J}(u)(x) v(x) dx \text{ for all } v \in L^2(O),$$

where $\nabla \mathscr{J}(u) \in L^2(O)$ and is given by

$$\nabla \mathscr{J}(u)(x) = -\int_O J(x, y, x-y)(u(y) - u(x)) dy.$$

The problem $(\mathscr{P}^{u_0,F}_{\mathscr{J}})$, which corresponds to $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$ with $\mathscr{E} = \mathscr{J}$, is a nonlocal reaction–diffusion problem of gradient flow type that is called "*Neumann–Cauchy homogeneous nonlocal reaction–diffusion problem*". Note that $(\mathscr{P}^{u_0,F}_{\mathscr{J}})$ can be rewritten as follows:

$$(\mathscr{P}^{u_0,F}_{\mathscr{J}}) \begin{cases} \dfrac{\partial u}{\partial t}(t,x) - \displaystyle\int_O J(x, y, x-y)(u(t,y) - u(t,x)) dy = F(t, u(t,x)) & \text{in } [0,T] \times O \\ u(0,\cdot) = u_0 \in L^2(O). \end{cases}$$

Remark 2.3 The term "*Neumann–Cauchy homogeneous nonlocal problem*" refers to homogeneous Neumann–Cauchy boundary conditions for local reaction–diffusion problems. Indeed, by suitably rescaling J and K, it can be established that the solutions of the rescaled corresponding problems converges to the solution of a "standard" local reaction–diffusion problem with the homogeneous Neumann boundary condition (see [8, Chapter 3, Sect. 3.1, p. 41] for $J = \overline{J}$ and $F = 0$).

2.1.2 Neumann–Cauchy nonhomogeneous nonlocal problems

Let $h \in L^1(\mathbb{R}^d \backslash O)$, let $K \in L^\infty(O \times \mathbb{R}^d)$ and let $\mathscr{N}_{h,K} : L^2(O) \to \mathbb{R}$ be defined by

$$\mathscr{N}_{h,K}(u) := \int_O \left(\int_{\mathbb{R}^d \backslash O} K(x, x-y) h(y) dy \right) u(x) dx.$$

It is easy to see that $\mathscr{N}_{h,K}$ is a continuous linear form and for every $u \in L^2(O)$, $\nabla \mathscr{N}_{h,K}(u) \in L^2(O)$ and is given by

$$\nabla \mathscr{N}_{h,K}(u)(x) = \int_{\mathbb{R}^d \backslash O} K(x, x-y) h(y) dy.$$

The problem $(\mathscr{P}^{u_0,F}_{\mathscr{J} - \mathscr{N}_{h,K}})$, which corresponds to $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$ with $\mathscr{E} = \mathscr{J} - \mathscr{N}_{h,K}$, is a nonlocal reaction–diffusion problem of gradient flow type that is called

"*Neumann–Cauchy nonhomogeneous nonlocal reaction–diffusion problem*". Note that $(\mathscr{P}^{u_0,F}_{\mathscr{J}-\mathscr{N}_{h,K}})$ can be rewritten as follows:

$$(\mathscr{P}^{u_0,F}_{\mathscr{J}-\mathscr{N}_{h,K}})\begin{cases}\dfrac{\partial u}{\partial t}(t,x)-\displaystyle\int_O J(x,y,x-y)(u(t,y)-u(t,x))dy\\ -\displaystyle\int_{\mathbb{R}^d\setminus O}K(x,x-y)h(y)dy=F(t,u(t,x)) & \text{in } [0,T]\times O\\ u(0,\cdot)=u_0\in L^2(O).\end{cases}$$

Remark 2.4 The term "*Neumann–Cauchy nonhomogeneous nonlocal problem*" refers to nonhomogeneous Neumann–Cauchy boundary conditions for local reaction–diffusion problems. Indeed, by suitably rescaling J and K, it can be established that the solutions of the rescaled corresponding problems converges to the solution of a "standard" local reaction–diffusion problem with the nonhomogeneous Neumann boundary condition $\frac{du}{d\mathbf{n}}=h$ where $\mathbf{n}$ denotes the unit outward normal to $\partial\Omega$ (see [8, Chapter 3, Sect. 3.2, p. 45] for $J=\overline{J}$ and $F=0$).

2.1.3 Dirichlet–Cauchy nonlocal problems

Set $O^J:=O+\mathrm{supp}(\overline{J})=O+\overline{B}_{R_J}(0)$, let $g\in L^2(O^J\setminus\overline{O})$ and let $\mathscr{D}_g:L^2(O)\to\mathbb{R}$ be defined by

$$\mathscr{D}_g(u):=\frac{1}{2}\int_O\int_{O^J\setminus\overline{O}}J(x,y,x-y)(g(y)-u(x))^2dxdy.$$

It is easy to see that $\mathscr{D}_g$ is convex and Fréchet-differentiable, and for every $u\in L^2(O)$, $\nabla\mathscr{D}_g(u)\in L^2(O)$ and is given by

$$\nabla\mathscr{D}_g(u)(x)=-\int_{O^J\setminus\overline{O}}J(x,y,x-y)(g(y)-u(x))dy.$$

The problem $(\mathscr{P}^{u_0,F}_{\mathscr{J}+\mathscr{D}_g})$, which corresponds to $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$ with $\mathscr{E}=\mathscr{J}+\mathscr{D}_g$, is a nonlocal reaction–diffusion problem of gradient flow type that is called "*Dirichlet–Cauchy nonlocal reaction–diffusion problem*". Note that $(\mathscr{P}^{u_0,F}_{\mathscr{J}+\mathscr{D}_g})$ can be rewritten as follows:

$$
(\mathscr{P}^{u_0,F}_{\mathscr{J}+\mathscr{D}_g})\begin{cases} \frac{\partial u}{\partial t}(t,x) - \int_O J(x,y,x-y)(u(t,y)-u(t,x))dy \\ -\int_{O^J\setminus\overline{O}} J(x,y,x-y)(g(y)-u(t,x))dy = F(t,u(t,x)) \quad \text{in } [0,T]\times O \\ u(0,\cdot) = u_0 \in L^2(O). \end{cases}
$$

Remark 2.5 In the spirit of Remarks 2.3–2.4, the term "*Dirichlet–Cauchy nonlocal problem*" refers to Dirichlet–Cauchy boundary conditions for local reaction–diffusion problems (see [8, Chapter 2, Sect. 2.1, p. 31] for $J = \overline{J}$ and $F = 0$).

From the above it is clear that $\mathscr{J}$, $\mathscr{J} - \mathscr{N}_{h,K}$ and $\mathscr{J} + \mathscr{D}_g$ are convex and Fréchet-differentiable. Hence, as a direct consequence of Theorem 2.1 we have the folowing result.

Corollary 2.6 *Under the hypotheses of Theorem 2.1, the same conclusions hold for* $\mathscr{E} \in \{\mathscr{J}, \mathscr{J} - \mathscr{N}_{h,K}, \mathscr{J} + \mathscr{D}_g\}$.

We are going to establish that the solutions are bounded, then possibly signed according to the initial conditions. We begin by establishing comparison principles. For each $u_0 \in L^2(O)$ and each $F \in \mathscr{F}_{(R_1)\text{-}(R_2)}$, we consider the following two problems:

$$
(\mathscr{P}^{u_0,F}_{\mathscr{E},\leq})\begin{cases} \frac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) \leq F(t,u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u(0) = u_0 \in L^2(O); \end{cases}
$$

$$
(\mathscr{P}^{u_0,F}_{\mathscr{E},\geq})\begin{cases} \frac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) \geq F(t,u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u(0) = u_0 \in L^2(O). \end{cases}
$$

Definition 2.7 A solution $u \in C([0,T]; L^2(O))$ of $(\mathscr{P}^{u_0,F}_{\mathscr{E},\leq})$ (resp. $(\mathscr{P}^{u_0,F}_{\mathscr{E},\geq})$) is called a sub-solution (resp. super-solution) of $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$. (If u is both a sub-solution and a super-solution of $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$ then u is solution of $(\mathscr{P}^{u_0,F}_{\mathscr{E}})$.)

Proposition 2.8 *Let* $u_{0,1}, u_{0,2} \in L^2(O)$ *and let* $F_1, F_2 \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ *be such*

$$
\begin{cases} F_1(t,u)(x) = f_1(t,x,u(x)) \\ F_2(t,u)(x) = f_2(t,x,u(x)) \end{cases}
$$

for all $(t,u,x) \in [0,T] \times L^2(O) \times O$*, where* $f_1, f_2 : [0,T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ *are two Borel measurable functions with* f_2 *Lipschitz continuous uniformly with respect to* (t,x)*, i.e. there exists* $L > 0$ *such that for every* $(\xi,\xi') \in \mathbb{R} \times \mathbb{R}$*,*

$$
|f_2(\cdot,\cdot,\xi) - f_2(\cdot,\cdot,\xi')| \leq L|\xi - \xi'|. \tag{2.1}
$$

Let $K \in L^\infty(O \times \mathbb{R}^d; [0, \infty[)$, let $h_1, h_2 \in L^1(\mathbb{R}^d \setminus O)$ and let $g_1, g_2 \in L^2(O^J \setminus \overline{O})$. If u_1 is a sub-solution of $(\mathscr{P}^{u_{0,1}, F_1}_{\mathscr{J} - \mathscr{N}_{h_1,K}})$ (resp. $(\mathscr{P}^{u_{0,1}, F_1}_{\mathscr{J} + \mathscr{D}_{g_1}})$) and if u_2 is a super-solution of $(\mathscr{P}^{u_{0,2}, F_2}_{\mathscr{J} - \mathscr{N}_{h_2,K}})$ (resp. $(\mathscr{P}^{u_{0,2}, F_2}_{\mathscr{J} + \mathscr{D}_{g_2}})$) then

$$\left.\begin{array}{l} u_{0,1} \leq u_{0,2} \\ h_1 \leq h_2 (resp. g_1 \leq g_2) \\ F_1 \leq F_2 \end{array}\right\} \Longrightarrow u_1(t) \leq u_2(t) \textit{ for all } t \in [0, T].$$

Proof of Proposition 2.8 We only give the proof in the Neumann–Cauchy case. (In the Dirichlet–Cauchy case, the proof follows by similar arguments.) Set $u := u_2 - u_1$, $u^+ := \max(u, 0)$ and $u^- := \max(-u, 0)$. To prove that $u_2(t) \leq u_1(t)$ for all $t \in [0, T]$, it suffices to show that

$$u^-(t) = 0 \text{ for all } t \in [0, T]. \tag{2.2}$$

First of all, it is clear that for $\mathscr{L}^1 \otimes \mathscr{L}^d$-a.e. $(t, x) \in [0, T] \times O$,

$$\begin{aligned} \frac{\partial u}{\partial t}(t, x) - \int_O J(x, y, x - y)(u(t, y) - u(t, x))dy \\ - \int_{\mathbb{R}^d \setminus O} K(x, x - y)(h_2(y) - h_1(y))dy \\ \geq f_2(t, x, u_2(t, x)) - f_1(t, x, u_1(t, x)). \end{aligned}$$

Then, by taking $u^- \in L^2(O)$ as a test function and by integrating over O,

$$\begin{aligned} \int_O \frac{\partial u}{\partial t}(t, x)u^-(t, x)dx - \int_O \left(\int_O J(x, y, x - y)(u(t, y) - u(t, x))dy\right) u^-(t, x)dx \\ - \int_O \left(\int_{\mathbb{R}^d \setminus O} K(x, x - y)(h_2(y) - h_1(y))dy\right) u^-(t, x)dx \\ \geq \int_O (f_2(t, x, u_2(t, x)) - f_1(t, x, u_1(t, x)))u^-(t, x)dx. \end{aligned}$$

But, taking (NL_1) into account, by an easy computation we see that

$$\begin{aligned} - \int_O \left(\int_O J(x, y, x - y)(u(t, y) - u(t, x))dy\right) u^-(t, x)dx \\ = \frac{1}{2} \int_O \int_O J(x, y, x - y)(u(t, y) - u(t, x))(u^-(t, y) - u^-(t, x))dxdy, \end{aligned}$$

and consequently, since $f_2 \geq f_1$,

$$\int_O \frac{\partial u}{\partial t}(t,x)u^-(t,x)dx + \frac{1}{2}\int_O\int_O J(x,y,x-y)(u(t,y)-u(t,x)) \times (u^-(t,y)-u^-(t,x))dxdy - \int_O\left(\int_{\mathbb{R}^d\setminus O} K(x,x-y)(h_2(y)-h_1(y))dy\right)u^-(t,x)dx \geq \int_O (f_2(t,x,u_2(t,x)) - f_2(t,x,u_1(t,x)))u^-(t,x)dx.$$

Noticing that:

- $u = u^+ - u^-$;
- $u^+u^- = \frac{\partial u^+}{\partial t}u^- = 0$;
- $-u^+(\cdot,x)u^-(\cdot,y) \leq 0$ for all $(x,y) \in O \times O$,

and using (2.1) we deduce that

$$-\int_O \frac{\partial u}{\partial t}(t,x)u^-(t,x)dx - \frac{1}{2}\int_O\int_O J(x,y,x-y)(u^-(t,y)-u^-(t,x))^2dxdy - \int_O\left(\int_{\mathbb{R}^d\setminus O} K(x,x-y)(h_2(y)-h_1(y))dy\right)u^-(t,x)dx \geq -L\int_O \left|u^-(t,x)\right|^2dx$$

with $L > 0$ given by (2.1). As $h_2 \geq h_1$ it follows that for $\mathscr{L}^1$-a.e. $t \in [0,T]$,

$$\frac{1}{2}\frac{d}{dt}\int_O |u^-(t,x)|^2dx = \int_O \frac{\partial u}{\partial t}(t,x)u^-(t,x)dx \leq L\int_O \left|u^-(t,x)\right|^2dx,$$

and so, by integrating over $s \in [0,T]$,

$$\int_O |u^-(s,x)|^2dx \leq \int_O |u^-(0,x)|^2dx + 2L\int_0^t\left(\int_O |u^-(t,x)|^2dx\right)dt \text{ for all } s \in [0,T].$$

Noticing that, since $u^- \in C([0,T];L^2(O))$, the function $[0,T] \ni s \mapsto \int_O |u^-(s,x)|^2dx$ is continuous, from Grönwall's lemma (see Lemma C.1 that we apply with $\phi(s) = \int_O |u^-(s,x)|^2dx$, $a = \int_O |u^-(0,x)|^2dx$ and $\mathrm{m}(t) = 2L$) we see that

$$\int_O |u^-(s,x)|^2dx \leq e^{2Ls}\int_O |u^-(0,x)|^2dx \text{ for all } s \in [0,T], \tag{2.3}$$

and (2.2) follows since $u^-(0) = 0$. □

The following class of reaction functionals, called the class of CP-structured reaction functionals and denoted by $\mathscr{F}_{\mathrm{CP}}$, was introduced in [3] (see also [4, Sect. 2.2.2, p. 27]).

Definition 2.9 A map $F : [0, T] \times L^2(O) \to L^2(O)$ is called a CP-structured reaction functional if

$$F(t, u)(x) = f(t, x, u(x))$$

for all $(t, u, x) \in [0, T] \times L^2(O) \times O$, where $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function satisfying the following three properties:

(CP_1) $f(t, x, \zeta)$ is locally Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$;
(CP_2) $f(\cdot, \cdot, 0) \in L^2([0, T]; L^2(O))$;
(CP_3) there exist $\underline{f}, \overline{f} : [0, T] \times \mathbb{R} \to \mathbb{R}$ with $\underline{f} \le 0 \le \overline{f}$ and $(\underline{\rho}, \overline{\rho}) \in \mathbb{R}^2$ with $\underline{\rho} \le \overline{\rho}$ such that each of the two following ordinary differential equations

$$(\underline{ODE}) \begin{cases} y'(t) = \underline{f}(t, y(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ y(0) = \underline{\rho} \end{cases}$$
$$(\overline{ODE}) \begin{cases} y'(t) = \overline{f}(t, y(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ y(0) = \overline{\rho} \end{cases}$$

admits at least a solution, $\underline{y}$ for $(\underline{ODE})$ and $\overline{y}$ for $(\overline{ODE})$, satisfying

$$\begin{cases} \underline{f}(t, \underline{y}(t)) \le f(t, x, \underline{y}(t)) \\ \overline{f}(t, \overline{y}(t)) \ge f(t, x, \overline{y}(t)) \end{cases} \tag{2.4}$$

for $\mathscr{L}^1 \otimes \mathscr{L}^1$-a.a. $(t, x) \in [0, T] \times \mathbb{R}$.

Remark 2.10 Note that $\mathscr{F}_{\mathrm{CP}} \subset \mathscr{F}_{(R_1)\text{-}(R_2)}$. From ($CP_3$) we see that $\underline{y}$ and $\overline{y}$ are decreasing and increasing respectively, and so $\underline{y}(T) \le \underline{y}(t) \le \underline{y}(0) = \underline{\rho} \le \overline{\rho} \le \overline{y}(0) \le \overline{y}(t) \le \overline{y}(T)$ for all $t \in [0, T]$.

For each $(a, b) \in \mathbb{R}^2$ with $a \le b$, we consider the following problem:

$$(\mathscr{P}^{u_0, F}_{\mathscr{E}, [a,b]}) \begin{cases} \dfrac{du}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(t, u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ \\ u(0) = u_0 \in L^2(O; [a, b]). \end{cases}$$

From Corollary 2.6 and Proposition 2.8 we can establish the existence and uniqueness of bounded solutions for nonlocal problems with CP-structured reaction functionals.

Corollary 2.11 *Let $F \in \mathscr{F}_{\mathrm{CP}}$ with $(\underline{f}, \overline{f})$, $(\underline{\rho}, \overline{\rho})$ and $(\underline{y}, \overline{y})$ given by (CP_3), let $u_0 \in L^2(O; [\underline{\rho}, \overline{\rho}])$ and let $g \in L^2(O^J \setminus \overline{O})$ be such that*:

$$
\begin{aligned}
I_g &:= \operatorname*{ess\,inf}_{x \in O^J \setminus \overline{O}} \frac{\int_{O^J \setminus \overline{O}} J(x, y, x-y) g(y) dy}{\int_{O^J \setminus \overline{O}} J(x, y, x-y) dy} > -\infty;\\
S_g &:= \operatorname*{ess\,sup}_{x \in O^J \setminus \overline{O}} \frac{\int_{O^J \setminus \overline{O}} J(x, y, x-y) g(y) dy}{\int_{O^J \setminus \overline{O}} J(x, y, x-y) dy} < \infty;\\
\underline{\rho} &\leq I_g \text{ and } \overline{\rho} \geq S_g.
\end{aligned}
\tag{2.5}
$$

Then $(\mathscr{P}^{u_0, F}_{\mathscr{J}, [\underline{\rho}, \overline{\rho}]})$ *(resp.* $(\mathscr{P}^{u_0, F}_{\mathscr{J}+\mathscr{D}_g, [\underline{\rho}, \overline{\rho}]})$*) admits a unique solution* $u \in AC([0, T]; L^2(O))$ *such that*

$$\underline{y}(T) \leq \underline{y}(t) \leq u(t) \leq \overline{y}(t) \leq \overline{y}(T) \text{ for all } t \in [0, T].$$

Moreover, if $F(\cdot, u(\cdot)) \in AC([0, T]; L^2(O))$ *then* u *admits a right derivative* $\frac{d^+u}{dt}(t)$ *at every* $t \in [0, T[$ *which satisfies* $\frac{d^+u}{dt}(t) + \nabla \mathscr{E}(u(t)) = F(t, u(t))$ *with* $\mathscr{E} = \mathscr{J}$ *(resp.* $\mathscr{E} = \mathscr{J} + \mathscr{D}_g$*).*

Proof of Corollary 2.11 We only give the proof in the Neumann–Cauchy case. (In the Dirichlet–Cauchy case, the proof follows by similar arguments, where in addition the inequalities in (2.5) are used for dealing with the concept of sub-solution and super-solution.) The proof is adapted from [3, Theorem 3.1] (see also [4, Corollary 2.1, p. 39]).

Firstly, let $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ be given by Definition 2.9. Taking (CP_1) into account, from McShane extension's theorem we can assert that there exists $\widehat{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ such that:

- $\widehat{f}(t, x, \zeta) = f(t, x, \zeta)$ for all $(t, x, \zeta) \in [0, T] \times \mathbb{R}^d \times [\underline{y}(T), \overline{y}(T)]$;
- $\widehat{f}(t, x, \zeta)$ is Lipschitz continuous in ζ uniformly with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$. (2.6)

Let $\widehat{F} : [0, T] \times L^2(O) \to L^2(O)$ be given by $\widehat{F}(t, u)(x) := \widehat{f}(t, x, u(x))$. Then, it is clear that $\widehat{F} \in \mathscr{F}_{(R_1)\text{-}(R_2)}$. Hence, by Theorem 2.1, $(\mathscr{P}^{u_0, \widehat{F}}_{\mathscr{J}, [\underline{\rho}, \overline{\rho}]})$ admits a unique solution $\widehat{u} \in AC([0, T]; L^2(O))$.

Secondly, by (CP_3) we see that $\underline{y}$ and $\overline{y}$ are decreasing and increasing respectively, so that, since $\underline{y}(0) = \underline{\rho} \leq \overline{\rho} = \overline{y}(0)$,

$$[\underline{y}(t), \overline{y}(t)] \subset [\underline{y}(T), \overline{y}(T)] \text{ for all } t \in [0, T]. \tag{2.7}$$

As $\underline{y}$ does not depend on the space variable we have $\nabla \mathscr{J}(\underline{y}(t)) = 0$ for all $t \in [0, T]$. Then, by using (2.4) and ($\underline{\mathrm{ODE}}$) in (CP_3), $\widehat{F}(t, \underline{y}(t)) = F(t, \underline{y}(t)) = f(t, \cdot, \underline{y}(t)) \geq \underline{f}(t, \underline{y}(t)) = \underline{y}'(t) + \nabla \mathscr{J}(\underline{y}(t))$ for $\mathscr{L}^1$-a.a. $t \in [0, T]$, and consequently, since $\underline{y}(0) = \underline{\rho}$, $\underline{y}$ is sub-solution of $(\mathscr{P}^{\underline{\rho}, \widehat{F}}_{\mathscr{J}})$. But, $\widehat{u}$ is a solution (and so a super-solution) of $(\mathscr{P}^{u_0,\widehat{F}}_{\mathscr{J}})$ and $\underline{\rho} \leq u_0$, hence from Proposition 2.8 (that we apply with $u_{0,1} = \underline{\rho}$, $u_{0,2} = u_0$, $h_1 = h_2 = 0$ and $F_1 = F_2 = \widehat{F}$) it follows that

$$\underline{y}(t) \leq \widehat{u}(t) \text{ for all } t \in [0, T]. \tag{2.8}$$

In the same manner we can see that

$$\overline{y}(t) \geq \widehat{u}(t) \text{ for all } t \in [0, T]. \tag{2.9}$$

From (2.7), (2.8) and (2.9) we deduce that

$$\widehat{u}(t) \in [\underline{y}(t), \overline{y}(t)] \subset [\underline{y}(T), \overline{y}(T)] \text{ for all } t \in [0, T]. \tag{2.10}$$

Finally, from (2.6) and (2.10) we see that $\widehat{F}(t, \widehat{u}(t)) = F(t, \widehat{u}(t))$ for all $t \in [0, T]$ so that $\widehat{u}$ is the unique solution of $(\mathscr{P}^{u_0,F}_{\mathscr{J},[\underline{\rho},\overline{\rho}]})$, and the proof is complete. □

Remark 2.12 Under additional assumptions on the structure of F, we automatically have $F(\cdot, u) \in AC([0, T]; L^2(O))$ (see [3] or [4, Sect. 2.2.2, p. 27] and the example treated in Sect. 5).

Remark 2.13 Roughly, the inequalities in (2.5) mean that $\underline{\rho}$ and $\overline{\rho}$ bound the proportion of g with respect to the density J in a neighborhood of the boundary $\partial\Omega$. Physically, this implies that there is no dissipation of the energy along the trajectories $\underline{y}$ and $\overline{y}$. Indeed, we can show that $\nabla\mathcal{E}(\underline{y}(t)) \leq 0$ (resp. $\nabla\mathcal{E}(\overline{y}(t)) \geq 0$) so that $\frac{d}{dt}\mathcal{E}(\underline{y}(t)) = \nabla\mathcal{E}(\underline{y}(t))\frac{d\underline{y}}{dt} \geq 0$ (resp. $\frac{d}{dt}\mathcal{E}(\overline{y}(t)) = \nabla\mathcal{E}(\overline{y}(t))\frac{d\overline{y}}{dt} \geq 0$) because $\underline{y}$ is decreasing (resp. $\overline{y}$ is increasing).

2.2 Invasion property for nonlocal problems with CP-structured autonomous reaction functionals

Let $F : L^2(O) \to L^2(O)$ be such that $F \in \mathscr{F}_{\mathrm{CP}}$ with $(\underline{f}, \overline{f})$, $(\underline{\rho}, \overline{\rho})$ and $(\underline{y}, \overline{y})$ given by (CP$_3$), let $u_0 \in L^2(O; [\underline{\rho}, \overline{\rho}])$ and let $g \in L^2(O^J \setminus \overline{O})$ be such that (2.5) holds. From Corollary 2.11 we can assert that $(\mathscr{P}^{u_0,F}_{\mathscr{J},[\underline{\rho},\overline{\rho}]})$ (resp. $(\mathscr{P}^{u_0,F}_{\mathscr{J}+\mathscr{D}_g,[\underline{\rho},\overline{\rho}]})$) admits a unique solution $u \in AC([0, T]; L^2(O))$ such that

$$\underline{y}(T) \leq \underline{y}(t) \leq u(t) \leq \overline{y}(t) \leq \overline{y}(T) \text{ for all } t \in [0, T].$$

Moreover, if $F(u(\cdot)) \in AC([0, T]; L^2(O))$ then:

- u admits a right derivative $\dfrac{d^+u}{dt}(t)$ at every $t \in [0, T[$; (2.11)
- $\dfrac{d^+u}{dt}(t) + \nabla\mathscr{E}(u(t)) = F(u(t))$ for all $t \in [0, T[$ (2.12)

with $\mathscr{E} = \mathscr{J}$ (resp. $\mathscr{E} = \mathscr{J} + \mathscr{D}_g$). The following theorem shows that under some conditions on F, the solution u of $(\mathscr{P}^{u_0,F}_{\mathscr{J},[\underline{\rho},\overline{\rho}]})$ (resp. $(\mathscr{P}^{u_0,F}_{\mathscr{J}+\mathscr{D}_g,[\underline{\rho},\overline{\rho}]})$) satisfies the invasion property, i.e. u grows over time.

Theorem 2.14 *If there exists* $f : \mathbb{R} \to \mathbb{R}$ *with* $f \in C^1([\underline{y}(T), \overline{y}(T)])$ *such that*

$$F(v)(x) = f(v(x))$$

for all $(v, x) \in L^2(O) \times O$ *and if*

$$\nabla \mathscr{J}(u_0) \le F(u_0) \text{ (resp. } \nabla(\mathscr{J} + \mathscr{D}_g)(u_0) \le F(u_0)\text{)},$$

then u *is differentiable at every* $t \in]0, T[$ *and*

$$\frac{du}{dt}(t) \ge 0 \text{ for all } t \in [0, T[\left(\text{with } \frac{du}{dt}(0) = \frac{d^+u}{dt}(0) \right). \tag{2.13}$$

Proof of Theorem 2.14 We only give the proof in the Neumann–Cauchy case (in the Dirichlet–Cauchy case, the proof follows by similar arguments). By assumption, we see that $F(u(\cdot)) \in AC([0, T]; L^2(O))$ and so (2.11) and (2.12) hold. Let $G_u : [0, T] \times L^2(O) \to L^2(O)$ be given by

$$G_u(t, v(t)) := f'(u(t))v(t)$$

for all $(t, v) \in [0, T] \times L^2(O)$ and consider the following problem:

$$(\mathscr{P}^u_{\mathscr{J}}) \begin{cases} \dfrac{dv}{dt}(t) + \nabla \mathscr{J}(v(t)) = G_u(t, v(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ v(0) = \frac{d^+u}{dt}(0). \end{cases}$$

It is easy to show that $G \in \mathscr{F}_{(R_1)\text{-}(R_2)}$. By Theorem 2.1 it follows that $(\mathscr{P}^u_{\mathscr{J}})$ admits a unique solution $v \in AC([0, T]; L^2(O))$. But, by taking time derivative in (2.12) with $\mathscr{E} = \mathscr{J}$ we see that $\frac{d^+u}{dt}$ is a solution of $(\mathscr{P}^u_{\mathscr{J}})$, hence $v = \frac{d^+u}{dt} \in AC([0, T]; L^2(O))$, and consequently u is differentiable at every $t \in]0, T[$ by [14, Proposition 3.3, p. 68], i.e.

$$v(t) = \frac{du}{dt}(t) \text{ for all } t \in [0, T[\left(\text{with } \frac{du}{dt}(0) = \frac{d^+u}{dt}(0) \right). \tag{2.14}$$

Set $v^+ := \max(v, 0)$ and $v^- := \max(-v, 0)$. Taking (2.14) into account, to prove (2.13) it suffices to show that

$$v^-(t) = 0 \text{ for all } t \in [0, T[. \tag{2.15}$$

By taking v^- as a test function in $(\mathscr{P}^u{}_{\mathscr{J}})$ and by integrating over O, we see that

$$\int_O \frac{\partial v}{dt}(t,x)v^-(t,x)dx - \int_O \left(\int_O J(x,y,x-y)(v(t,y)-v(t,x))dy\right) v^-(t,x)dx$$
$$= \int_O f'(u(t,x))v(t,x)v^-(t,x)dx \text{ in } [0,T].$$

But, taking (NL_1) into account, by an easy computation we have

$$-\int_O \left(\int_O J(x,y,x-y)(v(t,y)-v(t,x))dy\right) v^-(t,x)dx$$
$$= \frac{1}{2}\int_O\int_O J(x,y,x-y)(v(t,y)-v(t,x))(v^-(t,y)-v^-(t,x))dxdy,$$

hence

$$\int_O \frac{\partial v}{\partial t}(t,x)v^-(t,x)dx + \frac{1}{2}\int_O\int_O J(x,y,x-y)(v(t,y)$$
$$-v(t,x))(v^-(t,y)-v^-(t,x))dxdy$$
$$= \int_O f'(u(t,x))v(t,x)v^-(t,x)dx \text{ in } [0,T]. \tag{2.16}$$

Noticing that:

- $v = v^+ - v^-$;
- $v^+v^- = \frac{\partial v^+}{\partial t}v^- = 0$;
- $-v^+(\cdot,x)v^-(\cdot,y) \le 0$ for all $(x,y) \in O \times O$,

we see that

$$\int_O \frac{\partial v}{\partial t}(t,x)v^-(t,x)dx = -\int_O \frac{\partial v^-}{\partial t}(t,x)v^-(t,x)dx = -\frac{1}{2}\frac{d}{dt}\int_O |v^-(x,t)|^2 dx;$$
$$\frac{1}{2}\int_O\int_O J(x,y,x-y)(v(t,y)-v(t,x))(v^-(t,y)-v^-(t,x))dxdy \le 0;$$
$$\int_O f'(u(t,x))v(t,x)v^-(t,x)dx = -\int_O f'(u(t,x))|v^-(t,x)|^2 dx,$$

and, recalling that $f \in C^1([\underline{y}(T), \overline{y}(T)])$, from (2.16) we deduce that

$$\frac{1}{2}\frac{d}{dt}\int_O |v^-(x,t)|^2 dx \le C\int_O |v^-(t,x)|^2 dx \text{ in } [0,T]$$

with $C := \sup\{|f'(\xi)| : \xi \in [\underline{y}(T), \overline{y}(T)]\}$. Consequently, by integrating over $s \in [0,T]$,

$$\int_O \left|v^-(s,x)\right|^2 dx \le \int_O \left|v^-(0,x)\right|^2 dx + 2C\int_0^t \left(\int_O \left|v^-(t,x)\right|^2 dx\right) dt \text{ for all } s \in [0,T].$$

From Grönwall's lemma we obtain

$$\int_O |v^-(s,x)|^2 dx \le e^{2Cs} \int_O |v^-(0,x)|^2 dx \text{ for all } s \in [0,T]. \tag{2.17}$$

But $v(0) = \frac{d^+u}{dt}(0)$ hence, by using (2.12) with $\mathscr{E} = \mathscr{J}$ and the fact that $\nabla \mathscr{J}(u_0) \le F(u_0)$, we see that $v^-(0) \ge -\nabla \mathscr{J}(u_0) + F(u_0) \ge 0$. Thus (2.15) follows from (2.17). □

3 Main result

3.1 Probability setting and ergodic theory

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space and let $\{T_z\}_{z\in\mathbb{Z}^d}$ be satisfying the following three properties:

- (mesurability) $T_z : \Omega \to \Omega$ is $\mathscr{F}$-measurable for all $z \in \mathbb{Z}^d$;
- (group property) $T_z \circ T_{z'} = T_{z+z'}$ and $T_{-z} = T_z^{-1}$ for all $z, z' \in \mathbb{Z}^d$;
- (mass invariance) $\mathbb{P}(T_z A) = \mathbb{P}(A)$ for all $A \in \mathscr{F}$ and all $z \in \mathbb{Z}^d$.

Definition 3.1 Such a $\{T_z\}_{z\in\mathbb{Z}^d}$ is said to be a (discrete) group of $\mathbb{P}$-preserving transformation on $(\Omega, \mathscr{F}, \mathbb{P})$ and the quadruplet $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is called a (discrete) dynamical system.

Let $\mathscr{I} := \{A \in \mathscr{F} : \mathbb{P}(T_z A \Delta A) = 0 \text{ for all } z \in \mathbb{Z}^d\}$ be the σ-algebra of invariant sets with respect to $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$.

Definition 3.2 When $\mathbb{P}(A) \in \{0, 1\}$ for all $A \in \mathscr{I}$, the measurable dynamical system $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is said to be ergodic.

Remark 3.3 A sufficient condition to ensure the ergodicity of $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is the so-called mixing condition, i.e. for every $(E, F) \in \mathscr{F} \times \mathscr{F}$,

$$\lim_{|z|\to\infty} \mathbb{P}(T_z E \cap F) = \mathbb{P}(E)\mathbb{P}(F).$$

For each $X \in L^1_{\mathbb{P}}(\Omega)$, let $\mathbb{E}^{\mathscr{I}}(X)$ be the conditional mathematical expectation of X with respect to $\mathscr{I}$, i.e. the unique $(\mathscr{I}, \mathscr{B}(\overline{\mathbb{R}}))$-measurable function in $L^1_{\mathbb{P}}(\Omega)$ such that for every $E \in \mathscr{I}$,

$$\int_E \mathbb{E}^{\mathscr{I}}(X)(\omega) d\mathbb{P}(\omega) = \int_E X(\omega) d\mathbb{P}(\omega).$$

Remark 3.4 If $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ is ergodic then $\mathbb{E}^{\mathscr{I}}(X)$ is constant and equal to the mathematical expectation $\mathbb{E}(X)$ of X, i.e. $\mathbb{E}^{\mathscr{I}}(X) = \mathbb{E}(X) := \int_\Omega X(\omega) d\mathbb{P}(\omega)$.

Let $\mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$ be the class of bounded Borel subsets of $\mathbb{R}^d$ and let $\mathscr{I}(\mathbb{Z}^d)$ be the class of half-open intervals $[a, b[$ with $(a, b) \in \mathbb{Z}^d \times \mathbb{Z}^d$.

Definition 3.5 We say that $\mathcal{S} : \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \to L^1_{\mathbb{P}}(\Omega)$ is a subadditive process covariant (or stationary) with respect to $\{T_z\}_{z\in\mathbb{Z}^d}$ if the following four conditions hold:

- (subadditivity) for every $(A, B) \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \times \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$, if $A \cap B = \emptyset$ and $\mathscr{L}^d(\partial A) = \mathscr{L}^d(\partial B) = 0$ then

$$\mathcal{S}_{A\cup B} \le \mathcal{S}_A + \mathcal{S}_B;$$

- (covariance or stationarity) for every $A \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$ and every $z \in \mathbb{Z}^d$,

$$\mathcal{S}_{A+z} = \mathcal{S}_A \circ T_z;$$

- (domination) there exists $\Theta \in L^1_{\mathbb{P}}(\Omega; [0, \infty])$ such that for every $A \in \mathscr{B}(\mathbb{R}^d)$,

$$\mathcal{S}_A \le \Theta \mathscr{L}^d(A);$$

- (spatial constant property) $\gamma(\mathcal{S}) := \inf \left\{ \int_\Omega \frac{\mathcal{S}_I}{\mathscr{L}^d(I)} d\mathbb{P} : I \in \mathscr{I}(\mathbb{Z}^d) \right\} > -\infty.$

In order to study the pointwise convergence of subadditive processes introduced in the paper, we need the following notion of regularity for sequences of sets in $\mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$.

Definition 3.6 We say that $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$ is regular if there exists $\{I_\varepsilon\}_{\varepsilon>0} \subset \mathscr{I}(\mathbb{Z}^d)$ with $I_\varepsilon \subset I_{\varepsilon'}$ if $\varepsilon > \varepsilon'$ and $C > 0$ such that:

- $A_\varepsilon \subset I_\varepsilon$ for all $\varepsilon > 0$;
- $\sup\limits_{\varepsilon>0} \dfrac{\mathscr{L}^d(I_\varepsilon)}{\mathscr{L}^d(A_\varepsilon)} \le C.$

For each $A \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$, we set

$$\rho(A) := \sup \Big\{ r \ge 0 : B_r(0) \subset A \Big\}.$$

The following theorem can be found in [2, Theorem 12.4.3, p. 514] (see also [5, 17]).

Theorem 3.7 *Let $\mathcal{S} : \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d) \to L^1_{\mathbb{P}}(\Omega)$ be a subadditive process covariant with respect to $\{T_z\}_{z\in\mathbb{Z}^d}$ and let $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$ be such that*

$$\begin{cases} \{A_\varepsilon\}_{\varepsilon>0} \text{ is regular} \\ A_\varepsilon \text{ is convex for all } \varepsilon > 0 \\ \lim\limits_{\varepsilon\to 0} \rho(A_\varepsilon) = \infty. \end{cases}$$

Then, for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_{A_\varepsilon}(\omega)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}} \left(\frac{\mathcal{S}_{[0,k[^d}}{k^d} \right)(\omega).$$

If moreover $(\Omega, \mathscr{F}, \mathbb{P}, \{T_z\}_{z\in\mathbb{Z}^d})$ *is ergodic then, for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$,

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_{A_\varepsilon}(\omega)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}\left(\frac{\mathcal{S}_{[0,k[^d}}{k^d}\right) = \gamma(\mathcal{S}).$$

3.2 Random nonlocal reaction–diffusion problems of gradient flow type

Let $J : \Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty[$ be a $(\mathscr{F}\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{R}^d), \mathscr{B}(\mathbb{R}))$-measurable satisfying the following conditions:

(PNL$_1$) J is symmetric, i.e. for every (ω, x, y, ξ),

$$J(\omega, x, y, \xi) = J(\omega, y, x, \xi),$$

and J is bi-stationary with respect to $(T_z)_{z\in\mathbb{Z}^d}$, i.e. for every $z \in \mathbb{Z}^d$ and every $(\omega, x, y, \xi) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$J(\omega, x+z, y+z, \xi) = J(T_z\omega, x, y, \xi);$$

(PNL$_2$) there exist $\underline{J}, \overline{J} : \mathbb{R}^d \to [0,\infty[$ with

$$\begin{cases} \underline{J} \not\equiv 0 \\ \text{for every } (\xi,\zeta) \in \mathbb{R}^d\times\mathbb{R}^d,\ \text{if } |\xi| \le |\zeta| \text{ then } \underline{J}(\xi) \ge \underline{J}(\zeta) \\ \operatorname{supp}(\overline{J}) = \overline{B}_{R_J}(0) \text{ is compact with } R_J > 0, \end{cases} \tag{3.1}$$

such that for every $(\omega, x, y, \xi) \in \Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\underline{J}(\xi) \le J(\omega, x, y, \xi) \le \overline{J}(\xi).$$

Remark 3.8 The monotony condition (3.1) (firstly introduced in [9, Theorem 4 and Remark 4]) allows to obtain the strong compactness in $L^2(O)$ for sequences of solutions of nonlocal reaction–diffusion problems (see Lemma 4.2). This condition is also essential to show that Γ-convergence implies Mosco-convergence of the corresponding nonlocal energies.

Remark 3.9 From (3.1) we see that $\inf_{|\xi|\le\frac{R_J}{2}} \underline{J}(\xi) \ge \underline{J}(\frac{R_J}{2}) \ne 0$.

Fix any $\varepsilon > 0$. Let $O \subset \mathbb{R}^d$ be an open set and let $\mathscr{J}_\varepsilon : \Omega\times L^2(O) \to [0,\infty[$ be defined by

$$\mathscr{J}_\varepsilon(\omega, u) := \frac{1}{4\varepsilon^d}\int_O\int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{u(x)-u(y)}{\varepsilon}\right)^2 dxdy. \tag{3.2}$$

Given $T > 0$, let $F_\varepsilon : \Omega\times[0,T]\times L^2(O) \to L^2(O)$ be such that $F_\varepsilon(\omega,\cdot,\cdot) \in \mathscr{F}_{\rm CP}$ for all $\omega \in \Omega$. Given $T > 0$, for each $\omega \in \Omega$, let $(\underline{\rho}^\omega_\varepsilon, \overline{\rho}^\omega_\varepsilon)$ and $(\underline{y}^\omega_\varepsilon, \overline{y}^\omega_\varepsilon)$ be given by (CP$_3$) with $F = F_\varepsilon(\omega, \cdot, \cdot)$, where, taking Remark 2.10 into account, we further assume that

$$-\infty < \inf_{\varepsilon>0} \underline{y}_\varepsilon^\omega(T) \le \sup_{\varepsilon>0} \overline{y}_\varepsilon^\omega(T) < \infty, \tag{3.3}$$

and consider the Neumann–Cauchy homogeneous nonlocal problem $(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega}) := (\mathscr{P}^{u_0^\omega, F_\varepsilon(\omega,\cdot,\cdot)}_{\mathscr{J}_\varepsilon(\omega,\cdot),[\underline{\rho}_\varepsilon^\omega,\overline{\rho}_\varepsilon^\omega]})$, i.e.

$$(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega}) \begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t) + \nabla \mathscr{J}_\varepsilon(\omega, u_\varepsilon^\omega(t)) = F_\varepsilon(\omega, t, u_\varepsilon^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u_\varepsilon^\omega(0) = u_{0,\varepsilon}^\omega \in L^2(O; [\underline{\rho}_\varepsilon^\omega, \overline{\rho}_\varepsilon^\omega]). \end{cases}$$

Let $g \in H^1(O^J \setminus \overline{O})$ with $O^J := O + \mathrm{supp}(\overline{J}) = O + \overline{B}_{R_J}(0)$ be such that:

$$\begin{aligned} I_g^{\varepsilon,\omega} &:= \operatorname*{ess\,inf}_{x\in O^J\setminus\overline{O}} \frac{\int_{O^J\setminus\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) g(y)dy}{\int_{O^J\setminus\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) dy} > -\infty; \\ S_g^{\varepsilon,\omega} &:= \operatorname*{ess\,sup}_{x\in O^J\setminus\overline{O}} \frac{\int_{O^J\setminus\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) g(y)dy}{\int_{O^J\setminus\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) dy} < \infty; \\ \underline{\rho}_\varepsilon^\omega &\le I_g^{\varepsilon,\omega} \text{ and } \overline{\rho}_\varepsilon^\omega \ge S_g^{\varepsilon,\omega}. \end{aligned} \tag{3.4}$$

for all $x \in O$ and and all $\omega \in \Omega$, let $\mathscr{D}_g^\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ be defined by

$$\mathscr{D}_g^\varepsilon(\omega, u) := \frac{1}{2\varepsilon^d} \int_O \int_{O^J\setminus\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{g(y) - u(x)}{\varepsilon}\right)^2 dxdy \tag{3.5}$$

and consider the Dirichlet–Cauchy nonlocal problem $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega}) := (\mathscr{P}^{u_0^\omega, F_\varepsilon(\omega,\cdot,\cdot)}_{\mathscr{J}_\varepsilon(\omega,\cdot)+\mathscr{D}_g^\varepsilon(\omega,\cdot),[\underline{\rho}_\varepsilon^\omega,\overline{\rho}_\varepsilon^\omega]})$, i.e.

$$(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega}) \begin{cases} \dfrac{du_\varepsilon^\omega}{dt}(t) + \nabla \mathscr{J}_\varepsilon^g(\omega, u_\varepsilon^\omega(t)) = F_\varepsilon(\omega, t, u_\varepsilon^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u_\varepsilon^\omega(0) = u_{0,\varepsilon}^\omega \in L^2(O; [\underline{\rho}_\varepsilon^\omega, \overline{\rho}_\varepsilon^\omega]) \end{cases}$$

with $\mathscr{J}_\varepsilon^g := \mathscr{J}_\varepsilon + \mathscr{D}_g^\varepsilon$. The following result is a straightforward consequence of Corollary 2.11.

Corollary 3.10 *For each $\omega \in \Omega$ and each $\varepsilon > 0$, $(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega})$ (resp. $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$) admits a unique solution $u_\varepsilon^\omega \in AC([0, T]; L^2(O))$ such that*

$$\underline{y}_\varepsilon^\omega(T) \le \underline{y}_\varepsilon^\omega(t) \le u_\varepsilon^\omega(t) \le \overline{y}_\varepsilon^\omega(t) \le \overline{y}_\varepsilon^\omega(T)$$

for all $t \in [0,T]$. *Moreover, if* $F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon(\cdot)) \in AC([0,T];L^2(O))$ *then* u^ω_ε *admits a right derivative* $\frac{d^+u^\omega_\varepsilon}{dt}(t)$ *at every* $t \in [0,T[$ *which satisfies* $\frac{d^+u^\omega_\varepsilon}{dt}(t) + \nabla\mathscr{J}_\varepsilon(\omega,u^\omega_\varepsilon(t)) = F_\varepsilon(\omega,t,u^\omega_\varepsilon(t))$ *(resp.* $\frac{d^+u^\omega_\varepsilon}{dt}(t)+\nabla\mathscr{J}^g_\varepsilon(\omega,u^\omega_\varepsilon(t)) = F_\varepsilon(\omega,t,u^\omega_\varepsilon(t))$*).*

Our purpose is to look for the almost sure limit of $(\mathscr{P}^{\rm NH}_{\varepsilon,\omega})$ and $(\mathscr{P}^{\rm D}_{\varepsilon,\omega})$ as $\varepsilon \to 0$. This is the object of the next section.

3.3 Stochastic homogenization theorem

For each $\theta \in \mathbb{R}^d$, each $R > 0$ and each $A \in \mathscr{B}_{\rm b}(\mathbb{R}^d)$, set

$$L^2_{{\rm loc},\theta,R,A}(\mathbb{R}^d) := \Big\{u \in L^2_{\rm loc}(\mathbb{R}^d) : u = \ell_\theta \text{ in } \partial_R(A)\Big\}, \tag{3.6}$$

where $\ell_\theta : \mathbb{R}^d \to \mathbb{R}$ is the linear map defined by $\ell_\theta(x) = \theta x$ and $\partial_R(A)$ denotes the R-neighborhood of the boundary ∂A of A, i.e.

$$\partial_R(A) := \Big\{x \in \mathbb{R}^d : {\rm dist}(x,\partial A) < R\Big\}. \tag{3.7}$$

In what follows, we also set

$$A_R := \Big\{x \in A : {\rm dist}(x,\partial A) > R\Big\}. \tag{3.8}$$

Let $\mathcal{S}, \mathfrak{S}, \mathscr{K} : \mathscr{B}_{\rm b}(\mathbb{R}^d) \times \Omega \times \mathbb{R}^d \to [0,\infty[$ be defined by:

$$\mathcal{S}_A(\omega,\theta) := \inf\Big\{\mathscr{J}(\omega,u,\mathbb{R}^d,A) : u \in L^2_{{\rm loc},\theta,R_J,A}(\mathbb{R}^d)\Big\};$$
$$\mathfrak{S}_A(\omega,\theta) := \inf\Big\{\mathscr{J}(\omega,u,A,A) : u \in L^2_{{\rm loc},\theta,R_J,A}(\mathbb{R}^d)\Big\};$$
$$\mathscr{K}_A(\omega,\theta) := \inf\Big\{\mathscr{J}(\omega,u,A_{R_J},A_{R_J}) + \mathscr{D}_{\ell_\theta}(\omega,u,A_{R_J},A\backslash\overline{A}_{R_J}) : u \in L^2_{{\rm loc},\theta,R_J,A}(\mathbb{R}^d)\Big\},$$

where $R_J > 0$ is given by PNL$_2$ and $\mathscr{J}, \mathscr{D}_{\ell_\theta} : \Omega \times L^2_{\rm loc}(\mathbb{R}^d) \times \mathscr{B}_{\rm b}(\mathbb{R}^d) \times \mathscr{B}_{\rm b}(\mathbb{R}^d) \to [0,\infty[$ are defined by:

$$\mathscr{J}(\omega,u,A,B) := \frac{1}{4}\int_A\int_B J(\omega,x,y,x-y)(u(x)-u(y))^2dxdy;$$
$$\mathscr{D}_{\ell_\theta}(\omega,u,A,B) := \frac{1}{2}\int_A\int_B J(\omega,x,y,x-y)(\ell_\theta(y)-u(x))^2dxdy.$$

Remark 3.11 The random variational nonlocal functional $(u,A) \mapsto \mathscr{J}(\cdot,u,A_{R_J},A_{R_J}) + \mathscr{D}_{\ell_\theta}(\cdot,u,A_{R_J},A\backslash\overline{A}_{R_J})$ arising in the definition of the process $A \mapsto \mathscr{K}_A(\cdot,\theta)$ is the energy of the Dirichlet–Cauchy nonlocal problem introduced in Sect. 2.1.3 with $g = \ell_\theta$, $O^J = A$ and $O = A_{R_J}$. Consequently, the process $A \mapsto \mathscr{K}_A(\cdot,\theta)$ is the natural nonlocal version of the standard local process whose almost sure limit gives the homogenized density in standard stochastic homogenization. The processes $A \mapsto \mathcal{S}_A(\cdot, \theta)$ and $A \mapsto \mathfrak{S}_A(\cdot, \theta)$ are introduced for technical reasons.

The following lemma makes clear the link between $\mathcal{S}$, $\mathfrak{S}$ and $\mathscr{K}$.

Lemma 3.12 *For every $A \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$, every $\omega \in \Omega$ and every $\theta \in \mathbb{R}^d$, we have*:

$$0 \le \mathcal{S}_A(\omega, \theta) - \mathfrak{S}_A(\omega, \theta) \le \mathscr{L}^d(\partial_{R_J}(A)) \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi; \tag{3.9}$$

$$0 \le \mathfrak{S}_A(\omega, \theta) - \mathscr{K}_A(\omega, \theta) \le \mathscr{L}^d(\partial_{R_J}(A)) \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi. \tag{3.10}$$

Proof of Lemma 3.12 Fix $A \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$, $\omega \in \Omega$ and $\theta \in \mathbb{R}^d$.
Proof of (3.9). Fix any $\varepsilon > 0$. Let $u_\varepsilon \in L^2_{\mathrm{loc},\theta,R_J,A}(\mathbb{R}^d)$ be such that $\mathfrak{S}_A(\omega, A) > \mathscr{J}(\omega, u_\varepsilon, A, A) - \varepsilon$. Then, by using PNL$_2$,

$$\begin{aligned} \mathfrak{S}_A(\omega, \theta) - \mathscr{K}_A(\omega, \theta) &\le \mathscr{J}(\omega, u_\varepsilon, \mathbb{R}^d, A) - \mathscr{J}(\omega, u_\varepsilon, A, A) + \varepsilon \\ &= \mathscr{J}(\omega, u_\varepsilon, \mathbb{R}^d \backslash A, A) + \varepsilon \\ &\le \frac{1}{4} \int_{\mathbb{R}^d \backslash A} \int_A \overline{J}(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy + \varepsilon. \end{aligned}$$

But $\mathrm{supp}(\overline{J}) = \overline{B}_{R_J}(0)$ and $u_\varepsilon = \ell_\theta$ in $\partial_{R_J}(A)$, hence

$$\begin{aligned} \mathfrak{S}_A(\omega, \theta) - \mathscr{K}_A(\omega, \theta) &\le \frac{|\theta|^2}{4} \int_{\partial_{R_J}(A)} \int_{\partial_{R_J}(A)} \overline{J}(x-y)|x-y|^2 dxdy + \varepsilon \\ &\le \mathscr{L}^d(\partial_{R_J}(A)) \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi + \varepsilon, \end{aligned}$$

and (3.9) follows by letting $\varepsilon \to 0$.
Proof of (3.10). Fix any $\varepsilon > 0$. Let $u_\varepsilon \in L^2_{\mathrm{loc},\theta,R_J,A}(\mathbb{R}^d)$ be such that $\mathscr{K}_A(\omega, \theta) > \mathscr{J}(\omega, u_\varepsilon, A_{R_J}, A_{R_J}) + \mathscr{D}_{\ell_\theta}(\omega, u_\varepsilon, A_{R_J}, A \backslash \overline{A}_{R_J}) - \varepsilon$. Then, by using PNL$_2$,

$$\begin{aligned} \mathfrak{S}_A(\omega, \theta) - \mathscr{K}_A(\omega, \theta) \le\ & \mathscr{J}(\omega, u_\varepsilon, A, A) - \mathscr{J}(\omega, u_\varepsilon, A_{R_J}, A_{R_J}) \\ & - \mathscr{D}_{\ell_\theta}(\omega, u_\varepsilon, A_{R_J}, A \backslash \overline{A}_{R_J}) + \varepsilon \\ \le\ & \mathscr{J}(\omega, u_\varepsilon, A \backslash \overline{A}_{R_J}, A \backslash \overline{A}_{R_J}) + \varepsilon \\ \le\ & \frac{1}{4} \int_{A \backslash \overline{A}_{R_J}} \int_{A \backslash \overline{A}_{R_J}} \overline{J}(x-y)(u_\varepsilon(x) - u_\varepsilon(y))^2 dxdy + \varepsilon. \end{aligned}$$

But $A \backslash \overline{A}_{R_J} \subset \partial_{R_J}(A)$ and $u_\varepsilon = \ell_\theta$ in $\partial_{R_J}(A)$, hence...

$$\begin{aligned} \mathfrak{S}_A(\omega, \theta) - \mathscr{K}_A(\omega, \theta) &\le \frac{|\theta|^2}{4} \int_{\partial_{R_J}(A)} \int_{\partial_{R_J}(A)} \overline{J}(x-y)|x-y|^2 dxdy + \varepsilon \\ &\le \mathscr{L}^d(\partial_{R_J}(A)) \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi + \varepsilon, \end{aligned}$$

and (3.10) follows by letting $\varepsilon \to 0$. □

Remark 3.13 When A is a cube of size L it is easy to see that $\mathscr{L}^d(\partial_{R_J}(A)) \sim 2R_J\mathscr{H}^{d-1}(\partial A)$ for large L.

Proposition 3.14 *Let $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathscr{B}_{\rm b}(\mathbb{R}^d)$ be such that:*

$$\begin{cases} \{A_\varepsilon\}_{\varepsilon>0} \text{ is regular} \\ A_\varepsilon \text{ is convex for all } \varepsilon > 0 \\ \lim\limits_{\varepsilon\to 0} \rho(A_\varepsilon) = \infty; \end{cases} \tag{3.11}$$

$$\lim_{\varepsilon\to 0} \frac{\mathscr{L}^d(\partial_{R_J}(A_\varepsilon))}{\mathscr{L}^d(A_\varepsilon)} = 0. \tag{3.12}$$

Then, for every $\theta \in \mathbb{R}^d$ there exists $\Omega_\theta \in \mathscr{F}$ with $\mathbb{P}(\Omega_\theta) = 1$ such that for every $\omega \in \Omega_\theta$,

$$\lim_{\varepsilon\to 0} \frac{\mathscr{K}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathcal{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega).$$

Remark 3.15 Let $Q_\rho(x_0)$ be the cube of size $\rho > 0$ centered at $x_0 \in \mathbb{R}^d$. Then, it is easily seen that $\{\frac{1}{\varepsilon}Q_\rho(x_0)\}_{\varepsilon>0}$ satisfies (3.11)–(3.12).

Proof of Proposition 3.14 Let $\theta \in \mathbb{R}^d$. As $A \mapsto \mathcal{S}_A(\cdot,\theta)$ is clearly a subadditive process covariant with respect to $\{T_z\}_{z\in\mathbb{Z}^d}$, taking (3.11) into account, from Theorem 3.7 we can assert that there exists $\Omega_\theta \in \mathscr{F}$ with $\mathbb{P}(\Omega_\theta) = 1$ such that

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega) \text{ for all } \omega \in \Omega_\theta. \tag{3.13}$$

On the other hand, by Lemma 3.12, for every $\varepsilon > 0$ and every $\omega \in \Omega$, we have:

$$0 \le \frac{\mathcal{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} - \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} \le \frac{\mathscr{L}^d(\partial_{R_J}(A_\varepsilon))}{\mathscr{L}^d(A_\varepsilon)} \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi)d\xi; \tag{3.14}$$

$$0 \le \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} - \frac{\mathscr{K}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} \le \frac{\mathscr{L}^d(\partial_{R_J}(A_\varepsilon))}{\mathscr{L}^d(A_\varepsilon)} \frac{|\theta|^2}{4} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi)d\xi. \tag{3.15}$$

Consequently, from (3.12) and (3.13) we deduce that

$$\lim_{\varepsilon\to 0} \frac{\mathscr{K}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega) \text{ for all } \omega \in \Omega_\theta,$$

and the proof is complete. □

Now, we can define the homogenized density. First of all, it is not difficult to establish that for every $A \in \mathscr{B}_{\rm b}(\mathbb{R}^d)$, every $\omega \in \Omega$ and every $(\theta,\theta') \in \mathbb{R}^d \times \mathbb{R}^d$,

$$\left|\frac{\mathfrak{S}_A(\omega,\theta)}{\mathscr{L}^d(A)} - \frac{\mathfrak{S}_A(\omega,\theta')}{\mathscr{L}^d(A)}\right| \le C\left|\theta-\theta'\right|(|\theta|+|\theta'|) \tag{3.16}$$

with $C := \frac{1}{4}\int_{\mathbb{R}^d}|\xi|^2\overline{J}(\xi)d\xi$. Set

$$\Omega' := \bigcap_{\theta\in\mathbb{Q}^d}\Omega_\theta \tag{3.17}$$

with Ω_θ given by Proposition 3.14. Then $\Omega' \in \mathscr{F}$ and $\mathbb{P}(\Omega') = 1$. By using Proposition 3.14, from (3.16) we deduce that for every $\omega \in \Omega'$ and every $(\theta, \theta') \in \mathbb{Q}^d \times \mathbb{Q}^d$,

$$\left|\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega) - \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta')}{k^d}\right)(\omega)\right| \leq C\left|\theta-\theta'\right|(|\theta|+|\theta'|),$$

which allows to define $f_{\text{hom}} : \Omega' \times \mathbb{R}^d \to [0, \infty[$ by

$$f_{\text{hom}}(\omega,\theta) := \begin{cases} \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta)}{k^d}\right)(\omega) & \text{if } \theta\in\mathbb{Q}^d \\ \lim_{\mathbb{Q}^d\ni\zeta\to\theta}\inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\zeta)}{k^d}\right)(\omega) & \text{if } \theta\notin\mathbb{Q}^d. \end{cases}$$

Remark 3.16 It is easy to see that for every $\omega \in \Omega'$, $f_{\text{hom}}(\omega, \cdot)$ is quadratic, i.e. for every $\omega \in \Omega'$, there exists a symmetric $d \times d$ matrix A^{ω}_{hom} such that for every $\theta \in \mathbb{R}^d$,

$$f_{\text{hom}}(\omega,\theta) = \frac{1}{2}\langle A^{\omega}_{\text{hom}}\theta, \theta\rangle, \tag{3.18}$$

where $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^d$.

Proposition 3.17 *Let $\{A_\varepsilon\}_{\varepsilon>0} \subset \mathscr{B}_b(\mathbb{R}^d)$ be such that (3.11) and (3.12) hold. Then, for every $\omega \in \Omega'$, where Ω' is given by (3.17), and every $\theta \in \mathbb{R}^d$,*

$$\lim_{\varepsilon\to0}\frac{\mathscr{K}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to0}\frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to0}\frac{\mathcal{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = f_{\text{hom}}(\omega,\theta).$$

Proof of Proposition 3.17 Let $\omega \in \Omega'$ and let $\theta \in \mathbb{R}^d$. By density, there exists $\{\theta_n\}_{n\geq1} \subset \mathbb{Q}^d$ such that

$$\lim_{n\to\infty}|\theta-\theta_n| = 0. \tag{3.19}$$

(In particular $\sup_{n\geq1}|\theta_n| < \infty$.) Setting $C' := C(|\theta| + \sup_{n\geq1}|\theta_n|)$, by (3.16) we have

$$\frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta_n)}{\mathscr{L}^d(A_\varepsilon)} - C'|\theta-\theta_n| \leq \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} \leq \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta_n)}{\mathscr{L}^d(A_\varepsilon)} + C'|\theta-\theta_n| \tag{3.20}$$

for all $\varepsilon > 0$ and all $n \geq 1$. As $\omega \in \Omega'$ and $\{\theta_n\}_{n\geq1} \subset \mathbb{Q}^d$ we have $\omega \in \Omega_{\theta_n}$ for all $n \geq 1$, and so, by using Proposition 3.14,

$$\lim_{\varepsilon\to0}\frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta_n)}{\mathscr{L}^d(A_\varepsilon)} = \inf_{k\in\mathbb{N}^*}\mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta_n)}{k^d}\right)(\omega) \text{ for all } n\geq1.$$

Letting $\varepsilon \to 0$ in (3.20) we deduce that:

$$\inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta_n)}{k^d}\right)(\omega) - C'|\theta-\theta_n| \leq \underline{\lim}_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)}$$
$$\overline{\lim}_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} \leq \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta_n)}{k^d}\right)(\omega) + C'|\theta-\theta_n|,$$

and consequently, by letting $n \to \infty$ and using (3.19),

$$\lim_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{n\to\infty} \inf_{k\in\mathbb{N}^*} \mathbb{E}^{\mathscr{I}}\left(\frac{\mathcal{S}_{[0,k[^d}(\cdot,\theta_n)}{k^d}\right)(\omega) = f_{\text{hom}}(\omega,\theta).$$

On the other hand, by Lemma 3.12, (3.14) and (3.15) hold for all $\varepsilon > 0$, and so, taking (3.12) into account, we have:

$$\lim_{\varepsilon\to 0} \frac{\mathcal{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)};$$
$$\lim_{\varepsilon\to 0} \frac{\mathscr{K}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)} = \lim_{\varepsilon\to 0} \frac{\mathfrak{S}_{A_\varepsilon}(\omega,\theta)}{\mathscr{L}^d(A_\varepsilon)},$$

which completes the proof. □

Let $\mathscr{J}_{\text{hom}}, \mathscr{J}^g_{\text{hom}} : \Omega' \times L^2(O) \to [0,\infty]$ be defined by:

$$\mathscr{J}_{\text{hom}}(\omega,u) := \begin{cases} \int_O f_{\text{hom}}(\omega,\nabla u(x))dx & \text{if } u \in H^1(O) \\ \infty & \text{if } u \in L^2(O)\backslash H^1(O); \end{cases} \tag{3.21}$$

$$\mathscr{J}^g_{\text{hom}}(\omega,u) := \begin{cases} \int_O f_{\text{hom}}(\omega,\nabla u(x))dx & \text{if } u \in H^1_g(O) \\ \infty & \text{if } L^2(O)\backslash H^1_g(O) \end{cases} \tag{3.22}$$

with $H^1_g(O) := \{u \in H^1(O) : \gamma(u) = \gamma_J(g)\}$, where γ (resp. γ_J) is the trace operator $\gamma : H^1(O) \to L^2(\partial O)$ (resp. $\gamma_J : H^1(O^J\backslash\overline{O}) \to L^2(\partial O)$).

Remark 3.18 By Remark 3.16 we see that for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\mathscr{J}_{\text{hom}}(\omega,\cdot)$ (resp. $\mathscr{J}^g_{\text{hom}}(\omega,\cdot)$) is proper, convex and lower semicontinuous, and Fréchet-differentiable on $\text{dom}(\partial\mathscr{J}_{\text{hom}}(\omega,\cdot))$ (resp. $\text{dom}(\partial\mathscr{J}^g_{\text{hom}}(\omega,\cdot))$).

For $\mathbb{P}$-a.e. $\omega \in \Omega$, let $G^\omega : [0,T]\times L^2(O) \to L^2(O)$ be such that $G^\omega \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ and consider the following Neumann–Cauchy homogeneous local problem:

$$(\mathscr{P}^{\text{NH}}_{\text{hom},\omega}) \begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla\mathscr{J}_{\text{hom}}(\omega,u^\omega(t)) = G^\omega(t,u^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u^\omega(0) = u^\omega_0 \in \text{dom}(\mathscr{J}_{\text{hom}}(\omega,\cdot)) \end{cases}$$

and the following Dirichlet–Cauchy local problem:

$$(\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega})\begin{cases}\dfrac{du^\omega}{dt}(t)+\nabla\mathscr{J}^{g}_{\mathrm{hom}}(\omega,u^\omega(t))=G^\omega(t,u^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t\in[0,T]\\[2ex] u^\omega(0)=u^\omega_0\in\mathrm{dom}(\mathscr{J}^g_{\mathrm{hom}}(\omega,\cdot)).\end{cases}$$

Here is the main result of the paper.

Theorem 3.19 *For $\mathbb{P}$-a.e. $\omega\in\Omega$ and every $\varepsilon>0$, let u^ω_ε be the unique solution of $(\mathscr{P}^{\mathrm{NH}}_{\varepsilon,\omega})$ (resp. $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$), see Corollary 3.10, and assume that:*

(H^ω_1) $\sup\limits_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,u^\omega_{0,\varepsilon})<\infty$ *(resp.* $\sup\limits_{\varepsilon>0}\mathscr{J}^g_\varepsilon(\omega,u^\omega_{0,\varepsilon})<\infty$*);*
(H^ω_2) $u^\omega_{0,\varepsilon}\rightharpoonup u^\omega_0$ *in* $L^2(O)$*;*
(H^ω_3) $\sup\limits_{\varepsilon>0}\left\|F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon)\right\|_{L^2([0,T];L^2(O))}<\infty.$

Then, there exists $\widehat{\Omega}\in\mathscr{F}$ with $\mathbb{P}(\widehat{\Omega})=1$ such that for every $\omega\in\widehat{\Omega}$ there exists $u^\omega\in C([0,T];L^2(O))$ such that up to a subsequence:

$$u^\omega_\varepsilon\to u^\omega \text{ in } C([0,T];L^2(O)); \tag{3.23}$$
$$\frac{du^\omega_\varepsilon}{dt}\rightharpoonup\frac{du^\omega}{dt} \text{ in } L^2([0,T];L^2(O)). \tag{3.24}$$

Moreover, we have

$$\inf_{\varepsilon>0}\underline{y}^\omega_\varepsilon(T)\le u^\omega(t)\le\sup_{\varepsilon>0}\overline{y}^\omega_\varepsilon(T)\text{ for all } t\in[0,T].$$

Assume furthermore that

(H^ω_4) for every $v\in C([0,T];L^2(O))$,

$$u^\omega_\varepsilon\to v \text{ in } C([0,T];L^2(O))\Longrightarrow F_\varepsilon(\omega,\cdot,u^\omega_\varepsilon)\rightharpoonup G^\omega(\cdot,v) \text{ in } L^2([0,T];L^2(O)).$$

Then, (3.23)–(3.24) hold for the whole sequence ε and

$$u^\omega \text{ is the unique solution of } (\mathscr{P}^{\mathrm{NH}}_{\mathrm{hom},\omega})(\text{ resp. } (\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega})).$$

Moreover, $u^\omega_0\in H^1(O)\cap L^2(O;[\underline{\rho}^\omega,\overline{\rho}^\omega])$ (resp. $u^\omega_0\in H^1_g(O)\cap L^2(O;[\underline{\rho}^\omega,\overline{\rho}^\omega])$) where $\underline{\rho}^\omega:=\inf_{\varepsilon>0}\underline{\rho}^\omega_\varepsilon$ and $\overline{\rho}^\omega:=\sup_{\varepsilon>0}\overline{\rho}^\omega_\varepsilon$.

Remark 3.20 If (CP_1) is satisfied uniformly with respect to ε then a sufficient condition to ensure (H^ω_3) is that $\sup_{\varepsilon>0}\|F_\varepsilon(\omega,\cdot,0)\|_{L^2([0,T];L^2(O))}<\infty$. It is indeed a straightforward consequence of the uniform boundedness of u^ω_ε together with the local Lipschitz hypothesis on F_ε.

Remark 3.21 Taking (3.18) into account, for $\mathbb{P}$-a.e. $\omega \in \Omega$, we have:

$$\begin{cases} \mathrm{dom}(\partial \mathscr{J}_{\mathrm{hom}}(\omega,\cdot)) = \Big\{ v \in H^1(O) : \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla v) \in L^2(O) \text{ and } A^{\omega}_{\mathrm{hom}} \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial O \Big\} \\ \nabla \mathscr{J}_{\mathrm{hom}}(\omega,\cdot)(v) = -\mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla v) \text{ for all } v \in \mathrm{dom}(\partial \mathscr{J}_{\mathrm{hom}}(\omega,\cdot)); \end{cases}$$

$$\begin{cases} \mathrm{dom}(\partial \mathscr{J}^{g}_{\mathrm{hom}}(\omega,\cdot)) = \Big\{ v \in H^1_g(O) : \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla v) \in L^2(O) \Big\} \\ \nabla \mathscr{J}^{g}_{\mathrm{hom}}(\omega,\cdot)(v) = -\mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla v) \text{ for all } v \in \mathrm{dom}(\partial \mathscr{J}^{g}_{\mathrm{hom}}(\omega,\cdot)), \end{cases}$$

where $\mathbf{n}$ denotes the unit outward normal to ∂O. So, $(\mathscr{P}^{\mathrm{NH}}_{\mathrm{hom},\omega})$ and $(\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega})$ can be rewritten as follows:

$$(\mathscr{P}^{\mathrm{NH}}_{\mathrm{hom},\omega}) \begin{cases} \dfrac{du^{\omega}}{dt}(t) - \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla u^{\omega}(t)) = G^{\omega}(t) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u^{\omega}(0) = u^{\omega}_0 \in H^1(O) \cap L^2(O; [\underline{\rho}^{\omega}, \overline{\rho}^{\omega}]) \\ u^{\omega}(t) \in H^1(O) \text{ and } \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla u^{\omega}(t)) \in L^2(O) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ A^{\omega}_{\mathrm{hom}} \nabla u^{\omega}(t) \cdot \mathbf{n} = 0 \text{ on } \partial O \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T]; \end{cases}$$

$$(\mathscr{P}^{\mathrm{D}}_{\mathrm{hom},\omega}) \begin{cases} \dfrac{du^{\omega}}{dt}(t) - \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla u^{\omega}(t)) = G^{\omega}(t) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u^{\omega}(0) = u^{\omega}_0 \in H^1_g(O) \cap L^2(O; [\underline{\rho}^{\omega}, \overline{\rho}^{\omega}]) \\ u^{\omega}(t) \in H^1(O) \text{ and } \mathrm{div}(A^{\omega}_{\mathrm{hom}} \nabla u^{\omega}(t)) \in L^2(O) \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ \gamma(u^{\omega}(t)) = \gamma_J(g) \text{ on } \partial O \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0,T]. \end{cases}$$

Remark 3.22 By imposing additional structural conditions on $F_{\varepsilon}(\omega,\cdot,\cdot)$, the hypothesis ($\mathrm{H}^{\omega}_4$) is automatically fulfilled with a complete description of the limit G^{ω} (see Sect. 5 for an example in spatial population dynamics and also [3, Theorem 5.1], [4, Theorem 7.1, p. 205] for other general examples).

4 Proof of the main result

In this section we prove Theorem 3.19.

4.1 Convergence of reaction–diffusion problems of gradient flow type

Let $T > 0$, let $\{(\underline{a}_{\varepsilon}, \overline{a}_{\varepsilon})\}_{\varepsilon>0} \subset \mathbb{R} \times \mathbb{R}$ with $\underline{a}_{\varepsilon} \le \overline{a}_{\varepsilon}$ for all $\varepsilon > 0$ and

$$-\infty < \inf_{\varepsilon>0} \underline{a}_{\varepsilon} \le \sup_{\varepsilon>0} \overline{a}_{\varepsilon} < \infty,$$

let $\{(\underline{z}_\varepsilon, \overline{z}_\varepsilon)\}_{\varepsilon>0} \subset C([0,T];\mathbb{R}) \times C([0,T];\mathbb{R})$ be such that $\underline{z}_\varepsilon(T) \leq \underline{z}_\varepsilon \leq \overline{z}_\varepsilon \leq \overline{z}_\varepsilon(T)$ for all $\varepsilon > 0$ and

$$-\infty < \inf_{\varepsilon>0} \underline{z}_\varepsilon(T) \leq \sup_{\varepsilon>0} \overline{z}_\varepsilon(T) < \infty. \tag{4.1}$$

For each $\varepsilon > 0$, let $\mathscr{E}_\varepsilon : L^2(O) \to [0,\infty[$ be a convex and Fréchet-differentiable functional, let $F_\varepsilon : [0,T] \times L^2(O) \to L^2(O)$ and consider the following reaction–diffusion problem of gradient flow type:

$$(\mathscr{P}_\varepsilon) \begin{cases} \dfrac{du_\varepsilon}{dt}(t) + \nabla\mathscr{E}_\varepsilon(u_\varepsilon(t)) = F_\varepsilon(t, u_\varepsilon(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\[2ex] u_\varepsilon(0) = u_{0,\varepsilon} \in L^2(O; [\underline{a}_\varepsilon, \overline{a}_\varepsilon]). \end{cases}$$

Let $\mathscr{E}_0 : L^2(O) \to [0,\infty]$ be a proper, convex and lower semicontinuous functional, let $G : [0,T] \times L^2(O) \to L^2(O)$ be such that $G \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ and consider the following problem of gradient flow type:

$$(\mathscr{P}_0) \begin{cases} \dfrac{du}{dt}(t) + \partial\mathscr{E}_0(u(t)) \ni G(t, u(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\[2ex] u(0) = u_0 \in \mathrm{dom}(\mathscr{E}_0). \end{cases}$$

To establish the following result, which gives sufficient conditions for the convergence of $(\mathscr{P}_\varepsilon)$ to $(\mathscr{P}_0)$ as $\varepsilon \to 0$, we do not need the existence but only the uniqueness of the solution of $(\mathscr{P}_0)$, which is straightforward because $G \in \mathscr{F}_{(R_1)\text{-}(R_2)}$.

Theorem 4.1 *Assume that:*

- *(C_1)* $\sup\limits_{\varepsilon>0} \mathscr{E}_\varepsilon(u_{0,\varepsilon}) < \infty$;
- *(C_2)* $u_{0,\varepsilon} \rightharpoonup u_0$ *in* $L^2(O)$;
- *(C_3) for each* $\varepsilon > 0$, $(\mathscr{P}_\varepsilon)$ *admits a solution* $u_\varepsilon \in AC([0,T]; L^2(O))$ *with* $\underline{z}_\varepsilon(T) \leq \underline{z}_\varepsilon \leq u_\varepsilon \leq \overline{z}_\varepsilon \leq \overline{z}_\varepsilon(T)$ *and* $\sup_{\varepsilon>0} \|F_\varepsilon(\cdot, u_\varepsilon)\|_{L^2([0,T];L^2(O))} < \infty$;
- *(C_4) for every* $\{v_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, *if* $\sup\limits_{\varepsilon>0} \mathscr{E}_\varepsilon(v_\varepsilon) < \infty$ *then* $\{v_\varepsilon\}_{\varepsilon>0}$ *is relatively compact in* $L^2(O)$.

Then, there exists $u \in C([0,T]; L^2(O))$ *such that up to a subsequence*:

$$u_\varepsilon \to u \text{ in } C([0,T]; L^2(O)); \tag{4.2}$$

$$\frac{du_\varepsilon}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2([0,T]; L^2(O)). \tag{4.3}$$

Moreover, we have

$$\inf_{\varepsilon>0} \underline{z}_\varepsilon(T) \leq u(t) \leq \sup_{\varepsilon>0} \overline{z}_\varepsilon(T) \text{ for all } t \in [0,T].$$

Assume furthermore that:

(C_5) for every $v \in C([0,T]; L^2(O))$,

$$u_\varepsilon \to v \text{ in } C([0,T]; L^2(O)) \Longrightarrow F_\varepsilon(\cdot, u_\varepsilon) \rightharpoonup G(\cdot, v) \text{ in } L^2([0,T]; L^2(O));$$

(C_6) $\mathscr{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathscr{E}_0$[1]

Then, (4.2)–(4.3) hold for the whole sequence ε and

$$u^\omega \text{ is the unique solution of } (\mathscr{P}_0).$$

Moreover, $u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}])$ where $\underline{a} := \inf_{\varepsilon>0} \underline{a}_\varepsilon$ and $\overline{a} := \sup_{\varepsilon>0} \overline{a}_\varepsilon$.

Proof of Theorem 4.1 In what follows the scalar product in $L^2(O)$ is denoted by $\langle \cdot, \cdot \rangle$. The proof is divided into three steps.
Step 1: Bounds. First of all, from (C_3) and (4.1) we see that

$$-\infty < \inf_{\varepsilon>0} \underline{z}_\varepsilon(T) \le u_\varepsilon \le \sup_{\varepsilon>0} \overline{z}_\varepsilon(T) < \infty. \tag{4.4}$$

Hence

$$\sup_{\varepsilon>0} \|u_\varepsilon\|_{C([0,T];L^2(O))} < \infty. \tag{4.5}$$

Fix any $\varepsilon > 0$. From (C_3) and ($\mathscr{P}_\varepsilon$) we deduce that for $\mathscr{L}^1$-a.e. $t \in [0,T]$,

$$\left\| \frac{du_\varepsilon}{dt}(t) \right\|^2_{L^2(O)} + \left\langle \nabla \mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle = \left\langle F_\varepsilon(t, u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle,$$

and so, by integrating over $[0,T]$,

$$\int_0^T \left\| \frac{du_\varepsilon}{dt}(t) \right\|^2_{L^2(O)} dt + \int_0^T \left\langle \nabla \mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle dt = \int_0^T \left\langle F_\varepsilon(t, u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle dt.$$

But $\frac{d}{dt}\mathscr{E}_\varepsilon(u_\varepsilon(t)) = \left\langle \nabla \mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle$ for $\mathscr{L}^1$-a.a. $t \in [0,T]$ and $u_\varepsilon(0) = u_{0,\varepsilon}$ by ($\mathscr{P}_\varepsilon$), hence

$$\int_0^T \left\langle \nabla \mathscr{E}_\varepsilon(u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t) \right\rangle dt = \int_0^T \frac{d}{dt}\mathscr{E}_\varepsilon(u_\varepsilon(t)) dt = \mathscr{E}_\varepsilon(u_\varepsilon(T)) - \mathscr{E}_\varepsilon(u_{0,\varepsilon}),$$

[1] By $\mathscr{E}_\varepsilon \xrightarrow{\mathrm{M}} \mathscr{E}_0$ we mean that $\{\mathscr{E}_\varepsilon\}_{\varepsilon>0}$ Mosco-converges to $\mathscr{E}_0$, see B. for more details.

and consequently

$$\begin{aligned}\left\|\frac{du_\varepsilon}{dt}\right\|^2_{L^2([0,T];L^2(O))} &= \int_0^T \left\langle F_\varepsilon(t,u_\varepsilon(t)), \frac{du_\varepsilon}{dt}(t)\right\rangle dt + \mathscr{E}_\varepsilon(u_{0,\varepsilon}) - \mathscr{E}_\varepsilon(u_\varepsilon(T))\\ &\le \|F_\varepsilon(\cdot,u_\varepsilon)\|_{L^2([0,T];L^2(O))}\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + \mathscr{E}_\varepsilon(u_{0,\varepsilon}) - \mathscr{E}_\varepsilon(u_\varepsilon(T))\\ &\le \|F_\varepsilon(\cdot,u_\varepsilon)\|_{L^2([0,T];L^2(O))}\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + \mathscr{E}_\varepsilon(u_{0,\varepsilon}). \end{aligned}\tag{4.6}$$

Noticing that by (C_1) and (C_3) we have:

$$\begin{aligned} c_1 &:= \sup_{\varepsilon>0} \mathscr{E}_\varepsilon(u_{0,\varepsilon}) < \infty;\\ c_2 &:= \sup_{\varepsilon>0} \|F_\varepsilon(\cdot,u_\varepsilon)\|_{L^2([0,T];L^2(O))} < \infty, \end{aligned}\tag{4.7}$$

it follows that for every $\varepsilon > 0$,

$$\left\|\frac{du_\varepsilon}{dt}\right\|^2_{L^2([0,T];L^2(O))} \le C\left(\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} + 1\right)$$

with $C := \max(c_1, c_2)$, which implies that

$$\sup_{\varepsilon>0}\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))} < \infty. \tag{4.8}$$

Step 2: Compactness. By (4.5), $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $C([0,T];L^2(O))$. Moreover, For every $(s_1,s_2) \in [0,T]\times[0,T]$ with $s_1 < s_2$,

$$\begin{aligned}\|u_\varepsilon(s_1) - u_\varepsilon(s_2)\|_{L^2(O)} &\le \int_{s_1}^{s_2}\left\|\frac{du_\varepsilon}{dt}(t)\right\|_{L^2(O)} dt\\ &\le (s_2-s_1)^{\frac{1}{2}}\sup_{\varepsilon>0}\left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))},\end{aligned}$$

which, by (4.8), implies the equi-continuity of $\{u_\varepsilon\}_{\varepsilon>0}$. On the other hand, from (C_1) and (C_4) it is clear $\{u_\varepsilon(0)\}_{\varepsilon>0} = \{u_{0,\varepsilon}\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. Morever, if $s \in]0,T]$ then, by replacing T by s in (4.6), we have

$$\mathscr{E}_\varepsilon(u_\varepsilon(s)) \le \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))}(\|F_\varepsilon(\cdot,u_\varepsilon)\|_{L^2([0,T];L^2(O))} - \left\|\frac{du_\varepsilon}{dt}\right\|_{L^2([0,T];L^2(O))}) + \mathscr{E}_\varepsilon(u_{0,\varepsilon}).$$

From (C_1), (4.8) and (4.7), it follows that $\sup_{\varepsilon>0}\mathscr{E}_\varepsilon(u_\varepsilon(s)) < \infty$. Hence, by ($C_4$), $\{u_\varepsilon(s)\}_{\varepsilon>0}$ is relatively compact in $L^2(O)$. Consequently, by Arzelà-Ascoli's compactness theorem there exists $u \in C([0,T];L^2(O))$ such that, up to a subsequence,

$$u_\varepsilon \to u \text{ in } C([0,T];L^2(O)). \tag{4.9}$$

From (4.8) we deduce that

$$\frac{du_\varepsilon}{dt} \rightharpoonup \frac{du}{dt} \text{ in } L^2([0,T]; L^2(O)) \tag{4.10}$$

and from (C_3) and (4.9) it follows that $\inf_{\varepsilon>0} \underline{z}_\varepsilon(T) \le u(t) \le \sup_{\varepsilon>0} \overline{z}_\varepsilon(T)$ for all $t \in [0,T]$.

Step 3: Convergence to the solution of ($\mathscr{P}_0$). We are going to prove that u is a solution of ($\mathscr{P}_0$).

Step 3-1: Legendre–Fenchel transform of ($\mathscr{P}_\varepsilon$). Fix any $\varepsilon > 0$ and denote the Legendre–Fenchel conjugates of $\mathscr{E}_\varepsilon$ and $\mathscr{E}_0$ by $\mathscr{E}_\varepsilon^*$ and $\mathscr{E}_0^*$ respectively. From Fenchel's extremality relation (see Proposition A.4(b)) we see that ($\mathscr{P}_\varepsilon$) is equivalent to

$$\begin{cases} \mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t)) + \left\langle \frac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle = 0 \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u_\varepsilon(0) = u_{0,\varepsilon} \in L^2(O; [\underline{a}_\varepsilon, \overline{a}_\varepsilon]) \end{cases}$$

with $G_\varepsilon := F_\varepsilon(\cdot, u_\varepsilon)$. Using Legendre–Fenchel's inequality (see Theorem A.2(b)) it follows that

$$(\mathscr{P}_\varepsilon) \iff \begin{cases} \int_0^T \left[\mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t)) + \left\langle \frac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle \right] dt = 0 \\ u_\varepsilon(0) = u_{0,\varepsilon} \in L^2(O; [\underline{a}_\varepsilon, \overline{a}_\varepsilon]). \end{cases}$$

On the other hand, we have

$$\begin{aligned} \int_0^T \left\langle \frac{du_\varepsilon}{dt}(t) - G_\varepsilon(t), u_\varepsilon(t) \right\rangle dt &= \int_0^T \left[\frac{d}{dt}(\frac{1}{2}\|u_\varepsilon\|^2)(t) - \langle G_\varepsilon(t), u_\varepsilon(t) \rangle \right] dt \\ &= \frac{1}{2}(\|u_\varepsilon(T)\|^2 - \|u_{0,\varepsilon}\|^2) - \int_0^T \langle G_\varepsilon(t), u_\varepsilon(t) \rangle \, dt. \end{aligned}$$

Hence, for every $\varepsilon > 0$,

$$(\mathscr{P}_\varepsilon) \iff \begin{cases} \int_0^T \left[\mathscr{E}_\varepsilon(u_\varepsilon(t)) + \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t)) \right] dt + \frac{1}{2}(\|u_\varepsilon(T)\|^2 - \|u_{0,\varepsilon}\|^2) \\ - \int_0^T \langle G_\varepsilon(t), u_\varepsilon(t) \rangle \, dt = 0 \\ u_\varepsilon(0) = u_{0,\varepsilon} \in L^2(O; [\underline{a}_\varepsilon, \overline{a}_\varepsilon]). \end{cases} \tag{4.11}$$

Step 3-2: Passing to the limit. First of all, by (4.9) we have

$$u_\varepsilon(0) \to u(0) \text{ in } L^2(O).$$

From (C_1)–(C_2) and (C_4) we see that

$$u_\varepsilon(0) = u_{0,\varepsilon} \to u_0 \text{ in } L^2(O). \tag{4.12}$$

Hence:

$$u(0) = u_0; \tag{4.13}$$
$$\lim_{\varepsilon\to 0} \|u_{0,\varepsilon}\|^2_{L^2(O)} = \|u_0\|^2_{L^2(O)}. \tag{4.14}$$

Since $u_{0,\varepsilon} \in L^2(O; [\underline{a}_\varepsilon, \overline{a}_\varepsilon])$ for all $\varepsilon > 0$, $u_0 \in L^2(O; [\underline{a}, \overline{a}])$ by (4.12). Moreover, from (C_1), (4.12) and (C_6) we have $\mathscr{E}_0(u_0) \leq \underline{\lim}_{\varepsilon\to 0} \mathscr{E}_\varepsilon(u_{0,\varepsilon}) \leq \sup_{\varepsilon>0} \mathscr{E}_\varepsilon(u_{0,\varepsilon}) < \infty$, hence $u_0 \in \mathrm{dom}(\mathscr{E}_0)$ and consequently

$$u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}]). \tag{4.15}$$

Since $u_\varepsilon(T) = u_\varepsilon(0) + \int_0^T \frac{du_\varepsilon}{dt}(t)dt$ and $u(T) = u(0) + \int_0^T \frac{du}{dt}(t)dt$, from (4.10), (4.12) and (4.13) we deduce that

$$\underline{\lim}_{\varepsilon\to 0} \|u_\varepsilon(T)\|^2_{L^2(O)} \geq \|u(T)\|^2_{L^2(O)}. \tag{4.16}$$

Let $E_0, E_0^* : L^2([0,T]; L^2(O)) \to [0,\infty]$ be defined by

$$\begin{cases} E_0(u) := \displaystyle\int_0^T \mathscr{E}_0(u(t))dt \\ E_0^*(u) := \displaystyle\int_0^T \mathscr{E}_0^*(u(t))dt \end{cases}$$

and, for each $\varepsilon > 0$, let Let $E_\varepsilon : L^2([0,T]; L^2(O)) \to [0,\infty]$ be defined by

$$\begin{cases} E_\varepsilon(u) := \displaystyle\int_0^T \mathscr{E}_\varepsilon(u(t))dt \\ E_\varepsilon^*(u) := \displaystyle\int_0^T \mathscr{E}_\varepsilon^*(u(t))dt. \end{cases}$$

From (C_6) and Theorem B.4 we have $\mathscr{E}_\varepsilon^* \xrightarrow{M} \mathscr{E}_0^*$. Hence $E_\varepsilon \xrightarrow{M} E_0$ and $E_\varepsilon^* \xrightarrow{M} E_0^*$ by Theorem B.5. From (4.9), (C_5) and (4.10) it follows that:

$$\underline{\lim}_{\varepsilon\to 0} E_\varepsilon(u_\varepsilon) \geq E_0(u), \text{ i.e.}$$
$$\underline{\lim}_{\varepsilon\to 0} \int_0^T \mathscr{E}_\varepsilon(u_\varepsilon(t))dt \geq \int_0^T \mathscr{E}_0(u(t))dt; \tag{4.17}$$
$$\underline{\lim}_{\varepsilon\to 0} E_\varepsilon^*(G_\varepsilon - \frac{du_\varepsilon}{dt}) \geq E_0^*(G_0 - \frac{du}{dt}), \text{ i.e.}$$
$$\underline{\lim}_{\varepsilon\to 0} \int_0^T \mathscr{E}_\varepsilon^*(G_\varepsilon(t) - \frac{du_\varepsilon}{dt}(t))dt \geq \int_0^T \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t))dt \tag{4.18}$$

with $G_0 := G(\cdot, u)$. Taking (4.12), (4.13), (4.14), (4.15), (4.16), (4.17) and (4.18) into account, letting $\varepsilon \to 0$ in (4.11) we obtain

$$\begin{cases} \int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t))\right] dt + \frac{1}{2}(\|u(T)\|^2 - \|u_0\|^2) \\ -\int_0^T \langle G_0(t), u(t)\rangle \, dt \le 0 \\ u(0) = u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}]), \end{cases}$$

i.e.

$$\begin{cases} \int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t)\right\rangle\right] dt \le 0 \\ u(0) = u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}]) \end{cases}$$

But, by using again Legendre–Fenchel's inequality (see Theorem A.2(b)), we have

$$\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t)\right\rangle \ge 0 \text{ for } \mathscr{L}^1\text{-a.a. } t \in [0, T],$$

hence

$$\begin{cases} \int_0^T \left[\mathscr{E}_0(u(t)) + \mathscr{E}_0^*(G_0(t) - \frac{du}{dt}(t)) + \left\langle \frac{du}{dt}(t) - G_0(t), u(t)\right\rangle\right] dt = 0 \\ u(0) = u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}]). \end{cases} \tag{4.19}$$

Using again Fenchel's extremality relation (see Proposition A.4(b)) we see that (4.19) is equivalent to

$$\begin{cases} \frac{du}{dt}(t) + \partial\mathscr{E}_0(u(t)) \ni G_0(t) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u(0) = u_0 \in \mathrm{dom}(\mathscr{E}_0) \cap L^2(O; [\underline{a}, \overline{a}]), \end{cases}$$

which shows that u is a solution of $(\mathscr{P}_0)$, and the proof is complete because of the uniqueness of the solution of $(\mathscr{P}_0)$. □

4.2 Auxiliary lemmas

To prove almost sure Mosco-convergence of the energies (see Sect. 4.3), we will need the following lemmas. We begin with two compactness results.

Lemma 4.2 *Let* $\lambda : \mathbb{R}^d \to [0, \infty[$ *be such that*

$$\begin{cases} \lambda \not\equiv 0 \\ \textit{for every } (\xi, \zeta) \in \mathbb{R}^d \times \mathbb{R}^d, \ \textit{if } |\xi| \le |\zeta| \textit{ then } \lambda(\xi) \ge \lambda(\zeta) \\ \mathrm{supp}(\lambda) \textit{ is compact} \end{cases}$$

and, for each $\varepsilon > 0$*, let* $\lambda_\varepsilon : \mathbb{R}^d \to [0, \infty[$ *be defined by*

$$\lambda_\varepsilon(\xi) := \frac{1}{\varepsilon^d} \lambda\left(\frac{\xi}{\varepsilon}\right).$$

Let $U \subset \mathbb{R}^d$ *be a bounded open domain with Lipschitz boundary and let* $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(U)$ *be such that*

$$\sup_{\varepsilon>0} \frac{1}{\varepsilon^2} \int_U \int_U \lambda_\varepsilon(y - x) \, |u_\varepsilon(x) - u_\varepsilon(y)|^2 \, dxdy < \infty.$$

Then, there exists $u \in H^1(U)$ *such that, up to a subsequence,* $u_\varepsilon \to u$ *in* $L^2(U)$*.*

For a proof of Lemma 4.2 we refer to [8, Theorem 6.11, p. 128] (see also [9, Theorem 4 and Remark 4]). For each $\varepsilon > 0$, let $\mathscr{J}_\varepsilon : \Omega \times L^2_{\mathrm{loc}}(\mathbb{R}^d) \times \mathscr{B}(\mathbb{R}^d) \times \mathscr{B}(\mathbb{R}^d)$ be given by

$$\mathscr{J}_\varepsilon(\omega, u, A, B) := \frac{1}{4\varepsilon^d} \int_A \int_B J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) \left(\frac{u(x) - u(y)}{\varepsilon}\right)^2 dxdy. \tag{4.20}$$

Let $g \in H^1(O^J \setminus \overline{O})$ with $O^J := O + \mathrm{supp}(\overline{J}) = O + \overline{B}_{R_J}(0)$. For each $v \in L^2(O)$ we consider $v^g \in L^2(O^J)$ defined by

$$v^g(x) := \begin{cases} v(x) & \text{if } x \in O \\ g(x) & \text{if } x \in O^J \setminus \overline{O}. \end{cases}$$

As a consequence of Lemma 4.2 we obtain the second compactness lemma.

Lemma 4.3 *Let* $\omega \in \Omega$ *and let* $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ *be such that*

$$\sup_{\varepsilon>0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon^g, O^J, O^J) < \infty.$$

Then, there exists $u \in H^1_g(O)$ *such that* $u^g \in H^1(O^J)$ *and, up to a subsequence,*

$$\begin{cases} u_\varepsilon \to u \textit{ in } L^2(O) \\ u_\varepsilon^g \to u^g \textit{ in } L^2(O^J). \end{cases}$$

The following two lemmas are Poincaré type inequalities.

Lemma 4.4 *Let $R > 0$ and let Q be a cube of $\mathbb{R}^d$ of size $\eta > 0$. Then, there exists $C > 0$ such that*

$$\int_Q |u(x)|^2 dx \leq \frac{C\eta}{R^{d+2}} \int_Q \int_{\left[|\xi| \leq \frac{R}{2}\right]} |u(x+\xi) - u(x)|^2 dx d\xi$$

for all $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ such that $u = 0$ in $\partial_R(Q)$ and $u = 0$ in $\mathbb{R}^d \setminus Q$, where $\partial_R(Q)$ is defined in (3.7).

For a proof of Lemma 4.4 we refer to [12, Lemma 4.3] (see also [8, Proposition 6.25, p. 144]).

Lemma 4.5 *Let $\omega \in \Omega$ and let $A \subset O^J$ be an open subset with Lipschitz boundary and let $u \in H^1(A)$.*

(a) *There exists $C > 0$, which only depends on A, such that for every $u \in H^1(A)$,*

$$\sup_{\varepsilon > 0} \mathscr{J}_\varepsilon(\omega, u, A, O^J \setminus \overline{O}) \leq C \|u\|^2_{H^1(A)} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi.$$

(b) *Assume furthermore that $A \Subset O^J$ and let $\delta > 0$ be such that $A + B_\delta(0) \subset O^J$. Then, for every $\varepsilon > 0$ with $\varepsilon R_J < \delta$ and every $u \in H^1(A + B_\delta(0))$,*

$$\sup_{\varepsilon > 0} \mathscr{J}_\varepsilon(\omega, u, A, O^J \setminus \overline{O}) \leq \frac{1}{4} \int_{A + B_\delta(0)} |\nabla u(x)|^2 dx \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi. \quad (4.21)$$

Proof of Lemma 4.5 (a) Let $P : H^1(A) \to H^1(\mathbb{R}^d)$ be a continuous extension operator. Then, there exists $C_A > 0$ such that $\|Pu\|_{H^1(\mathbb{R}^d)} \leq C_A \|u\|_{H^1(A)}$ for all $u \in H^1(A)$. Hence, if we establish

$$\sup_{\varepsilon > 0} \mathscr{J}_\varepsilon(\omega, u, A, O^J \setminus \overline{O}) \leq \frac{1}{4} \|Pu\|^2_{H^1(\mathbb{R}^d)} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi \quad (4.22)$$

for all $u \in H^1(A)$ then (a) will follow with $C = \frac{1}{4} C_A$. Let $u \in H^1(A)$ and let $\varepsilon > 0$. By changing of scale ($\xi = \frac{x-y}{\varepsilon}$ with x fixed) and by using PNL_2 and Fubini's theorem, we see that

$$\sup_{\varepsilon > 0} \mathscr{J}_\varepsilon(\omega, u, A, O^J \setminus \overline{O}) \leq \frac{1}{4} \int_{B_{R_J}(0)} \overline{J}(\xi) \left(\int_{\mathbb{R}^d} \left| \frac{Pu(x) - Pu(x + \varepsilon\xi)}{\varepsilon} \right|^2 dx \right) d\xi. \quad (4.23)$$

On the other hand, for every $\xi \in \mathbb{R}^d$ and $\mathscr{L}^d$-a.e. $x \in \mathbb{R}^d$, we have

$$\frac{Pu(x) - Pu(x + \varepsilon\xi)}{\varepsilon} = |\xi| \fint_0^{\varepsilon|\xi|} \nabla Pu \left(x + t \frac{\xi}{|\xi|} \right) \frac{\xi}{|\xi|} dt,$$

hence, by using Jensen's inequality,

$$\left|\frac{Pu(x)-Pu(x+\varepsilon\xi)}{\varepsilon}\right|\leq|\xi|^2\fint_0^{\varepsilon|\xi|}\left|\nabla Pu\left(x+t\frac{\xi}{|\xi|}\right)\right|^2dt,$$

and consequently, by using Fubini's theorem and by changing of variable ($y=x+t\frac{\xi}{|\xi|}$ with t fixed),

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\frac{Pu(x)-Pu(x+\varepsilon\xi)}{\varepsilon}\right|dx&\leq|\xi|^2\int_{\mathbb{R}^d}|\nabla Pu(y)|^2dy\\&\leq|\xi|^2\|Pu\|^2_{H^1(\mathbb{R}^d)}.\end{aligned}\tag{4.24}$$

From (4.23) and (4.24) we deduce that

$$\sup_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,u,A,O^J\backslash\overline{O})\leq\frac{1}{4}\|Pu\|^2_{H^1(\mathbb{R}^d)}\int_{B_{R_j}(0)}|\xi|^2\overline{J}(\xi)d\xi,$$

and (4.22) follows because $\mathrm{supp}(\overline{J})=\overline{B}_{R_J}(0)$ by PNL_2.

(b) In the same way, for every $u\in H^1(A+B_\delta(0))$, we have

$$\sup_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,u,A,O^J\backslash\overline{O})\leq\frac{1}{4}\int_{B_{R_j}(0)}\overline{J}(\xi)\left(\int_A\left|\frac{u(x)-u(x+\varepsilon\xi)}{\varepsilon}\right|^2dx\right)d\xi,$$

where, for every $\xi\in\mathbb{R}^d$ and $\mathscr{L}^d$-a.e. $x\in A$,

$$\int_A\left|\frac{u(x)-u(x+\varepsilon\xi)}{\varepsilon}\right|dx\leq|\xi|^2\int_{A+B_\delta(0)}|\nabla u(y)|^2dy,$$

which implies (4.22). □

For each $x_0\in O$ and each $u\in H^1(O)$, we consider the affine function $u_{x_0}:O\to\mathbb{R}$ given by

$$u_{x_0}(x):=u(x_0)+\nabla u(x_0)(x-x_0).$$

By [25, Theorem 3.4.2, p. 129] there exists $N_1\subset O$ with $\mathscr{L}^d(N_1)=0$ such that for every $x_0\in O\backslash N_1$,

$$\fint_{Q_\rho(x_0)}\left|u(x)-u_{x_0}(x)\right|^2dx=o(\rho^2)\text{ as }\rho\to 0.\tag{4.25}$$

By using (4.25) we can establish the following lemma.

Lemma 4.6 *Let $u\in H^1(O)$ and let $\{u_\varepsilon\}_{\varepsilon>0}\subset L^2(O)$ be such that $u_\varepsilon\to u$ in $L^2(O)$ and, for each $x_0\in O\backslash N_1$, each $\rho>0$ with $Q_\rho(x_0)\subset O$, each $\varepsilon>0$ and each*

$\delta \in]0,1[$, *let* $u^{\rho,\delta}_{\varepsilon,x_0} \in L^2(Q_\rho(x_0))$ *be defined by*

$$u^{\rho,\delta}_{\varepsilon,x_0}(x) := \begin{cases} u_\varepsilon & \text{if } x \in \left[Q_\rho(x_0)\right]_{2\rho\delta} \\ u_{x_0}(x) & \text{if } x \in Q_\rho(x_0)\backslash\left[Q_\rho(x_0)\right]_{2\rho\delta} \end{cases} \tag{4.26}$$

with $\left[Q_\rho(x_0)\right]_{2\rho\delta} := \left\{x \in Q_\rho(x_0) : \mathrm{dist}(x, \partial Q_\rho(x_0)) > 2\rho\delta\right\} = Q_{2\rho\delta}(x_0)$ *(see (3.8)). Then:*

$$u^{\rho,\delta}_{\varepsilon,x_0} \to u \text{ in } L^2(Q_\rho(x_0)) \text{ as } \varepsilon \to 0;$$

$$\overline{\lim_{\rho\to 0}}\ \overline{\lim_{\varepsilon\to 0}} \left[\frac{\mathscr{J}_\varepsilon(\omega, u^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0)) - \mathscr{J}_\varepsilon(\omega, u_\varepsilon, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))}\right]$$
$$\le o(1) \text{ as } \delta \to 0.$$

Proof of Lemma 4.6 Arguing as in the proof of [13, Proposition 2.2] we can assert that

$$\begin{aligned}\overline{\lim_{\varepsilon\to 0}} &\left[\frac{\mathscr{J}_\varepsilon(\omega, u^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0)) - \mathscr{J}_\varepsilon(\omega, u_\varepsilon, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))}\right] \\ &\le \frac{CN^2}{(\delta\rho)^2} ⨍_{Q_\rho(x_0)} |u - u_{x_0}|^2 dx \\ &\quad + R^{\rho,\delta}_\varepsilon + \frac{C}{N},\end{aligned}$$

where $C > 0$, N is the number of slides of $Q_\rho(x_0)\backslash\left[Q_\rho(x_0)\right]_{2\rho\delta}$ and

$$\begin{aligned}R^{\rho,\delta}_\varepsilon := \frac{C}{(\rho\varepsilon)^d} &\int_{Q_\rho(x_0)\backslash[Q_\rho(x_0)]_{2\rho\delta}} \int_{Q_\rho(x_0)\backslash[Q_\rho(x_0)]_{2\rho\delta}} \\ &J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) |\nabla u(x_0)|^2 \left|\frac{x-y}{\varepsilon}\right|^2 dxdy.\end{aligned}$$

But, by changing of scale ($\xi = \frac{x-y}{\varepsilon}$ with x fixed) and by using PNL_2,

$$R^{\rho,\delta}_\varepsilon \le C(1-(1-2\delta)^d)|\nabla u(x_0)|^2 \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi)d\xi = o(1) \text{ as } \delta \to 0$$

and, by using (4.25),

$$\frac{CN^2}{(\delta\rho)^2} ⨍_{Q_\rho(x_0)} |u - u_{x_0}|^2 dx = \frac{N^2}{\delta^2} o(1) \text{ as } \rho \to 0,$$

hence

$$\overline{\lim_{\rho\to 0}}\ \overline{\lim_{\varepsilon\to 0}}\left[\frac{\mathscr{J}_\varepsilon(\omega, u^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0)) - \mathscr{J}_\varepsilon(\omega, u_\varepsilon, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))}\right]$$
$$\leq o(1) + \frac{C}{N} \text{ as } \delta \to 0,$$

and the conclusion follows by letting $N \to \infty$. □

Finally, the proof of the following lemma can be found in [13, Proposition 2.2].

Lemma 4.7 *Let $\omega \in \Omega$, let $U \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary, let $u \in H^1(U)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(U)$ be such that $u_\varepsilon \to u$ in $L^2(U)$. Then, for every $\delta > 0$ there exists $\{u^\delta_\varepsilon\}_{\varepsilon>0} \subset L^2(U)$ such that:*

- $\begin{cases} u^\delta_\varepsilon = u \text{ in } U\backslash U_\delta \\ u^\delta_\varepsilon = u_\varepsilon \text{ in } U_{2\delta}\end{cases}$
- $u^\delta_\varepsilon \to u$ *in* $L^2(U)$;
- $\overline{\lim_{\varepsilon\to 0}}(\mathscr{J}_\varepsilon(\omega, u^\delta_\varepsilon, U, U) - \mathscr{J}_\varepsilon(\omega, u_\varepsilon, U, U)) \leq o(1)$ *as* $\delta \to 0$.

4.3 Almost sure Mosco-convergence of the energies

Here, we establish the almost sure Mosco-convergence of $\{\mathscr{J}_\varepsilon\}_{\varepsilon>0}$ and $\{\mathscr{J}^g_\varepsilon := \mathscr{J}_\varepsilon + \mathscr{D}^\varepsilon_g\}_{\varepsilon>0}$, where $\mathscr{J}_\varepsilon : \Omega \times L^2(O) \to [0, \infty[$ and $\mathscr{D}^\varepsilon_g : \Omega \times L^2(O) \to [0, \infty[$ are defined by (3.2) and (3.5) respectively.

Theorem 4.8 *Let $\Omega' \in \mathscr{F}$ be such that $\mathbb{P}(\Omega') = 1$ given by Proposition 3.17. Then, for every $\omega \in \Omega'$, $\{\mathscr{J}_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$ (resp. $\{\mathscr{J}^g_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$) Mosco-convergence to $\mathscr{J}_{\text{hom}}(\omega, \cdot)$ (resp. $\mathscr{J}^g_{\text{hom}}(\omega, \cdot)$).*

Proof of Theorem 4.8 Let $\omega \in \Omega'$. According to Lemma 4.2 (resp. Lemma 4.3) and Proposition B.3, it is equivalent to prove that $\{\mathscr{J}_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$ (resp. $\{\mathscr{J}^g_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$) Γ-convergen-ce with respect to the strong convergence in $L^2(O)$ to $\mathscr{J}_{\text{hom}}(\omega, \cdot)$ (resp. $\mathscr{J}^g_{\text{hom}}(\omega, \cdot)$). To do this, the proof is divided into three steps.
Step 1: Γ-limit inf. We have to prove that:

$$\mathscr{J}_{\text{hom}}(\omega, \cdot) \leq \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\ \mathscr{J}_\varepsilon(\omega, \cdot); \tag{4.27}$$

$$\mathscr{J}^g_{\text{hom}}(\omega, \cdot) \leq \Gamma\text{-}\underline{\lim}_{\varepsilon\to 0}\ \mathscr{J}^g_\varepsilon(\omega, \cdot). \tag{4.28}$$

Proof of (4.27). According to Definition B.1 it is equivalent to prove that for every $u \in L^2(O)$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, if $u_\varepsilon \to u$ in $L^2(O)$ then

$$\mathscr{J}_{\text{hom}}(\omega, u) \leq \underline{\lim}_{\varepsilon\to 0}\ \mathscr{J}_\varepsilon(\omega, u_\varepsilon) \tag{4.29}$$

Let $u \in L^2(O)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ be such that

$$u_\varepsilon \to u \text{ in } L^2(O). \tag{4.30}$$

Without loss of generality we can assume that $\underline{\lim}_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon) = \lim_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon) < \infty$, and so

$$\sup_{\varepsilon>0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon) < \infty. \tag{4.31}$$

Taking PNL$_2$ into account, from (4.31) and Lemma 4.2 there exists $\widehat{u} \in H^1(O)$ such that, up to a subsequence, $u_\varepsilon \to \widehat{u}$ in $L^2(O)$. By (4.30) it follows that $u \in H^1(O)$. Hence, to prove (4.29) it is sufficient to establish that

$$\int_O f_{\hom}(\omega, \nabla u(x))dx \le \underline{\lim}_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon). \tag{4.32}$$

For each $\varepsilon > 0$, we define the (positive) Radon measure μ_ε on O by

$$\begin{aligned}\mu_\varepsilon(A) &:= \frac{1}{4\varepsilon^d}\int_A\int_A J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{u_\varepsilon(x)-u_\varepsilon(y)}{\varepsilon}\right)^2 dxdy \\ &= \mathscr{J}_\varepsilon(\omega, u_\varepsilon, A, A)\end{aligned}$$

for all $A \in \mathscr{B}(O)$. From (4.31) we see that $\sup_{\varepsilon>0} \mu_\varepsilon(O) < \infty$ and so there exists a (positive) Radon measure μ on O such that, up to a subsequence, $\mu_\varepsilon \rightharpoonup \mu$ weakly in the sense of measures. By Lebesgue's decomposition theorem, we have $\mu = \mu^a + \mu^s$ where μ^a and μ^s are (positive) Radon measures on O such that $\mu^a \ll \mathscr{L}^d$ and $\mu^s \perp \mathscr{L}^d$. Thus, to prove (4.32) it suffices to show that

$$f_{\hom}(\omega, \nabla u(\cdot))\mathscr{L}^d \le \mu^a. \tag{4.33}$$

From Radon–Nikodym's theorem and Alexandrov's theorem, there exists $N_0 \subset O$ with $\mathscr{L}^d(N_0) = 0$ such that

$$\begin{cases} \mu^a = g\mathscr{L}^d \text{ with } g \in L^1(O; [0, \infty[) \\ g(x) = \lim_{\rho\to 0}\lim_{\varepsilon\to 0} \dfrac{\mu_\varepsilon(Q_\rho(x))}{\mathscr{L}^d(Q_\rho(x))} \text{ for all } x \in O\backslash N_0. \end{cases}$$

Let $N_1 \subset O$ (with $\mathscr{L}^d(N_1) = 0$) be given by (4.25) (and used in Lemma 4.6). From the above we see that to prove (4.33) it is sufficient to establish that for every $x_0 \in O\backslash(N_0 \cup N_1)$,

$$f_{\text{hom}}(\omega, \nabla u(x_0)) \leq g(x_0),$$

i.e., by using (4.20),

$$f_{\text{hom}}(\omega, \nabla u(x_0)) \leq \lim_{\rho\to 0} \lim_{\varepsilon\to 0} \frac{\mathscr{J}_\varepsilon(\omega, u_\varepsilon, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))}. \tag{4.34}$$

Let $x_0 \in O\backslash(N_0 \cup N_1)$. From Lemma 4.6 we deduce as $\delta \to 0$,

$$\lim_{\rho\to 0} \lim_{\varepsilon\to 0} \frac{\mathscr{J}_\varepsilon(\omega, u_\varepsilon, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} \geq \underline{\lim}_{\rho\to 0}\, \underline{\lim}_{\varepsilon\to 0} \frac{\mathscr{J}_\varepsilon(\omega, u^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} + o(1) \tag{4.35}$$

with $u^{\rho,\delta}_{\varepsilon,x_0} \in L^2(Q_\rho(x_0))$ given by (4.26). As $\mathscr{J}_\varepsilon(\cdot, v + c, \cdot, \cdot) = \mathscr{J}_\varepsilon(\cdot, v, \cdot, \cdot)$ for all $v \in L^2_{\text{loc}}(\mathbb{R}^d)$ and all $c \in \mathbb{R}$, in (4.35) we can replace $u^{\rho,\delta}_{\varepsilon,x_0}$ by $\widetilde{u}^{\rho,\delta}_{\varepsilon,x_0}$ given by

$$\widetilde{u}^{\rho,\delta}_{\varepsilon,x_0}(x) := \begin{cases} u_\varepsilon & \text{if } x \in \left[Q_\rho(x_0)\right]_{2\rho\delta} \\ \ell_{\nabla u(x_0)}(x) & \text{if } x \in Q_\rho(x_0)\backslash\left[Q_\rho(x_0)\right]_{2\rho\delta} \end{cases}$$

with $\ell_{\nabla u(x_0)} : \mathbb{R}^d \to \mathbb{R}$ the linear map defined by $\ell_{\nabla u(x_0)}(x) = \nabla u(x_0)x$, i.e.

$$\frac{\mathscr{J}_\varepsilon(\omega, u^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} = \frac{\mathscr{J}(\omega, \widetilde{u}^{\rho,\delta}_{\varepsilon,x_0}, \frac{1}{\varepsilon}Q_\rho(x_0), \frac{1}{\varepsilon}Q_\rho(x_0))}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_\rho(x_0))}. \tag{4.36}$$

On the other hand, by change of scale and function, i.e. $(x', y') = (\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ and $\widehat{u}^{\rho,\delta}_{\varepsilon,x_0}(x') = \frac{1}{\varepsilon}\widetilde{u}^{\rho,\delta}_{\varepsilon,x_0}(\varepsilon x')$, we have:

$$\begin{aligned} &\frac{\mathscr{J}_\varepsilon(\omega, \widetilde{u}^{\rho,\delta}_{\varepsilon,x_0}, Q_\rho(x_0), Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} = \frac{\mathscr{J}(\omega, \widehat{u}^{\rho,\delta}_{\varepsilon,x_0}, \frac{1}{\varepsilon}Q_\rho(x_0), \frac{1}{\varepsilon}Q_\rho(x_0))}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_\rho(x_0))}; \\ &\widehat{u}^{\rho,\delta}_{\varepsilon,x_0} = \ell_{\nabla u(x_0)} \text{ in } \frac{1}{\varepsilon}Q_\rho(x_0)\backslash\left[\frac{1}{\varepsilon}Q_\rho(x_0)\right]_{\frac{2\rho\delta}{\varepsilon}}. \end{aligned} \tag{4.37}$$

For each $\rho > 0$ there exists $\varepsilon_\rho > 0$ such that $\frac{2\rho\delta}{\varepsilon} > R_J$ for all $\varepsilon \in]0, \varepsilon_\rho]$ (with $R_J > 0$ given by PNL_2). Hence

$$\widehat{u}^{\rho,\delta}_{\varepsilon,x_0} = \ell_{\nabla u(x_0)} \text{ in } \frac{1}{\varepsilon}Q_\rho(x_0)\backslash\left[\frac{1}{\varepsilon}Q_\rho(x_0)\right]_{R_J} \text{ for all } \rho > 0 \text{ and all } \varepsilon \in]0, \varepsilon_\rho],$$

and so, by extending $\widehat{u}^{\rho,\delta}_{\varepsilon,x_0}$ by $\ell_{\nabla u(x_0)}$ outside $\frac{1}{\varepsilon}Q_\rho(x_0)$,

$$\widehat{u}^{\rho,\delta}_{\varepsilon,x_0} = \ell_{\nabla u(x_0)} \text{ in } \partial_{R_j}\left(\frac{1}{\varepsilon}Q_\rho(x_0)\right) \text{ for all } \rho > 0 \text{ and all } \varepsilon \in]0, \varepsilon_\rho].$$

Thus $\widehat{u}^{\rho,\delta}_{\varepsilon,x_0} \in L^2_{\mathrm{loc},\ell_{\nabla u(x_0)},R_j,\frac{1}{\varepsilon}Q_\rho(x_0)}(\mathbb{R}^d)$ for all $\rho > 0$ and all $\varepsilon \in]0,\varepsilon_\rho]$, where $L^2_{\mathrm{loc},\theta,R,A}(\mathbb{R}^d)$ is defined by (3.6) with $\theta = \ell_{\nabla u(x_0)}$, $R = R_j$ and $A = \frac{1}{\varepsilon}Q_\rho(x_0)$. From (4.35), (4.36) and (4.37) it follows that

$$\lim_{\rho\to 0}\varliminf_{\varepsilon\to 0}\frac{\mathscr{J}_\varepsilon(\omega,u_\varepsilon,Q_\rho(x_0),Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} \geq \varliminf_{\rho\to 0}\varliminf_{\varepsilon\to 0}\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_\rho(x_0)}(\omega,\ell_{\nabla u(x_0)})}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_\rho(x_0))} + o(1) \text{ as } \delta\to 0.$$

Hence, by Proposition 3.17 (and Remark 3.15),

$$\lim_{\varepsilon\to 0}\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_\rho(x_0)}(\omega,\ell_{\nabla u(x_0)})}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_\rho(x_0))} = f_{\mathrm{hom}}(\omega,\nabla u(x_0)),$$

and consequently

$$\lim_{\rho\to 0}\lim_{\varepsilon\to 0}\frac{\mathscr{J}_\varepsilon(\omega,u_\varepsilon,Q_\rho(x_0),Q_\rho(x_0))}{\mathscr{L}^d(Q_\rho(x_0))} \geq f_{\mathrm{hom}}(\omega,\nabla u(x_0)) + o(1) \text{ as } \delta\to 0,$$

which gives (4.34) by letting $\delta \to 0$.

Proof of (4.28). As in the proof of (4.27) it is equivalent to prove that for every $u \in L^2(O)$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$, if $u_\varepsilon \to u$ in $L^2(O)$ then

$$\mathscr{J}^g_{\mathrm{hom}}(\omega,u) \leq \varliminf_{\varepsilon\to 0}\mathscr{J}^g_\varepsilon(\omega,u_\varepsilon). \tag{4.38}$$

Let $u \in L^2(O)$ and let $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ be such that $u_\varepsilon \to u$ in $L^2(O)$. Without loss of generality we can assume that $\varliminf_{\varepsilon\to 0}\mathscr{J}^g_\varepsilon(\omega,u_\varepsilon) = \lim_{\varepsilon\to 0}\mathscr{J}^g_\varepsilon(\omega,u_\varepsilon) < \infty$, hence

$$\sup_{\varepsilon>0}\mathscr{J}^g_\varepsilon(\omega,u_\varepsilon) < \infty,$$

and consequently

$$\sup_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega,u_\varepsilon) < \infty; \tag{4.39}$$

$$\sup_{\varepsilon>0}\mathscr{D}^\varepsilon_g(\omega,u_\varepsilon) < \infty, \tag{4.40}$$

where $\mathscr{J}_\varepsilon(\omega,\cdot)$ and $\mathscr{D}^\varepsilon_g(\omega,\cdot)$ are defined in (3.2) and (3.5) respectively. Fix any $\varepsilon > 0$ and consider $u^g_\varepsilon \in L^2(O^J)$ defined by

$$u^g_\varepsilon(x) := \begin{cases} u_\varepsilon(x) & \text{if } x \in O \\ g(x) & \text{if } x \in O^J\setminus\overline{O}. \end{cases}$$

By using PNL_1 and Fubini's theorem, it is easy to see that

$$\begin{aligned}\mathscr{J}_\varepsilon\left(\omega, u_\varepsilon^g, O^J, O^J\right) &= \mathscr{J}_\varepsilon(\omega, u_\varepsilon, O, O) + \mathscr{J}_\varepsilon\left(\omega, g, O^J\backslash\overline{O}, O^J\backslash\overline{O}\right)\\ &\quad+\frac{1}{4\varepsilon^d}\int_O\int_{O^J\backslash\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{u(x)-g(y)}{\varepsilon}\right)^2 dxdy\\ &\quad+\frac{1}{4\varepsilon^d}\int_{O^J\backslash\overline{O}}\int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{g(x)-u(y)}{\varepsilon}\right)^2 dxdy\\ &= \mathscr{J}_\varepsilon(\omega, u_\varepsilon) + \mathscr{J}_\varepsilon\left(\omega, g, O^J\backslash\overline{O}, O^J\backslash\overline{O}\right)\\ &\quad+\frac{1}{2\varepsilon^d}\int_O\int_{O^J\backslash\overline{O}} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{u(x)-g(y)}{\varepsilon}\right)^2 dxdy\\ &= \mathscr{J}_\varepsilon(\omega, u_\varepsilon) + \mathscr{J}_\varepsilon\left(\omega, g, O^J\backslash\overline{O}, O^J\backslash\overline{O}\right) + \mathscr{D}_g^\varepsilon(\omega, u_\varepsilon).\end{aligned}$$

and so, by using Lemma 4.5(a) (with $A = O^J\backslash\overline{O}$),

$$\mathscr{J}_\varepsilon(\omega, u_\varepsilon^g, O^J, O^J) \leq \mathscr{J}_\varepsilon(\omega, u_\varepsilon) + C\|g\|_{H^1(O^J\backslash\overline{O})}\int_{\mathbb{R}^d}|\xi|^2\overline{J}(\xi)d\xi + \mathscr{D}_g^\varepsilon(\omega, u_\varepsilon),$$

where $C > 0$, which only depends on $O^J\backslash\overline{O}$, is given by Lemma 4.5(a). Recalling that $g \in H^1(O^J\backslash\overline{O})$ and using PNL_2, (4.39) and (4.40) we deduce that

$$\sup_{\varepsilon>0}\mathscr{J}_\varepsilon(\omega, u_\varepsilon^g, O^J, O^J) < \infty,$$

hence, by using Lemma 4.3, there exists $\widehat{u} \in H_g^1(O)$ such that, up to a subsequence, $u_\varepsilon \to \widehat{u}$ in $L^2(O)$, and consequently $\widehat{u} = u$ because $u_\varepsilon \to u$ in $L^2(O)$. Thus

$$u \in H_g^1(O). \tag{4.41}$$

On the other hand, from (4.27) we have

$$\mathscr{J}_{\mathrm{hom}}(\omega, u) \leq \underline{\lim}_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\omega, u_\varepsilon). \tag{4.42}$$

Moreover, it is clear that

$$\underline{\lim}_{\varepsilon\to 0}\mathscr{J}_\varepsilon(\omega, u_\varepsilon) \leq \underline{\lim}_{\varepsilon\to 0}\mathscr{J}_\varepsilon^g(\omega, u_\varepsilon), \tag{4.43}$$

and, since $H_g^1(O) \subset H^1(O)$, from (4.41) and the definitions of $\mathscr{J}_{\mathrm{hom}}(\omega, \cdot)$ and $\mathscr{J}_{\mathrm{hom}}^g(\omega, \cdot)$ in (3.21) and (3.22) respectively, we see that

$$\mathscr{J}_{\mathrm{hom}}(\omega, u) = \mathscr{J}_{\mathrm{hom}}^g(\omega, u) = \int_O f_{\mathrm{hom}}(\omega, \nabla u(x))dx. \tag{4.44}$$

Consequently, (4.38) follows from (4.42), (4.43) and (4.42).
Step 2: Γ-limit sup. We have to prove that:

$$\mathscr{J}_{\text{hom}}(\omega,\cdot) \geq \Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\ \mathscr{J}_{\varepsilon}(\omega,\cdot); \tag{4.45}$$

$$\mathscr{J}^{g}_{\text{hom}}(\omega,\cdot) \geq \Gamma\text{-}\overline{\lim_{\varepsilon\to 0}}\ \mathscr{J}^{g}_{\varepsilon}(\omega,\cdot). \tag{4.46}$$

Proof of (4.45). According to Definition B.1 it is equivalent to prove that for every $u \in L^2(O)$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to u \text{ in } L^2(O) \\ \mathscr{J}_{\text{hom}}(\omega,u) \geq \overline{\lim}_{\varepsilon\to 0}\ \mathscr{J}_{\varepsilon}(\omega,u_\varepsilon). \end{cases} \tag{4.47}$$

Let $u \in L^2(O)$. By definition of $\mathscr{J}_{\text{hom}}(\omega,\cdot)$ in (3.21), without loss of generality we can assume that $u \in H^1(O)$, and to prove (4.47) it suffices to show that there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to u \text{ in } L^2(O) \\ \displaystyle\int_O f_{\text{hom}}(\omega,\nabla u(x))dx \geq \overline{\lim_{\varepsilon\to 0}}\ \mathscr{J}_{\varepsilon}(\omega,u_\varepsilon). \end{cases} \tag{4.48}$$

As $\text{Aff}(O)$ is dense in $H^1(O)$ and, since $f_{\text{hom}}(\omega,\cdot)$ is quadratic, $u \mapsto \int_O f_{\text{hom}}(\omega,\nabla u(x))dx$ is continuous with respect to the norm of $H^1(O)$, it is sufficient to prove (4.48) for u affine, i.e. for $u = \ell_\theta$ with $\theta \in \mathbb{R}^d$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to \ell_\theta \text{ in } L^2(O) \\ f_{\text{hom}}(\omega,\theta)\mathscr{L}^d(O) \geq \overline{\lim_{\varepsilon\to 0}}\ \mathscr{J}_{\varepsilon}(\omega,u_\varepsilon). \end{cases} \tag{4.49}$$

As O is regular, for every $\eta > 0$ there exist two finite sets I_η and J_η with $I_\eta \subset J_\eta$ and a family $\{Q_i\}_{i\in J_\eta}$ of cubes of size η with disjoint interiors such that:

- $$\bigcup_{i\in I_\eta} Q_i \subset O \subset \bigcup_{i\in J_\eta} \overline{Q_i}; \tag{4.50}$$
- $$\mathscr{L}^d(O\setminus \bigcup_{i\in I_\eta} Q_i) = 0; \tag{4.51}$$
- $$\lim_{\eta\to 0} \mathscr{L}^d(\bigcup_{i\in J_\eta\setminus I_\eta} Q_i) = 0. \tag{4.52}$$

Fix any $\eta > 0$, any $\varepsilon > 0$ and any $i \in J_\eta$. Let $u^\eta_{i,\varepsilon} \in L^2_{\text{loc},\theta,R_J,\frac{1}{\varepsilon}Q_i}(\mathbb{R}^d)$ be such that

$$\mathscr{J}(\omega,u^\eta_{i,\varepsilon},\frac{1}{\varepsilon}Q_i,\frac{1}{\varepsilon}Q_i) = \mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta). \tag{4.53}$$

By change of scale and function, i.e. $(x', y') = (\frac{x}{\varepsilon}, \frac{y}{\varepsilon})$ and $\widehat{u}^{\eta}_{i,\varepsilon}(x') = \frac{1}{\varepsilon} u^{\eta}_{i,\varepsilon}(\varepsilon x')$, we have $\widehat{u}^{\eta}_{i,\varepsilon} \in L^2_{\mathrm{loc},\theta,\varepsilon R_J,Q_i}(\mathbb{R}^d)$ and, by (4.53),

$$\mathscr{J}_\varepsilon(\omega, \widehat{u}^{\eta}_{i,\varepsilon}, Q_i, Q_i) = \mathscr{L}^d(Q_i)\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta)}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_i)}. \tag{4.54}$$

Let $u^{\eta}_{\varepsilon} \in L^2_{\mathrm{loc}}(\mathbb{R}^d)$ be defined by

$$u^{\eta}_{\varepsilon}(x) := \begin{cases} \widehat{u}^{\eta}_{i,\varepsilon}(x) & \text{if } x \in Q_i \text{ with } i \in J_\eta \\ \ell_\theta & \text{otherwise.} \end{cases}$$

From (4.54) we see that

$$\mathscr{J}_\varepsilon(\omega, u^{\eta}_{\varepsilon}) \le \sum_{i\in J_\eta} \mathscr{L}^d(Q_i)\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta)}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_i)} + R_{\varepsilon,\eta} \tag{4.55}$$

with

$$R_{\varepsilon,\eta} := \sum_{J_\eta \ni i \ne j \in J_\eta} \int_{Q_i}\int_{Q_j} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)\left(\frac{\widehat{u}^{\eta}_{i,\varepsilon}(x) - \widehat{u}^{\eta}_{j,\varepsilon}(x)}{\varepsilon}\right)^2 dxdy.$$

On the other hand, by using PNL_2 we have

$$R_{\varepsilon,\eta} \le \sum_{J_\eta \ni i \ne j \in J_\eta} \int_{(Q_i\times Q_j)\cap[|x-y|\le\varepsilon R_J]} \overline{J}\left(\frac{x-y}{\varepsilon}\right)\left(\frac{\widehat{u}^{\eta}_{i,\varepsilon}(x) - \widehat{u}^{\eta}_{j,\varepsilon}(x)}{\varepsilon}\right)^2 dxdy,$$

and noticing that if $i \ne j$ and $|x-y| \le \varepsilon R_J$ then:

- $x, y \in \partial_{\varepsilon R_j}(Q_i)$;
- $\dfrac{x-y}{\varepsilon} \in \overline{B}_{R_j}(0)$;
- $\dfrac{\widehat{u}^{\eta}_{i,\varepsilon}(x) - \widehat{u}^{\eta}_{j,\varepsilon}(x)}{\varepsilon} = \dfrac{\theta(x-y)}{\varepsilon}$,

we deduce that

$$\begin{aligned} R_{\varepsilon,\eta} &\le \sum_{i\in J_\eta} \theta^2 \int_{\partial_{\varepsilon R_j}(Q_i)} dx \sum_{i\ne j\in J_\eta} \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi)d\xi \\ &\le \mathscr{L}^d(\partial_{\varepsilon R_j}(]0,\eta[^d))(\theta \mathrm{card}(J_\eta))^2 \int_{\mathbb{R}^d} |\xi|^2\overline{J}(\xi)d\xi. \end{aligned} \tag{4.56}$$

Since $\mathscr{L}^d(\partial_{\varepsilon R_j}(]0,\eta[^d)) = o(1)$ as $\varepsilon \to 0$, from (4.54) it follows that

$$\lim_{\varepsilon\to 0} R_{\varepsilon,\eta} = 0 \text{ for all } \eta > 0. \tag{4.57}$$

From (4.55), (4.57), Remark 3.15 and Proposition 3.17 we deduce that for every $\eta > 0$,

$$\overline{\lim_{\varepsilon\to 0}}\, \mathscr{J}_\varepsilon(\omega, u_\varepsilon^\eta) \le \sum_{i\in J_\eta} \mathscr{L}^d(Q_i) f_{\rm hom}(\omega,\theta) = f_{\rm hom}(\omega,\theta)\mathscr{L}^d(\underset{i\in J_\eta}{\cup} \overline{Q_i}),$$

hence, by using (4.50) and (4.51),

$$\begin{aligned}
\overline{\lim_{\varepsilon\to 0}}\, \mathscr{J}_\varepsilon(\omega, u_\varepsilon^\eta) &\le \left[\mathscr{L}^d(\underset{i\in I_\eta}{\cup} Q_i) + \mathscr{L}^d(\underset{i\in J_\eta}{\cup} \overline{Q_i}\setminus \underset{i\in I_\eta}{\cup} Q_i)\right] f_{\rm hom}(\omega,\theta)\\
&= \left[\mathscr{L}^d(\underset{i\in I_\eta}{\cup} Q_i) + \mathscr{L}^d(\underset{i\in I_\eta}{\cup} \partial Q_i) + \mathscr{L}^d(\underset{i\in J_\eta\setminus I_\eta}{\cup} \overline{Q_i})\right] f_{\rm hom}(\omega,\theta)\\
&= \left[\mathscr{L}^d(\underset{i\in I_\eta}{\cup} Q_i) + \mathscr{L}^d(\underset{i\in J_\eta\setminus I_\eta}{\cup} Q_i)\right] f_{\rm hom}(\omega,\theta)\\
&= \left[\mathscr{L}^d(O) + \mathscr{L}^d(\underset{i\in J_\eta\setminus I_\eta}{\cup} Q_i)\right] f_{\rm hom}(\omega,\theta).
\end{aligned}$$

Consequently, letting $\eta \to 0$ and using (4.52), we obtain

$$\overline{\lim_{\eta\to 0}}\,\overline{\lim_{\varepsilon\to 0}}\, \mathscr{J}_\varepsilon(\omega, u_\varepsilon^\eta) \le \mathscr{L}^d(O) f_{\rm hom}(\omega,\theta). \tag{4.58}$$

We are going to establish that

$$\overline{\lim_{\eta\to 0}}\,\overline{\lim_{\varepsilon\to 0}} \int_O |u_\varepsilon^\eta - \ell_\theta|^2 dx = 0. \tag{4.59}$$

Applying Lemma 4.4 with $R = \varepsilon R_j$, $Q = Q_i$ (whose size is $\eta > 0$) and $u = (u_\varepsilon^\eta - \ell_\theta)\mathbb{1}_{Q_i}$ (verifying $u \in L^2_{\rm loc}(\mathbb{R}^d)$, $u = 0$ in $\partial_{\varepsilon R_J}(Q_i)$ and $u = 0$ in $\mathbb{R}^d\setminus Q_i$), there exists $C > 0$ such that for each $\eta > 0$, each $\varepsilon > 0$ and each $i \in I_\eta$,

$$\begin{aligned}
\int_{Q_i} |u_\varepsilon^\eta - \ell_\theta|^2 dx &\le \frac{C\eta}{R_J^{d+2}\varepsilon^d} \int_{Q_i}\int_{\left[|x-y|\frac{\varepsilon R_J}{2}\right]\cap Q_i} \left(\frac{\left(u_\varepsilon^\eta - \ell_\theta\right)(x) - \left(u_\varepsilon^\eta - \ell_\theta\right)(y)}{\varepsilon}\right)^2 dxdy\\
&\le \frac{2C\eta}{R_J^{d+2}\varepsilon^d} \int_{Q_i}\int_{\left[|x-y|\le\frac{\varepsilon R_J}{2}\right]\cap Q_i} \left(\frac{u_\varepsilon^\eta(x) - u_\varepsilon^\eta(y)}{\varepsilon}\right)^2 dxdy\\
&\quad + \frac{2C\eta\theta^2}{R_J^{d+2}\varepsilon^d} \int_{Q_i}\int_{\left[|x-y|\le\frac{\varepsilon R_J}{2}\right]\cap Q_i} \left(\frac{x-y}{\varepsilon}\right)^2 dxdy.
\end{aligned} \tag{4.60}$$

Taking Remark 3.9, PNL_2 and (4.54) into account, we see that

$$\begin{aligned}
&\int_{Q_i}\int_{\left[|x-y|\le\frac{\varepsilon R_J}{2}\right]\cap Q_i}\left(\frac{u_\varepsilon^\eta(x)-u_\varepsilon^\eta(y)}{\varepsilon}\right)^2dxdy\\
&\le\frac{1}{\underline{J}\left(\frac{R_J}{2}\right)}\int_{Q_i}\int_{Q_i}\underline{J}\left(\frac{x-y}{\varepsilon}\right)\left(\frac{u_\varepsilon^\eta(x)-u_\varepsilon^\eta(y)}{\varepsilon}\right)^2dxdy\\
&\le\frac{4\varepsilon^d}{\underline{J}\left(\frac{R_J}{2}\right)}\mathscr{J}_\varepsilon\left(\omega,\widehat{u}_{i,\varepsilon}^\eta,Q_i,Q_i\right)\\
&=\frac{4\varepsilon^d}{\underline{J}\left(\frac{R_J}{2}\right)}\mathscr{L}^d(Q_i)\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta)}{\mathscr{L}^d\left(\frac{1}{\varepsilon}Q_i\right)}.
\end{aligned}\tag{4.61}$$

Moreover, we have

$$\begin{aligned}
\int_{Q_i}\int_{\left[|x-y|\le\frac{\varepsilon R_J}{2}\right]\cap Q_i}\left(\frac{x-y}{\varepsilon}\right)^2dxdy&=\varepsilon^d\int_{Q_i}\int_{\left[|\xi|\le\frac{R_J}{2}\right]\cap Q_i}|\xi|^2d\xi\\
&\le\mathscr{L}^d(Q_i)\frac{\varepsilon^dR_J^{d+2}}{4}.
\end{aligned}\tag{4.62}$$

From (4.60), (4.61) and (4.62) we deduce that for every $\eta>0$, every $\varepsilon>0$ and every $i\in I_\eta$,

$$\int_{Q_i}|u_\varepsilon^\eta-\ell_\theta|^2dx\le C'\eta\mathscr{L}^d(Q_i)\left(\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta)}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_i)}+1\right)$$

with $C':=C\max\left\{\frac{8}{R_J^{d+2}\underline{J}(\frac{R_J}{2})},\frac{\theta^2}{2}\right\}$. From (4.51) it follows that

$$\int_O|u_\varepsilon^\eta-\ell_\theta|^2dx\le C'\eta\sum_{i\in I_\eta}\mathscr{L}^d(Q_i)\left(\frac{\mathfrak{S}_{\frac{1}{\varepsilon}Q_i}(\omega,\theta)}{\mathscr{L}^d(\frac{1}{\varepsilon}Q_i)}+1\right)$$

for all $\eta>0$ and $\varepsilon>0$. Letting $\varepsilon\to0$ and using Proposition 4.56 [(and again (4.51)] we see that

$$\overline{\lim_{\varepsilon\to0}}\int_O|u_\varepsilon^\eta-\ell_\theta|^2dx\le C'\eta\sum_{i\in I_\eta}\mathscr{L}^d(Q_i)(f_{\mathrm{hom}}(\omega,\theta)+1)=C'\eta\mathscr{L}^d(O)(f_{\mathrm{hom}}(\omega,\theta)+1)$$

for all $\eta > 0$, and (4.59) follows. According to (4.58) and (4.59), by diagonalization there exists a mapping $\varepsilon \mapsto \eta(\varepsilon)$, with $\eta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that

$$\begin{cases} \varlimsup_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon^{\eta(\varepsilon)}) \le \mathscr{L}^d(O) f_{\text{hom}}(\omega, \theta) \\ u_\varepsilon^{\eta(\varepsilon)} \to \ell_\theta \text{ in } L^2(O), \end{cases}$$

which gives (4.49) with $u_\varepsilon := u_\varepsilon^{\eta(\varepsilon)}$.
Proof of (4.46). As in the proof of (4.45) it is equivalent to prove that for every $u \in L^2(O)$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to u \text{ in } L^2(O) \\ \mathscr{J}^g_{\text{hom}}(\omega, u) \ge \varlimsup_{\varepsilon\to 0} \mathscr{J}^g_\varepsilon(\omega, u_\varepsilon). \end{cases}$$

Let $u \in L^2(O)$. By definition of $\mathscr{J}_{\text{hom}}(\omega, \cdot)$ in (3.22), without loss of generality we can assume that $u \in H^1_g(O)$, and so we have to prove that there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to u \text{ in } L^2(O) \\ \displaystyle\int_O f_{\text{hom}}(\omega, \nabla u(x))dx \ge \varlimsup_{\varepsilon\to 0} \mathscr{J}^g_\varepsilon(\omega, u_\varepsilon). \end{cases} \tag{4.63}$$

By (4.45) there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that

$$\begin{cases} u_\varepsilon \to u \text{ in } L^2(O) \\ \displaystyle\int_O f_{\text{hom}}(\omega, \nabla u(x))dx \ge \varlimsup_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u_\varepsilon). \end{cases} \tag{4.64}$$

Fix any $\delta > 0$. From Lemma 4.7 (that we apply with $U = O$) there exists $\{u^\delta_\varepsilon\}_{\varepsilon>0} \subset L^2(O)$ such that:

- $\begin{cases} u^\delta_\varepsilon = u \text{ in } O\backslash O_\delta \\ u^\delta_\varepsilon = u_\varepsilon \text{ in } O_{2\delta} \end{cases}$ (4.65)
- $u^\delta_\varepsilon \to u$ in $L^2(O)$; (4.66)
- $\varlimsup_{\varepsilon\to 0}(\mathscr{J}_\varepsilon(\omega, u^\delta_\varepsilon) - \mathscr{J}_\varepsilon(\omega, u_\varepsilon)) \le o(1)$ as $\delta \to 0$. (4.67)

By (4.67) and the inequality in (4.64) we see that

$$\varlimsup_{\varepsilon\to 0} \mathscr{J}_\varepsilon(\omega, u^\delta_\varepsilon) \le \int_O f_{\text{hom}}(\omega, \nabla u(x))dx + o(1) \text{ as } \delta \to 0. \tag{4.68}$$

Fix any $\varepsilon \in \left]0, \frac{\delta}{R_j}\right[$. Then, taking (4.65) into account, $u_\varepsilon^\delta = u$ in $O\backslash O_{\varepsilon R_j}$ and, noticing that $J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right) = 0$ if $|x-y| > \varepsilon R_j$, we see that

$$\mathscr{D}_g^\varepsilon(\omega, u_\varepsilon^\delta) = 2\mathscr{J}_\varepsilon(\omega, u^g, O\backslash O_{\varepsilon R_J}, O^J\backslash\overline{O}), \tag{4.69}$$

where $u^g \in H^1(O^J\backslash\overline{O}_{2\delta})$ is defined by

$$u^g(x) := \begin{cases} u(x) & \text{if } x \in O\backslash O_{2\delta} \\ g(x) & \text{if } x \in O^J\backslash\overline{O}. \end{cases}$$

(Note that $u^g \in H^1(O^J\backslash\overline{O}_{2\delta})$ because $u \in H_g^1(O)$.) On the other hand, as $\delta > \varepsilon R_j$ we have $O\backslash O_{\varepsilon R_J} \subset O\backslash\overline{O}_\delta$, hence

$$\mathscr{J}_\varepsilon(\omega, u^g, O\backslash O_{\varepsilon R_J}, O^J\backslash\overline{O}) \leq \mathscr{J}_\varepsilon(\omega, u^g, O\backslash\overline{O}_\delta, O^J\backslash\overline{O}).$$

Moreover, it is easy to see that $O\backslash\overline{O}_\delta + B_\delta(0) \subset O^J\backslash\overline{O}_{2\delta}$ so that $u \in H^1(O\backslash\overline{O}_\delta + B_\delta(0))$. Consequently, taking PNL_2 into account, by Lemma 4.5(b) (that we apply with $A = O\backslash\overline{O}_\delta$) it follows that for every $\varepsilon \in]0, \frac{\delta}{R_J}[$,

$$\begin{aligned}\mathscr{J}_\varepsilon(\omega, u^g, O\backslash O_{\varepsilon R_J}, O^J\backslash\overline{O}) &\leq \int_{O\backslash\overline{O}_\delta+B_\delta(0)} |\nabla u^g(x)|^2 dx \int_{\mathbb{R}^d} |\xi|^2 \overline{J}(\xi) d\xi \\ &= o(1) \text{ as } \delta \to 0.\end{aligned}$$

Hence, by using (4.69),

$$\overline{\lim_{\varepsilon\to 0}}\, \mathscr{D}_g^\varepsilon(\omega, u_\varepsilon^\delta) \leq o(1) \text{ as } \delta \to 0. \tag{4.70}$$

From (4.66) and (4.68) together with (4.70) we deduce that

$$\begin{cases} \lim\limits_{\delta\to 0}\lim\limits_{\varepsilon\to 0} \|u_\varepsilon^\delta - u\|_{L^2(O)} = 0 \\ \overline{\lim\limits_{\delta\to 0}}\,\overline{\lim\limits_{\varepsilon\to 0}}\, \mathscr{J}_\varepsilon^g(\omega, u_\varepsilon^\delta) \leq \int_O f_{\text{hom}}(\omega, \nabla u(x)) dx. \end{cases} \tag{4.71}$$

From (4.71), by diagonalization, there a mapping $\varepsilon \mapsto \delta(\varepsilon)$, with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, such that...

$$\begin{cases} u_\varepsilon^{\delta(\varepsilon)} \to u \text{ in } L^2(O) \\ \overline{\lim\limits_{\varepsilon\to 0}}\, \mathscr{J}_\varepsilon^g(\omega, u_\varepsilon^{\delta(\varepsilon)}) \leq \int_O f_{\text{hom}}(\omega, \nabla u(x)) dx, \end{cases}$$

which gives (4.63) with $u_\varepsilon = u_\varepsilon^{\delta(\varepsilon)}$.

Step 3: End of the proof. From (4.27) and (4.45) (resp. (4.28) and (4.46)) we deduce that the Γ-convergence of $\{\mathscr{J}_\varepsilon(\omega, \cdot)\}_{\varepsilon>0}$ (resp. $\{\mathscr{J}_\varepsilon^g(\omega, \cdot)\}_{\varepsilon>0}$) to $\mathscr{J}_{\text{hom}}(\omega, \cdot)$ (resp.

$\mathscr{J}^g_{\rm hom}(\omega,\cdot)$) with respect to the strong convergence in $L^2(O)$, which finishes the proof. □

4.4 Proof of Theorem 3.19

Let $\Omega'' \in \mathscr{F}$ be such that $\mathbb{P}(\Omega'') = 1$ and (H^ω_1)–(H^ω_4) (in Theorem 3.19) hold. Set $\widehat{\Omega} = \Omega' \cap \Omega''$ where $\Omega' \in \mathscr{F}$, with $\mathbb{P}(\Omega') = 1$, is given by Proposition 3.17 and Theorem 4.8. Then $\widehat{\Omega} \in \mathscr{F}$ and $\mathbb{P}(\widehat{\Omega}) = 1$. Fix $\omega \in \widehat{\Omega}$. We are going to apply Theorem 4.1.

Firstly, it is easy to see that (C_1)–(C_5) hold with $u_{0,\varepsilon} = u^\omega_{0,\varepsilon}$, $u_0 = u^\omega_0$, $u_\varepsilon = u^\omega_\varepsilon$, $\underline{z}_\varepsilon = \underline{y}^\omega_\varepsilon$, $\overline{z}_\varepsilon = \overline{y}^\omega_\varepsilon$, $\underline{a}_\varepsilon = \underline{\rho}^\omega_\varepsilon$ and $\underline{a} = \underline{\rho}^\omega$, $\overline{a}_\varepsilon = \overline{\rho}^\omega_\varepsilon$ and $\overline{a} = \overline{\rho}^\omega$, $F_\varepsilon = F_\varepsilon(\omega,\cdot,\cdot)$, $G = G^\omega$ and $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}^g_\varepsilon(\omega,\cdot)$). Note that ($C_4$) is verified with $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}^g_\varepsilon(\omega,\cdot)$) by using Lemma 4.2 (resp. Lemma 4.3). Secondly, by Theorem 4.8, (C_6) is satisfied with $\mathscr{E}_\varepsilon = \mathscr{J}_\varepsilon(\omega,\cdot)$ and $\mathscr{E}_0 = \mathscr{J}_{\rm hom}(\omega,\cdot)$ (resp. $\mathscr{E}_\varepsilon = \mathscr{J}^g_\varepsilon(\omega,\cdot)$ and $\mathscr{E}_0 = \mathscr{J}^g_{\rm hom}(\omega,\cdot)$), and the conclusion of Theorem 3.19 follows by applying Theorem 4.1 and noticing that $\partial\mathscr{J}_{\rm hom}(\omega,\cdot) = \{\nabla\mathscr{J}_{\rm hom}(\omega,\cdot)\}$ (resp. $\partial\mathscr{J}^g_{\rm hom}(\omega,\cdot) = \{\nabla\mathscr{J}^g_{\rm hom}(\omega,\cdot)\}$). □

5 Application to spatial population dynamics

Here we apply Theorem 3.19 to a model coming from spatial population dynamics.

5.1 Heuristic derivation of the model

Let $T > 0$ and let $O \subset \mathbb{R}^d$ (with $d = 1, 2$ or 3) be a bounded open domain with Lipschitz boundary. The state of the population is represented by its density $u(t,x)$ at time $t \in [0,T]$ and located at $x \in O$. Although, for each $x \in O$, $u(\cdot,x)$ is intrinsically discrete, as the population is assumed to be very large $u(\cdot,x)$ is considered as a real function, i.e.

$$u : [0,T] \times O \to \mathbb{R}.$$

To precise the model we need to specify what the population flux is and how the population growth is regulated.

We assume that the environment in which the population evolves is randomly heterogeneous and we denote the density of population by u^ω_ε where $\varepsilon > 0$ represents the (small) size of the heterogeneities of the environment and $\omega \in \Omega$ its randomness with $(\Omega, \mathscr{F}, \mathbb{P})$ a suitable complete probability space.

The population flux at (t,x) is given by

$$\begin{aligned}\mathfrak{F}^\omega_\varepsilon\left(u^\omega_\varepsilon(t,x)\right) &= \frac{1}{4\varepsilon^{d+2}}\int_O J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)\left(u^\omega_\varepsilon(t,y) - u^\omega_\varepsilon(t,x)\right)dy\\ &\quad+\frac{1}{2\varepsilon^{d+2}}\int_{O^J\setminus O} J\left(\omega,\frac{x}{\varepsilon},\frac{y}{\varepsilon},\frac{x-y}{\varepsilon}\right)\left(g(y) - u^\omega_\varepsilon(t,x)\right)dy \qquad (5.1)\end{aligned}$$

where $J : \Omega\times\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}^d \to [0,\infty[$ satisfies PNL_1–PNL_2 and $O^J := O+\mathrm{supp}(\overline{J})$ with $\overline{J}$ given by PNL_2. Roughly, the first (resp. second) term in (5.1) accounts for the number of individuals at time t in O (resp. outside O, i.e. in $O^J\setminus O$) which jump from y to x. Note that the scaling $\frac{1}{\varepsilon^{d+2}}$ together with the scaling $\frac{1}{\varepsilon}$ with respect to third variable of J is introduced to provide a local limit model of divergence form as $\varepsilon \to 0$.

The regulation of the population growth at (t,x) is governed by

$$\mathfrak{R}^\omega_\varepsilon(t,u^\omega_\varepsilon(t,x)) = f\left(\omega,t,\frac{x}{\varepsilon},u^\omega_\varepsilon(t,x)\right),$$

where $f : \Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is the density of a CP-structured reaction functional (see Definition 2.9).

Let $D\subset O$ be an arbitrary domain. The time rate of change of the number of individuals in D is equal to the rate that the population is grown in D plus the rate that the population flows in D, i.e. the balance law for u^ω_ε is given by

$$\frac{d}{dt}\int_D u^\omega_\varepsilon(t,x)dx = \int_D \mathfrak{R}^\omega_\varepsilon(t,u^\omega_\varepsilon(t,x))dx + \int_D \mathfrak{F}^\omega_\varepsilon(u^\omega_\varepsilon(t,x))dx.$$

Hence, assuming that u^ω_ε is sufficiently regular,

$$\int_D \frac{\partial u^\omega_\varepsilon}{\partial t}(t,x)dx - \int_D \mathfrak{F}^\omega_\varepsilon(u^\omega_\varepsilon(t,x))dx = \int_D \mathfrak{R}^\omega_\varepsilon(t,u^\omega_\varepsilon(t,x))dx.$$

Then, the arbitrariness of D implies the differential form of the balance law:

$$\frac{\partial u^\omega_\varepsilon}{\partial t}(t,x) - \mathfrak{F}^\omega_\varepsilon(u^\omega_\varepsilon(t,x)) = \mathfrak{R}^\omega_\varepsilon(t,u^\omega_\varepsilon(t,x)) \text{ for } \mathscr{L}^1\otimes\mathscr{L}^d\text{-a.a. } (t,x)\in[0,T]\times O. \tag{5.2}$$

Noticing that $\nabla\mathscr{J}^g_\varepsilon(\omega,u^\omega_\varepsilon(t))(x) = -\mathfrak{F}^\omega_\varepsilon(u^\omega_\varepsilon(t,x))$ with $\mathscr{J}^g_\varepsilon := \mathscr{J}_\varepsilon + \mathscr{D}^\varepsilon_g$, where $\mathscr{J}_\varepsilon : \Omega\times L^2(O)\to[0,\infty[$ and $\mathscr{D}^\varepsilon_g : \Omega\times L^2(O)\to[0,\infty[$ are defined by (3.2) and (3.5) respectively, and setting $F_\varepsilon(\omega,t,u^\omega_\varepsilon(t))(x) = \mathfrak{R}^\omega_\varepsilon(t,u^\omega_\varepsilon(t,x))$ with $F_\varepsilon : \Omega\times[0,T]\times L^2(O)\to L^2(O)$, we see that (5.2) can be rewritten as follows:

$$\frac{du^\omega_\varepsilon}{dt}(t) + \nabla\mathscr{J}^g_\varepsilon(\omega,u^\omega_\varepsilon(t)) = F_\varepsilon(\omega,t,u^\omega_\varepsilon(t)) \text{ for } \mathscr{L}^1\text{-a.a. } t\in[0,T],$$

which gives $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$ in Sect. 3.2 by adding a suitable initial condition.

5.2 Mathematical description of the model

In what follows we consider the logistic model with a growth rate whose environmental carrying capacity depending on time and in which a percentage of the population density is subtracted (reflecting a reduction of the population due to hunting or capturing individuals). More precisely, for each $\varepsilon>0$, $F_\varepsilon : \Omega\times[0,T]\times L^2(O)\to L^2(O)$

is given by

$$F_\varepsilon(\omega, t, u)(x) := f(\omega, t, \frac{x}{\varepsilon}, u(x)), \tag{5.3}$$

where $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f(\omega, t, x, \xi) := r(\omega, t, x)\xi \left(1 - \frac{\xi}{K(\omega, t, x)}\right) - h\xi, \tag{5.4}$$

with $h \geq 0$ and $r, K \in L^\infty(\Omega \times [0, T] \times \mathbb{R}^d)$ such that $r > 0$ and $K \geq \gamma > 0$, where r is the growth rate, K is the carrying capacity and h the percentage of harversting. (In practice, the challenge is to evaluate reasonable values, or at least to have a good statistical knowledge, for the growth rate r and the carrying capacity K in heterogeneous environments.)

Remark 5.1 It is easy to see that $f : \Omega \times [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ defined in (5.4) can be rewritten as follows:

$$f(\omega, t, x, \xi) = \langle a(\omega, t, x), b(\xi)\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product, with $a : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^3$ and $b : \mathbb{R} \to \mathbb{R}^3$ given by

$$\begin{cases} a(\omega, t, x) := (r(\omega, t, x), -\frac{r(\omega,t,x)}{K(\omega,t,x)}, -h) \\ b(\xi) := (\xi, \xi^2, \xi). \end{cases} \tag{5.5}$$

Thus, for every $\omega \in \Omega$ and every $\varepsilon > 0$, $F_\varepsilon(\omega, \cdot, \cdot)$ satisfies the special structure of CP-structured reaction functionals as introduced in [4, Definition 2.1, p. 27]. This special structure[2] allows to pass to the weak limit in the reaction term (see Lemma 5.6).

In what follows, we consider $\underline{r}, \overline{r}, \underline{K}, \overline{K} \in [0, \infty[$ given by:

- $\underline{r} := \operatorname*{ess\,inf}_{(\omega,t,x)} r(\omega, t, x)$;
- $\overline{r} := \operatorname*{ess\,sup}_{(\omega,t,x)} r(\omega, t, x)$;
- $\underline{K} := \operatorname*{ess\,inf}_{(\omega,t,x)} K(\omega, t, x)$;
- $\overline{K} := \operatorname*{ess\,sup}_{(\omega,t,x)} K(\omega, t, x)$,

[2] By the class of special CP-structured reaction functionals we mean the subclass of $\mathscr{F}_{\mathrm{CP}}$ (see Definition 2.9) for which $f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ given by (CP_3) is of the form: $f(t, x, \xi) = \langle a(t, x), b(\xi)\rangle + c(t, x)$ with $a \in L^\infty([0, T] \times \mathbb{R}^d; \mathbb{R}^m)$, $c \in L^2([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^d))$ and $b : \mathbb{R} \to \mathbb{R}^m$ locally Lipschitz continuous, where $m \in \mathbb{N}^*$.

and we assume that

$$\begin{cases} g \geq 0 \\ \underset{x\in\overline{O\setminus O_{R_J}}}{\operatorname{ess\,inf}} \frac{\int_{O^J\setminus\overline{O}} \underline{J}(x-y)g(y)dy}{\int_{O^J\setminus\overline{O}} \overline{J}(x-y)dy} \geq 0 \\ \underset{x\in\overline{O\setminus O_{R_J}}}{\operatorname{ess\,sup}} \frac{\int_{O^J\setminus\overline{O}} \overline{J}(x-y)g(y)dy}{\int_{O^J\setminus\overline{O}} \underline{J}(x-y)dy} < \infty \\ \overline{r} > h. \end{cases} \tag{5.6}$$

Lemma 5.2 *Every $F_\varepsilon(\omega,\cdot,\cdot)$ satisfies (CP_1)–(CP_3) with $f(\omega,\cdot,\frac{\cdot}{\varepsilon},\cdot)$, where $f:\Omega\times[0,T]\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ is given by (5.4), and so $F_\varepsilon(\omega,\cdot,\cdot)\in\mathscr{F}_{\rm CP}$ for all $\omega\in\Omega$ and all $\varepsilon>0$. More precisely, $(\underline{\rho}^\omega_\varepsilon,\underline{f}^\omega_\varepsilon,\underline{y}^\omega_\varepsilon)=(0,0,0)$ and $(\overline{\rho}^\omega_\varepsilon,\overline{f}^\omega_\varepsilon,\overline{y}^\omega_\varepsilon)=(\overline{\rho},\overline{f},\overline{y})$ does not depend on (ω,ε). Moreover, $(\underline{\rho}^\omega_\varepsilon,\overline{\rho}^\omega_\varepsilon)=(0,\overline{\rho})$ verifies (3.4) and, since $(\underline{y}^\omega_\varepsilon,\overline{y}^\omega_\varepsilon)=(0,\overline{y})$ does not depend on ε, it is clear that (3.3) holds.*

Proof of Lemma 5.2 Fix $\omega\in\Omega$ and $\varepsilon>0$. It is clear that we can take $(\underline{\rho}^\omega_\varepsilon,\underline{f}^\omega_\varepsilon,\underline{y}^\omega_\varepsilon)=(0,0,0)$. Moreover, from (5.6) we see that (3.4) is satisfied. To find a suitable triple $(\overline{\rho}^\omega_\varepsilon,\overline{f}^\omega_\varepsilon,\overline{y}^\omega_\varepsilon)$ we need to consider $\mu\in\mathbb{R}$ given by

$$\mu := \nu - \underset{x\in\overline{O\setminus O_{R_J}}}{\operatorname{ess\,sup}} \frac{\int_{O^J\setminus\overline{O}} \overline{J}(x-y)g(y)dy}{\int_{O^J\setminus\overline{O}} \underline{J}(x-y)dy}$$

with $\nu := (\overline{r}-h)\frac{K}{\underline{r}}$. If $\mu\leq 0$ then we can take

$$\overline{\rho}^\omega_\varepsilon \geq \underset{x\in\overline{O\setminus O_{R_J}}}{\operatorname{ess\,sup}} \frac{\int_{O^J\setminus\overline{O}} \overline{J}(x-y)g(y)dy}{\int_{O^J\setminus\overline{O}} \underline{J}(x-y)dy},$$

$\overline{f}^\omega_\varepsilon=0$ and $\overline{y}^\omega_\varepsilon=\overline{\rho}^\omega_\varepsilon$. Indeed, $\overline{\rho}^\omega_\varepsilon$ satisfies (3.4) by (5.6) and, since $\overline{\rho}^\omega_\varepsilon\geq\nu$,

$$f(\omega,t,\frac{x}{\varepsilon},\overline{y}^\omega_\varepsilon(t)) = f(\omega,t,\frac{x}{\varepsilon},\overline{\rho}^\omega_\varepsilon) \leq -\frac{\underline{r}}{K}(\overline{\rho}^\omega_\varepsilon)^2+(\overline{r}-h)\overline{\rho}^\omega_\varepsilon \leq 0 = \overline{f}^\omega_\varepsilon(t,\overline{y}^\omega_\varepsilon(t)).$$

If $\mu>0$ then we consider $\overline{\rho}^\omega_\varepsilon$ such that

$$\underset{x\in\overline{O\setminus O_{R_J}}}{\operatorname{ess\,sup}} \frac{\int_{O^J\setminus\overline{O}} \overline{J}(x-y)g(y)dy}{\int_{O^J\setminus\overline{O}} \underline{J}(x-y)dy} \leq \overline{\rho}^\omega_\varepsilon \leq \nu$$

and we set $\overline{f}^\omega_\varepsilon(t,\xi) := -\frac{\underline{r}}{K}\xi^2+(\overline{r}-h)\xi$. Then, $\overline{\rho}^\omega_\varepsilon$ satisfies (3.4) by (5.6), and by a standard calculation we see that

$$\overline{y}^\omega_\varepsilon(t) := \frac{1}{(\frac{1}{\overline{\rho}^\omega_\varepsilon}-\frac{1}{\nu})e^{-t(\overline{r}-h)}+\frac{1}{\nu}}$$

solves ($\overline{ODE}$) in Definition 2.9 with $\overline{\rho} = \overline{\rho}^{\omega}_{\varepsilon}$ and $\overline{f} = \overline{f}^{\omega}_{\varepsilon}$. Moreover, $\overline{y}^{\omega}_{\varepsilon}(t) \geq 0$ for all $t \in [0, T]$ because $\overline{\rho}^{\omega}_{\varepsilon} \leq \nu$, and $f(\omega, t, \frac{x}{\varepsilon}, \overline{y}^{\omega}_{\varepsilon}(t)) \leq \overline{f}^{\omega}_{\varepsilon}(t, \overline{y}^{\omega}_{\varepsilon}(t))$ for all $t \in [0, T]$ and all $x \in \mathbb{R}^d$, which completes the proof. □

Given $\{u^{\omega}_{0,\varepsilon}\}_{\varepsilon>0} \subset L^2(O)$ we consider the Dirichlet–Cauchy nonlocal reaction–diffusion problem of gradient flow type:

$$(\mathscr{P}^{\mathrm{DL}}_{\varepsilon,\omega}) \begin{cases} \dfrac{du^{\omega}_{\varepsilon}}{dt}(t) + \nabla \mathscr{J}^{g}_{\varepsilon}(\omega, u^{\omega}_{\varepsilon}(t)) = F_{\varepsilon}(\omega, t, u^{\omega}_{\varepsilon}(t)) \quad \text{for } \mathscr{L}^1\text{-a.a. } t \in [0, T] \\ u^{\omega}_{\varepsilon}(0) = u^{\omega}_{0,\varepsilon} \in L^2(O; [0, \overline{\rho}]). \end{cases}$$

This problem, which corresponds to the problem $(\mathscr{P}^{\mathrm{D}}_{\varepsilon,\omega})$ in Sect. 3.2 with $F_{\varepsilon} : \Omega \times [0, T] \times L^2(O) \to L^2(O)$ defined by (5.3)–(5.4), is called "*Dirichlet–Cauchy nonlocal reaction–diffusion Logistic growth problem*" and can be rewritten as follows:

$$(\mathscr{P}^{\mathrm{DL}}_{\varepsilon,\omega}) \begin{cases} \dfrac{\partial u^{\omega}_{\varepsilon}}{\partial t}(x, t) - \dfrac{1}{4\varepsilon^{d+2}} \displaystyle\int_O J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)(u^{\omega}_{\varepsilon}(t, y) - u^{\omega}_{\varepsilon}(t, x))dy \\ -\dfrac{1}{2\varepsilon^{d+2}} \displaystyle\int_{O^J \setminus O} J\left(\omega, \frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \frac{x-y}{\varepsilon}\right)(g(y) - u^{\omega}_{\varepsilon}(t, x))dy \\ = r\left(\omega, t, \dfrac{x}{\varepsilon}\right) u^{\omega}_{\varepsilon}(t, x)\left(1 - \dfrac{u^{\omega}_{\varepsilon}(t, x)}{K\left(\omega, t, \frac{x}{\varepsilon}\right)}\right) - h u^{\omega}_{\varepsilon}(t, x) \text{ in } O \times [0, T] \\ u^{\omega}_{\varepsilon}(0, \cdot) = u^{\omega}_{0,\varepsilon} \in L^2(O; [0, \overline{\rho}]). \end{cases}$$

Taking (5.6) into account, as a consequence of Lemma 5.2 and corollary 2.11 we obtain the following result.

Corollary 5.3 *For every $\omega \in \Omega$ and every $\varepsilon > 0$, $(\mathscr{P}^{\mathrm{DL}}_{\varepsilon,\omega})$ admits a unique solution $u^{\omega}_{\varepsilon} \in AC([0, T]; L^2(O))$ such that*

$$0 \leq u^{\omega}_{\varepsilon}(t) \leq \overline{y}(t) \leq \overline{y}(T) \text{ for all } t \in [0, T].$$

Moreover, if $F_{\varepsilon}(\omega, \cdot, u^{\omega}_{\varepsilon}) \in AC([0, T]; L^2(O))$ then u^{ω}_{ε} admits a right derivative $\frac{d^+ u^{\omega}_{\varepsilon}}{dt}(t)$ at every $t \in [0, T[$ which satisfies $\frac{d^+ u^{\omega}_{\varepsilon}}{dt}(t) + \nabla \mathscr{J}^{g}_{\varepsilon}(\omega, u^{\omega}_{\varepsilon}(t)) = F_{\varepsilon}(\omega, t, u^{\omega}_{\varepsilon}(t))$.

Remark 5.4 When $r(\omega, \cdot)$ and $\frac{r(\omega,\cdot,\cdot)}{K(\omega,\cdot,\cdot)}$ are sufficiently regular, i.e, H^1 is replaced by $W^{1,1}$ in (A − 2) below, we automatically have $F_{\varepsilon}(\omega, \cdot, u^{\omega}_{\varepsilon}) \in AC([0, T]; L^2(O))$.

Remark 5.5 From (5.3)–(5.4) it is easy to see that $\sup_{\varepsilon>0} \left\| F_{\varepsilon}(\omega, \cdot, u^{\omega}_{\varepsilon}) \right\|_{L^2([0,T]; L^2(O))} < \infty$, i.e. the hypothesis (H^{ω}_3) of Theorem 3.19 is verified.

5.3 The homogenized model

Here, by using Theorem 3.19, we study the almost sure limit of $(\mathscr{P}^{\mathrm{DL}}_{\varepsilon,\omega})$ as $\varepsilon \to 0$ (see Corollary 5.8). To do this we need the following additional assumptions:

(A$_1$) $r(\omega, t, x+z) = r(T_z\omega, t, x)$ and $K(\omega, t, x+z) = K(T_z\omega, t, x)$ for all $z \in \mathbb{Z}^d$, all $t \in [0, \infty[$, all $x \in \mathbb{R}^d$ and all $\omega \in \Omega$;

(A$_2$) $r(\omega, \cdot, \cdot) \in H^1([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^d))$ and $\dfrac{r(\omega, \cdot, \cdot)}{K(\omega, \cdot, \cdot)} \in H^1([0, T]; L^2_{\mathrm{loc}}(\mathbb{R}^d))$ for all $\omega \in \Omega$;

(A$_3$) for every $B \in \mathscr{B}_{\mathrm{b}}(\mathbb{R}^d)$ and every $t \in [0, T]$, the functions $\omega \longmapsto \|r(\omega, t, \cdot)\|^2_{L^2(B)}$, $\omega \longmapsto \left\|\frac{r(\omega,t,\cdot)}{K(\omega,t,\cdot)}\right\|_{L^2(B)}$, $\omega \longmapsto \int_0^T \left\|\frac{dr}{ds}(\omega, s, \cdot)\right\|_{L^2(B)} ds$ and $\omega \longmapsto \int_0^T \left\|\frac{d(\frac{r}{K})}{ds}(\omega, s, \cdot)\right\|_{L^2(B)} ds$ belong to $L^1_{\mathbb{P}}(\Omega)$.

The following Lemma, allows to establish the assumption (H^ω_4) of Theorem 3.19 and gives a formula for the homogenized reaction functional.

Lemma 5.6 *If (A$_1$)–(A$_3$) hold then there exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that for each $\omega \in \Omega'$, (H^ω_4) is satisfied with $G^\omega = F_{\mathrm{hom}}(\omega, \cdot, \cdot) : [0, T] \times L^2(O) \to L^2(O)$ defined by*

$$F_{\mathrm{hom}}(\omega, t, u)(x) := f_{\mathrm{hom}}(\omega, t, u(x)), \tag{5.7}$$

where $f_{\mathrm{hom}}(\omega, \cdot, \cdot) : [0, T] \times \mathbb{R} \to \mathbb{R}$ is given by

$$f_{\mathrm{hom}}(\omega, t, \xi) := \left\langle \mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} a(\cdot, t, y)dy \right)(\omega), b(\xi) \right\rangle$$

with $a : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^3$ and $b : \mathbb{R} \to \mathbb{R}^3$ given by (5.5). More precisely, we have

$$f_{\mathrm{hom}}(\omega, t, \xi) = r_{\mathrm{hom}}(\omega, t)\xi(1 - \frac{\xi}{K_{\mathrm{hom}}(\omega, t)}) - h\xi, \tag{5.8}$$

where $r_{\mathrm{hom}}(\omega, \cdot) : [0, T] \to [0, \infty[$ and $K_{\mathrm{hom}}(\omega, \cdot) : [0, T] \to [0, \infty[$ are defined by

$$\begin{cases} r_{\mathrm{hom}}(\omega, t) := \mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} r(\cdot, t, y)dy \right)(\omega) \\ \\ K_{\mathrm{hom}}(\omega, t) := \dfrac{\mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} r(\cdot, t, y)dy \right)(\omega)}{\mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} \frac{r(\cdot,t,y)}{K(\cdot,t,y)}dy \right)(\omega)}. \end{cases} \tag{5.9}$$

Moreover $F_{\mathrm{hom}}(\omega, \cdot, \cdot) \in \mathscr{F}_{(R_1)\text{-}(R_2)}$ for all $\omega \in \Omega'$.

Proof of Lemma 5.6 By [4, Lemma 7.2, p. 208] there exists $\Omega' \in \mathscr{F}$ with $\mathbb{P}(\Omega') = 1$ such that for every $\omega \in \Omega'$,

$$a(\omega, t, \frac{\cdot}{\varepsilon}) \rightharpoonup \mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} a(\cdot, t, y)dy \right)(\omega) \text{ in } L^2(O; \mathbb{R}^3) \text{ for all } t \in [0, T],$$

hence, arguing as in the proof of [4, Theorem 7.1, pp. 209–210],

$$a(\omega, \cdot, \frac{\cdot}{\varepsilon}) \rightharpoonup \mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} a(\cdot, \cdot, y) dy \right) (\omega) \text{ in } L^2([0,T]; L^2(O; \mathbb{R}^3)).$$

Let $v \in C([0,T]; L^2(O))$ be such that $u_\varepsilon^\omega \to v$. By using similar arguments as in the proof of [4, Lemma 7.2, p. 60] from the above we deduce that

$$\left\langle a(\omega, \cdot, \cdot), b(u_\varepsilon^\omega) \right\rangle \rightharpoonup \left\langle \mathbb{E}^{\mathscr{I}} \left(\int_{]0,1[^d} a(\cdot, \cdot, y) dy \right) (\omega), b(v) \right\rangle \text{ in } L^2([0,T]; L^2(O)),$$

and the proof is complete. □

Remark 5.7 In the formula of the homogenized reaction functional, the homogenized carrying capacity K_{hom} is given by a mixture between carrying capacity and growth rate.

Taking Corollary 5.3, Remark 5.5 and Lemma 5.6 into account, from Theorem 3.19 we deduce the following stochastic homogenization result.

Corollary 5.8 *Let assumptions* (A_1)*–*(A_3) *hold and for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, assume that*:

- $\sup_{\varepsilon > 0} \mathscr{I}_\varepsilon^g(\omega, u_{0,\varepsilon}^\omega) < \infty$;
- *there exists* $u_0^\omega \in L^2(O)$ *such that* $u_{0,\varepsilon}^\omega \rightharpoonup u_0^\omega$ *in* $L^2(O)$.

Then, for $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, there exists* $u^\omega \in AC([0,T]; L^2(O))$ *such that*:

- $u_\varepsilon^\omega \to u^\omega$ *in* $C([0,T]; L^2(O))$;
- $\dfrac{du_\varepsilon^\omega}{dt} \rightharpoonup \dfrac{du^\omega}{dt}$ *in* $L^2([0,T]; L^2(O))$;
- $0 \le u^\omega(t) \le \overline{y}(T)$ *for all* $t \in [0,T]$;
- u^ω *is the unique solution of the following Dirichlet–Cauchy local reaction–diffusion problem of gradient flow type*:

$$\begin{cases} \dfrac{du^\omega}{dt}(t) + \nabla \mathscr{I}_{\text{hom}}^g(\omega, u^\omega(t)) = F_{\text{hom}}(\omega, t, u^\omega(t)) & \text{for } \mathscr{L}^1\text{-a.a. } t \in [0,T] \\ u^\omega(0) = u_0^\omega \in \operatorname{dom}(\mathscr{I}_{\text{hom}}^g(\omega, \cdot)). \\ u_0^\omega \in H_g^1(O) \cap L^2(O; [0, \overline{\rho}]). \end{cases}$$

  with $F_{\text{hom}}(\omega, \cdot, \cdot)$ *given by* (5.7)–(5.9).

Funding This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

Data Availability There is no data.

Declarations

Conflict of interest The authors declare no conflict of interest.

A. Elements of Legendre–Fenchel calculus

Let X be a normed space and let X^* be its topological dual. In what follows, for any $u \in X$ and any $u^* \in X^*$, we write $u^*(u) = \langle u^*, u\rangle$. We begin with the following definition.

Definition A.1 Let $\Phi : X \to]-\infty, \infty]$ be a proper[3] function. The Legendre–Fenchel conjugate (or the conjugate) of Φ is the function $\Phi^* : X^* \to]-\infty, \infty]$ defined by

$$\Phi^*(u^*) := \sup\left\{\langle u^*, u\rangle - \Phi(u) : u \in X\right\}.$$

(As Φ is proper and $\Phi > -\infty$ we have $\Phi^* > -\infty$.) The Legendre–Fenchel biconjugate (or the biconjugate) of Φ is the function $\Phi^{**} : X \to [-\infty, \infty]$ defined by

$$\Phi^{**}(u) := \sup\left\{\langle u^*, u\rangle - \Phi^*(u^*) : u^* \in X^*\right\}.$$

(Since $\Phi^* > -\infty$, $u^* \in \mathrm{dom}(\Phi^*)$ if and only if there exists $\alpha \in \mathbb{R}$ such that $\Phi^*(u^*) \le \alpha$, i.e. $\Phi(u) \ge \langle u^*, u\rangle - \alpha$ for all $u \in X$. Hence, if Φ admits a continuous affine minorant function[4] then Φ^* is proper and $\Phi^{**} > -\infty$.) The following theorem gives the main properties of the Legendre–Fenchel conjugate and biconjugate (see [2, Sect. 9.3, p. 343] for more details).

Theorem A.2 *Let $\Phi : X \to]-\infty, \infty]$ be a proper function.*

(a) *If Φ is convex and lower semicontinuous then Φ^* is proper, convex and lower semicontinuous.*
(b) *(Legendre–Fenchel's inequality.) For every $u \in X$ and every $u^* \in X^*$,*

$$\Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle \ge 0.$$

(c) *(Fenchel–Moreau–Rockafellar's theorem.) If Φ is convex and lower semicontinuous then*

$$\Phi^{**} = \Phi.$$

(d) *If Φ is convex and admits a continuous affine minorant function then*

$$\Phi^{**} = \overline{\Phi},$$

where $\overline{\Phi}$ denotes the lower semicontinuous envelope of Φ.

Here is the definition of the subdifferential of a function.

Definition A.3 Let $\Phi : X \to]-\infty, \infty]$ be a proper function. The subdifferential of Φ is the multivalued operator $\partial\Phi : X \rightrightarrows X^*$ defined by

$$\partial\Phi(u) := \left\{u^* \in X^* : \Phi(v) \ge \Phi(u) + \langle u^*, v - u\rangle \text{ for all } v \in X\right\}.$$

[3] We say that $\Phi : X \to]-\infty, \infty]$ is proper if (its effective domain) $\mathrm{dom}(\Phi) := \{u \in X : \Phi(u) < \infty\} \neq \emptyset$.

[4] This is true if $\Phi : X \to]-\infty, \infty]$ is a proper, convex and lower semicontinuous function, because Φ is then equal to the supremum of all its continuous affine minorant functions.

(Note that $\mathrm{dom}(\Phi) \supset \mathrm{dom}(\partial\Phi) := \{u \in X : \partial\Phi(u) \neq \emptyset\}$.)

For the subdifferentials of convex functions we have the following result (see [2, Sect. 9.5, p. 355 and Lemma 17.4.1, p. 737] for more details).

Proposition A.4 *Let $\Phi : X \to]-\infty, \infty]$ be a proper and convex function.*

(a) *If Φ is Fréchet-differentiable at $u \in X$ then*

$$\partial\Phi(u) = \{\nabla\Phi(u)\}.$$

(b) *(Fenchel's extremality relation.) If Φ is lower semicontinuous then*

$$u^* \in \partial\Phi(u) \iff \Phi(u) + \Phi^*(u^*) - \langle u^*, u\rangle = 0.$$

(c) *(Brønsted–Rockafellar's lemma) If Φ is lower semicontinuous then*

$$\overline{\mathrm{dom}(\partial\Phi)} = \overline{\mathrm{dom}(\Phi)}.$$

B. Mosco-convergence

Let X be a Banach space and let X^* be its topological dual. In what follows, "$\to$" (resp. "$\rightharpoonup$") denotes the strong (resp. the weak) convergence. We begin with the definition of De Giorgi Γ-convergence (see [10, 15, 16] for more details).

Definition B.1 Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ strongly Γ-converges (resp. weakly Γ-converges) to Φ, and we write

$$\Phi = \Gamma_{\mathrm{s}}\text{-}\lim_{\varepsilon\to0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{s}}} \Phi \quad (\text{resp. } \Phi = \Gamma_{\mathrm{w}}\text{-}\lim_{\varepsilon\to0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\Gamma_{\mathrm{w}}} \Phi),$$

if the following two assertions hold:

- for every $u \in X$, $\Gamma_{\mathrm{s}}\text{-}\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u) \geq \Phi(u)$(resp. $\Gamma_{\mathrm{w}}\text{-}\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u) \geq \Phi(u)$) with

$$\Gamma_{\mathrm{s}}\text{-}\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u) := \inf\left\{\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$
$$(\text{resp. } \Gamma_{\mathrm{w}}\text{-}\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u) := \inf\left\{\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

  or equivalently, for every $u \in X$ and every $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, if $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) then

$$\varliminf_{\varepsilon\to0} \Phi_\varepsilon(u_\varepsilon) \geq \Phi(u);$$

- for every $u \in X$, $\Gamma_s\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \leq \Phi(u)$(resp. $\Gamma_w\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) \leq \Phi(u)$) with

$$\Gamma_s\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \to u\right\}$$

$$(\text{resp. } \Gamma_w\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u) := \inf\left\{\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) : u_\varepsilon \rightharpoonup u\right\})$$

or equivalently, for every $u \in X$ there exists $\{u_\varepsilon\}_{\varepsilon>0} \subset X$ such that $u_\varepsilon \to u$ (resp. $u_\varepsilon \rightharpoonup u$) and

$$\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon(u_\varepsilon) \leq \Phi(u).$$

From Γ-convergence we can define Mosco-convergence (which was introduced by Mosco, see [19]).

Definition B.2 Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. We say that $\{\Phi_\varepsilon\}_{\varepsilon>0}$ Mosco-converges to Φ, and we write

$$\Phi = \text{M-}\lim_{\varepsilon\to 0} \Phi_\varepsilon \text{ or } \Phi_\varepsilon \xrightarrow{\text{M}} \Phi,$$

if $\Phi = \Gamma_s\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon = \Gamma_w\text{-}\lim_{\varepsilon\to 0} \Phi_\varepsilon$ or equivalently $\Gamma_s\text{-}\overline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon \leq \Phi \leq \Gamma_w\text{-}\underline{\lim}_{\varepsilon\to 0} \Phi_\varepsilon$.

From Definition B.2, it is easy to see that under a suitable compactness condition strong Γ-convergence is equivalent to Mosco-convergence.

Proposition B.3 *Let $\Phi : X \to]-\infty, \infty]$ and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$. Assume that the following compactness condition hold*:

- *for every $\{u_\varepsilon\}_{\varepsilon>0} \subset X$, if $\sup_{\varepsilon>0} \Phi_\varepsilon(u_\varepsilon) < \infty$ then $\{u_\varepsilon\}_{\varepsilon>0}$ is strongly relatively compact in X.*

Then, $\Phi_\varepsilon \xrightarrow{\Gamma_s} \Phi$ if and only if $\Phi_\varepsilon \xrightarrow{\text{M}} \Phi$.

As stated in the following theorem due to Mosco (see [19, Theorem 1]), in the reflexive case and for lower semicontinuous, convex and proper functions, the Legendre–Fenchel transform is continuous with respect to Mosco-convergence.

Theorem B.4 *Let $\Phi : X \to]-\infty, \infty]$ be a proper, convex and lower semicontinuous function and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to]-\infty, \infty]$ be a proper, convex and lower semicontinuous function. If X is reflexive then $\Phi_\varepsilon \xrightarrow{\text{M}} \Phi$ if and only if $\Phi^*_\varepsilon \xrightarrow{\text{M}} \Phi^*$.*

The following result allows to pass from Mosco-convergence in X to Mosco-convergence in $L^2([0, T]; X)$ (see [4, Lemma 2.6, p. 50] for a proof).

Theorem B.5 *Fix $T > 0$ and assume that X is a Hilbert space. Let $\Phi : X \to [0, \infty]$ be a proper, convex and lower semicontinuous function, let $\Theta : L^2([0, T]; X) \to [0, \infty]$ be defined by*

$$\Theta(u) := \int_0^T \Phi(u(t))dt$$

and, for each $\varepsilon > 0$, let $\Phi_\varepsilon : X \to [0, \infty]$ be a lower semicontinuous, proper and convex function and let $\Theta_\varepsilon : L^2([0, T]; X) \to [0, \infty]$ be defined by

$$\Theta_\varepsilon(u) := \int_0^T \Phi_\varepsilon(u(t))dt.$$

If $\Phi_\varepsilon \xrightarrow{\mathrm{M}} \Phi$ then $\Theta_\varepsilon \xrightarrow{\mathrm{M}} \Theta$.

C. Grönwall's lemma

In the paper we use the following version of the so-called Grönwall's lemma (for a proof we refer to [4, Lemma A.1, p. 277]).

Lemma C.1 *Let $T > 0$, let $a \in [0, \infty[$, let $\mathrm{m} \in L^1([0, T])$ be such that $\mathrm{m}(s) \geq 0$ for $\mathscr{L}^1$-a.a. $s \in [0, T]$ and let $\phi \in C([0, T]; \mathbb{R})$ be such that $\phi(s) \leq a + \int_0^s \phi(t)\mathrm{m}(t)dt$ for all $s \in [0, T]$. Then $\phi(s) \leq a e^{\int_0^s \mathrm{m}(t)dt}$ for all $s \in [0, T]$.*

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