



# From anomalous to classical diffusion in a nonlinear heat equation

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## Abstract

In this work, we consider a heat equation with a nonlinear term of polynomial type and with two different cases in the diffusion term. The first case (anomalous diffusion) concerns to the fractional Laplacian operator with parameter  $1 < \alpha < 2$ , while the second case (classical diffusion) involves the classical Laplacian operator. When  $\alpha \rightarrow 2^-$ , we prove the *uniform convergence* of strong solutions of the anomalous diffusion case to a strong solution of the classical diffusion case. Moreover, we rigorously derive a convergence rate (when  $\alpha \rightarrow 2^-$ ) which highlights some phenomenological effects closely related to the structure of solutions.

**Keywords** Nonlinear heat equation · Fractional Laplacian operator · Asymptotic behavior of solutions depending on the diffusion parameter · From non local to local behavior

**Mathematics Subject Classification** 35B40 · 35B30

## 1 Introduction

In this paper, we consider the following multi-dimensional, nonlinear and anomalous diffusion heat equation in the whole space  $\mathbb{R}^n$  with  $n \geq 1$ :

$$\partial_t u + (-\Delta)^{\alpha/2} u + \eta \cdot \nabla(u^b) = 0, \quad 1 < \alpha < 2, \quad b \in \mathbb{N} \quad \text{with} \quad b \geq 2. \quad (1)$$

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Here, the function  $u : [0, +\infty[ \times \mathbb{R}^n \rightarrow \mathbb{R}$  is the solution, and  $(-\Delta)^{\alpha/2} u$  is the *anomalous* diffusion term which is given by the fractional Laplacian operator  $(-\Delta)^{\alpha/2}$ . We recall that this operator is defined in the Fourier level by the expression

$$\widehat{(-\Delta)^{\alpha/2} u}(t, \xi) = c_{n,\alpha} |\xi|^\alpha \widehat{u}(t, \xi).$$

Moreover, in the spatial variable, the fractional Laplacian operator is defined as the following non-local operator:

$$(-\Delta)^{\alpha/2} u(t, x) = c_{n,\alpha} \mathbf{p.v.} \int_{\mathbb{R}^n} \frac{u(t, x) - u(t, y)}{|x - y|^{n+\alpha}} dy,$$

where  $\mathbf{p.v.}$  denotes the principal value and  $c_{n,\alpha} > 0$  is a constant depending on the dimension  $n$  and the parameter  $\alpha$ . Finally,  $\eta \in \mathbb{R}^n$  is a fixed vector, and moreover, the parameter  $b \in \mathbb{N}$  in the nonlinear term verifies  $b \geq 2$ .

We may observe that this *highly nonlinear term* essentially behaves as the derivative of a polynomial of degree  $b$  in the variable  $u$ . Thus, this term agrees with the classical assumption for the non-linearity in the qualitative study of the heat equation. See, for instance, [3–6, 12] and the references therein.

Nonlinear evolution PDEs involving the fractional Laplacian, which describe the *anomalous* or  $\alpha$ -Lévy stable diffusion, have been extensively studied in the physical and mathematical points of view. From the physical point of view, and for  $b = 2$ , the equation (1) deals with a generalized Burgers-type equation [4] which has been largely used to model a variety of physical phenomena such as, for example, the anomalous homogeneous turbulence [10], applications to hydrodynamics and statistical mechanics [17], and moreover, applications to molecular biology in the modeling of growth of molecular interfaces [20]. In the latter application, the general algebraic non-linear term  $u^b$ , with  $b \in \mathbb{N}^*$  and  $b \geq 2$ , provides a good model for multi-particle interactions. For more references, see the book [16].

From the mathematical point of view, when the solution  $u(t, \cdot)$  is considered as the density of a probability distribution for every  $t > 0$ , the equation (1) has an important probabilistic interpretation in the theory of nonlinear Markov processes and propagation of chaos. See, e.g., the works [11], [14] and the references therein.

Getting back to the expression (1), we observe that for each value of the parameter  $1 < \alpha < 2$  in the fractional Laplacian operator  $(-\Delta)^{\alpha/2}$  we get a corresponding fractional PDE. We thus denote by  $u_\alpha(t, x)$  the corresponding solution of each equation and the main objective of this paper is to understand the asymptotic behavior of the family of functions  $u_\alpha(t, x)$  when the parameter  $\alpha$  goes to 2. This question was pointed out from the experimental point of view in [10, 20] and has some interesting applications in these physical and biological models. Our aim is then to provide a rigorous mathematical framework to give an answer.

Formally, we may observe that if in the expression (1) we set  $\alpha = 2$ , then we get a *classical diffusion equation* involving the Laplacian operator:

$$\partial_t u - \Delta u + \eta \cdot \nabla(u^b) = 0. \quad (2)$$

Consequently, if  $u_2(t, x)$  denotes a solution of the equation above, we are interested in providing a rigorous understanding of the expected convergence  $u_\alpha(t, x) \rightarrow u_2(t, x)$ , when  $\alpha \rightarrow 2$ . It is worth mentioning although this problem is easily formulated, it is not a trivial study since for each value of the parameter  $\alpha$  we have a different fractional PDE depending on this parameter.

In the particular case of the following *linear equation* in a smooth and bounded domain  $\Omega \subset \mathbb{R}^n$ :

$$\partial_t u_\alpha + (-\Delta)^{\alpha/2} u_\alpha = f_\alpha, \quad 0 < \alpha < 2, \tag{3}$$

and where the function  $f_\alpha(t, x)$  does not depend on the solution  $u_\alpha$ , this convergence problem was studied by U. Biccari & V. Hernández-Santamaría in [2]. For a time  $0 < T < +\infty$ , the authors consider a family of functions  $f_\alpha \in L^2(0, T, H^{-\alpha}(\Omega))$ , which is uniformly bounded with respect to the parameter  $\alpha$ :  $\|f_\alpha(t, \cdot)\|_{H^{-\alpha}(\Omega)} \leq C$ , and such that when  $\alpha \rightarrow 2$  we have the convergence  $f_\alpha(t, \cdot) \rightarrow f(t, \cdot)$  in the weak topology of the space  $H^{-1}(\Omega)$ . Then, by using a compactness argument (due to the boundness of the domain  $\Omega$ ) it is shown that *weak solutions* of equation (3) converge in the strong topology of the space  $L^2(0, T, H_0^{1-\delta}(\Omega))$  (with  $0 < \delta \leq 1$ ) to a *weak solution* of the corresponding linear heat equation with datum  $f$ .

On the other hand, L. Ignat & J.D. Rossi studied in [12], among other things, that *weak solutions*  $u(t, x)$  fo the nonlinear heat equation (2) can be obtained as the limit (when  $\varepsilon \rightarrow 0^+$ ) of the *weak solutions* to the following nonlocal convection-diffusion equation in the whole space  $\mathbb{R}^n$ :

$$\partial_t u_\varepsilon + \frac{1}{\varepsilon^2} (J_\varepsilon * u_\varepsilon - u_\varepsilon) + \frac{1}{\varepsilon} (G_\varepsilon * u_\varepsilon^b - u_\varepsilon^b) = 0, \quad \varepsilon > 0. \tag{4}$$

This equation has the same scaling properties of the equation (2) and here, for suitable non-negative functions  $J \in \mathcal{S}(\mathbb{R}^n)$  and  $G \in \mathcal{S}(\mathbb{R}^n)$ , we have  $J_\varepsilon(x) = \frac{1}{\varepsilon^n} J(x/\varepsilon)$  and  $G_\varepsilon(x) = \frac{1}{\varepsilon^n} G(x/\varepsilon)$  respectively. Moreover,  $J$  is a radially symmetric function and the key assumption is that its Fourier transform  $\widehat{J}(\xi)$  satisfies the following condition:

$$\frac{1}{2} \partial_{\xi_i}^2 \widehat{J}_\varepsilon(0) = 1, \quad i = 1, \dots, n, \tag{5}$$

which is similarly satisfied for the symbol  $|\xi|^2$  of the classical Laplacian operator. In this setting, by using some sharp estimates of the kernel associated to the linear problem, and moreover, by setting the vector  $\eta = (\eta_1, \dots, \eta_n)$  in the equation (2) as  $\eta_i = \int_{\mathbb{R}^n} x_i G(x) dx$ , for all time  $0 < T < +\infty$  it is proven the following convergence result in the natural framework (due to the Plancherel’s identity) of the Lebesgue space  $L^2(\mathbb{R}^n)$ :

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{0 \leq t \leq T} \|u_\varepsilon(t, \cdot) - u(t, \cdot)\|_{L^2(\mathbb{R}^n)} = 0. \tag{6}$$

Nevertheless, we remark that this result cannot be applied to the case of the equation (1). Indeed, since the symbol  $|\xi|^\alpha$  of the operator  $(-\Delta)^{\alpha/2}$  does not verify the key condition (5) the nonlocal diffusion operator  $1/\varepsilon^2(J_\varepsilon * (\cdot) - I_d)$  (where  $I_d$  is the identity operator) *does not contain* the fractional Laplacian operator  $(-\Delta)^{\alpha/2}$  as a particular case. Moreover, one the main property of the approximated equation (4) is the same scaling of the equation (2), which is not the case of the equation (1) due to the different scaling provided by the fractional Laplacian operator.

In this work, we will use a different approach. For any time  $0 < T < +\infty$ , in the framework the space  $L^\infty([0, T] \times \mathbb{R}^n)$  we shall study the convergence (in the strong topology) of the *strong (mild) solutions*  $u_\alpha(t, x)$  for the anomalous diffusion equation (1) (given in the expression (14)) to a *strong solution*  $u_2(t, x)$  for the classical diffusion equation (2) (given in the expression (15)). See our main result given in Theorem 2.2 for the details. This uniform convergence also allows us to prove a strong convergence in the  $L_t^p L_x^q$  spaces (see the Corollary 2.1).

Our method is based on two key ideas. On the one hand, we study the convergence of the fundamental solution  $p_\alpha(t, x)$  associated with the fractional linear heat equation (see the expression (16) for a definition) to the heat kernel  $h(t, x)$ . On the other hand, we prove some uniform estimates with respect to the parameter  $\alpha$  for the family of functions  $u_\alpha(t, x)$ .

Finally, we think that in a further research our method could be adapted to the case when the fractional Laplacian operator  $(-\Delta)^{\alpha/2}$  in the equation (1) is substituted by a more general Lévy-type operator  $\mathcal{L}^\alpha$ . For a definition and some well-known properties of this latter operator, we refer to the book [13].

## 2 The main result

Let us consider the Cauchy problem for both *anomalous* (when  $1 < \alpha < 2$ ) and *classical* (when  $\alpha = 2$ ) nonlinear heat equation:

$$\begin{cases} \partial_t u_\alpha + (-\Delta)^{\alpha/2} u_\alpha + \eta \cdot \nabla(u_\alpha^b) = 0, & 1 < \alpha \leq 2, \\ u_\alpha(0, \cdot) = u_{0,\alpha}. \end{cases} \quad (7)$$

Well-posedness (WP) issues for this equation have been studied in several works [3, 8, 9] and it is well-known that for an initial datum  $u_0 \in L^1(\mathbb{R}^n)$  the initial value problem (7) has a unique solution  $u_\alpha \in \mathcal{C}([0, +\infty[, L^1(\mathbb{R}^n))$  which verifies

$$\|u_\alpha(t, \cdot)\|_{L^1} \leq \|u_0\|_{L^1}. \quad (8)$$

Moreover, for  $1 \leq p \leq +\infty$  this solution also verifies  $u_\alpha \in \mathcal{C}([0, +\infty[, W^{1,p}(\mathbb{R}^n))$ , and the following estimate holds:

$$\|u_\alpha(t, \cdot)\|_{L^p} \leq C t^{-\frac{n}{\alpha}(1-\frac{1}{p})} \|u_0\|_{L^1}.$$

Finally, under the additional assumption on the initial datum:  $u_0 \in L^1 \cap L^p(\mathbb{R}^n)$  the corresponding solution verifies  $u_\alpha \in \mathcal{C}([0, +\infty[, L^p(\mathbb{R}^n))$ , and for all time  $t \geq 0$  we have the estimate

$$\|u_\alpha(t, \cdot)\|_{L^p} \leq \|u_0\|_{L^p}.$$

As mentioned, our aim is to study the convergence (when  $\alpha \rightarrow 2$ ) of mild solutions of the equation (7). For technical reasons, principally due to the study of the limit concerning the highly nonlinear term  $\nabla(u_\alpha^b) \rightarrow \nabla(u_2^b)$  (recall that  $b$  is any integer such that  $b \geq 2$ ), we shall need more regularity than the one given by the space  $W^{1,p}(\mathbb{R}^n)$ . For this, we shall consider initial data belonging to the space  $L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$  with  $s > n/2$ .

The global-well posedness in the setting of the space  $L^1(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$  is rather standard but, to our knowledge, this fact has not been proven before. Consequently, only for the completeness of this paper, we state and we will give a proof of the following theorem. We emphasize that the only novelty is the gain of regularity given in the expression (9) below, which follows from the additional hypothesis  $u_0 \in H^s(\mathbb{R}^n)$  with  $s > n/2$ .

**Theorem 2.1** *Let  $1 < \alpha \leq 2$ . For  $s > n/2$ , let  $u_0 \in L^1 \cap H^s(\mathbb{R}^n)$  be an initial datum. Then there exists a unique mild solution*

$$u_\alpha \in \mathcal{C}([0, +\infty[, L^1 \cap H^s(\mathbb{R}^n)),$$

*of the equation (7). Moreover, this solution is regular:*

$$u_\alpha \in \mathcal{C}^1([0, +\infty[, \mathcal{C}^\infty(\mathbb{R}^n)), \tag{9}$$

*and it verifies the equation (7) in the classical sense.*

We study now the convergence of mild solutions for the equation (7) when  $\alpha \rightarrow 2^-$ . For each  $1 < \alpha < 2$  we consider  $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$  an initial datum and we shall denote by  $u_\alpha(t, x)$  its corresponding arising solution of the equation (7), which is given by Theorem 2.1. Moreover, for the classical case (when  $\alpha = 2$ ) we similarly consider an initial datum and its corresponding solution  $u_{0,2} \in L^1 \cap H^s(\mathbb{R}^n)$  and  $u_2(t, x)$  respectively.

We shall assume the following strong convergence on the initial data:

$$u_{0,\alpha} \rightarrow u_{0,2}, \quad \alpha \rightarrow 2^-, \quad \text{in } L^1 \cap H^s(\mathbb{R}^n). \tag{10}$$

By the Sobolev embedding  $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$  (since  $s > n/2$ ) we also have  $u_{\alpha,0} \in L^\infty(\mathbb{R}^n)$ ,  $u_{0,2} \in L^\infty(\mathbb{R}^n)$  and the convergence above also holds true in the space  $L^\infty(\mathbb{R}^n)$ . Thus, for the corresponding family of solutions  $(u_\alpha)_{1 < \alpha \leq 2}$  we will study the *uniform convergence* for any  $0 < T < +\infty$ :

$$u_\alpha(t, x) \rightarrow u_2(t, x), \quad \alpha \rightarrow 2^-, \quad \text{in } L^\infty([0, T] \times \mathbb{R}^n). \tag{11}$$

Moreover, we are also interested in studying the convergence rate in (11). For this, we introduce a parameter  $\gamma > 0$  and we shall assume the estimate (12) below, which is a given convergence rate of the initial data in the space  $L^\infty(\mathbb{R}^n)$ . Our aim is then to study when the family of solutions follows this prescribed convergence rate. In this setting, our main result reads as follows:

**Theorem 2.2** *Let  $(u_{0,\alpha})_{1 < \alpha \leq 2}$  be a family of initial data such that for all  $1 < \alpha \leq 2$  we have  $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ . Let  $(u_\alpha)_{1 < \alpha \leq 2}$  be corresponding family of solutions to the equation (7) given by Theorem 2.1.*

*We assume the convergence given in (10), and moreover, for a parameter  $\gamma > 0$  we assume the estimate*

$$\|u_{0,\alpha} - u_{0,2}\|_{L^\infty} \leq \mathbf{c} (2 - \alpha)^\gamma, \tag{12}$$

where  $\mathbf{c} > 0$  is a given generic constant. Then, there exists  $0 < \varepsilon \ll 1$ , and there exists a constant  $\mathbf{C} > 0$ , which depends on the parameters  $\eta$  and  $b$  in the equation (7), the initial data  $u_{0,2}$ , the quantity  $\varepsilon$  and the constant  $\mathbf{c}$ , such that the following estimate holds:

$$\sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} \leq \mathbf{C} (1 + T + T^2) \max\left((2 - \alpha)^\gamma, 2 - \alpha\right), \tag{13}$$

for all  $1 + \varepsilon < \alpha < 2$  and for all  $0 < T < +\infty$ .

Some remarks are in order here. First note that our approach allows us obtain a uniform convergence (in both the temporal and the spatial variables) which is not studied in the previous related works [2] and [12]. Moreover, it is interesting to observe that the convergence rate given in the estimate (13) is determined by a competition between the quantities  $(2 - \alpha)^\gamma$  and  $(2 - \alpha)$ .

In order to make a deeper discussion of this fact, let us briefly explain the general idea of the proof. As pointed out, we shall consider mild solutions of the equation (7). Thus, for  $1 < \alpha < 2$  we have

$$u_\alpha(t, \cdot) = p_\alpha(t, \cdot) * u_{0,\alpha} + \int_0^t p_\alpha(t - s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds, \tag{14}$$

where the kernel  $p_\alpha(t, x)$  is given in (16), while for  $\alpha = 2$  we have

$$u_2(t, \cdot) = h(t, \cdot) * u_{0,2} + \int_0^t h(t - s, \cdot) * \eta \cdot \nabla(u_2^b)(s, \cdot) ds, \tag{15}$$

where  $h(t, x)$  denotes the well-known heat kernel. The estimate (13) is then obtained by the following estimates

$$\|p_\alpha(t, \cdot) * u_{0,\alpha} - h(t, \cdot) * u_{0,2}\|_{L^\infty} \lesssim \max\left((2 - \alpha)^\gamma, 2 - \alpha\right),$$

and

$$\begin{aligned} & \left\| \int_0^t p_\alpha(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds \int_0^t h(t-s, \cdot) * \eta \cdot \nabla(u_2^b)(s, \cdot) ds \right\|_{L^\infty} \\ & \lesssim \max \left( (2-\alpha)^\gamma, 2-\alpha \right), \end{aligned}$$

on the linear and the nonlinear terms respectively. For the sake of simplicity, we will only explain more in detail the estimates on the linear term. Of course the estimates for the nonlinear term are much more delicate, but they follow some similar ideas. We split the linear term as

$$\begin{aligned} & \left\| p_\alpha(t, \cdot) * u_{0,\alpha} - h(t, \cdot) * u_{0,2} \right\|_{L^\infty} \leq \left\| (p_\alpha(t, \cdot) - h(t, \cdot)) * u_{0,\alpha} \right\|_{L^\infty} \\ & + \left\| h(t, \cdot) * (u_{0,\alpha} - u_{0,2}) \right\|_{L^\infty}, \end{aligned}$$

where we have

$$\left\| (p_\alpha(t, \cdot) - h(t, \cdot)) * u_{0,\alpha} \right\|_{L^\infty} \lesssim (2-\alpha),$$

and

$$\left\| h(t, \cdot) * (u_{0,\alpha} - u_{0,2}) \right\|_{L^\infty} \lesssim (2-\alpha)^\gamma.$$

Here, the quantity  $(2-\alpha)^\gamma$  is the convergence rate assumed for the initial data, while the quantity  $(2-\alpha)$  is the convergence rate of the kernels  $p_\alpha(t, x) \rightarrow h(t, x)$  (when  $\alpha \rightarrow 2^-$ ), which is rigorously proven in Lemma 5.1 below.

Since we have  $1 < \alpha < 2$  and therefore  $0 < 2-\alpha < 1$ , the estimate (13) yields the following conclusions by considering two cases of the parameter  $\gamma$ .

- **The case  $0 < \gamma \leq 1$ .** Here we have  $\max \left( (2-\alpha)^\gamma, 2-\alpha \right) = (2-\alpha)^\gamma$ , and consequently, the solutions  $u_\alpha(t, x)$  converge to the solution  $u_2(t, x)$  with the same convergence rate as that of the initial data.
- **The case  $\gamma > 1$ .** In this case we have  $\max \left( (2-\alpha)^\gamma, 2-\alpha \right) = 2-\alpha$ . Then, it is interesting to observe that the convergence rate of the solutions does not follow the one of initial data. More precisely, the solutions  $u_\alpha(t, x)$  converge to the solution  $u_2(t, x)$  with a rate of order  $2-\alpha$ , which is *slower* than the convergence rate of the initial data  $(2-\alpha)^\gamma$ .

Summarizing, the increasing of the parameter parameter  $\gamma$  makes the assumption (12) strong but not the result given in (13). This is an interesting *phenomenological effect*, which is given by the convergence rate of the kernels  $p_\alpha(t, \cdot) \rightarrow h(t, \cdot)$ .

On the other hand, as mentioned in the introduction, the convergence result given in Theorem 2.2 also allows us to study the convergence (11) in the following Lebesgue spaces.

**Corollary 2.1** *With the same hypothesis of Theorem 2.2, for all  $1 \leq p \leq +\infty$ ,  $1 < q < +\infty$  and for all  $1 + \varepsilon < \alpha < 2$  we have the estimate:*

$$\|u_\alpha - u_2\|_{L^p((0,T],L^q(\mathbb{R}^n))} \leq C_{p,q} (1 + T + T^2)^{1-\frac{1}{q}} T^{\frac{1}{p}} \max\left((2 - \alpha)^\gamma \left(1 - \frac{1}{q}\right), (2 - \alpha)^{1-\frac{1}{q}}\right).$$

We observe that in the framework of  $L_t^p L_x^q$ -spaces, the convergence rate is only driven by the parameter  $q$ , which describes the decaying properties of solutions in the spatial variable. Moreover, by setting the parameter  $\gamma = 1$  and with the particular values  $p = q = 2$ , we obtain the following convergence rate:  $\|u_\alpha - u_2\|_{L_t^2 L_x^2} \lesssim (2 - \alpha)^{1/2}$ , which is similar to the one experimentally obtained in [2] for the time independent version of the equation (3):  $(-\Delta)^{\alpha/2} u_\alpha = f_\alpha$ .

To close this section, let us make the following final comments. First note that in this work we have restricted ourselves in the case when the parameter  $\alpha$  verifies  $1 < \alpha < 2$ , however, our results are also valid for the case  $\alpha > 2$  with minor technical modifications.

The lower constraint  $1 + \varepsilon < \alpha$  (with  $0 < \varepsilon \ll 1$ ) given in Theorem 2.2 is essentially technical, due to estimates involving the expression  $\frac{1}{1-1/\alpha}$  (see, for instance, the estimate in Proposition 4.2 below). Consequently, our result left open the convergence problem when  $\alpha \rightarrow 1^+$  which is also interesting and could be a matter of further research.

Finally, we think that our method explained above could be also adapted to study the convergence given in (11) within the framework of other functional spaces, provided that we assume some natural hypothesis on the initial data.

**Organization of the paper.** In Sect. 3 we recall some well-known facts on the linear fractional heat equation that we will use in the next sections. Section 4 is devoted to the proof of Theorem 2.1, while in Sect. 5 we give a proof of Theorem 2.2 and Corollary 2.1.

### 3 Some well-known facts

In this section, for the completeness of this paper, we summarize some well-known facts on the linear and homogeneous fractional heat equation:

$$\partial_t p_\alpha + (-\Delta)^{\alpha/2} p_\alpha = 0, \quad 1 < \alpha < 2, \quad t > 0.$$

The fundamental solution of this equation, denoted by  $p_\alpha(t, x)$ , can be computed via the Fourier transform by

$$\widehat{p}_\alpha(t, \xi) = e^{-t |\xi|^\alpha}.$$



Moreover, in the spatial variable the fundamental solution  $p_\alpha$  is given by

$$p_\alpha(t, x) = \frac{1}{t^{\frac{1}{\alpha}}} P_\alpha \left( \frac{x}{t^{\frac{1}{\alpha}}} \right), \tag{16}$$

where the function  $P_\alpha$  is the inverse Fourier transform of  $e^{-|\xi|^\alpha}$ . See [13, Chapter 13] for more details. It is well-known that for  $1 < \alpha < 2$  the functions  $P_\alpha$  are smooth and positive. In addition, they verify the following pointwise inequalities

$$0 < P_\alpha(x) \leq \frac{C}{(1 + |x|)^{n+\alpha}}, \quad |\nabla P_\alpha(x)| \leq \frac{C}{(1 + |x|)^{n+\alpha+1}},$$

for a constant  $C > 0$  and for all  $x \in \mathbb{R}^n$ . These inequalities allow us to derive the following estimates.

**Proposition 3.1** ( *$L^p$ -estimates*) For  $1 \leq p \leq +\infty$ , there exists a constant  $C_{n,p} > 0$ , which depends on the dimension  $n \in \mathbb{N}^*$  and the parameter  $p$ , such that for every  $1 < \alpha < 2$  and for every  $t > 0$ , we have

1.  $\|p_\alpha(t, \cdot)\|_{L^p} \leq C_{n,p} t^{-\frac{n}{\alpha} \left(1 - \frac{1}{p}\right)},$
2.  $\|\nabla p_\alpha(t, \cdot)\|_{L^p} \leq C_{n,p} t^{-\frac{1+n \left(1 - \frac{1}{p}\right)}{\alpha}}.$

Moreover we have:

**Proposition 3.2** ( *$L^p$ -continuity*) Let  $1 \leq p < +\infty$ . For every  $\varphi \in L^p(\mathbb{R}^n)$ , we have  $\lim_{t \rightarrow 0^+} \|p_\alpha(t, \cdot) * \varphi - \varphi\|_{L^p} = 0$ .

On the other hand, by the identity  $\widehat{p_\alpha}(t, \xi) = e^{-t|\xi|^\alpha}$  we have the following known results in the setting of the Sobolev spaces:

**Proposition 3.3** ( *$\dot{H}^s$  and  $H^s$  estimates*) Let  $s_1, s_2 \geq 0$ . Then, there is a constant  $C_{n,s_2} > 0$ , which depends on the dimension  $n \in \mathbb{N}^*$  and the parameter  $s_2$ , such that for every  $1 < \alpha \leq 2$  and for every  $t > 0$ , we have:

1.  $\|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq C_{n,s_2} t^{-\frac{s_2}{\alpha}} \|\varphi\|_{\dot{H}^{s_1}}.$
2.  $\|p_\alpha(t, \cdot) * \varphi\|_{H^{s_1+s_2}} \leq C_{n,s_2} \left(1 + t^{-\frac{s_2}{\alpha}}\right) \|\varphi\|_{H^{s_1}}.$

**Proof** In order to verify the first point, we just write:

$$\begin{aligned} \|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}}^2 &= \int_{\mathbb{R}^n} |\xi|^{2(s_1+s_2)} e^{-2t|\xi|^\alpha} |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq t^{-\frac{2s_2}{\alpha}} \left( \sup_{\xi \in \mathbb{R}^n} |t^{1/\alpha} \xi|^{2s_2} e^{-2|t^{1/\alpha} \xi|^\alpha} \right) \int_{\mathbb{R}^n} |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi. \end{aligned}$$

To verify the second point, let us start by writing

$$\|p_\alpha(t, \cdot) * \varphi\|_{H^{s_1+s_2}} = \|p_\alpha(t, \cdot) * \varphi\|_{L^2} + \|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}}.$$

Then, for the first term on the right-hand side, by the Young’s inequalities and the point 1 in Proposition 3.1, we obtain

$$\|p_\alpha(t, \cdot) * \varphi\|_{L^2} \leq \|p_\alpha(t, \cdot)\|_{L^1} \|\varphi\|_{L^2} \leq c \|\varphi\|_{L^2} \leq c \|\varphi\|_{H^{s_1}}, \tag{17}$$

while for the second term on the right-hand side, by the point 1 proven above we can write:

$$\|p_\alpha(t, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq c_{n,s_2} t^{-\frac{s_2}{\alpha}} \|\varphi\|_{H^{s_1}}. \tag{18}$$

Thus, the desired estimate follows directly from (17) and (18). □

**Proposition 3.4** (*H<sup>s</sup> – and  $\dot{H}^s$ –continuity*) *Let  $s_1, s_2 \geq 0$  and  $\varepsilon > 0$ . There exists a constant  $C_{n,s_2,\varepsilon} > 0$ , which depends on the dimension  $n \in \mathbb{N}^*$ , the parameters  $s_2$  and  $\varepsilon$ , such that for every  $1 < \alpha < 2$  and for every  $t_1, t_2 > \varepsilon$ , we have*

1.  $\|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}} \leq C_{n,s_2,\varepsilon} |t_1 - t_2|^{1/2} \|\varphi\|_{\dot{H}^{s_1}}$ ,
2.  $\|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{H^{s_1+s_2}} \leq C_{n,s_2,\varepsilon} |t_1 - t_2|^{1/2} \|\varphi\|_{H^{s_1}}$ .

**Proof** To verify the first point, without loss of generality we shall assume that  $t_1 > t_2 > \varepsilon$ . Then we write

$$\begin{aligned} & \|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}}^2 \\ &= \int_{\mathbb{R}^n} |\xi|^{2(s_1+s_2)} |e^{-t_1|\xi|^\alpha} - e^{-t_2|\xi|^\alpha}|^2 |\widehat{\varphi}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2s_2} e^{-2t_2|\xi|^\alpha} |e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq t_2^{-\frac{2s_2}{\alpha}} \left( \sup_{\xi \in \mathbb{R}^n} |t_2^{1/\alpha} \xi|^{2s_2} e^{-|t_2^{1/\alpha} \xi|^\alpha} \right) \int_{\mathbb{R}^n} e^{-t_2|\xi|^\alpha} |e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq \varepsilon^{-\frac{2s_2}{\alpha}} C_{n,s_2} \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha} |e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq C_{n,s_2,\varepsilon} \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha} |e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi. \end{aligned}$$

We study now the expression  $|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2$ . First, we remark that since we have  $t_1 > t_2$  then the expression  $|e^{-(t_1-t_2)|\xi|^\alpha} - 1|$  is uniformly bounded and we can write

$$|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 = |e^{-(t_1-t_2)|\xi|^\alpha} - 1| |e^{-(t_1-t_2)|\xi|^\alpha} - 1| \leq C |e^{-(t_1-t_2)|\xi|^\alpha} - 1|.$$

Now, by the mean value theorem in the temporal variable we have

$$|e^{-(t_1-t_2)|\xi|^\alpha} - 1| \leq C |\xi|^\alpha |t_1 - t_2|.$$

Thus, gathering these estimates we get

$$|e^{-(t_1-t_2)|\xi|^\alpha} - 1|^2 \leq C |\xi|^\alpha |t_1 - t_2|.$$

Getting back to the last integral we finally have:

$$\begin{aligned} \|p_\alpha(t_1, \cdot) * \varphi - p_\alpha(t_2, \cdot) * \varphi\|_{\dot{H}^{s_1+s_2}}^2 &\leq C_{n,s_2,\varepsilon} |t_1 - t_2| \int_{\mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha} |\xi|^\alpha |\xi|^{2s_1} |\widehat{\varphi}(\xi)|^2 d\xi \\ &\leq C_{n,s_2,\varepsilon} |t_1 - t_2| \left( \sup_{\xi \in \mathbb{R}^n} e^{-\varepsilon|\xi|^\alpha} |\xi|^\alpha \right) \|\varphi\|_{\dot{H}^{s_1}}^2 \\ &\leq C_{n,s_2,\varepsilon} |t_1 - t_2| \|\varphi\|_{\dot{H}^{s_1}}^2, \end{aligned}$$

hence, the first point is verified. The second point essentially follows these same lines. □

### 4 Global well-posedness and regularity: proof of Theorem 2.1

Let  $1 < \alpha \leq 2$  fixed, and let  $u_0 \in L^1 \cap H^s(\mathbb{R}^n)$  be an initial datum. The result stated in Theorem 2.1 is well-known for the case  $\alpha = 2$ , see for instance [3], [8] and [9]. Consequently, we just consider the range  $1 < \alpha < 2$ . As mentioned, the proof of this theorem is rather standard but, for the reader’s convenience, we shall detail some technical estimates.

**Step 1: Local well-posedness.** We consider the (equivalent) mild formulation given in (14), where the nonlinear term defines a multi-linear form in the variable  $u$  (see the expression (22) below). In order to construct a solution of the equation (14) we will use the *Picard’s contraction principle* for a time  $0 < T < +\infty$  small enough. We thus consider the Banach space

$$E_T = \mathcal{C}([0, T], L^1(\mathbb{R}^n)) \cap \mathcal{C}([0, T], H^s(\mathbb{R}^n)), \tag{19}$$

endowed with the norm

$$\|u\|_{E_T} = \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{L^1} + \sup_{0 \leq t \leq T} \|u(t, \cdot)\|_{H^s}. \tag{20}$$

Then, we will prove the following::

**Proposition 4.1** *Let  $s > n/2$  and let  $u_0 \in L^1 \cap H^s(\mathbb{R}^n)$  be an initial datum. Moreover, let  $1 < \alpha < 2$ . Then, there exists a time given by:*

$$T = \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^b c |\eta| \left( \|u_0\|_{L^1} + \|u_0\|_{H^s} \right)^{b-1}} \right]^{\frac{\alpha}{\alpha-1}}, \tag{21}$$

where  $c > 0$  is a numerical constant, and moreover, there exists a function  $u_\alpha \in E_T$  which is a solution of the equation (14).

**Proof** We start by estimating the linear term in the equation (14).

**Lemma 4.1** *Let  $p_\alpha(t, x)$  be the kernel given in (16). Then we have  $\|p_\alpha(t, \cdot) * u_{0,\alpha}\|_{E_T} \leq c (\|u_0\|_{L^1} + \|u_0\|_{H^s})$ .*

**Proof** We first observe that, due to Proposition 3.2 and the first point in Proposition 3.3, the quantities  $\|p_\alpha(t, \cdot) * u_0\|_{L^1}$  and  $\|p_\alpha(t, \cdot) * u_0\|_{H^s}$  are continuous in the temporal variable.

On the other hand, by the Young’s inequalities and the point 1 in Proposition 3.1 (with  $p = 1$ ) we write

$$\|p_\alpha(t, \cdot) * u_0\|_{L^1} \leq \|p_\alpha(t, \cdot)\|_{L^1} \|u_0\|_{L^1} \leq c \|u_0\|_{L^1}.$$

We also write

$$\|p_\alpha(t, \cdot) * u_0\|_{H^s} \leq \|\widehat{p_\alpha}(t, \cdot)\|_{L^\infty} \|u_0\|_{H^s} \leq \|p_\alpha(t, \cdot)\|_{L^1} \|u_0\|_{H^s} \leq c \|u_0\|_{H^s}.$$

to obtain the wished estimate. □

We study now the nonlinear term in the equation (14). For  $b \in \mathbb{N}$  with  $b \geq 2$ , we denote the multi-linear form

$$M_b(u) = \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau, \tag{22}$$

where, to simplify our writing, we have written the function  $u$  instead of  $u_\alpha$ . Then, we have the following estimate

**Lemma 4.2** *For  $u \in E_T$  we have  $M_b(u) \in E_T$ . Moreover, the following estimate holds:*

$$\|M_b(u)\|_{E_T} \leq c |\eta| \frac{T^{1-\frac{1}{\alpha}}}{1-\frac{1}{\alpha}} \|u\|_{E_T}^b.$$

**Proof** By [3] we have  $M_b(u) \in \mathcal{C}([0, T], L^1(\mathbb{R}^n))$ , so it remains to prove that  $M_b(u) \in \mathcal{C}([0, T], H^s(\mathbb{R}^n))$ . Indeed, let  $t_1, t_2 > 0$  and without loss of generality we assume that  $0 < t_1 < t_2 \leq T$ . Then we write

$$\begin{aligned} & \left\| \int_0^{t_1} p_\alpha(t_1 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau - \int_0^{t_2} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \\ & \leq \left\| \int_0^{t_1} p_\alpha(t_1 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau - \int_0^{t_1} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \\ & \quad + \left\| \int_0^{t_1} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau - \int_0^{t_2} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^{t_1} \|p_\alpha(t_1 - \tau, \cdot) * \eta u^b(\tau, \cdot) - p_\alpha(t_2 - \tau, \cdot) * \eta u^b(\tau, \cdot)\|_{H^{s+1}} d\tau \\
 &\quad + \int_{t_1}^{t_2} \|\nabla p_\alpha(t_2 - \tau, \cdot) * \eta u^b(\tau, \cdot)\|_{H^s} d\tau \\
 &= R_{\alpha,1}(t_1, t_2) + R_{\alpha,2}(t_1, t_2).
 \end{aligned}
 \tag{23}$$

For the first term on the right-hand side, by the point 2 in Proposition 3.4 (with  $s_1 = s$  and  $s_2 = 1$ ) and as  $s > n/2$ , by the product laws in the Sobolev spaces we obtain:

$$\begin{aligned}
 R_{\alpha,1}(t_1, t_2) &\leq c \int_0^{t_1} |t_1 - t_2|^{1/2} |\eta| \|u^b(\tau, \cdot)\|_{H^s} d\tau \\
 &\leq c |\eta| |t_1 - t_2|^{1/2} \int_0^{t_1} \|u(\tau, \cdot)\|_{H^s}^b d\tau \\
 &\leq c |\eta| |t_1 - t_2|^{1/2} T \|u\|_{E_T}^b.
 \end{aligned}$$

Hence, we have  $\lim_{t_1 \rightarrow t_2} R_{\alpha,1}(t_1, t_2) = 0$ . On the other hand, for the second term on the right-hand side we write

$$\begin{aligned}
 R_{\alpha,2}(t_1, t_2) &= \int_{t_1}^{t_2} \|\nabla p_\alpha(t_2 - \tau, \cdot) * \eta u^b(\tau, \cdot)\|_{L^2} + \|\nabla p_\alpha(t_2 - \tau, \cdot) * \eta u^b(\tau, \cdot)\|_{\dot{H}^s} d\tau \\
 &\leq |\eta| \int_{t_1}^{t_2} \|\nabla p_\alpha(t_2 - \tau, \cdot)\|_{L^1} \|u^b(\tau, \cdot)\|_{L^2} d\tau \\
 &\quad + |\eta| \int_{t_1}^{t_2} \|p_\alpha(t_2 - \tau, \cdot) * u^b(\tau, \cdot)\|_{\dot{H}^{s+1}} d\tau \\
 &= R_{\alpha,2,1}(t_1, t_2) + R_{\alpha,2,2}(t_1, t_2).
 \end{aligned}$$

In order to estimate the term  $R_{\alpha,2,1}(t_1, t_2)$ , by the Hölder inequalities, the second point in Proposition 3.1, and moreover, the product laws in the Sobolev spaces, we write:

$$\begin{aligned}
 R_{\alpha,2,1}(t_1, t_2) &\leq c |\eta| \int_{t_1}^{t_2} (t_2 - \tau)^{-1/\alpha} \|u^b(\tau, \cdot)\|_{L^2} d\tau \\
 &\leq c |\eta| \int_{t_1}^{t_2} (t_2 - \tau)^{-1/\alpha} \|u^b(\tau, \cdot)\|_{H^s} d\tau \\
 &\leq c |\eta| \|u\|_{E_T}^b \frac{|t_2 - t_1|^{1-\frac{1}{\alpha}}}{1 - \frac{1}{\alpha}}.
 \end{aligned}$$

In addition, in order to estimate  $R_{\alpha,2,2}(t_1, t_2)$ , by Proposition 3.3 (with  $s_1 = s$  and  $s_2 = 1$ ), and by using again the product laws in the Sobolev spaces, we can write

$$R_{\alpha,2,2}(t_1, t_2) \leq c |\eta| \int_{t_1}^{t_2} (t_2 - \tau)^{-1/\alpha} \|u^b(\tau, \cdot)\|_{\dot{H}^s} d\tau \leq c |\eta| \|u\|_{E_T}^b \frac{|t_2 - t_1|^{1-\frac{1}{\alpha}}}{1 - \frac{1}{\alpha}}.$$

By gathering the estimates made for the terms  $R_{\alpha,2,1}(t_1, t_2)$  and  $R_{\alpha,2,2}(t_1, t_2)$ , we obtain  $\lim_{t_1 \rightarrow t_2} R_{\alpha,2}(t_1, t_2) = 0$ . We thus have  $M_b(u) \in \mathcal{C}((0, T], H^s(\mathbb{R}^n))$ . Now, we must prove the continuity at  $t = 0$ . For this we will verify the estimate

$$\left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \leq c |\eta| \|u\|_{E_T}^b \frac{t^{1-1/\alpha}}{1 - 1/\alpha}. \tag{24}$$

Indeed, by the Young’s inequalities, the second point in Proposition 3.1, Proposition 3.3, and moreover, the product laws in the Sobolev spaces we can write:

$$\begin{aligned} & \left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \\ & \leq \int_0^t \|\nabla p_\alpha(t - \tau, \cdot)\|_{L^1} \left\| \eta u^b(\tau, \cdot) \right\|_{L^2} + \left\| p_\alpha(t - \tau, \cdot) * \eta u^b(\tau, \cdot) \right\|_{\dot{H}^{s+1}} d\tau \\ & \leq \int_0^t c(t - \tau)^{-\frac{1}{\alpha}} |\eta| \left\| u^b(\tau, \cdot) \right\|_{L^2} + c(t - \tau)^{-\frac{1}{\alpha}} |\eta| \left\| u^b(\tau, \cdot) \right\|_{\dot{H}^s} d\tau \\ & \leq c |\eta| \|u\|_{E_T}^b \int_0^t (t - \tau)^{-\frac{1}{\alpha}} d\tau \leq c |\eta| \|u\|_{E_T}^b \frac{t^{1-1/\alpha}}{1 - 1/\alpha}. \end{aligned}$$

Once we have  $M_b(u) \in E_T$ , we verify now the estimate stated in Lemma 4.2. First note that by the estimate (24) we can write

$$\sup_{t \in [0, T]} \left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{H^s} \leq c |\eta| \frac{T^{1-1/\alpha}}{1 - 1/\alpha} \|u\|_{E_T}^b. \tag{25}$$

On the other hand, by applying the Young inequalities and the point 2 in Proposition 3.1 we have

$$\begin{aligned} \left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{L^1} & \leq \int_0^t \left\| \nabla p_\alpha(t - \tau, \cdot) * \eta u^b(\tau, \cdot) \right\|_{L^1} d\tau \\ & \leq c |\eta| \int_0^t (t - \tau)^{-\frac{1}{\alpha}} \left\| u^b(\tau, \cdot) \right\|_{L^1} d\tau \\ & \leq c |\eta| \int_0^t (t - \tau)^{-\frac{1}{\alpha}} \|u(\tau, \cdot)\|_{L^\infty}^{b-1} \|u(\tau, \cdot)\|_{L^1} d\tau. \end{aligned}$$

Since  $s > n/2$  we have the embedding  $H^s(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)$ , and thus we can write

$$\begin{aligned} & c|\eta| \int_0^t (t - \tau)^{-\frac{1}{\alpha}} \|u(\tau, \cdot)\|_{L^\infty}^{b-1} \|u(\tau, \cdot)\|_{L^1} d\tau \\ & \leq c|\eta| \int_0^t (t - \tau)^{-\frac{1}{\alpha}} \|u(\tau, \cdot)\|_{H^s}^{b-1} \|u(\tau, \cdot)\|_{L^1} d\tau \\ & \leq c|\eta| \left( \sup_{\tau \in [0, T]} \|u(\tau, \cdot)\|_{H^s}^{b-1} \right) \left( \sup_{\tau \in [0, T]} \|u(\tau, \cdot)\|_{L^1} \right) \frac{T^{1-1/\alpha}}{1 - 1/\alpha} \\ & \leq c|\eta| \frac{T^{1-1/\alpha}}{1 - 1/\alpha} \|u\|_{E_T}^b. \end{aligned}$$

Then, we have

$$\begin{aligned} \sup_{t \in [0, T]} \left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u^b)(\tau, \cdot) d\tau \right\|_{L^1} & \leq \int_0^t \left\| \nabla p_\alpha(t - \tau, \cdot) * \eta u^b(\tau, \cdot) \right\|_{L^1} d\tau \\ & \leq c|\eta| \frac{T^{1-1/\alpha}}{1 - 1/\alpha} \|u\|_{E_T}^b. \end{aligned} \tag{26}$$

Finally, by (25) and (26) we obtain the desired estimate. This lemma is proven.  $\square$

Once we have Lemmas 4.1 and 4.2 at our disposal, the rest of the proof of Proposition 4.1 follows from standard arguments.  $\square$

**Step 2: Regularity.** We define the space  $H^\infty(\mathbb{R}^n)$  as  $H^\infty(\mathbb{R}^n) = \bigcap_{s \geq 0} H^s(\mathbb{R}^n)$ .

**Proposition 4.2** *Let  $u_\alpha \in E_T$  be the unique solution of the equation (14) given by Proposition 4.1. This solution satisfies  $u_\alpha \in \mathcal{C}((0, T], H^\infty(\mathbb{R}^n))$ . Moreover, we have  $u_\alpha \in C^1((0, T], C^\infty(\mathbb{R}^n))$ ; and for  $0 < t \leq T$  the solution  $u_\alpha$  verifies the differential equation (7) in the classical sense.*

**Proof** We will verify that each term on the right-hand side in the equation (14) belongs to the space  $\mathcal{C}([0, T], H^\infty(\mathbb{R}^n))$ . For the first (linear) term, by the second point in Proposition 3.3, and moreover, by the second point in Proposition 3.4, we directly have  $p_\alpha * u_{0,\alpha} \in \mathcal{C}([0, T], H^\infty(\mathbb{R}^n))$ .

For the second (nonlinear) term, we recall that by (24) for all time  $0 < t \leq T$  we have  $\int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \in H^s(\mathbb{R}^n)$ . Then, we will prove that for  $\sigma > 0$  small enough we also have:  $\int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \in H^{s+\sigma}(\mathbb{R}^n)$ . Indeed, by using the second point in Proposition 3.3, for  $\sigma > 0$  (which we shall set later) we write

$$\begin{aligned} & \left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \right\|_{H^{s+\sigma}} \\ & \leq c|\eta| \int_0^t \left\| p_\alpha(t - \tau, \cdot) * u_\alpha^b(\tau, \cdot) \right\|_{H^{s+\sigma+1}} d\tau \end{aligned}$$

$$\begin{aligned} &\leq c |\eta| \int_0^t \left[ 1 + (t - \tau)^{-\frac{(\sigma+1)}{\alpha}} \right] \|u_\alpha^b(\tau, \cdot)\|_{H^s} d\tau \\ &\leq c |\eta| \|u_\alpha\|_{E_T}^b \int_0^t \left[ 1 + (t - \tau)^{-\frac{(\sigma+1)}{\alpha}} \right] d\tau. \end{aligned}$$

We thus set  $0 < \sigma < \alpha - 1$  (recall that we have  $1 < \alpha < 2$ ) to obtain that the last integral above computes down as

$$\int_0^t 1 + (t - \tau)^{-\frac{(\sigma+1)}{\alpha}} d\tau = t + \frac{t^{1-\frac{(\sigma+1)}{\alpha}}}{1 - \frac{(\sigma+1)}{\alpha}}.$$

Then, for all time  $0 < t \leq T$  we obtain the estimate:

$$\left\| \int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \right\|_{H^{s+\sigma}} \leq c |\eta| \|u_\alpha\|_{E_T}^b \left[ t + \frac{t^{1-\frac{(\sigma+1)}{\alpha}}}{1 - \frac{(\sigma+1)}{\alpha}} \right].$$

We will show now that we have

$$\int_0^t p_\alpha(t - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \in \mathcal{C}((0, T], H^{s+\sigma}(\mathbb{R}^n)).$$

Let  $0 < t_1, t_2 < T$ , where, always without loss of generality we shall assume that  $t_1 < t_2$ . Then we write:

$$\begin{aligned} &\left\| \int_0^{t_2} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau - \int_0^{t_1} p_\alpha(t_1 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \right\|_{H^{s+\sigma}} \\ &\leq \left\| \int_0^{t_2} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau - \int_0^{t_1} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \right\|_{H^{s+\sigma}} \\ &\quad + \left\| \int_0^{t_1} p_\alpha(t_2 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau - \int_0^{t_1} p_\alpha(t_1 - \tau, \cdot) * \eta \cdot \nabla(u_\alpha^b)(\tau, \cdot) d\tau \right\|_{H^{s+\sigma}} \\ &= \tilde{R}_{\alpha,1}(t_1, t_2) + \tilde{R}_{\alpha,2}(t_1, t_2), \tag{27} \end{aligned}$$

where, we must study the terms  $\tilde{R}_{\alpha,1}(t_1, t_2)$  and  $\tilde{R}_{\alpha,2}(t_1, t_2)$ . For the term  $\tilde{R}_{\alpha,1}(t_1, t_2)$ , by the second point in Proposition 3.3 we can write:

$$\begin{aligned} \tilde{R}_{\alpha,1}(t_1, t_2) &\leq \int_{t_1}^{t_2} \left\| p_\alpha(t_2 - \tau, \cdot) * \eta (u_\alpha^b)(\tau, \cdot) \right\|_{H^{s+\sigma+1}} d\tau \\ &\leq C \int_{t_1}^{t_2} \left[ 1 + (t_2 - \tau)^{-\frac{(\sigma+1)}{\alpha}} \right] \left\| \eta (u_\alpha^b)(\tau, \cdot) \right\|_{H^s} d\tau. \end{aligned}$$

Since  $0 < \sigma < \alpha - 1$  the integral above computes down as

$$\int_{t_1}^{t_2} 1 + (t_2 - \tau)^{-\frac{(\sigma+1)}{\alpha}} d\tau = (t_2 - t_1) + \frac{(t_2 - t_1)^{1-\frac{(\sigma+1)}{\alpha}}}{1 - \frac{(\sigma+1)}{\alpha}}.$$



Hence, we have:

$$\tilde{R}_{\alpha,1}(t_1, t_2) \leq c |\eta| \|u\|_{E_T}^b \left[ (t_2 - t_1) + \frac{(t_2 - t_1)^{1 - \frac{(\sigma+1)}{\alpha}}}{1 - \frac{(\sigma+1)}{\alpha}} \right]. \tag{28}$$

For the term  $\tilde{R}_{\alpha,2}(t_1, t_2)$ , always by the second point in Proposition 3.4, we can write:

$$\begin{aligned} \tilde{R}_{\alpha,2}(t_1, t_2) &\leq c |\eta|, \int_0^{t_1} \left\| p_{\alpha}(t_2 - \tau, \cdot) \eta(u_{\alpha}^b)(\tau, \cdot) - p_{\alpha}(t_1 - \tau, \cdot) * (u_{\alpha}^b)(\tau, \cdot) \right\|_{H^{s+\sigma+1}} d\tau \\ &\leq c |\eta| |t_1 - t_2|^{1/2} \int_0^{t_1} \|u_{\alpha}^b(\tau, \cdot)\|_{H^s} d\tau \leq c |\eta| |t_1 - t_2|^{1/2} T \|u\|_{E_T}^b. \end{aligned} \tag{29}$$

Therefore, for  $0 < \sigma < \alpha - 1$ , by (28) and (29) we have

$$\int_0^t p_{\alpha}(t - \tau, \cdot) * \eta \cdot \nabla(u_{\alpha}^b)(\tau, \cdot) d\tau \in \mathcal{C}((0, T], H^{s+\sigma}(\mathbb{R}^n)).$$

At this point, we have proven that  $u_{\alpha} \in \mathcal{C}((0, T_0], H^{s+\delta}(\mathbb{R}^n))$  and by repeating this process (in order to obtain a gain of regularity for the nonlinear term) we conclude that  $u_{\alpha} \in \mathcal{C}((0, T], H^{\infty}(\mathbb{R}^n))$ .

With this information at our disposal, we can verify now that for all  $0 < t \leq T$  and for all multi-index  $\mathbf{a} \in \mathbb{N}^n$  we have  $\partial_x^{\mathbf{a}} u_{\alpha}(t, \cdot) \in \mathcal{C}((0, T], \mathcal{C} \cap L^{\infty}(\mathbb{R}^n))$ . Indeed, let  $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$  be a multi-index, where we denote by  $|\mathbf{a}| = a_1 + \dots + a_n$  its size. Then, for  $\frac{n}{2} < s_1 < \frac{n}{2} + 1$  we set  $s = |\mathbf{a}| + s_1$ . Since we have  $u_{\alpha} \in \mathcal{C}((0, T], H^{\infty}(\mathbb{R}^n))$  then we get  $\partial_x^{\mathbf{a}} u_{\alpha}(t, \cdot) \in H^{s_1}(\mathbb{R}^n)$ . Moreover, since  $\frac{n}{2} < s_1$  we have the continuous embedding  $H^{s_1}(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n)$ , hence we conclude that  $\partial_x^{\mathbf{a}} u_{\alpha}(t, \cdot) \in L^{\infty}(\mathbb{R}^n)$ .

On the other hand, we recall that we have the identification  $H^{s_1}(\mathbb{R}^n) = B_{2,2}^{s_1}(\mathbb{R}^n)$  (where  $B_{2,2}^{s_1}(\mathbb{R}^n)$  denotes a non-homogeneous Besov space [1]). Moreover, we also have the continuous embedding  $B_{2,2}^{s_1}(\mathbb{R}^n) \subset B_{\infty,\infty}^{s_1-n/1}(\mathbb{R}^n) \subset \dot{B}_{\infty,\infty}^{s_1-n/2}(\mathbb{R}^n)$ .

We thus have  $\partial_x^{\mathbf{a}} u_{\alpha}(t, \cdot) \in \dot{B}_{\infty,\infty}^{s_1-n/2}(\mathbb{R}^n)$ . But, since  $\frac{n}{2} < s_1 < \frac{n}{2} + 1$  then we have  $0 < s_1 - \frac{n}{2} < 1$ , and thereafter, by definition of the homogeneous Besov space  $\dot{B}_{\infty,\infty}^{s_1-n/2}(\mathbb{R}^n)$  (see always [1]) we get that  $\partial_x^{\mathbf{a}} u_{\alpha}(t, \cdot)$  is a  $\beta$ -Hölder continuous functions with parameter  $\beta = s_1 - \frac{n}{2} \in (0, 1)$ .

We have proven that  $u_{\alpha} \in \mathcal{C}((0, T], C^{\infty}(\mathbb{R}^n))$ , and we write  $\partial_t u_{\alpha} = -(-\Delta)^{\alpha/2} u_{\alpha} - \eta \cdot \nabla(u_{\alpha}^b)$  to obtain that  $\partial_t u_{\alpha} \in \mathcal{C}((0, T], C^{\infty}(\mathbb{R}^n))$ . Finally, we conclude that  $u_{\alpha} \in \mathcal{C}^1((0, T], C^{\infty}(\mathbb{R}^n))$ . Proposition 4.2 is proven.  $\square$

**Step 4: Global in time existence.** By following similar arguments of [7] (see the proof of Theorem 2, page 9) we have the following result.

**Proposition 4.3** *Let  $u_0 \in L^1 \cap H^s(\mathbb{R}^n)$  be an initial data and let  $T^* > 0$  be the maximal time of existence of the unique corresponding arising solution  $u_{\alpha} \in E_{T^*}$  (given by Proposition 4.1) to the problem (14). Then we have  $T^* = +\infty$ .*

**Proof** Let us briefly explain the general idea of the proof. We assume that  $T^* < +\infty$ . Then we will extend the solution  $u_\alpha$  beyond the time  $T^*$  to obtain a contradiction. We thus conclude that  $T^* = +\infty$ .

We start by defining the following function  $T(\cdot) : [0, +\infty[ \rightarrow [0, +\infty[$  such that for each initial datum  $w_0 \in L^1 \cap H^s(\mathbb{R}^n)$  the quantity  $T(\|w_0\|_{L^1})$  is given by the expression

$$T(\|w_0\|_{L^1}) = \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^{bc|\eta|} (\|w_0\|_{L^1} + \|w_0\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}}.$$

We recall that  $T(\|w_0\|_{L^1})$  is precisely the first time of the existence of the solution  $w_\alpha$  to the equation (14), which is given by Proposition 4.1. Additionally, the *key remark* is that this function is decreasing in the variable  $\|w_0\|_{L^1}$ .

On the other hand, by [3] we know that for every initial datum  $w_0 \in L^1 \cap H^s(\mathbb{R}^n)$  we have a unique solution  $w_\alpha \in C([0, +\infty[, L^1(\mathbb{R}^n))$  of the equation (14). Moreover, for every time  $t > 0$  we have the estimate

$$\|w_\alpha(t, \cdot)\|_{L^1} \leq \|w_0\|_{L^1}. \tag{30}$$

Since the function  $T(\cdot)$  defined above is decreasing in the variable  $\|w_0\|_{L^1}$ , for the initial datum  $u_0 \in L^1 \cap H^s(\mathbb{R}^n)$  we can set a time  $0 < T_1 < T^*$  such that for all  $w_0 \in L^1 \cap H^s(\mathbb{R}^n)$  with  $\|w_0\|_{L^1} \leq \|u_0\|_{L^1}$  we have

$$T(\|w_0\|_{L^1}) \geq T_1. \tag{31}$$

Then, for  $0 < \varepsilon < T_1$  small enough, we consider the time  $T^* - \varepsilon > 0$  and we set the initial datum  $w_0 = u_\alpha(T^* - \varepsilon, \cdot)$ . We shall denote by  $w_\alpha$  its corresponding arising solution, which exists at least until the time  $T(\|w_0\|_{L^1})$ . We thus observe that the function

$$\tilde{w}_\alpha(t, \cdot) = \begin{cases} u_\alpha(t, \cdot), & t \in [0, T^* - \varepsilon], \\ w_\alpha(t, \cdot), & t \in [T^* - \varepsilon, T^* - \varepsilon + T(\|v_0\|_{L^1})], \end{cases}$$

is a solution of the equation (14) associated to the initial datum  $u_0$ . Moreover, we observe that this function is defined in the interval of time  $[0, T^* - \varepsilon + T(\|v_0\|_{L^1})]$ . But, by (30) we have  $\|w_0\|_{L^1} = \|u_\alpha(T^* - \varepsilon, \cdot)\|_{L^1} \leq \|u_0\|_{L^1}$  and consequently by (31) we get  $T(\|w_0\|_{L^1}) \geq T_1$ .

Finally, we can write  $T^* - \varepsilon + T_1 \leq T^* - \varepsilon + T(\|v_0\|_{L^1})$ ; and since  $0 < \varepsilon < T_1$  we obtain  $T^* < T^* - \varepsilon + T_1$ , which is a contradiction with the definition of the time  $T^*$ . Proposition 4.3 is proven. □

Once we have proven the Propositions 4.1, 4.2 and 4.3, we can finish with the proof of Theorem 2.1. □

## 5 From anomalous to classical diffusion

### 5.1 Proof of Theorem 2.2

For  $1 < \alpha < 2$ , let  $u_\alpha$  be the mild solution of the equation (7) given by the expression (14). Moreover, for  $\alpha = 2$  let  $u_2$  be the mild solution of the equation (7), which is given by the expression (15). Then, for a time  $0 < T < +\infty$  fixed we write

$$\begin{aligned} & \sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} \\ & \leq \sup_{0 \leq t \leq T} \|p_\alpha(t, \cdot) * u_{0,\alpha} - h(t, \cdot) * u_{0,2}\|_{L^\infty} \\ & \quad + \sup_{0 \leq t \leq T} \left\| \int_0^t p_\alpha(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds \right. \\ & \quad \left. - \int_0^t h(t-s, \cdot) * \eta \cdot \nabla(u_2^b)(s, \cdot) ds \right\|_{L^\infty} \\ & = I_\alpha + J_\alpha, \end{aligned} \tag{32}$$

where we must estimate each term on the right-hand side. For the term  $I_\alpha$  we write

$$\begin{aligned} I_\alpha & \leq \sup_{0 \leq t \leq T} \|(p_\alpha(t, \cdot) - h(t, \cdot)) * u_{0,\alpha}\|_{L^\infty} + \sup_{0 \leq t \leq T} \|h(t, \cdot) * (u_{0,\alpha} - u_{0,2})\|_{L^\infty} \\ & = I_{\alpha,1} + I_{\alpha,2}. \end{aligned} \tag{33}$$

In order to estimate the term  $I_{\alpha,1}$ , we apply the Bessel potential operators  $(1 - \Delta)^{-s/2}$  and  $(1 - \Delta)^{s/2}$  to obtain:

$$I_{\alpha,1} = \sup_{0 \leq t \leq T} \|(1 - \Delta)^{-s/2} (p_\alpha(t, \cdot) - h(t, \cdot)) * (1 - \Delta)^{s/2} u_{0,\alpha}\|_{L^\infty} = (a).$$

Then, by applying the Young inequalities (with  $1 + 1/\infty = 1/2 + 1/2$ ) we have

$$\begin{aligned} (a) & \leq c \sup_{0 \leq t \leq T} \left( \|(1 - \Delta)^{-s/2} (p_\alpha(t, \cdot) - h(t, \cdot))\|_{L^2} \|(1 - \Delta)^{s/2} u_{0,\alpha}\|_{L^2} \right) \\ & \leq c \left( \sup_{0 \leq t \leq T} \|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}} \right) \left( \sup_{1 < \alpha < 2} \|u_{0,\alpha}\|_{H^s} \right), \end{aligned} \tag{34}$$

where we shall control each term above separately. For the first term on the right-hand side we have the following technical result:

**Lemma 5.1** *For  $s > n/2$  there exists a constant  $C = C(s) > 0$  such that for all  $1 < \alpha < 2$  we have:*

$$\sup_{0 \leq t \leq T} \|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}} \leq C T |2 - \alpha|.$$

**Proof** First, we verify that the quantity  $\|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}}^2$  is continuous in the temporal variable  $t$ . Indeed, for  $0 \leq t_0, t \leq T$  we have

$$\begin{aligned} & \|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}}^2 - \|p_\alpha(t_0, \cdot) - h(t_0, \cdot)\|_{H^{-s}}^2 \\ &= \int_{\mathbb{R}^n} \left| e^{-|\xi|^\alpha t} - e^{-|\xi|^2 t} \right|^2 \frac{d\xi}{(1 + |\xi|^2)^s} - \int_{\mathbb{R}^n} \left| e^{-|\xi|^\alpha t_0} - e^{-|\xi|^2 t_0} \right|^2 \frac{d\xi}{(1 + |\xi|^2)^s} \\ &= \int_{\mathbb{R}^n} \left( \left| e^{-|\xi|^\alpha t} - e^{-|\xi|^2 t} \right|^2 - \left| e^{-|\xi|^\alpha t_0} - e^{-|\xi|^2 t_0} \right|^2 \right) \frac{d\xi}{(1 + |\xi|^2)^s}. \end{aligned}$$

As  $s > n/2$  we have  $\int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|^2)^s} < +\infty$ ; and then, we can apply the dominated convergence theorem to obtain that

$$\lim_{t \rightarrow t_0} \left( \|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}}^2 - \|p_\alpha(t_0, \cdot) - h(t_0, \cdot)\|_{H^{-s}}^2 \right) = 0.$$

Thereafter, by the continuity of the quantity  $\|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}}^2$  with respect to the variable  $t$ , there exists a time  $0 < t_1 \leq T$  such that

$$\sup_{0 \leq t \leq T} \|p_\alpha(t, \cdot) - h(t, \cdot)\|_{H^{-s}} = \|p_\alpha(t_1, \cdot) - h(t_1, \cdot)\|_{H^{-s}}.$$

Now, we will prove the estimate  $\|p_\alpha(t_1, \cdot) - h(t_1, \cdot)\|_{H^{-s}} \leq C T |2 - \alpha|$ . For this we write:

$$\|p_\alpha(t_1, \cdot) - h(t_1, \cdot)\|_{H^{-s}}^2 = \int_{\mathbb{R}^n} \left| e^{-|\xi|^\alpha t_1} - e^{-|\xi|^2 t_1} \right|^2 \frac{d\xi}{(1 + |\xi|^2)^s}. \tag{35}$$

Here, for  $\xi \in \mathbb{R}^n \setminus \{0\}$  fixed, and for  $1 < \alpha < 2 + \delta$  (with  $\delta > 0$ ) we define the function

$$f_\xi(\alpha) = e^{-t_1 |\xi|^\alpha}, \tag{36}$$

where, by computing its derivative with respect to the variable  $\alpha$  we get

$$f'_\xi(\alpha) = -t_1 e^{-t_1 |\xi|^\alpha} |\xi|^\alpha \ln(|\xi|).$$

Thus, by the mean value theorem (in the variable  $\alpha$ ) we can write

$$|f_\xi(\alpha) - f_\xi(2)| \leq \|f'_\xi\|_{L^\infty((1, 2+\delta))} |2 - \alpha|.$$

Moreover, we can also prove the uniform estimate with respect to the variable  $\xi$ :

$$\left\| \|f'_\xi\|_{L^\infty((1, 2+\delta))} \right\|_{L^\infty(\mathbb{R}^n)} \leq c T. \tag{37}$$

The proof of this estimate is not difficult and it is given in detail at the Appendix 6. We thus have,

$$|f_\xi(\alpha) - f_\xi(2)| \leq c T |2 - \alpha|.$$

Then, we get back to the identity (35) and we can write

$$\begin{aligned} \|p_\alpha(t_1, \cdot) - h(t_1, \cdot)\|_{H^{-s}}^2 &= \int_{\mathbb{R}^n} |f_\xi(\alpha) - f_\xi(2)|^2 \frac{d\xi}{(1 + |\xi|^2)^s} \\ &\leq c T^2 |2 - \alpha|^2 \int_{\mathbb{R}^n} \frac{d\xi}{(1 + |\xi|^2)^s} \leq C(s) T^2 |2 - \alpha|^2. \end{aligned}$$

□

On the other hand, to control the second term on the right-hand side in the expression (34), we recall that by hypothesis (10) the family  $(u_{0,\alpha})_{1 < \alpha < 2}$  is bounded in  $H^s(\mathbb{R}^n)$ .

Thus, for the term  $I_{\alpha,1}$  given in (33) we can write:

$$I_{\alpha,1} \leq C T |2 - \alpha|. \tag{38}$$

We study now the term  $I_{\alpha,2}$ , which is also given in (33). By the Young inequalities (with  $1 + 1/\infty = 1 + 1/\infty$ ), the well-known properties of the heat kernel, and moreover, by the assumption give in (12), we have:

$$I_{\alpha,2} \leq \mathbf{c}(2 - \alpha)^\gamma. \tag{39}$$

Consequently, with the estimates (38) and (39) above we obtain:

$$I_\alpha \leq \mathbf{C} (1 + T) \max \left( (2 - \alpha)^\gamma, 2 - \alpha \right). \tag{40}$$

We study now the term  $J_\alpha$  given in the expression (32). For this we write

$$\begin{aligned} J_\alpha &\leq \sup_{0 \leq t \leq T} \left\| \int_0^t p_\alpha(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds - \int_0^t h_\alpha(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds \right\|_{L^\infty} \\ &\quad + \sup_{0 \leq t \leq T} \left\| \int_0^t h(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds - \int_0^t h(t-s, \cdot) * \eta \cdot \nabla(u_2^b)(s, \cdot) ds \right\|_{L^\infty} \\ &\leq \sup_{0 \leq t \leq T} \left\| \int_0^t (p_\alpha(t-s, \cdot) - h(t-s, \cdot)) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) ds \right\|_{L^\infty} \\ &\quad + \sup_{0 \leq t \leq T} \left\| \int_0^t h(t-s, \cdot) * \eta \cdot \nabla(u_\alpha^b - u_2^b)(s, \cdot) ds \right\|_{L^\infty} = J_{\alpha,1} + J_{\alpha,2}, \end{aligned} \tag{41}$$

where we will study the terms  $J_{\alpha,1}$  and  $J_{\alpha,2}$  separately. For the term  $J_{\alpha,1}$ , we apply first the operators  $(1 - \Delta)^{-s/2}$  and  $(1 - \Delta)^{s/2}$ , and moreover, by the Young inequalities

(with  $1 + 1/\infty = 1/2 + 1/2$ ) we have

$$\begin{aligned}
 J_{\alpha,1} &\leq \sup_{0 \leq t \leq T} \left( \int_0^t \left\| \left( p_\alpha(t-s, \cdot) - h(t-s, \cdot) \right) * \eta \cdot \nabla(u_\alpha^b)(s, \cdot) \right\|_{L^\infty} ds \right) \\
 &\leq |\eta| \sup_{0 \leq t \leq T} \left( \int_0^t \|\nabla p_\alpha(t-s, \cdot) - \nabla h(t-s, \cdot)\|_{H^{-s}} \|u_\alpha^b(s, \cdot)\|_{H^s} ds \right) \\
 &\leq |\eta| T \left( \sup_{0 \leq t \leq T} \|\nabla p_\alpha(t, \cdot) - \nabla h(t, \cdot)\|_{H^{-s}} \right) \left( \sup_{0 \leq t \leq T} \|u_\alpha^b(s, \cdot)\|_{H^s} \right).
 \end{aligned}
 \tag{42}$$

In order to control the first term on the right-hand side, we follow the same lines in the proof of Lemma 5.1 with the function  $f_\xi(\alpha) = i\xi_j e^{-t|\xi|^\alpha}$ , with  $j = 1, 2, \dots, n$ . Then we have

$$\sup_{0 \leq t \leq T} \|\nabla p_\alpha(t, \cdot) - \nabla h(t, \cdot)\|_{H^{-s}} \leq CT |2 - \alpha|.
 \tag{43}$$

Thereafter, to control the second term on the right-hand side we shall need the following:

**Lemma 5.2** *There exists  $0 < \varepsilon \ll 1$ , and there exists a constant*

$$C = C(\varepsilon, T, b, \|u_{0,2}\|_{L^1}, \|u_{0,2}\|_{H^s}) > 0,$$

such that for all  $1 + \varepsilon < \alpha < 2$  we have:

$$\sup_{0 \leq t \leq T} \|u_\alpha^b(t, \cdot)\|_{H^s} \leq C.
 \tag{44}$$

**Proof** For any the initial data  $u_{0,\alpha} \in L^1 \cap H^s(\mathbb{R}^n)$ , with  $1 < \alpha < 2$ , we recall that by Proposition 4.1 there exists a time  $T_\alpha$  (depending on  $\alpha$ ) defined by (21) as

$$T_\alpha = \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^{bc} |\eta| \left( \|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s} \right)^{b-1}} \right]^{\frac{\alpha}{\alpha-1}},$$

and there exists a (unique) solution  $u_\alpha \in E_{T_\alpha}$  of the equation (14). Our starting point is to obtain a lower bound for the time  $T_\alpha$  which does not depend on  $\alpha$ .

By our hypothesis given in (10) we can set  $0 < \varepsilon \ll 1$  such that for all  $1 + \varepsilon < \alpha < 2$  we have:

$$\left| \left( \|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s} \right) - \left( \|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s} \right) \right| \leq \frac{1}{2} \left( \|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s} \right),$$

hence we get the control

$$\left( \|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s} \right) \leq \frac{3}{2} \left( \|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s} \right), \quad 1 + \varepsilon < \alpha < 2, \quad (45)$$

and then, we can write:

$$\frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}} \leq T_\alpha.$$

Moreover, since  $1 + \varepsilon < \alpha < 2$  then the expression on the left-hand can be estimated from below by the following quantity:

$$T_0 = \max \left( \frac{1}{2} \left[ \frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{2/\varepsilon}, \frac{1}{2} \left[ \frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{1+\varepsilon} \right). \quad (46)$$

At the Appendix 7 we verify in detail this estimate. Then, for all  $1 + \varepsilon < \alpha < 2$  we have  $T_0 \leq T_\alpha$ .

Once we have the lower estimate  $T_0 \leq T_\alpha$ , we remark that for all  $1 + \varepsilon < \alpha < 2$  the solution  $u_\alpha$  of the equation (14), which is constructed in the Proposition 4.1 by the Picard’s fixed point argument, verifies  $u_\alpha \in E_{T_0}$  and moreover we have the estimate  $\|u_\alpha\|_{E_{T_0}} \leq c_0 (\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s})$ .

We also remark that by Proposition 4.3 the solution  $u_\alpha$  is extended to a global in time solution by a well-known iterative argument: for every interval  $[kT_0, (k + 1)T_0]$  (with  $k \in \mathbb{N}^*$ ) we set the initial datum  $u_\alpha(kT_0, \cdot)$  and we apply again the Picard’s fixed point schema to obtain a (unique) solution  $u_\alpha \in E_{[kT_0, (k+1)T_0]}$  (recall that the space  $E_{[kT_0, (k+1)T_0]}$  is defined in (19) and (20)). Moreover, there exists a constant  $c_k > 0$  such that we have

$$\|u_\alpha\|_{E_{[kT_0, (k+1)T_0]}} \leq c_k (\|u_\alpha(kT_0, \cdot)\|_{L^1} + \|u_\alpha(kT_0, \cdot)\|_{H^s}).$$

We study now the expression  $c_k (\|u_\alpha(kT_0, \cdot)\|_{L^1} + \|u_\alpha(kT_0, \cdot)\|_{H^s})$ . For the quantity  $\|u_\alpha(kT_0, \cdot)\|_{L^1}$ , by (30) we have  $\|u_\alpha(kT_0, \cdot)\|_{L^1} \leq \|u_{0,\alpha}\|_{L^1}$ . Then, we can write

$$c_k (\|u_\alpha(kT_0, \cdot)\|_{L^1} + \|u_\alpha(kT_0, \cdot)\|_{H^s}) \leq c_k (\|u_{0,\alpha}\|_{L^1} + \|u_\alpha(kT_0, \cdot)\|_{H^s}).$$

On the other hand, for the quantity  $\|u_\alpha(kT_0, \cdot)\|_{H^s}$ , we remark that we have

$$\begin{aligned} \|u_\alpha(kT_0, \cdot)\|_{H^s} &\leq \sup_{(k-1)T_0 \leq t \leq kT_0} \|u_\alpha(t, \cdot)\|_{H^s} \leq \|u_\alpha\|_{E_{[(k-1)T_0, kT_0]}} \\ &\leq c_{k-1} (\|u_{0,\alpha}\|_{L^1} + \|u_\alpha((k-1)T_0, \cdot)\|_{H^s}). \end{aligned}$$

Thus, we can iterate these estimates and for a constant  $C_k > 0$  big enough (in particular we must have  $C_k > \prod_{j=0}^k c_j$ ) we obtain  $\|u_\alpha(kT_0, \cdot)\|_{H^s} \leq C_k(\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s})$ . Consequently, for all  $k \in \mathbb{N}^*$  we have

$$\|u_\alpha\|_{E_{[kT_0, (k+1)T_0]}} \leq C_k(\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s}). \tag{47}$$

Now, we are able to prove the estimate (44) stated in this lemma. For the time  $T$  there exists  $k_T \in \mathbb{N}$  (which depends on  $T$ ) such that we have  $k_T T_0 \leq T \leq (k_T + 1)T_0$ . Then, as  $s > n/2$  by the product laws in the Sobolev spaces we can write

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u_\alpha^b(t, \cdot)\|_{H^s} &\leq \sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot)\|_{H^s}^b \leq \left( \sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot)\|_{H^s} \right)^b \\ &\leq \left( \sum_{j=0}^{k_T} \sup_{jT_0 \leq (j+1)T_0} \|u_\alpha(t, \cdot)\|_{H^s} \right)^b \leq \left( \sum_{j=0}^{k_T} \|u_\alpha\|_{E_{[jT_0, (j+1)T_0]}} \right)^b. \end{aligned}$$

Then, by the control given in (47) and the control given in (45) we have

$$\begin{aligned} \left( \sum_{j=0}^{k_T} \|u_\alpha\|_{E_{[jT_0, (j+1)T_0]}} \right)^b &\leq \left( \sum_{j=0}^{k_T} C_j(\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s}) \right)^b \\ &\leq \left( \sum_{j=0}^{k_T} C_j \right)^b (\|u_{0,\alpha}\|_{L^1} + \|u_{0,\alpha}\|_{H^s})^b \\ &\leq \left( \sum_{j=0}^{k_T} C_j \right)^b c (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^b \\ &= \mathcal{C}(\varepsilon, b, \|u_{0,2}\|_{L^1}, \|u_{0,2}\|_{H^s}, T). \end{aligned}$$

To finish the proof of this lemma, we just remark that the constant  $\mathcal{C}$  defined above also depends on the parameter  $\varepsilon$ , since the  $k_T$  depends on  $T$  and  $T_0$ ; and the time  $T$  given in (46) depends on  $\varepsilon$ . □

With the estimates (43) and (44) at our disposal, we get back to the estimate (42) to write

$$J_{\alpha,1} \leq C |\eta| T^2 |2 - \alpha| \leq \mathbf{C} |\eta| T^2 \max\left((2 - \alpha)^\gamma, 2 - \alpha\right). \tag{48}$$



On the other hand, We study now the term  $J_{\alpha,2}$  given in (41). For this, by the Young inequalities we write:

$$J_{\alpha,2} \leq c |\eta| \sup_{0 \leq t \leq T} \int_0^t \|\nabla h(t-s, \cdot)\|_{L^1} \|u_\alpha^b(s, \cdot) - u_2^b(s, \cdot)\|_{L^\infty} ds = (a).$$

Here, by the well-known properties of the heat kernel  $h(t, \cdot)$  we have the estimate  $\|\nabla h(t-s, \cdot)\|_{L^1} \leq c(t-s)^{-1/2}$ . Thereafter, in order to estimate the term  $\|u_\alpha^b(s, \cdot) - u_2^b(s, \cdot)\|_{L^\infty}$ , since  $s > n/2$  and by Lemma 5.1 we can write

$$\begin{aligned} \|u_\alpha^b(s, \cdot) - u_2^b(s, \cdot)\|_{L^\infty} &= \left\| (u_\alpha(s, \cdot) - u_2(s, \cdot)) \sum_{j=0}^{b-1} u_\alpha^{b-1-j}(s, \cdot) u_2^j(s, \cdot) \right\|_{L^\infty} \\ &\leq \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \sum_{j=0}^{b-1} \|u_\alpha(s, \cdot)\|_{L^\infty}^{b-1-j} \|u_2(s, \cdot)\|_{L^\infty}^j \\ &\leq \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \sum_{j=0}^{b-1} \|u_\alpha(s, \cdot)\|_{H^s}^{b-1-j} \|u_2(s, \cdot)\|_{H^s}^j \\ &\leq C \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty}, \end{aligned}$$

where the constant  $C > 0$  does not depend on  $\alpha$ . With these estimates we obtain

$$\begin{aligned} (a) &\leq C |\eta| \sup_{0 \leq t \leq T} \int_0^t (t-s)^{-1/2} \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} ds \\ &\leq C |\eta| T^{1/2} \left( \sup_{0 \leq s \leq T} \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \right), \end{aligned}$$

and consequently we have

$$J_{\alpha,2} \leq C |\eta| T^{1/2} \left( \sup_{0 \leq s \leq T} \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \right). \tag{49}$$

Once we estimated the terms  $I_\alpha$ ,  $J_{\alpha,1}$  and  $J_{\alpha,2}$  in (40), (48) and (49) respectively, we get back to (32) we can write

$$\begin{aligned} \sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} &\leq I_\alpha + J_{\alpha,1} + J_{\alpha,2} \leq I_\alpha + J_{\alpha,1} \\ &\quad + C |\eta| T^{1/2} \left( \sup_{0 \leq s \leq T} \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \right). \end{aligned} \tag{50}$$

In this estimate, first we set a time  $0 < T_1 < T$  small enough such that it verifies

$$C |\eta| T_1^{1/2} \leq \frac{1}{2},$$

we get:

$$\sup_{0 \leq t \leq T_1} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1} + \frac{1}{2} \left( \sup_{0 \leq s \leq T_1} \|u_\alpha(s, \cdot) - u_2(s, \cdot)\|_{L^\infty} \right),$$

and then we can write

$$\frac{1}{2} \sup_{0 \leq t \leq T_1} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} \leq I_\alpha + J_{\alpha,1}.$$

Hence, by (40) and (48) we obtain

$$\begin{aligned} \sup_{0 \leq t \leq T_1} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} &\leq C(1 + T_1 + T_1^2) \max((2 - \alpha)^\gamma, (2 - \alpha)) \\ &\leq C(1 + T + T^2) \max((2 - \alpha)^\gamma, (2 - \alpha)). \end{aligned}$$

Finally, we iterate this argument on the intervals  $[kT_1, (k + 1)T_1]$ , with  $k \in \mathbb{N}$ , and then, for the time  $0 < T < +\infty$  we have

$$\sup_{0 \leq t \leq T} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty} \leq C(1 + T + T^2) \max((2 - \alpha)^\gamma, 2 - \alpha).$$

Theorem 2.2 is now proven. □

### 5.2 Proof of Corollary 2.1

For  $0 < T < +\infty$  fixed, and moreover, for  $1 \leq p < +\infty$  and  $1 < q < +\infty$ , by the interpolation inequalities in the spatial variable (with  $\theta = 1/q$ ) we write

$$\begin{aligned} &\left( \int_0^T \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^q}^p dt \right)^{1/p} \\ &\leq \left( \int_0^T \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^1}^{p\theta} \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty}^{p(1-\theta)} dt \right)^{1/p} = (a). \end{aligned} \tag{51}$$

By the hypothesis (10), we can set  $M > 0$  (small enough) such that for  $1 + \varepsilon < \alpha < 2$  we have  $\|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^1} \leq M$ . Then we obtain

$$(a) \leq M^\theta \left( \int_0^T \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty}^{p(1-\theta)} dt \right)^{1/p} \leq M^\theta \|u_\alpha(t, \cdot) - u_2(t, \cdot)\|_{L^\infty}^{(1-\theta)} T^{1/p}.$$

Thus, the wished estimate follows from the inequality (13) proven in Theorem 2.2. Moreover, the case  $p = +\infty$  follows the same lines above with the obvious modifications.  $\square$

**Data Availability** Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

### Declarations

**Conflicts of interest** This work has not received any financial support. In addition, the authors declare that they have no conflicts of interest.

### Appendix

We prove here the estimate (37). We recall the expression

$$f'_\xi(\alpha) = -t_1 e^{-t_1 |\xi|^\alpha} |\xi|^\alpha \ln(|\xi|), \quad 1 < \alpha < 2 + \delta, \quad 0 < t_1 \leq T.$$

Then, we write

$$\begin{aligned} \left\| \|f'_\xi\|_{L^\infty([1, 2+\delta])} \right\|_{L^\infty(\mathbb{R}^n)} &\leq \left\| \|f'_\xi\|_{L^\infty([1, 2+\delta])} \right\|_{L^\infty(|\xi| \leq 1)} \\ &\quad + \left\| \|f'_\xi\|_{L^\infty([1, 2+\delta])} \right\|_{L^\infty(|\xi| > 1)} \\ &= A + B, \end{aligned} \tag{52}$$

where, we shall estimate the terms  $A$  and  $B$  separately. For the term  $A$ , as we have  $|\xi| \leq 1$ ,  $1 < \alpha < 2 + \delta$ , and moreover, as we have  $\lim_{|\xi| \rightarrow 0^+} |\xi| \ln(|\xi|) = 0$ , then we can write:

$$A \leq T \left( \sup_{\xi \in \mathbb{R}^n} e^{-t_1 |\xi|^{2+\delta}} |\xi| \ln(|\xi|) \right) \leq C T.$$

For the term  $B$ , since  $|\xi| > 1$  then we can write

$$B \leq T \left( \sup_{\xi \in \mathbb{R}^n} e^{-t_1 |\xi|} |\xi|^{2+\delta} \ln(|\xi|) \right) \leq C T.$$

### Appendix

Here we give a proof of the estimate

$$\begin{aligned}
 T_0 &= \max \left( \frac{1}{2} \left[ \frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{2/\varepsilon}, \right. \\
 &\quad \left. \frac{1}{2} \left[ \frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{1+\varepsilon} \right) \\
 &\leq \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}}.
 \end{aligned}$$

First, as we have  $1 + \varepsilon < \alpha < 2$ , then we get  $1 - \frac{1}{1+\varepsilon} < 1 - \frac{1}{\alpha}$ , and we can write

$$\frac{1}{2} \left[ \frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}} \leq \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}}.$$

Thereafter, by the sake of simplicity, we denote

$$\frac{1 - \frac{1}{1+\varepsilon}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} = (a),$$

and we have

$$\frac{1}{2} [(a)]^{\frac{\alpha}{\alpha-1}} \leq \frac{1}{2} \left[ \frac{1 - \frac{1}{\alpha}}{2^b c |\eta| (\|u_{0,2}\|_{L^1} + \|u_{0,2}\|_{H^s})^{b-1}} \right]^{\frac{\alpha}{\alpha-1}}.$$

We study now the expression  $\frac{\alpha}{\alpha-1}$ . Since we have  $1 + \varepsilon < \alpha < 2$  then we get  $1 + \varepsilon < \frac{\alpha}{\alpha-1} < \frac{2}{\varepsilon}$ . Thus, on the one hand, if the quantity  $(a)$  above verifies  $(a) < 1$  then we have  $\frac{1}{2} [(a)]^{\frac{2}{\varepsilon}} \leq \frac{1}{2} [(a)]^{\frac{\alpha}{\alpha-1}}$ . On the other hand, if the quantity  $(a)$  verifies  $(a) \geq 1$  then we have  $\frac{1}{2} [(a)]^{1+\varepsilon} \leq \frac{1}{2} [(a)]^{\frac{\alpha}{\alpha-1}}$ .

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