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Existence of solutions for a class of nonlinear elliptic problems with measure data in the setting of Musielak–Orlicz –Sobolev spaces

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Abstract

We study the existence of solutions for some nonlinear elliptic problems of the type $-\mathrm{div}(b(x,u,\nabla u)+F(x,u))=\nu$ in Ω, in the setting of Musielak–Orlicz spaces. The lower order term F verifies the natural growth condition, no Δ_2-condition is assumed on the Musielak function, and the datum ν is assumed to belong to $L^1(\Omega)+W^{-1}E_{\psi}(\Omega)$.

Keywords Musielak–Orlicz–Sobolev spaces · Elliptic equation · Renormalized solutions · Truncations

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1 Introduction and basic assumptions

In this note we will prove an existence of a renormalized solutions for the following nonlinear boundary value problem :

$$\begin{cases} B(u) - div\Big(F(x,u)\Big) = \nu & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N \geq 2$, $B(u) = -\text{div}(b(x,u,\nabla u))$ is a Leray-Lions operator defined from the space $W_0^1 L_\varphi(\Omega)$ into its dual $W^{-1}L_{\overline{\varphi}}(\Omega)$, with φ and $\overline{\varphi}$ are two complementary Musielak-Orlicz functions and where b is a function satisfying the following conditions:

$$b \, : \, \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N \text{ is a Carath éodory function.} \tag{1.2}$$

There exist two Musielak-Orlicz functions φ and P such that $P << \varphi$, a positive function $d(x) \in E_{\overline{\varphi}}(\Omega)$, $\alpha > 0$ and $k_i > 0$ for $i = 1, \cdots, 4$, such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$ and all $\xi,\ \xi' \in \mathbb{R}^N$, $\xi \neq \xi'$:

$$|b(x,s,\xi)| \leq k_1 \Big(d(x) + \overline{\varphi}_x^{-1}\big(P\big(x, k_2|s|\big)\big) + \overline{\varphi}_x^{-1}\big(\varphi\big(x, k_3|\xi|\big)\big) \Big) \tag{1.3}$$

$$\big(b(x,s,\xi) - b\big(x,s,\xi'\big)\big)\big(\xi - \xi'\big) > 0, \tag{1.4}$$

$$b(x,s,\xi).\xi \geq \alpha\varphi(x,|\xi|). \tag{1.5}$$

The lower order term F is a Carathéodory function satisfying, for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, the following condition:

$$|F(x,s)| \leq c(x)\overline{\varphi}_x^{-1}\varphi(x,\alpha_0|s|), \tag{1.6}$$

where $c(.) \in L^\infty(\Omega)$ such that

$$\|c(.)\|_{L^\infty} \leq \min\left(\frac{\alpha}{\alpha_0 + 1}; \frac{\alpha}{2\big(\alpha_0 + 1\big)} \right) \text{and } 0 < \alpha_0 < 1. \tag{1.7}$$

The right hand side of (1.1) is assumed to satisfy

$$\nu \in L^1(\Omega) + W^{-1}E_{\overline{\varphi}}(\Omega) \, : \, \nu = f - \text{div}(\phi) \text{ with } f \in L^1(\Omega) \text{ and } \phi \in \big(E_{\overline{\varphi}}(\Omega)\big)^N. \tag{1.8}$$

In the usual Sobolev spaces, the concept of renormalized solutions was introduced by Diperna and Lions in [22] for the study of the Boltzmann equations, this notion of solutions was then adapted to the study of the problem (1.1) by Boccardo et al. in [21] when the right hand side is in $W^{-1,p'}(\Omega)$ and in the case where the nonlinearity g depends only on x and u, this work was then studied by Rakotoson in [31] when the right hand side is in $L^1(\Omega)$, and finally by DalMaso et al. in [23] for the case in which the right hand side is general measure data.

On Orlicz-Sobolev spaces and in variational case, Benkirane and Bennouna have studied in [8] the problem (1.1) where $\Phi(x,u) \equiv \Phi(u)$, and the nonlinearity g depends only on x and u under the restriction that the N-function satisfies the Δ_2-condition, this work was then extended in [4] by Aharouch, Bennouna and Touzani for N-function not satisfying necessarily the Δ_2-condition and $\Phi(x,u) \equiv \Phi(u)$. If g depends also on ∇u, the problem (1.1) has been solved by Aissaoui Fqayeh, Benkirane, El Moumni and Youssfi in [5] where $\Phi(x,u) \equiv \Phi(u)$, and without assuming the Δ_2-condition on the N-function.

In the framework of variable exponent Sobolev spaces, Bendahmane and Wittbold have treated in [7] the nonlinear elliptic equation (1.1) where $a(x,u,\nabla u) = |\nabla u|^{p(x)-2}\nabla u$, $\Phi \equiv 0$, $g \equiv 0$ and where $f \in L^1(\Omega)$, they proved the existence and uniqueness of a renormalized solution in Sobolev space with variable exponents $W_0^{1,p(x)}(\Omega)$.

In the variational case of Musielak-Orlicz spaces and in the case where $g \equiv 0$ and $\Phi \equiv 0$, an existence result for (1.1) has been proved by Benkirane and Sidi El Vally in [10] a when the non-linearity g depends only on x and u. If g depends also on ∇u, the problem (1.1) has recently been solved by N. El Amarty, B. El Haji and M. El Moumni in [18] where $\Phi(x,u) \equiv \Phi(u)$.

and several researches deals with the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts (see [6, 11–16, 18–20] for more details).

The paper is organized as follows: In Sect. 2, we give some preliminaries and background. Section 3 is devoted to some technical lemmas which can be used to our result. In Sect. 4, we state our main result and in Sect. 5 we give the proof of an existence solution .

2 Some preliminaries and background

Here we give some definitions and properties that concern Musielak-Orlicz spaces (see [17]). Let Ω be an open subset of $\mathbb{R}^N$, a Musielak-Orlicz function φ is a real-valued function defined in $\Omega \times \mathbb{R}^+$ such that

a) $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, nondecreasing, continuous, $\varphi(x,0) = 0$, $\varphi(x,t) > 0$ for all $t > 0$ and $\lim_{t\to 0}\sup_{x\in\Omega} \frac{\varphi(x,t)}{t} = 0$ and $\lim_{t\to\infty}\inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$).

b) $\varphi(.,t)$ is a measurable function for all $t \geq 0$.

For a Musielak–Orlicz function φ, let $\varphi_x(t) = \varphi(x,t)$ and let φ_x^{-1} be the nonnegative reciprocal function with respect to t, i.e. the function that satisfies

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi\left(x, \varphi_x^{-1}(t)\right) = t.$$

The Musielak–Orlicz function φ is said to satisfy the Δ_2 -condition if for some $k > 0$, and a nonnegative function h, integrable in Ω, we have

$$\varphi(x, 2t) \leq k\varphi(x, t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \tag{2.1}$$

When (2.1) holds only for $t \geq t_0 > 0$, then φ is said to satisfy the Δ_2-condition near infinity. Let φ and γ be two Musielak–Orlicz functions, we say that φ dominate γ and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants c and t_0 such that for a.e. $x \in \Omega$:

$$\gamma(x, t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \text{ (resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that γ grows essentially less rapidly than φ at 0 (resp. near infinity) and we write $\gamma \prec\prec \varphi$ if for every positive constant c we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0, \quad \text{(resp. } \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\gamma(x, ct)}{\varphi(x, t)}\right) = 0).$$

Remark 1 (see [33]) If $\gamma \prec\prec \varphi$ near infinity, then $\forall\varepsilon > 0$ there exists a nonnegative integrable function h, such that

$$\gamma(x, t) \leq \varphi(x, \varepsilon t) + h(x) \quad \text{for all } t \geq 0 \text{ and for a.e. } x \in \Omega. \tag{2.2}$$

For a Musielak-Orlicz function φ and a measurable function $u : \Omega \longrightarrow \mathbb{R}$, we define the functional

$$\rho_{\varphi,\Omega}(u) = \int_\Omega \varphi(x, |u(x)|)\, dx.$$

The set $K_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable/ } \rho_{\varphi,\Omega}(u) < \infty\right\}$ is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \left\{u : \Omega \longrightarrow \mathbb{R} \text{ measurable/ } \rho_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}$$

For a Musielak-Orlicz function φ we put:

$$\overline{\varphi}(x, s) = \sup_{t>0}\{st - \varphi(x, t)\},$$

Note that $\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ (or conjugate of φ) in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\|u\|_{\varphi,\Omega} = \inf\left\{\lambda > 0/ \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}$$

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$|||u|||_{\varphi,\Omega} = \sup_{\|v\|_{\overline{\varphi}}\leq 1} \int_\Omega |u(x)v(x)|\,dx$$

where $\overline{\varphi}$ is the Musielak-Orlicz function complementary to φ. These two norms are equivalent (see [17]). The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$, It is a separable space (see [17], Theorem 7.10).

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\varphi,\Omega}\Big(\frac{u_n - u}{\lambda}\Big) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \Big\{ u \in L_\varphi(\Omega) / \ \forall |\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega) \Big\}$$

and

$$W^m E_\varphi(\Omega) = \Big\{ u \in E_\varphi(\Omega) / \ \forall |\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega) \Big\}$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ with nonnegative integers α_i, $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak-Orlicz Sobolev space. Let for $u \in W^m L_\varphi(\Omega)$:

$$\overline{\rho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \rho_{\varphi,\Omega}(D^\alpha u) \text{ and } \|u\|^m_{\varphi,\Omega} = \inf\Big\{ \lambda > 0 / \ \overline{\rho}_{\varphi,\Omega}\Big(\frac{u}{\lambda}\Big) \leq 1 \Big\}$$

these functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\Big(W^m L_\varphi(\Omega), \|.\|^m_{\varphi,\Omega}\Big)$ is a Banach space if φ satisfies the following condition (see [17]):

$$\text{There exist a constant } c_0 > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x,1) \geq c_0. \tag{2.3}$$

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closed.

The space $W^m_0 L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$, and the space $W^m_0 E_\varphi(\Omega)$ as the closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_{\overline{\varphi}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$. The following spaces of distributions will also be used:

$$W^{-m} L_{\overline{\varphi}}(\Omega) = \Big\{ f \in \mathcal{D}'(\Omega) / f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\varphi(\Omega) \Big\}$$

and

$$W^{-m}E_{\overline{\varphi}}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega)/ f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^{\alpha} f_{\alpha} \text{ with } f_{\alpha} \in E_{\varphi}(\Omega) \right\}.$$

We say that a sequence of functions $u_n \in W^m L_{\varphi}(\Omega)$ is modular convergent to $u \in W^m L_{\varphi}(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \overline{\rho}_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

We recall that

$$\varphi(x,t) \leq t\overline{\varphi}^{-1}(\varphi(x,t)) \leq 2\varphi(x,t) \quad \text{for all } t \geq 0. \tag{2.4}$$

For φ and her complementary function $\overline{\varphi}$, the following inequality is called the Young inequality (see [17]):

$$ts \leq \varphi(x,t) + \overline{\varphi}(x,s), \quad \forall t, s \geq 0, \text{ a.e. } x \in \Omega. \tag{2.5}$$

This inequality implies that

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) + 1 \tag{2.6}$$

In $L_{\varphi}(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi,\Omega} \leq \rho_{\varphi,\Omega}(u) \quad \text{if } \|u\|_{\varphi,\Omega} > 1 \tag{2.7}$$

and

$$\|u\|_{\varphi,\Omega} \geq \rho_{\varphi,\Omega}(u) \quad \text{if } \|u\|_{\varphi,\Omega} \leq 1 \tag{2.8}$$

For two complementary Musielak-Orlicz functions φ and $\overline{\varphi}$, let $u \in L_{\varphi}(\Omega)$ and $v \in L_{\overline{\varphi}}(\Omega)$, then we have the Hölder inequality (see [17]):

$$\left| \int_{\Omega} u(x)v(x)\,dx \right| \leq \|u\|_{\varphi,\Omega} |||v|||_{\overline{\varphi},\Omega}. \tag{2.9}$$

3 Some technical lemmas

This section concern some technical lemmas that will be used in our main result.

Definition 3.1 We say that a Musielak function φ verifies the log-Hölder continuity hypothesis on Ω if there exists $A > 0$ such that

$$\frac{\varphi(x,t)}{\varphi(y,t)} \leq t\left(\frac{A}{\log\left(\frac{1}{|x-y|}\right)}\right)$$

$\forall t \geq 1$ and $\forall x, y \in \Omega$ with $|x-y| \leq \frac{1}{2}$

Lemma 3.1 [2] *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function verifying the log-Hölder continuity such that*

$$\bar{\varphi}(x,1) \leq c_1 \quad \text{a.e in } \Omega \text{ for some } c_1 > 0 \tag{3.1}$$

Then $\mathfrak{D}(\Omega)$ is dense in $L_\varphi(\Omega)$ and in $W_0^1 L_\varphi(\Omega)$ for the modular convergence.

Remark 2 Note that if $\lim_{t\to\infty} \inf_{x\in\Omega} \frac{\varphi(x,t)}{t} = \infty$, then (3.1) holds (see [2]).

Example 3.1 Let $p \in P(\Omega)$ a bounded variable exponent on Ω, such that there exists a constant $A > 0$ such that for all points $x, y \in \Omega$ with $|x-y| < \frac{1}{2}$, we have the inequality

$$|p(x) - p(y)| \leq \frac{A}{\log\left(\frac{1}{|x-y|}\right)}$$

We can show that the Musielak function defined by $\varphi(x,t) = t^{p(x)} \log(1+t)$ satisfies the hypothesis of Lemma 3.1.

Proof (see [2]). □

Lemma 3.2 [2] (*Poincare's inequality*: *Integral form*) *Let Ω be a bounded Lipschitz domain of $R^N (N \geq 2)$ and let φ be a Musielak function satisfying the hypothesis of Lemma 3.1. Then there exists $\beta, \eta > 0$ and $\lambda > 0$ depending only on Ω and φ such that*

$$\int_\Omega \varphi(x,|v|)dx \leq \beta + \eta \int_\Omega \varphi(x, \lambda|\nabla v|)dx \text{ for all } v \in W_0^1 L_\varphi(\Omega). \tag{3.2}$$

□

Corollary 3.3 [2] (*Poincare's inequality*) *Let Ω be a bounded Lipchitz domain of $\mathbb{R}^N (N \geq 2)$ and let φ be a Musielak function satisfying the same hypothesis of Lemma 3.2. Then there exists $C > 0$ such that*

$$\|v\|_\varphi \leq C\|\nabla v\|_\varphi \quad \forall v \in W_0^1 L_\varphi(\Omega).$$

Lemma 3.4 ([30]) *Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let φ be a Musielak-Orlicz function and let $u \in W_0^1 L_\varphi(\Omega)$. Then $F(u) \in W_0^1 L_\varphi(\Omega)$.*

Hawever, if the set D of discontinuity points of F' is finite, we obtain

$$\frac{\partial F(u)}{\partial x_i} = \begin{cases} F'(u)\frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \in D\} \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \notin D\}. \end{cases}$$

Lemma 3.5 [1] (*Poincare's inequality*). *Let φ a Musielak-Orlicz function which satisfies the hypothesis of Lemma 3.1, let $\varphi(x,t)$ decreases with respect of one of coordinate of x, then, that exists $c>0$ depends only of Ω such that*

$$\int_\Omega \varphi(x,|v|)\,dx \le \int_\Omega \varphi(x,c|\nabla v|)\,dx \quad \forall u \in W_0^1L_\varphi(\Omega).$$

Lemma 3.6 [9] *Let Ω satisfies the segment property and suppose that $u \in W_0^1L_\varphi(\Omega)$. Then, there exists a sequence $\left(u_n\right) \subset \mathcal{D}(\Omega)$ such that*

$$u_n \to u \text{ for modular convergence in } W_0^1L_\varphi(\Omega).$$

In addition to this, if $u \in W_0^1L_\varphi(\Omega) \cap L^\infty(\Omega)$ then $\|u_n\|_\infty \le (N+1)\|u\|_\infty$.

Lemma 3.7 *Suppose that $\left(g_n\right),\ g \in L^1(\Omega)$ such that*

(*i*) $g_n \ge 0$ *a.e in* Ω,

(*ii*) $g_n \longrightarrow g$ *a.e in* Ω,

(*iii*) $\int_\Omega g_n(x)\,dx \longrightarrow \int_\Omega g(x)\,dx.$

Then $g_n \longrightarrow g$ strongly in $L^1(\Omega)$.

Lemma 3.8 [10] *If a sequence $h_n \in L_\varphi(\Omega)$ converges in measure to a measurable function h and if h_n remains bounded in $L_\varphi(\Omega)$, then $h \in L_\varphi(\Omega)$ and $h_n \rightharpoonup h$ for $\sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right)$.*

Lemma 3.9 [10] *Let $v_n,\ v \in L_\varphi(\Omega)$. If $v_n \to v$ with respect to the modular convergence, then $v_n \to v$ for $\sigma\left(L_\varphi(\Omega), L_{\overline{\varphi}}(\Omega)\right)$.*

Lemma 3.10 [25] *If $\gamma \prec \varphi$ and $u_n \to u$ for the modular convergence in $L_\varphi(\Omega)$ then $u_n \to u$ strongly in $E_\gamma(\Omega)$.*

Lemma 3.11 (*The Nemytskii Operator*). *Suppose that Ω be an open subset of $\mathbb{R}^N$ with finite measure and let φ and ψ be two Musielak Orlicz functions. Suppose that $g : \Omega \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:*

$$|g(x,s)| \le c(x) + k_1\psi_x^{-1}\varphi\left(x, k_2|s|\right)$$

where k_1 and k_2 are real positives constants and $c(.) \in E_\psi(\Omega)$. Then the Nemytskii Operator N_g defined by $N_g(u)(x) = g(x,u(x))$ is continuous from

$$\mathcal{P}\left(E_M(\Omega), \frac{1}{k_2}\right)^p = \prod \left\{ u \in L_M(\Omega) : d\left(u, E_M(\Omega)\right) < \frac{1}{k_2} \right\}$$

into $\left(L_\psi(\Omega)\right)^q$ *for the modular convergence. However if* $c(\cdot) \in E_\gamma(\Omega)$ *and* $\gamma << \psi$ *then* N_g *is strongly continuous from* $\mathcal{P}\Big(E_M(\Omega), \frac{1}{k_2}\Big)^p$ *to* $\left(E_\gamma(\Omega)\right)^q$.

4 Main result

We now give the definition of a renormalized solution of (1.1).

Definition 4.1 A measurable function $u : \Omega \to \mathbb{R}$ is called a renormalized solution of (1.1) if:

$$T_k(u) \in W_0^1 L_\varphi(\Omega) \quad \text{and} \quad b(x, u, \nabla u) \in \left(L_{\overline{\varphi}}(\Omega)\right)^N, \tag{4.1}$$

$$\lim_{m \to +\infty} \int_{\{x \in \Omega: \quad m \le |u(x)| \le m+1\}} b(x, u, \nabla u) \nabla u \, dx = 0, \tag{4.2}$$

and for every function $h \in C_c^1(\mathbb{R})$ such that

$$-\text{div}\Big(b(x, u, \nabla u) h(u)\Big) - \text{div}\Big(F(x, u) h(u)\Big) + h'(u) F(x, u) \nabla u \tag{4.3}$$

$$= fh(u) - \text{div}(\phi h(u)) + h'(u) \phi \nabla u \quad \text{ in } \mathcal{D}'(\Omega).$$

Remark 3 Every term in equation (4.3) is meaningful in the distributional sense. Indeed, for $h \in C_c^1(\mathbb{R})$ and $u \in W_0^1 L_\varphi(\Omega)$, then $h(u) \in W^1 L_\varphi(\Omega)$ and for V in $\mathcal{D}(\Omega)$ the function $Vh(u) \in W_0^1 L_\varphi(\Omega)$. Since $\text{div}\Big(b(x, u, \nabla u)\Big) \in W^{-1} L_{\overline{\varphi}}(\Omega)$, we have for every $V \in \mathcal{D}(\Omega)$:

$$\Big\langle \text{div}\Big(b(x, u, \nabla u)\Big) h(u) \; ; \; V \Big\rangle_{\mathcal{D}'(\Omega), \mathcal{D}(\Omega)} = \Big\langle \text{div}\Big(b(x, u, \nabla u)\Big) \; ; \; Vh(u) \Big\rangle_{W^{-1} L_{\overline{\varphi}}(\Omega), W_0^1 L_\varphi(\Omega)}$$

Finally, $F(x, u) h(u) \in (L^\infty(\Omega))^N$, $F(x, u) h'(u) \in (L^\infty(\Omega))^N$, $\text{div}\Big(F(x, u) h(u)\Big) \in W^{-1} L_{\overline{\varphi}}(\Omega)$ and $F(x, u) h'(u) \nabla u \in L_\varphi(\Omega)$.

Our main result is the following

Theorem 4.1 *Under assumptions* (1.2)-(1.8) *there exists at least a renormalized solution of Problem* (1.1).

Remark 4 Actually the original equation (1.1) will be recovered whenever $h(u) \equiv 1$ but unfortunately this cannot happen in general strong additional requirements on u. Therefore, (4.3) is to be viewed as a weaker form of (1.1).

Remark 5 Generalized Orlicz spaces (Musielak-Orlicz-sobolev spaces), Orlicz spaces and $L^{p(\cdot)}$-spaces have different nature, and neither of them is a subset of the other.

Let us list some techniques from the classical case which do not work in $L^{p(\cdot)}$-spaces and some additional ones that do not work in the generalized Orlicz case. Orlicz spaces are similar to L^p-spaces in many regards, but some differences exist.

- Exponents cannot be moved outside the Φ-function, i.e. $\varphi(t^\gamma) \neq \varphi(t)^\gamma$ in general.
- The formula $\varphi^{-1}\left(\int_\Omega \varphi(|f|)dx\right)$ does not define a norm. Techniques which do not work in $L^{p(\cdot)}$-spaces (from [24], pp. 9–10]):
- The space $L^{p(\cdot)}$ is not rearrangement invariant; the translation operator $T_h : L^{p(\cdot)} \to L^{p(\cdot)}, T_h f(x) := f(x+h)$ is not bounded; Young's convolution inequality $\|f * g\|_{p(\cdot)} \leqslant c\|f\|_1 \|g\|_{p(\cdot)}$ does not hold [24], Section 3.6].
- The formula

$$\int_\Omega |f(x)|^p dx = p \int_0^\infty t^{p-1} |\{x \in \Omega : |f(x)| > t\}| dt$$

  has no variable exponent analogue.
- Maximal, Poincaré, Sobolev, etc., inequalities do not hold in a modular form. For instance, A. Lerner showed that the inequality

$$\int_{\mathbb{R}^n} |Mf|^{p(x)} dx \leqslant c \int_{\mathbb{R}^n} |f|^{p(x)} dx$$

  holds if and only if $p \in (1, \infty)$ is constant [29], Theorem 1.1]. For the Poincaré inequality see [24], Example 8.2.7] and the discussion after it.
- Interpolation is not so useful, since variable exponent spaces never result as an interpolant of constant exponent spaces (see Sect. 5.5).
- Solutions of the $p(\cdot)$-Laplace equation are not scalable, i.e. λu need not be a solution even if u is [24], Example 13.1.9]. New obstructions in generalized Orlicz spaces:
- We cannot estimate $\varphi(x,t) \lesssim \varphi(y,t)^{1+\epsilon} + 1$ even when $|x-y|$ is small, because of lack of polynomial growth. This complicates e.g. the use of higher integrability in PDE proofs.
- It is not always the case that $\chi_E \in L^\varphi(\Omega)$ when $|E| < \infty$.

5 Proof of Theorem 4.1

Throughout the paper, T_k denotes the truncation function at height $k \geq 0$:

$$T_k(s) = \max(-k, \min(k, s))$$

5.1 Approximate problem

For $n \in \mathbb{N}^*$, let define the following approximations of f and Φ. Let f_n be a sequence of $L^\infty(\Omega)$ functions that converge strongly to f in $L^1(\Omega)$, and $\|f_n\|_{L^1} \leq \|f\|_{L^1}$. Let $F_n(x,s) = F\big(x, T_n(s)\big)$. Then we consider the approximate Eq. (1.1) for $n \geq 1$: $u_n \in W_0^1 L_\varphi(\Omega)$

$$-\text{div}\Big(b\big(x,u_n,\nabla u_n\big)\Big)+\text{div}\Big(F_n\big(x,u_n\big)\Big)=f_n-\text{div}(\phi)\quad \text{in}\mathcal{D}'(\Omega). \tag{5.1}$$

there exists at last one solution $u_n\in W_0^1L_\varphi(\Omega)$ of (5.1) (see [26]).

5.2 A priori estimates

Choosing $T_k(u_n)$ as a test function in (5.1), we get

$$\begin{aligned}&\int_\Omega b_n(x,u_n,\nabla u_n)\nabla T_k\big(u_n\big)\,dx+\int_\Omega F_n(x,u_n)\nabla T_k\big(u_n\big)\,dx\\&\quad\leq k\|f_n\|_{L^1(\Omega)}+\int_\Omega \phi\nabla T_k(u_n)\,dx.\end{aligned} \tag{5.2}$$

By (1.6), Lemma 3.5 and Young inequality, we obtain:

$$\begin{aligned}&\int_\Omega F_n(x,u_n)\nabla T_k\big(u_n\big)\,dx\\&\leq \|c(.)\|_{L^\infty(\Omega)}\left[\alpha_0\int_\Omega \varphi\big(x,u_n\big)T_k\big(u_n\big)\,dx+\int_\Omega \varphi\big(x,\big|\nabla u_n\big|\big)T_k\big(u_n\big)dx\right].\\&\leq \|c(.)\|_{L^\infty}\big(\alpha_0+1\big)\int_{Q_\tau}\varphi\Big(x,\Big|\nabla T_k\big(u_n\big)\Big|\Big)dxdt.\end{aligned} \tag{5.3}$$

Recall that

$$\int_\Omega \phi\nabla T_k\big(u_n\big)\,dx\leq\frac{\alpha}{2}\int_\Omega\varphi\big(x,\big|\nabla T_k(u_n)\big|\big)\,dx+c(\Omega,N,\alpha,\phi). \tag{5.4}$$

return to (5.2) and using (5.3) and (5.4) we get

$$\begin{aligned}&\int_\Omega b_n\big(x,u_n,\nabla u_n\big)\nabla T_k\big(u_n\big)dxdt\leq k\|f_n\|_{L^1(\Omega)}+\left[\|c(.)\|_{L^\infty}\big(\alpha_0+1\big)+\frac{\alpha}{2}\right]\\&\quad\int_\Omega\varphi\Big(x,\Big|\nabla T_k\big(u_n\big)\Big|\Big)dxdt\end{aligned} \tag{5.5}$$

by using (1.5) we get

$$\begin{aligned}&\int_\Omega b_n\big(x,u_n,\nabla u_n\big)\nabla T_k\big(u_n\big)dxdt\leq\frac{\left[\|c(.)\|_{L^\infty}\big(\alpha_0+1\big)+\frac{\alpha}{2}\right]}{\alpha}\\&\quad\int_\Omega b_n\big(x,u_n,\nabla u_n\big)\nabla T_k\big(u_n\big)dxdt+k\|f_n\|_{L^1(\Omega)},\end{aligned}$$

thus

$$\left[\frac{1}{2}-\frac{\big[\|c(.)\|_{L^\infty}\big(\alpha_0+1\big)\big]}{\alpha}\right]\int_\Omega b\big(x,u_n,\nabla u_n\big)\nabla\big(T_k\big(u_n\big)\big)\,dx\leq kc_1,$$

We take $\frac{1}{c_2} = \left[\frac{1}{2} - \frac{\left[\|c(.)\|_{L^\infty}\left(\alpha_0+1\right)\right]}{\alpha}\right]$. Then we deduce that

By (1.7) we have $c_2 > 0$ where $C = c_1 c_2$. And by (1.5) we have

$$\int_\Omega \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right)dx \le kC. \tag{5.6}$$

So it follows that $\left(T_k\left(u_n\right)\right)_n$ *is* bounded in $W_0^1L_\varphi(\Omega)$, then there exists some $v_k \in W_0^1L_\varphi(\Omega)$ such that

$$\begin{cases} T_k\left(u_n\right) \rightharpoonup v_k & \text{weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right) \\ T_k\left(u_n\right) \longrightarrow v_k & \text{strongly in } E_{\overline{\varphi}}(\Omega). \end{cases} \tag{5.7}$$

On the other hand, using (5.6), we have

$$\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\delta}\right)\text{meas}\left\{\left|u_n\right| > k\right\} \le \int_{\left\{\left|u_n\right|>k\right\}} \varphi\left(x, \frac{\left|T_k\left(u_n\right)\right|}{\delta}\right)dx$$
$$\le \int_\Omega \varphi\left(x, \left|\nabla T_k\left(u_n\right)\right|\right) dx \le kC.$$

Then

$$\text{meas}\left\{\left|u_n\right| > k\right\} \le \frac{kC}{\inf_{x\in\Omega} \varphi\left(x, \frac{k}{\delta}\right)}$$

for all $n \ge 1$ and for all $k \ge 1$. Assuming that there exists a positive function $\overline{\varphi}$ such that $\lim_{t\to\infty} \frac{\overline{\varphi}(t)}{t} = +\infty$ and $\overline{\varphi}(t) \le ess \inf_{x\in\Omega} \varphi(x,t)$, $\forall t \ge 0$. Thus, we get

$$\lim_{k\to\infty} \text{meas}\left\{\left|u_n\right| > k\right\} = 0. \tag{5.8}$$

Let $\eta > 0$ and $\epsilon > 0$ then

$$\text{meas}\left\{\left|u_n - u_m\right| > \eta\right\} \le \text{meas}\left\{\left|u_n\right| > k\right\} + \text{meas}\left\{\left|u_m\right| > k\right\} + \text{meas}\left\{\left|T_k\left(u_n\right) - T_k\left(u_m\right)\right| > \eta\right\}$$

then, by using (5.7) one suppose that $\left(T_k\left(u_n\right)\right)_n$ is a Cauchy sequence in measure in Ω, Let $\varepsilon > 0$, then by (5.8) there exists some $k = k(\varepsilon) > 0$ such that

$$\text{meas}\left\{\left|u_n - u_m\right| > \eta\right\} < \varepsilon, \quad \text{for all } n,\ m\ \ge h_0(k(\varepsilon), \eta),$$

which means that $\left(u_n\right)_n$ is a Cauchy sequence in measure in Ω, thus converge almost every where to u. Consequently

$$\begin{cases} u_n \rightharpoonup u \text{ weakly in } W_0^1L_\varphi(\Omega) \text{ for } \sigma\left(\Pi L_\varphi, \Pi E_{\overline{\varphi}}\right) \\ u_n \longrightarrow u \text{ strongly in } E_{\overline{\varphi}}(\Omega). \end{cases} \tag{5.9}$$

5.3 Boundedness of $\left(b\left(x,u_n,\nabla u_n\right)\right)_n$ in $\left(L_{\overline{\varphi}}(\Omega)\right)^N$

Let $\vartheta \in \left(E_{\varphi}(\Omega)\right)^N$ such that
$\|\vartheta\|_{\varphi,\Omega} = 1$, we have

$$\int_{\Omega}\left[b\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right) - b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right]\left[\nabla T_k\left(u_n\right) - \frac{\vartheta}{k_3}\right]dx \geq 0.$$

This implies that

$$\begin{aligned}
&\int_{\Omega}\frac{1}{k_3}b\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\vartheta\,dx \\
&\leq \int_{\Omega}b\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\nabla T_k\left(u_n\right)dx - \int_{\Omega}b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\left(\nabla T_k\left(u_n\right) - \frac{\vartheta}{k_3}\right)dx \\
&\leq kC_1 + C_2 - \int_{\Omega}b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\nabla T_k\left(u_n\right)dx + \frac{1}{k_3}\int_{\Omega}b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\vartheta\,dx.
\end{aligned} \tag{5.10}$$

By using Young's inequality in the last two terms of the last side and (5.6) we get

$$\begin{aligned}
&\int_{\Omega}b\left(x,T_k\left(u_n\right),\nabla T_k\left(u_n\right)\right)\vartheta\,dx \\
&\leq k_3(kC_1 + C_2) + 3k_1\left(1+k_3\right)\int_{\Omega}\overline{\varphi}\left(x,\frac{\left|b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right)dx \\
&\quad + 3k_1k_3\int_{\Omega}\varphi\left(x,\left|\nabla T_k\left(u_n\right)\right|\right)dx + 3k_1\int_{\Omega}\varphi(x,|\vartheta|)\,dx \\
&\leq k_3(kC_1 + C_2) + 3k_1k_3(kC_1 + C_2) + 3k_1 + 3k_1\left(1+k_3\right)\int_{\Omega}\overline{\varphi}\left(x,\frac{\left|b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right)dx
\end{aligned} \tag{5.11}$$

Now, by using (1.3) and the convexity of $\overline{\varphi}$ we get

$$\overline{\varphi}\left(x,\frac{\left|b\left(x,T_k\left(u_n\right),\frac{\vartheta}{k_3}\right)\right|}{3k_1}\right) \leq \frac{1}{3}\left(\overline{\varphi}(x,d(x)) + P\left(x,k_2\left|T_k\left(u_n\right)\right|\right) + \varphi(x,|\vartheta|)\right) \tag{5.12}$$

Thanks to Remark 1 there exists $h \in L^1(\Omega)$ such that

$$P\left(x,k_2\left|T_k\left(u_n\right)\right|\right) \leq P\left(x,k_2k\right) \leq \varphi(x,1) + h(x)$$

then by integrating over Ω we deduce that

$$\int_\Omega \overline{\varphi}\left(x, \frac{\left|b\left(x, T_k(u_n), \frac{v}{k_3}\right)\right|}{3k_1}\right) dx$$
$$\leq \frac{1}{3}\left(\int_\Omega \overline{\varphi}(x, c(x))\, dx + \int_\Omega h(x)\, dx + \int_\Omega \varphi(x, 1)\, dx + \int_\Omega \varphi(x, |\vartheta|)\, dx\right) \leq c'_k, \tag{5.13}$$

where c'_k is a constant depending on k, then $\forall \vartheta \in \left(E_\varphi(\Omega)\right)^N$ with $\|\vartheta\|_{\varphi,\Omega} = 1$ we have $\int_\Omega b\left(x, T_k(u_n), \nabla T_k(u_n)\right)\vartheta dx \leq c'_k$, and thus $\left\|b\left(x, T_k(u_n), \nabla T_k(u_n)\right)\right\|_{\overline{\varphi},\Omega} \leq c'_k$, which implies that

$$\left(b\left(x, T_k(u_n), \nabla T_k(u_n)\right)\right)_n \text{ is bounded in } L_{\overline{\varphi}}(\Omega)^N. \tag{5.14}$$

5.4 Renormalization identity for the approximate solutions

Consider the function $Z_m(u_n) = T_1(u_n - T_m(u_n))$ and by taking $Z_m(u_n)$ as test function in (5.1) we obtain

$$\int_\Omega b_n(x, u_n, \nabla u_n)\nabla Z_m(u_n)\, dx + \int_\Omega F_n(x, u_n)\nabla Z_m(u_n)\, dx = \int_\Omega f_n Z_m(u_n) dx + \int_\Omega \phi \nabla Z_m(u_n)\, dx. \tag{5.15}$$

By the same argument used in a priori estimates, we get

$$\int_\Omega \varphi\left(x, \left|\nabla Z_m(u_n)\right|\right) dx \leq C\left[\int_\Omega f_n Z_m(u_n) dx + \int_\Omega \overline{\varphi}\left(x, \frac{|\phi|}{\epsilon_1}\right) Z_m(u_n) dx\right] + C\int_{\{m \leq u_n \leq m+1\}} \overline{\varphi}\left(x, \frac{|\phi|}{\epsilon_1}\right) dx \tag{5.16}$$

where $\frac{1}{C} = \left[\frac{1}{2} - \left(\frac{\|c(\cdot)\|_{L^\infty(\Omega)} + \epsilon_1}{\alpha}\right)\right]$. In order to pass to the limit in (5.16) as $n \to +\infty$, we use the pointwise convergence of u_n and strongly convergence in $L^1(\Omega)$ of f_n, we get

$$\lim_{n \to +\infty} \int_\Omega \varphi\left(x, \left|\nabla Z_m(u_n)\right|\right) dx \leq C\left[\int_\Omega f Z_m(u) dx + \int_\Omega \overline{\varphi}\left(x, \frac{|\phi|}{\epsilon_1}\right) Z_m(u) dx\right] C\int_{\{m \leq u \leq m+1\}} \overline{\varphi}\left(x, \frac{|\phi|}{\epsilon_1}\right) dx \tag{5.17}$$

Thanks to Lebesgue's theorem and passing to the limit as $m \to +\infty$, in every term of the right-hand side of the previous inequalities, we obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega \varphi\Big(x,\Big|\nabla Z_m\big(u_n\big)\Big|\Big)\,dx=0. \tag{5.18}$$

Using (1.6) and Young inequality, for $n>m+1$ we have

$$\begin{aligned}\int_\Omega \Big|F_n\big(x,u_n\big)\nabla Z_m\big(u_n\big)\Big|dx &\le \int_{\{m\le u_n\le m+1\}} \varphi\Big(x,\alpha_0\Big|T_{m+1}\big(u_n\big)\Big|\Big)dx\\ &+\int_\Omega \varphi\Big(x,\Big|\nabla Z_m\big(u_n\big)\Big|\Big)dx.\end{aligned} \tag{5.19}$$

Thanks to Lebesgue's theorem, and by the pointwise convergence of u_n we can have

$$\begin{aligned}\lim_{n\to+\infty}\int_\Omega \Big|F_n\big(x,u_n\big)\nabla Z_m\big(u_n\big)\Big|\,dx &\le \int_{\{m\le u\le m+1\}} \varphi\big(x,\alpha_0\big|T_{m+1}(u)\big|\big)dx\\ &+\lim_{n\to+\infty}\int_\Omega \varphi\big(x,\big|\nabla Z_m(u)\big|\big)\,dx.\end{aligned} \tag{5.20}$$

Passing to the limit in (5.20) as $m\to+\infty$, we obtain

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_\Omega F_n\big(x,u_n\big)\nabla Z_m\big(u_n\big)\,dx=0.$$

Finally passing to the limit in (5.16), we get

$$\lim_{m\to+\infty}\lim_{n\to+\infty}\int_{\{m\le u_n\le m+1\}} b_n\big(x,u_n,\nabla u_n\big)\nabla u_n\,dx=0. \tag{5.21}$$

5.5 Almost everywhere convergence of the gradients

Let $v_j\in\mathcal{D}(\Omega)$ be a sequence such that $v_j\to u$ in $W_0^1L_\varphi(\Omega)$ for the modular convergence. For $m\ge k$, we define the function ϱ_m by

$$\varrho_m(s)=\begin{cases}1 & \text{if } |s|\le m\\ m+1-|s| & \text{if } m\le|s|\le m+1\\ 0 & \text{if } |s|\ge m+1\end{cases}$$

We denote by $\epsilon(n,\eta,j,m)$ all quantities (possibly different) such that

$$\lim_{m\to+\infty}\lim_{j\to+\infty}\lim_{\eta\to+\infty}\lim_{n\to+\infty}\epsilon(n,\eta,j,m)=0.$$

For fixed $k\ge0$, let $W_\eta^{n,j}=T_\eta\big(T_k\big(u_n\big)-T_k\big(v_j\big)\big)$ and $W_\eta^j=T_\eta\big(T_k(u)-T_k\big(v_j\big)\big)$. Multiplying the approximating equation by $W_\eta^{n,j}\varrho_m\big(u_n\big)$, we obtain

$$\begin{aligned}&\int_\Omega b_n\big(x,u_n,\nabla u_n\big)\nabla W_\eta^{n,j}\varrho_m\big(u_n\big)\,dx-\int_\Omega F_n\big(x,u_n\big)\nabla W_\eta^{n,j}\varrho_m\big(u_n\big)\,dx\\ &\le\int_\Omega f_n W_\eta^{n,j}\varrho_m\big(u_n\big)dx+\int_\Omega \phi\nabla W_\eta^{n,j}\varrho_m\big(u_n\big)dx.\end{aligned} \tag{5.22}$$

Remark that if we take $n>m+1$, we obtain

$$F_n(x,u_n)\varrho_m(u_n) = F(x,T_{m+1}(u_n))\varrho_m(T_{m+1}(u_n)),$$

then $F_n(x,u_n)$ is bounded in $L_{\overline{\varphi}}(\Omega)$, thus, by using the pointwise convergence of u_n and Lebesgue's theorem we obtain $F_n(x,u_n)$ converges to $F(x, u)$ with the modular convergence as $n \to +\infty$, then

$$F_n(x,u_n)\varrho_m(u_n) \longrightarrow F(x,u)\varrho_m(u) \text{ for } \sigma(\Pi L_\varphi, \Pi L_\varphi).$$

In the other hand for $0 \le T_k(u_n) - T_k(v_j) \le \eta$ then $\nabla W_\eta^{n,j} = \nabla\Big(T_k(u_n) - T_k(v_j)\Big)$ converges to $\nabla\Big(T_k(u) - T_k(v_j)\Big)$ weakly in $(L_\varphi(\Omega))^N$ as n tends to $+\infty$, then

$$\lim_{n\to+\infty}\int_\Omega F_n(x,u_n)\varrho_m(u_n)\nabla W_\eta^{n,j}\,dx = \int_\Omega F(x,u)\varrho_m(u)\nabla W_\eta^j\,dx.$$

By using the modular convergence of W_η^j as $j \to +\infty$ and letting μ tends to infinity, we get

$$\int_\Omega F_n(x,u_n)\varrho_m(u_n)\nabla W_\eta^{n,j}\,dx = \epsilon(n,j) \quad \text{for any } m \ge 1. \tag{5.23}$$

In the other hand for $n > m + 1 > k$, we have $\nabla u_n\varrho_m'(u_n) = \nabla T_{m+1}(u_n)$ a.e. in Ω. By the almost every where convergence of u_n we have $W_\eta^{n,j} \to W_\eta^j$ in $L^\infty(\Omega)$ weak- $*$ and since the sequence $\big(F_n(x,T_{m+1}(u_n))\big)_n$ converge strongly in$E_{\overline{\varphi}}(\Omega)$ then

$$F_n(x,T_{m+1}(u_n))W_\eta^{n,j} \to F(x,T_{m+1}(u))W_\eta^j$$

converge strongly in $E_{\overline{\varphi}}(\Omega)$ as $n \to +\infty$. By virtue of $\nabla T_{m+1}(u_n) \to \nabla T_{m+1}(u)$ weakly in $(L_\varphi(\Omega))^N$ as $n \to +\infty$ we have

$$\begin{aligned}\lim_{n\to+\infty}&\int_{\{m\le|u_n|\le m+1\}} F_n(x,T_{m+1}(u_n))\nabla u_n\varrho_m'(u_n)W_\eta^{n,j}\,dx\\ &= \int_{\{m\le|u|\le m+1\}} F(x,u)\nabla u\varrho_m'(u)W_\eta^j\,dx\end{aligned} \tag{5.24}$$

with the modular convergence of W_η^j as $j \to +\infty$, we get

$$\int_\Omega F_n(x,u_n)\nabla u_n\varrho_m'(u_n)W_\eta^{n,j}\,dx = \epsilon(n,j) \quad \text{for any } m \ge 1 \tag{5.25}$$

Concerning the first term of (5.22) we have

$$\begin{aligned}\int_\Omega b_n(x,u_n,\nabla u_n)\varrho_m'(u_n)W_\eta^{n,j}\,dx &= \int_{\{m\le|u_n|\le m+1\}} b_n(x,u_n,\nabla u_n)\varrho_m'(u_n)\nabla u_n W_\eta^{n,j}\,dx\\ &\le \eta C\int_{\{m\le|u_n|\le m+1\}} b_n(x,u_n,\nabla u_n)\nabla u_n\,dx,\end{aligned} \tag{5.26}$$

thus

$$\int_\Omega b_n(x,u_n,\nabla u_n)\varrho_m'(u_n)W_\eta^{n,j}dx \le \epsilon(n,m). \tag{5.27}$$

The weakly convergence of $T_k(u_n)$ to $T_k(v_j)$ in $W^{0,1}L_\varphi(\Omega)$ as n tends to $+\infty$, the bounded character of $W_\eta^{n,j}$, we obtain

$$\int_\Omega f_n\varrho_m(u_n)W_\eta^{n,j}\,dx = \epsilon(n,\eta), \tag{5.28}$$

and

$$\int_\Omega \phi\nabla W_\eta^{n,j}\varrho_m(u_n)\,dx = \epsilon(n,\eta). \tag{5.29}$$

Appealing now (1.5), we get

$$\begin{aligned}&\left|\int_\Omega \phi\nabla u_n\varrho_m'(u_n)W_\eta^{n,j}\,dx\right| \le \epsilon_1\\ &\quad\int_\Omega \overline{\varphi}\left(x,\frac{\phi}{\epsilon_1}\right)W_\eta^{n,j}\,dx + \epsilon_1\eta\int_{\{m\le|u_n|\le m+1\}} b_n(x,u_n,\nabla u_n)\nabla u_n dx \le \epsilon(n,m,j,\eta).\end{aligned} \tag{5.30}$$

In the other hand we have

$$\begin{aligned}&\int_\Omega b_n(x,u_n,\nabla u_n)\varrho_m(u_n)\nabla W_\eta^{n,j}\,dx\\ &\quad= \int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} b_n(x,T_k(u_n),\nabla T_k(u_n))\varrho_m(u_n)\\ &\quad\times(\nabla T_k(u_n)-\nabla T_k(v_j))\,dx\\ &\quad-\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} b_n(x,u_n,\nabla u_n)\varrho_m(u_n)\nabla T_k(v_j)\,dx.\end{aligned} \tag{5.31}$$

Since $b_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n))$ is bounded in $(L_{\overline{\varphi}}(\Omega))^N$, there exist some $\varpi_{k+\eta}\in(L_{\overline{\varphi}}(\Omega))^N$ such that $b_n(x,T_{k+\eta}(u_n),\nabla T_{k+\eta}(u_n)) \rightharpoonup \varpi_{k+\eta}$ weakly in $(L_{\overline{\varphi}}(\Omega))^N$. Thus:

$$\begin{aligned}&\int_{\{|u_n|>k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} b_n(x,u_n,\nabla u_n)\varrho_m(u_n)\nabla T_k(v_j)\,dx\\ &\quad= \int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j))\le\eta\}} \varrho_m(u)\varpi_{k+\eta}\nabla T_k(v_j)\,dx + \epsilon(n),\end{aligned} \tag{5.32}$$

By letting $j\to+\infty$, we get

$$\int_{\{|u|>k\}\cap\{0\le T_k(u)-T_k(v_j))\le\eta\}} \varrho_m(u)\nabla T_k(v_j)\varpi_{k+\eta}\,dx = \int_{\{|u|>k\}\}} \varrho_m(u)\nabla T_k(u)\varpi_{k+\eta}dx + \epsilon(n,j) = \epsilon(n,j). \tag{5.33}$$

Thanks to (5.23)–(5.33), one has

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j))\le\eta\}} b_n\big(x,T_k(u_n),\nabla T_k(u_n)\big)\varrho_m(u_n) \times\big(\nabla T_k(u_n)-\nabla T_k(v_j)\big)\,dx \le C\eta+\epsilon(n,j,m). \tag{5.34}$$

Since $\exp\big(G(u_n)\big)\ge 1$ and $\varrho_m(u_n)=1$ for $|u_n|\le k$ then

$$\int_{\{|u_n|\le k\}\cap\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b_n\big(x,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\big)\,dx \le C\eta+\epsilon(n,j,m). \tag{5.35}$$

Finally we show that,

$$\int_\Omega\Big(b\big(x,T_k(u_n),\nabla T_k(u_n)\big)-b\big(x,T_k(u_n),\nabla T_k(u)\big)\Big)\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big)\,dx\to 0. \tag{5.36}$$

For $s>0$, denoting by $\Omega^s=\big\{x\in\Omega:\big|\nabla T_k(u)\big|\le s\big\}$ and $\Omega^s_j=\Big\{x\in\Omega:\Big|\nabla T_k(v_j)\Big|\le s\}$ then by χ^s and χ^s_j the characteristic functions of Ω^s and Ω^s_j respectively, letting $0<\delta<1$, define

$$\Theta_{n,k}=\Big(b\big(x,T_k(u_n),\nabla T_k(u_n)\big)-b\big(x,T_k(u_n),\nabla T_k(u)\big)\Big)\Big(\nabla T_k(u_n)-\nabla T_k(u)\Big).$$

For $s>0$, we have

$$0\le\int_{\Omega^s}\Theta^\delta_{n,k}\,dx=\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}dx+\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}\,dx.$$

The first term of the right-side hand, with the Hölder inequality we obtain

$$\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\le\left(\int_{\Omega^*}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}dx\right)^\delta\left(\int_{\Omega^*}dx\right)^{1-\delta} \le C_1\left(\int_{\Omega^s}\Theta_{n,k}\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}\,dx\right)^\delta. \tag{5.37}$$

For the second term of the right-side hand by the Hölder inequality we have

$$\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}dx\le\left(\int_{\Omega^s}\Theta_{n,k}\,dx\right)^\delta\left(\int_{\{T_k(u_n)-T_k(v_j)>\eta\}}dx\right)^{1-\delta}, \tag{5.38}$$

since $a\big(x,T_k(u_n),\nabla T_k(u_n)\big)$ is bounded in $\big(L_{\overline{\varphi}}(\Omega)\big)^N$, while $\nabla T_k(u_n)$ is bounded in $\big(L_\varphi(\Omega)\big)^N$ then

$$\int_{\Omega^s}\Theta^\delta_{n,k}\chi_{\{T_k(u_n)-T_k(v_j)>\eta\}}\,dx\le C_2\text{meas}\big\{x\in\Omega: T_k(u_n)-T_k(v_j)>\eta\big\}^{1-\delta} \tag{5.39}$$

We obtain

$$\int_{\Omega^g} \Theta_{n,k}^{\delta} dx \leq C_1 \left(\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx \right)^{\delta} + C_2 \text{meas}\{x \in \Omega : T_k(u_n) - T_k(v_j) > \eta\}^{1-\delta} \tag{5.40}$$

On the other hand

$$\begin{aligned}
&\int_{\Omega^s} \Theta_{n,k} \chi_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \, dx \\
&\leq \int_{\{0 \leq T_k(u) - T_k(v_j) \leq \eta\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u)\chi_s) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s \Big) \, dx.
\end{aligned} \tag{5.41}$$

For each $s, \ r \in \mathbb{R}^+$ with $s > r$ one has

$$\begin{aligned}
0 &\leq \int_{\Omega^r \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta)\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u)) \Big) \\
&\quad \times \big(\nabla T_k(u_n) - \nabla T_k(u) \big) \, dx \\
&\leq \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta)\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u)) \Big) \\
&\quad \times \big(\nabla T_k(u_n) - \nabla T_k(u) \big) \, dx \\
&= \int_{\Omega^s \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u)\chi_s) \Big) \\
&\quad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi_s \big) \, dx \\
&\leq \int_{\Omega \cap \{0 \leq T_k(u_n) - T_k(v_j) \leq \eta)\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u)\chi^s) \Big) \\
&\quad \times \big(\nabla T_k(u_n) - \nabla T_k(u)\chi^s \big) \, dx \\
&= \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b\Big(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s\Big) \Big) \\
&\quad \times \Big(\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s \Big) \, dx \\
&\quad + \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} b(x, T_k(u_n), \nabla T_k(u_n)) \Big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s \Big) \, dx \\
&\quad + \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} \Big(b\Big(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s\Big) - b(x, T_k(u_n), \nabla T_k(u)\chi^s) \Big) \nabla T_k(u_n) \, dx \\
&\quad - \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} b\Big(x, T_k(u_n), \nabla T_k(v_j)\chi_j^s\Big) \nabla T_k(v_j)\chi_j^s) \, dx \\
&\quad + \int_{\{0 \leq T_k(u_n) - T_k(v_j) \leq \eta\}} b(x, T_k(u_n), \nabla T_k(u)\chi^s) \nabla T_k(u)\chi^s) \, dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{5.42}$$

In the sequel we pass to the limit in I_i when $n, \ j, \ \mu$, and $s \to +\infty$. We have

$$I_1 = \int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\big(x,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\big)dx$$
$$-\int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(v_j)\chi_j^s-\nabla T_k(v_j)\Big)dx$$
$$-\int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\Big(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big)dx$$

Thanks to (5.35), the first term of the right hand side in I_1, we get

$$\int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\big(x,T_k(u_n),\nabla T_k(u_n)\big)\big(\nabla T_k(u_n)-\nabla T_k(v_j)\big)dx$$
$$\le C\eta+\epsilon(n,m,j,s)-\int_{\{|u|>k\cap 0\le T_k(u)-T_k(v_j)\le\eta\}} b\big(x,T_k(u),0\big)\nabla T_k(v_j)\,dx$$
$$\le C\eta+\epsilon(n,m,j).$$

Since $b\big(x,T_k(u_n),\nabla T_k(u_n)\big)$ is bounded in $\big(L_{\overline{\varphi}}(\Omega)\big)^N$, there exist some $\varpi_k\in\big(L_{\overline{\varphi}}(\Omega)\big)^N$ such that (for a subsequence still denoted by u_n):

$$b\big(x,T_k(u_n),\nabla T_k(u_n)\big)$$
$$\rightarrow\varpi_k\quad\text{in}\quad\big(L_\varphi(\Omega)\big)^N\quad\text{for}\quad\sigma\big(\Pi L_\varphi,\Pi E_\varphi\big)$$

By using in the fact $\Big(\nabla T_k(v_j)\chi_j^s-\nabla T_k(v_j)\Big)\chi_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}}$ strongly converges to $\Big(\nabla T_k(v_j)\chi_j^s-\nabla T_k(v_j)\Big)\chi_{\{0\le T_k(u)-T_k(v_j)\le\eta\}}$ in $\big(E_\varphi(\Omega)\big)^N$ as $n\to+\infty$.

The second term of the right hand side of I_1 tends to

$$\int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\Big(x,T_k(u_n),\nabla T_k(u_n)\Big)\Big(\nabla T_k(v_j)\chi_j^s-\nabla T_k(v_j)\Big)\,dx$$
$$=\int_{\{0\le T_k(u)-T_k(v_j)\le\eta\}} \varpi_k\Big(\nabla T_k(v_j)\chi_j^s-\nabla T_k(v_j)\Big)\,dx+\epsilon(n).$$

The third term of the right-hand side tends to

$$\int_{\{0\le T_k(u_n)-T_k(v_j)\le\eta\}} b\Big(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s\Big)\Big(\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s\Big)\,dx$$
$$=\int_{\{0\le T_k(u)-T_k(v_j)\le\eta\}} b\Big(x,T_k(u),\nabla T_k(v_j)\chi_j^s\Big)\Big(\nabla T_k(u)-\nabla T_k(v_j)\chi_j^s\Big)\,dx+\epsilon(n),$$

Letting $j\to+\infty$ and $\mu\to+\infty$ of I_1, it possible to conclude that

$$I_1\le C\eta+\epsilon(n,j,s).$$

Concerning I_2, by letting $n\to+\infty$, we obtain

$$I_2 \to \int_{\{0 \le T_k(u) - T_k(v_j) \le \eta\}} \varpi_k \Big(\nabla T_k(v_j) \chi_j^s - \nabla T_k(u) \chi^s \Big)\, dx.$$

Since $b\big(x, T_k(u_n), \nabla T_k(u_n)\big) \to \varpi_k$ in $\big(L_{\overline{\varphi}}(\Omega)\big)^N$, for $\sigma\big(\Pi L_{\overline{\varphi}}, \Pi E_\varphi\big)$ while

$$\Big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\Big)\chi_{\{0 \le T_k(u) - T_k(v_j) \le \eta\}} \to \Big(\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\Big)\chi_{\{0 \le T_k(u) - T_k(v_j) \le \eta\}}$$

strongly in $\big(E_\varphi(\Omega)\big)^N$. Now, letting $j \to +\infty$, and thanks to Lebesgue's theorem, we obtain

$$\begin{aligned} I_2 &= \epsilon(n,j), \\ I_3 &= \epsilon(n,j), \\ I_4 &= \int_{\{0 \le T_k(u) - T_k(v_j) \le \eta\}} b\big(x, T_k(u), \nabla T_k(u)\big) \nabla T_k(u) dx + \epsilon(n,j,s,m), \end{aligned}$$

and

$$I_5 = \int_{\{0 \le T_k(u) - T_k(v_j) \le \eta\}} b\big(x, T_k(u), \nabla T_k(u)\big) \nabla T_k(u) dx + \epsilon(n,j,s,m).$$

Consequently, we obtain

$$\int_{\Omega^s} \Theta_{n,k} dx \le C_1 (C\eta + \epsilon(n,\eta,m))^\delta + C_2(\epsilon(n,))^{1-\delta}.$$

Which leads to

$$\begin{aligned} \int_{\{T_\eta(T_k(u_n) - T_k(v_j)) \ge 0\} \cap \Omega^r} &\Big[\Big(b(x, T_k(u_n), \nabla T_k(u_n)) - b(x, T_k(u_n), \nabla T_k(u))\Big) \\ &\times (\nabla T_k(u_n) - \nabla T_k(u))\Big]^\delta dx = \epsilon(n). \end{aligned} \tag{5.43}$$

By taking $W_\eta^{n,j} = T_\eta\big(T_k(u_n) - T_k(v_j)\big)^-$ and $W_\eta^j = T_\eta\big(T_k(u) - T_k(v_j)\big)^-$, then testing the approximating equation by $\exp\big(G(u_n)\big) W_\eta^{n,j} \varrho_m(u_n)$, we obtain

$$\begin{aligned} \int_{\{T_\eta(T_k(u_n) - T_k(v_j)) \le 0\} \cap \Omega^r} &\Big[\Big(b\big(x, T_k(u_n), \nabla T_k(u_n)\big) - b\big(x, T_k(u_n), \nabla T_k(u)\big)\Big) \\ &\times \big(\nabla T_k(u_n) - \nabla T_k(u)\big)\Big]^\delta dx = \epsilon(n). \end{aligned} \tag{5.44}$$

Thanks to (5.43) and (5.44) we have

$$\int_{\Omega^r} \Big[\big(b\big(x, T_k(u_n), \nabla T_k(u_n)\big) - b\big(x, T_k(u_n), \nabla T_k(u)\big)\big)\big(\nabla T_k(u_n) - \nabla T_k(u)\big)\Big]^\delta dx = \epsilon(n)$$

As a consequence, since r is arbitrary:

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega, \tag{5.45}$$

and for all $k \geq 0$, we have

$$b\big(x, T_k(u_n), \nabla T_k(u_n)\big) \rightharpoonup b\big(x, T_k(u), \nabla T_k(u)\big) \text{ weakly in } \big(L_\psi(\Omega)\big)^N, \tag{5.46}$$

$$\varphi\Big(x, \Big|\nabla T_k(u_n)\Big|\Big) \to \varphi\big(x, |\nabla T_k(u)|\big) \text{ strongly in } L^1(\Omega). \tag{5.47}$$

5.6 Renormalization identity for the solutions

We show that The limit u of the approximate solution u_n of (5.1) satisfies:

$$\lim_{m\to\infty} \int_{\{m\leq|u|\leq m+1\}} b(x, u, \nabla u)\nabla u dx = 0. \tag{5.48}$$

To this end, remark that for any $m > 0$ one has

$$\begin{aligned}
\int_{\{m\leq|u_n|\leq m+1\}} b(x, u_n, \nabla u_n)\nabla u_n\, dx &= \int_\Omega b(x, u_n, \nabla u_n)\big(\nabla T_{m+1}(u_n) - \nabla T_m(u_n)\big)\, dx \\
&= \int_\Omega b\Big(x, T_{m+1}(u_n), \nabla T_{m+1}(u_n)\Big)\nabla T_{m+1}(u_n)\, dx \\
&- \int_\Omega b\Big(x, T_m(u_n), \nabla T_m(u_n)\Big)\nabla T_m(u_n)\, dx.
\end{aligned} \tag{5.49}$$

According to (5.46), (5.47) one is at liberty to pass to the limit as n tends to infinity for fixed m and to obtain

$$\begin{aligned}
\lim_{n\to\infty} \int_{\{m\leq|u_n|\leq m+1\}} b(x, u_n, \nabla u_n)\nabla u_n\, dx &= \int_\Omega b\Big(x, T_{m+1}(u), \nabla T_{m+1}(u)\Big)\nabla T_{m+1}(u)\, dx \\
&- \int_\Omega b\Big(x, T_m(u), \nabla T_m(u)\Big)\nabla T_m(u)\, dx. = \int_{\{m\leq|u|\leq m+1\}} b(x, u, \nabla u)\nabla u\, dx
\end{aligned} \tag{5.50}$$

Taking the limit as m tends to $+\infty$ and using the estimate (5.21) show that u satisfies (5.48).

5.7 Passing to the limit

Let $h \in C_c^1(\mathbb{R})$ and $V \in \mathcal{D}(\Omega)$. Using the admissible test function $h(u_n)V$ in (5.1) leads to

$$\int_{\Omega} b(x,u_n,\nabla u_n)\nabla u_n h'(u_n) V dx + \int_{\Omega} b(x,u_n,\nabla u_n)\nabla V h(u_n) dx$$
$$+ \int_{\Omega} F_n(x,u_n)\nabla\big(h(u_n)V\big) dx = \int_{\Omega} f_n h(u_n) V dx + \int_{\Omega} \phi \nabla\big(h(u_n)V\big) dx. \tag{5.51}$$

We shall pass to the limit in each term in the previous equality, to this end, remark that since h and h' have a compact support in h, there exists $K > 0$ such that $supp(h) \subset [-K,K]$. For n large enough, we have:

$$F_n(x,t)h(t) = F_n\big(x,T_n(t)\big)h(t) = F\big(x,T_K(t)\big)h(t)$$
$$F_n(x,t)h'(t) = F_n\big(x,T_n(t)\big)h'(t) = F\big(x,T_K(t)\big)h'(t)$$

Let us start by the third integral of the left-hand side and the right hand-side of (5.51). Since $h \in C_c^1(\mathbb{R})$ and $V \in \mathcal{D}(\Omega)$, then there exists two positive constants c_1 and c_1' such that $\left\| h\big(T_K(u_n)\big)\nabla V \right\|_{\infty} \leq c_1$ and $\left\| h'(t)\big(T_K(u_n)V\nabla T_K(u_n)\right\|_{\infty} \leq c_1'$ Now since $T_K(u_n)$ is bounded in $W_0^1 L_{\varphi}(\Omega)$, then there exists two positive constant λ_0 and λ such that $\int_{\Omega} \varphi\left(x, \frac{\left|\nabla T_K(u_n)\right|}{\lambda}\right) dx \leq \lambda_0$. Using the convexity and monotonicity of φ, for η large enough, we can write

$$\begin{aligned}
&\int_{\Omega} \varphi\left(x, \frac{\nabla\big(h\big(T_K(u_n)\big)V\big)}{\eta}\right) dx \\
&= \int_{\Omega} \varphi\left(x, \frac{h\big(T_K(u_n)\big)\nabla V + h'(t)\Big(T_K(u_n)V\left|\nabla T_K(u_n)\right|}{\eta}\right) dx \\
&\leq \int_{\Omega} \varphi\left(x, \frac{c_1 + c_1'\lambda \frac{\left|\nabla T_K(u_n)\right|}{\lambda}}{\eta}\right) dx \\
&\leq \int_{\Omega} \varphi\left(x, \frac{c_1}{\eta}\right) dx + \frac{c_1'\lambda}{\eta}\int_{\Omega} \varphi\left(x, \frac{\left|\nabla T_K(u_n)\right|}{\lambda}\right) dx \\
&\leq C_{\eta,c_1} + \frac{c_1'\lambda\lambda_0}{\eta} \quad \text{where } C_{\eta,c_1} = \int_{\Omega} \varphi\left(x, \frac{c_1}{\eta}\right) dx < \infty.
\end{aligned}$$

Then the sequence $\left\{\nabla\big(h\big(T_K(u_n)\big)V\big)\right\}$ is bounded in $\big(L_{\varphi}(\Omega)\big)^N$, as a consequence, we deduce

$$h(u_n)V \rightharpoonup h(u)V \text{ weakly in } W_0^1 L_{\varphi}(\Omega) \text{ for } \sigma\big(\Pi L_{\varphi}, \Pi E_{\psi}\big). \tag{5.52}$$

Moreover, since $F\big(x,T_K(u_n)\big)$ is bounded in $L_{\psi}(\Omega)$, we have from Lemma 3.10

$$F\big(x,T_K(u_n)\big) \to F\big(x,T_K(u)\big) \quad \text{strongly in } E_{\psi}(\Omega).$$

By (5.52), we get

$$\lim_{n\to\infty}\int_\Omega F_n\big(x,u_n\big)\nabla\big(h\big(u_n\big)V\big)dx=\int_\Omega F\big(x,T_K(u)\big)\nabla(h(u)V)dx.$$

Moreover we have

$$\lim_{n\to\infty}\int_\Omega f_n h\big(u_n\big)Vdx=\int_\Omega fh(u)Vdx,$$
$$\lim_{n\to\infty}\int_\Omega \phi\nabla h\big(u_n\big)Vdx=\int_\Omega \phi\nabla h(u)Vdx.$$

Concerning the first integral of (5.51), while supp $h'\subset[-K,K]$, we obtain

$$h'\big(u_n\big)Vb\big(x,u_n,\nabla u_n\big)\nabla u_n=h'\big(u_n\big)Vb\big(x,T_K\big(u_n\big),\nabla T_K\big(u_n\big)\big)\nabla T_K\big(u_n\big)\quad \text{a.e. in } \Omega.$$

The pointwise convergence of u_n to u, the bounded character of $h'V$, (5.46) and (5.47) imply that

$$h'\big(u_n\big)Vb\big(x,u_n,\nabla u_n\big)\nabla u_n\rightharpoonup h'(u)Vb\big(x,T_K(u),\nabla T_K(u)\big)\nabla T_K(u)\text{ weakly in } L^1(\Omega).$$

The term $h'(u)Vb\big(x,T_K(u),\nabla T_K(u)\big)\nabla T_K(u)$ is identified with $h'(u)Vb(x,u,\nabla u)\nabla u$.

Now since $h\big(u_n\big)Vb\big(x,u_n,\nabla u_n\big)=h\big(u_n\big)Vb\big(x,T_K\big(u_n\big),\nabla T_K\big(u_n\big)\big)$ a.e. in Ω, and using the strongly convergence of $h\big(u_n\big)\nabla V$ to $h(u)\nabla V$ in $\big(E_\varphi(\Omega)\big)^N$, and using the weakly convergence of $b\big(x,T_K\big(u_n\big),\nabla T_K\big(u_n\big)\big)$ to $b\big(x,T_K(u),\nabla T_K(u)\big)$ in $\big(L_\psi(\Omega)\big)^N$ for $\sigma\big(\Pi L_\psi,\Pi E_\varphi\big)$, then

$$\lim_{n\to\infty}\int_\Omega b\big(x,u_n,\nabla u_n\big)\nabla Vh\big(u_n\big)dx=\int_\Omega b(x,u,\nabla u)\nabla Vh(u)\,dx.$$

As a consequence of the above convergence results, we are in a position to pass to the limit as n tends to $+\infty$ in (5.51) and to conclude that u satisfies (4.3). As a conclusion of Step 5.1 to Step 5.7, the proof of Theorem 4.1 is complete.

Remark 6

(1) It is possible to extend this result to the following parabolic equation

$$\begin{cases}\frac{\partial u}{\partial t}-\operatorname{div}(a(x,t,u,\nabla u))+F(x,t,u)=\mu & \text{in } \Omega\times(0,T),\\ u=0 & \text{on } \partial\Omega\times(0,T),\\ u(x,0)=u_0(x) & \text{in } \Omega.\end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N, N\geq 1, T>0$ and Q_T is the cylinder $\Omega\times(0,T)$. The operator $A(u)=-\operatorname{div}(a(x,t,u,\nabla u))$ is a Leray-Lions operator lefined in $W_0^{1,x}L_\varphi\big(Q_T\big)$. The lower order term F verifies the natural growth condition, no Δ_2-condition is assumed on the Musielak function, and the datum μ is assumed to belong to $L^1(Q_T)+W^{-1}E_\psi(Q_T)$.

(2) In the case of $F\equiv 0$, the problem (1.1) admits a unique solution.

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