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# Existence and regularity of solutions for degenerate elliptic equations with variable growth

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## Abstract

We prove the existence and regularity of solutions to a degenerate nonlinear elliptic problem with boundary conditions of the Dirichlet type $-\operatorname{div}\ b(x, v, \nabla v) = g$ in $\Omega$, where $\Omega$ is a bounded open set with smooth boundary in $\mathbb{R}^N$, $(N \geq 2)$ and $b(\cdot, v, \nabla v)$ is a Carathéodory function and the second member $g$ belongs to $L^1(\Omega)$. The main tools used are a priori estimates in Marcinkiewicz space with variable exponent.


**Keywords** Elliptic problem · Marcinkiewicz space · Weak and entropy solutions

**Mathematics Subject Classification** 46E35 · 35J60 · 35D30

## 1 Introduction

We consider the degenerate nonlinear elliptic problem

$$(\mathcal{P})\begin{cases} -\operatorname{div}\ b(x, v, \nabla v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $b : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for $\xi \neq \xi', \alpha > 0$ and $0 \leq \theta(\cdot) \ll p(\cdot) - 1, (\theta \in \mathcal{C}(\overline{\Omega}))$, the function $b$ satisfies

$$b(x, s, \xi)\xi \geq \alpha \frac{|\xi|^{p(x)}}{(1 + |s|)^{\theta(x)}}, \quad \text{and} \quad (b(x, s, \xi) - b(x, s, \xi'))(\xi - \xi') > 0, \tag{1}$$

and the natural growth

$$|b(x, s, \xi)| \leq \beta |\xi|^{p(x)-1}, \tag{2}$$

where $\beta > 0$.

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Our goal in this paper is first to prove the existence of entropy solutions when the second member $g$ belongs to $L^1(\Omega)$, moreover we prove that such solutions are also weak solutions under the hypothesis $p(\cdot) \gg \theta^+ + 2 - \frac{1}{N}$. In a second step, we will deal with the regularity of entropy solutions when $g$ belongs to $L^{q(\cdot)}(\Omega)$, with $q(x) \geq 1, \forall x \in \Omega$, and $q^- < \frac{p_*^-(p^- - \theta^-)}{p_*^-(p^- - \theta^-) - p^-}$. Noting that the last condition is equivalent to $q < (p_*)'$ if $\theta = 0$ and $p(\cdot) = p$ in the problem $(\mathcal{P})$.

We recall that in the classical case, i.e., when $p(\cdot) = p$ and $\theta(\cdot) = \theta$ the problem $(\mathcal{P})$, was studied for example in [23], where the authors established the regularity of entropy solutions when the second member $g$ belongs to the Marcinkiewicz spaces $M^m(\Omega)$ using the generalized Stampacchia Lemma. We also refer to [12], where the authors proved the existence of a distributive or entropic solution for a degenerate problem according to the growth assumptions on a lower order term. For more results in this topic, see for example [12, 13, 24].

For $\theta = 0$ in (1), i.e., when the principal part of problem $(\mathcal{P})$ is coercive, we have $\gamma = (p^-)', m_0(\cdot) = \frac{p_*(\cdot)}{(p^-)'}$ and $m_1(\cdot) = \frac{m_0(\cdot)}{m_0(\cdot)+1} p(\cdot)$ which are the same quantities obtained in [27, 29], where the authors established the existence and uniqueness of an entropy solution to the obstacle problem for nonlinear elliptic equations with variable growth and a second member $L^1$. For more details in the framework of Sobolev spaces with variable exponent, see [2–5, 10, 19, 20, 31].

The difficulty presented for studying this problem is that the coercivity can degenerate when $u$ is too big, so we cannot apply the standard Leray–Lions surjectivity theorem for the establishment of existence of solutions. To overcome this difficulty, we consider the approximate problem $(\mathcal{P}_n)$ of which the differential operator on $W_0^{1,p(\cdot)}(\Omega)$ is coercive and we can establish a priori estimates on approximating solutions. Then the existence of entropy solutions to problem $(\mathcal{P})$ can be obtained by passing to the limit in the approximate problem. Moreover the entropy solutions are also weak solutions under additional assumptions on exponent $p(\cdot)$. The method using the approximate problem is widely studied in the literature, see for example [1, 8, 9, 14–16, 22, 29].

The study of the Partial Differential Equations with non-standard (variable exponent) growth received wide attention in recent years due to their applications in image processing, elasticity theory and fluid mechanics. In fact, when $b(x, v, \nabla v) = b(x, \nabla v)$, the equation is studied (in a broader framework) in [28], where Lewy–Stampacchia inequalities are used to derive regularity of solutions under coercivity and truncated monotonicity (T-monotonicity) conditions. Moreover, in [18] also are established estimates on the second order derivatives of solutions. A more particular case, that is, when $b(x, v, \nabla v) = b(\nabla v)$ is studied in [17]. The study of problems governed by these type of operators goes as far as understanding the limit case with rapidly oscillating coefficients (homogenization), as in [26, 33].

The paper is organized as follows. In Sect. 2, we recall some rearrangement properties and the definitions of Sobolev and Marcinkiewicz spaces with variable exponent. In Sect. 3, we obtain a priori estimates, the existence of entropy solutions and then the weak solutions are proved. In the last section we prove some regularity results of solutions.

## 2 Preliminaries

### 2.1 Sobolev spaces with variable exponent

Let $p : \overline{\Omega} \to \mathbb{R}$ a real-valued continuous function and $c > 0$. If

$$-|p(x) - p(y)| \log |x - y| < c, \quad \forall x, y \in \overline{\Omega} \text{ such that } |x - y| < \frac{1}{2},$$

we say that $p(\cdot)$ verifies the log-Lipschitz condition.

We denote

$$\mathcal{C}_+(\overline{\Omega}) = \left\{ \text{log-Lipschitz } p : \overline{\Omega} \to \mathbb{R} \text{ with } 1 < p^- \leq p^+ < N \right\},$$

where $\rho^- = \inf_{x\in\Omega} \rho(x)$ and $\rho^+ = \sup_{x\in\Omega} \rho(x)$ $\forall \rho \in \mathcal{C}(\overline{\Omega})$.

For $\rho_1(\cdot)$ and $\rho_2(\cdot)$ in $\mathcal{C}(\overline{\Omega})$

$$\text{we means by } \rho_1(\cdot) \ll \rho_2(\cdot) \text{ that } \inf_{x\in\Omega}(\rho_2(x) - \rho_1(x)) > 0. \tag{3}$$

Let $\rho$ be the function defined by

$$\rho(v) = \int_\Omega |v(x)|^{p(x)}\, dx \qquad \forall v \in L^{p(\cdot)}(\Omega).$$

For $p \in \mathcal{C}_+(\overline{\Omega})$, we define the Lebesgue space with variable exponent by

$$L^{p(\cdot)}(\Omega) = \{ v : \Omega \to \mathbb{R} \text{ measurable } : \rho(v) < \infty \},$$

the space $(L^{p(\cdot)}(\Omega), \|v\|_{p(\cdot)})$ is reflexive with

$$\|v\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \rho\left(\frac{v}{\lambda}\right) \leq 1 \right\}.$$

We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, for all $x \in \Omega$.

**Proposition 2.1** (Hölder inequality [21, 32])

(i) *For all* $(v, v') \in L^{p(\cdot)}(\Omega) \times L^{p'(\cdot)}(\Omega)$, *we have*

$$\left| \int_\Omega v(x)v'(x)\, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'^-} \right) \|v\|_{p(\cdot)} \|v'\|_{p'(\cdot)}.$$

(ii) *For all* $p_1, p_2 \in \mathcal{C}_+(\overline{\Omega})$ *such that* $p_1(x) \leq p_2(x)$ *for all* $x$ *in* $\Omega$, *we have*

$$L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega),$$

*moreover the embedding is continuous.*

**Proposition 2.2** [21, 32] *The following assertions hold*

(1) $\|v\|_{p(\cdot)} < 1$ (*resp.* $= 1, > 1$) $\Longleftrightarrow \rho(v) < 1$ (*resp.* $= 1, > 1$);
(2) *We have the following implication*

$$\|v\|_{p(\cdot)} > 1 \Longrightarrow \|v\|_{p(\cdot)}^{p^-} \leq \rho(v) \leq \|v\|_{p(\cdot)}^{p^+},$$
$$\|v\|_{p(\cdot)} < 1 \Longrightarrow \|v\|_{p(\cdot)}^{p^+} \leq \rho(v) \leq \|v\|_{p(\cdot)}^{p^-}.$$

(3) *The following equivalents hold true*

$$\|v\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho(v) \to 0,$$
$$\|v\|_{p(\cdot)} \to \infty \Longleftrightarrow \rho(v) \to \infty.$$

We define Sobolev space with variable exponent by

$$W^{1,p(\cdot)}(\Omega) = \left\{ v \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla v| \in L^{p(\cdot)}(\Omega) \right\},$$

with the norm

$$\|v\|_{1,p(\cdot)} = \|v\|_{p(\cdot)} + \|\nabla v\|_{p(\cdot)} \quad \forall v \in W^{1,p(\cdot)}(\Omega).$$

We denote by $W_0^{1,p(\cdot)}(\Omega)$ the closure of $\mathcal{D}(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$, and we define the Sobolev exponent by $p^*(x) = \frac{Np(x)}{N-p(x)}$ with $p(x) < N$.

**Proposition 2.3** [21]

(i) *The spaces* $W^{1,p(\cdot)}(\Omega)$ *and* $W_0^{1,p(\cdot)}(\Omega)$ *are Banach spaces, separable and reflexive.*
(ii) *The embedding* $W_0^{1,p(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^{m(\cdot)}(\Omega)$ *is continuous and compact, if* $m(x) < p^*(x), \forall x \in \Omega$.
(iii) (*Poincaré inequality*). *For all* $v \in W_0^{1,p(\cdot)}(\Omega)$ *there exists a constant* $c > 0$, *such that* $\|v\|_{p(\cdot)} \leq c\|\nabla v\|_{p(\cdot)}$.
(iv) (*Sobolev–Poincaré inequality*). *For all* $v \in W_0^{1,p(\cdot)}(\Omega)$ *there exists a constant* $c > 0$, *such that* $\|v\|_{p_*(\cdot)} \leq c\|\nabla v\|_{p(\cdot)}$.

***Remark* 1** We conclude that the norms $\|\nabla v\|_{p(\cdot)}$ and $\|v\|_{1,p(\cdot)}$ are equivalents in $W_0^{1,p(\cdot)}(\Omega)$ using (iii) of Proposition 2.3.

The truncation function $T_k : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(r) = \begin{cases} r & \text{if } |r| \leq k \\ k \cdot \text{ sign } (r) & \text{if } |r| > k. \end{cases}$$

## 2.2 Marcinkiewicz spaces

In this section we review some properties of rearrangements and Marcinkiewicz spaces with variable exponents, for more details, see [7, 25, 29, 30].

First we recall the definition of decreasing rearrangement of functions. Let $v : \Omega \to \mathbb{R}$ a measurable function.

**Definition 2.4** We define the distribution function of $v$ as follows

$$\mu_v(t) = \text{ meas } \{x \in \Omega : |v(x)| > t\}, \quad t \geq 0.$$

$\mu_v$ is right continuous and decreasing function.

**Definition 2.5** We define the decreasing rearrangement of $v$ as follows

$$v_*(s) := \sup\{t \geq 0 : \mu_v(t) > s\}, \quad s \geq 0.$$

**Definition 2.6** A measurable function $v : \Omega \to \mathbb{R}$ belongs to the Marcinkiewicz space $M^p(\Omega)$(or weak-$L^p$) if

$$\mu_v(t) \leq \frac{c}{t^r}, \ \forall t > 0, \quad \text{or} \quad v_*(s) \leq \frac{c}{s^{1/r}}, \ \forall s > 0,$$

for some constant c.

Let $m(\cdot)$ be a measurable function such that $m^- > 0$. We say that a measurable function $v$ belongs to the Marcinkiewicz space $M^{m(\cdot)}$ if there exists a positive constant $C$ such that

$$\int_{\{|v|>t\}} t^{m(x)} \, dx \leq C, \quad \text{for all } t > 0.$$

When $m(\cdot)$ is constant i.e. $m(\cdot) \equiv m$ this definition is coincides with the classical definition of the Marcinkiewicz space $M^m(\Omega)$. Moreover we have

$$\int_{\{|v|>t\}} t^{m(x)} \, dx \leq \int_\Omega |v|^{m(x)} \, dx, \quad \text{for all } t > 0.$$

Thus if $|v|^{m(\cdot)} \in L^1(\Omega)$, we have $v \in M^{m(\cdot)}(\Omega)$ and $L^{m(\cdot)}(\Omega) \subset M^{m(\cdot)}(\Omega)$, for all $m(\cdot) \geq 1$.

In the Marcinkiewicz space with constant exponent, if $v \in M^r(\Omega)$, then $|v|^m \in L^1(\Omega)$, for all $0 < m < r$.

This claim is extended to the nonconstant setting by the following lemma, whose proof is given in [29].

**Lemma 2.7** *Let $r(\cdot)$ and $m(\cdot)$ be bounded functions such that $0 \ll m(\cdot) \ll r(\cdot)$ and let $\epsilon := (r - m)^- > 0$. If $v \in M^{r(\cdot)}(\Omega)$, then*

$$\int_{\Omega} |v|^{m(x)}\,dx \leq 2|\Omega| + c\frac{r^+ - \epsilon}{\epsilon},$$

*where c is a positive constant. In particular,* $M^{r(\cdot)}(\Omega) \subset L^{m(\cdot)}(\Omega)$ *for all* $1 \leq m(\cdot) \ll r(\cdot)$.

# 3 Main results

## 3.1 A priori estimate

**Definition 3.1** A measurable function $v$ is an entropy solution of problem $(\mathcal{P})$ if for every $t > 0, T_t(v) \in W_0^{1,p(\cdot)}(\Omega)$ and

$$\int_{\Omega} b(x, v, \nabla v)\nabla T_t(v - \varphi)\,dx \leq \int_{\Omega} g(x)T_t(v - \varphi)\,dx, \tag{4}$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$.

It is well known in [11], that for a measurable function $v$ such that $T_t(v) \in W_0^{1,p(\cdot)}(\Omega)$ there exists a unique measurable function $w : \Omega \to \mathbb{R}^N$ such that $w\chi_{\{|v|\leq t\}} = \nabla T_t(v)$ for a.e. $x$ in $\Omega$ and for all $t > 0$. We will define the gradient of $v$ as the function $w$, and we will denote it by $w = \nabla v$.

**Theorem 3.2** *Under assumptions* (1) *and* (2). *If v is an entropy solution of problem* $(\mathcal{P})$, *then there exists a positive constant* $C$, *depending only on* $p^{\pm}, N$, *and* $\Omega$, *such that for all* $t > 0$

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\,dx \leq C, \qquad \text{with } \gamma = \left(\frac{p(\cdot)}{p(\cdot) - (\theta(\cdot) + 1)}\right)^+.$$

***Proof*** We denote by $C$ a constant that varies from line to line.

Since $v$ is an entropy solution to the problem $(\mathcal{P})$, for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^{\infty}(\Omega)$ we have

$$\int_{\Omega} b(x, v, \nabla v)\nabla T_t(v - \varphi)\,dx \leq \int_{\Omega} g(x)T_t(v - \varphi)\,dx,$$

for $\varphi = 0$, we obtain

$$\int_\Omega b(x,v,\nabla v)\nabla T_t(v)\,dx \le t\int_\Omega |g(x)|\,dx, \tag{5}$$

by using (1) it follows that

$$\int_\Omega \frac{|\nabla T_t(v)|^{p(x)}}{(1+|v|)^{\theta(x)}}\,dx \le t\frac{\|g\|_1}{\alpha}. \tag{6}$$

Using the Sobolev inequality in Propositions 2.3 and 2.2, we have for $t>1$

$$\begin{aligned}
\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\,dx &= \int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\left|\frac{T_t(v)}{t}\right|^{p_*(x)}dx\\
&\le \int_\Omega \left(t^{\frac{1}{\gamma}-1}|T_t(v)|\right)^{p_*(x)}dx\\
&\le \|t^{\frac{1}{\gamma}-1}T_t(v)\|^{\alpha_1}_{p_*(\cdot)}\\
&\le C\|\nabla(t^{\frac{1}{\gamma}-1}T_t(v))\|^{\alpha_1}_{p(\cdot)}\\
&\le C\left(\int_\Omega |\nabla(t^{\frac{1}{\gamma}-1}T_t(v))|^{p(x)}\,dx\right)^{\frac{\alpha_1}{\alpha_2}},
\end{aligned}$$

using (6) we get

$$\begin{aligned}
\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\,dx &\le C\left(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)}\frac{|\nabla T_t(v)|^{p(x)}}{(1+|v|)^{\theta(x)}}(1+|v|)^{\theta(x)}\,dx\right)^{\frac{\alpha_1}{\alpha_2}}\\
&\le C\left(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)+1}\frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}(1+t)^{\theta(x)}\,dx\right)^{\frac{\alpha_1}{\alpha_2}}\\
&\le C\left(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)+1}\frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}(2t)^{\theta(x)}\,dx\right)^{\frac{\alpha_1}{\alpha_2}}\\
&\le 2^{\theta^+}C\left(\int_\Omega t^{(\frac{1}{\gamma}-1)p(x)+\theta(x)+1}\frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}\,dx\right)^{\frac{\alpha_1}{\alpha_2}}\\
&\le C\left(\frac{\|g\|_1}{\alpha}\right)^{\frac{\alpha_1}{\alpha_2}},
\end{aligned}$$

where

$$\alpha_1 = \begin{cases} (p_*)^+ & \text{if } \|t^{\frac{1}{\gamma}-1}T_t(v)\|_{p_*(\cdot)} \ge 1\\ (p_*)^- & \text{if } \|t^{\frac{1}{\gamma}-1}T_t(v)\|_{p_*(\cdot)} \le 1\end{cases} \quad\text{and}\quad \alpha_2 = \begin{cases} p^+ & \text{if } \|t^{\frac{1}{\gamma}-1}\nabla T_t(v)\|_{p(\cdot)} \le 1\\ p^- & \text{if } \|t^{\frac{1}{\gamma}-1}\nabla T_t(v)\|_{p(\cdot)} \ge 1.\end{cases}$$

For $t \le 1$ we have

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma}}\, dx \le |\Omega|.$$

By combining the estimates in both cases, the result follows. □

***Remark* 2** Let $m_0(\cdot) = \frac{p_*(\cdot)}{\gamma}$, from Theorem 3.2 we have $u \in M^{m_0(\cdot)}(\Omega)$. For the constant exponent case ($p(\cdot) \equiv p$) we have

$$u \in M^{m_0}(\Omega) \quad \text{with } m_0 = \frac{N(p-1-\theta)}{N-p},$$

which the same regularity obtained by the authors in [23].

**Theorem 3.3** *Assume that the assumptions* (1) *and* (2) *hold true, if* $v$ *is an entropy solution of a problem* ($\mathcal{P}$), *then* $|v|^{m(\cdot)} \in L^1(\Omega)$ *for all* $m(\cdot)$ *such that* $0 \ll m(\cdot) \ll m_0(\cdot)$.

***Proof*** Let $m(\cdot)$ and $m_0(\cdot)$ such that $0 \ll m(\cdot) \ll m_0(\cdot)$ and $\epsilon = (m_0(\cdot) - m(\cdot))^- > 0$. By Theorem 3.2, we get

$$\int_{\{|v|>t\}} t^{m_0(x)}\, dx \le c, \quad \text{for all } t > 0.$$

From Lemma 2.7, we conclude that

$$\int_\Omega |v|^{m(x)}\, dx \le 2|\Omega| + c\left(\frac{m_0(\cdot)-\epsilon}{\epsilon}\right)^+,$$

which implies that $|v|^{m(\cdot)} \in L^1(\Omega)$. □

**Theorem 3.4** *Let* $\alpha(x) = \frac{p(x)}{\theta(x)+m_0(x)+1}$ *for all* $x$ *in* $\Omega$, *under assumptions* (1) *and* (2), *if* $v$ *an entropy solution of a problem* ($\mathcal{P}$), *then* $|\nabla v|^{\alpha(\cdot)} \in M^{m_0(\cdot)}(\Omega)$. *Moreover*

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_0(x)}\, dx \le c, \quad \text{for all } t > 0.$$

***Proof*** Using Theorem 3.2, for $t > 1$ we have

$$\begin{aligned}
\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_0(x)}\,dx &\le \int_{\{|\nabla v|^{\alpha(\cdot)}>t\}\cap\{|v|\le t\}} t^{m_0(x)}\,dx + \int_{\{|v|>t\}} t^{m_0(x)}\,dx \\
&\le \int_{\{|v|\le t\}} t^{m_0(x)} \left(\frac{|\nabla v|^{\alpha(x)}}{t}\right)^{p(x)/\alpha(x)} dx + c \\
&= \int_{\{|v|\le t\}} t^{m_0(x)+1-\frac{p(x)}{\alpha(x)}} \frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}(1+|v|)^{\theta(x)}\,dx + c \\
&\le \int_{\{|v|\le t\}} t^{m_0(x)+1-\frac{p(x)}{\alpha(x)}} \frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}(1+t)^{\theta(x)}\,dx + c \\
&\le \int_\Omega \left(\frac{t+1}{t}\right)^{\theta(x)} \frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}\,dx + c \\
&\le 2^{\theta^+} \int_\Omega \frac{|\nabla T_t(v)|^{p(x)}}{t(1+|v|)^{\theta(x)}}\,dx + c \le C',
\end{aligned}$$

where $c$ and $C'$ are positive constants, for $t \le 1$ we have

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_0(x)}\,dx \le |\Omega|,$$

which gives the required result according to the cases $t > 1$ and $t \le 1$. □

**Theorem 3.5** *Let* $m_1(x) = \frac{m_0(x)p(x)}{m_0(x)+\theta(x)+1}$ *for all* $x$ *in* $\Omega$, *under assumptions stated in Theorem* 3.4, *we have*

$$|\nabla v|^{m(\cdot)} \in L^1(\Omega) \quad \text{for all } m(\cdot) \text{ such that } 0 \ll m(\cdot) \ll m_1(\cdot).$$

***Proof*** By Theorem 3.4, we have

$$|\nabla v|^{\alpha(\cdot)} \in M^{m_0(\cdot)}(\Omega), \quad \textit{with } \alpha(\cdot) = \frac{p(\cdot)}{m_0(\cdot)+\theta(\cdot)+1}.$$

Let $0 \ll m(\cdot) \ll m_1(\cdot)$ and $r(\cdot) = m(\cdot)/\alpha(\cdot)$ then $r(\cdot) \ll m_0(\cdot)$.

By using Lemma 2.7 we obtain

$$\int_\Omega |\nabla v|^{m(x)}\,dx = \int_\Omega |\nabla v|^{\alpha(x)r(x)}\,dx \le C.$$

□

### 3.2 Existence of entropy solutions

In this section we prove the existence and regularity of entropy solutions, extending some results known in the constant exponent case.

**Theorem 3.6** *Under assumptions* (1), (2) *and* $g \in L^1(\Omega)$. *There exists an entropy solutions* $v$ *of problem* $(\mathcal{P})$. *Moreover*

(1) $|v|^{m(\cdot)} \in L^1(\Omega)$ for $0 \ll m(\cdot) \ll m_0(\cdot)$, with $m_0(\cdot) = \frac{p_*(\cdot)}{\gamma}$.

(2) $|\nabla v|^{m(\cdot)} \in L^1(\Omega)$ for $0 \ll m(\cdot) \ll m_1(\cdot)$, with $m_1(\cdot) = \frac{m_0(\cdot)p(\cdot)}{m_0(\cdot)+\theta(\cdot)+1}$,

*where*

$$\gamma = \left(\frac{p(\cdot)}{p(\cdot) - (\theta(\cdot) + 1)}\right)^+.$$

***Remark* 3** In the case $p(\cdot) = p$ and $\theta(\cdot) = \theta$, the exponents $m_0(\cdot)$ and $m_1(\cdot)$ are respectively of the form $m_0 = \frac{N(p-(\theta+1))}{N-p}$ and $m_1 = \frac{N(p-(\theta+1))}{N-(\theta+1)}$ which are the same quantities obtained by the authors in [23].

Let $(g_n)_n \subset L^\infty(\Omega)$ a sequence that converge strongly to $g$ in $L^1(\Omega)$, and $\|g_n\|_1 \leq \|g\|_1$, for all $n$. Let $(\mathcal{P}_n)$ the approximate problem defined by

$$(\mathcal{P}_n)\begin{cases} -\operatorname{div}\ b_n(x, v, \nabla v) = g_n, & \text{in } \Omega, \\ v = 0, & \text{on } \partial\Omega, \end{cases}$$

where $b_n(x, s, \xi) = b(x, T_n(s), \xi)$.

We remark that the operator $b_n$ is coercive. Indeed we have

$$\begin{aligned} b_n(x, s, \xi) \cdot \xi &= b(x, T_n(s), \xi) \cdot \xi \\ &\geq \alpha \frac{|\xi|^{p(x)}}{(1 + |T_n(s)|)^{\theta(x)}} \\ &\geq \frac{\alpha}{(1 + n)^{\theta^+}} |\xi|^{p(x)}. \end{aligned}$$

The problem $(\mathcal{P}_n)$ has a weak energy solutions $v_n \in W_0^{1,p(\cdot)}(\Omega)$ as a result of a standard modification of the arguments in [21]. Our goal is to prove that $v_n$ tend to a measurable function $v$ as $n$ tend to infinity, and we prove that $v$ is an entropy solution of problem $(\mathcal{P})$. We will divide the proof in two steps and we employ the a priori estimates for $v_n$ and its gradient derived in the preceding section as our main tool. We follow the standard method used in the several paper as [6, 11, 27].

We prove in first step the almost everywhere convergence of the gradient.

First we prove that the sequence $(v_n)_n$ of solutions to problem $(\mathcal{P}_n)$ converges in measure to a measurable function $v$.

Define the $\mathcal{I}_1, \mathcal{I}_2$, and $\mathcal{I}_3$ sets as follows

$$\mathcal{I}_1 = \{|v_n| > t\}, \quad \mathcal{I}_2 = \{|v_m| > t\}, \quad \text{and} \quad \mathcal{I}_3 = \{|T_t(v_n) - T_t(v_m)| > s\},$$

for $s > 0$ and $t > 0$. Since

$$\{|v_n - v_m| > s\} \subset \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3,$$

it follows that

$$\text{meas}\ \{|v_n - v_m| > s\} \leq \text{ meas } (\mathcal{I}_1) + \text{ meas } (\mathcal{I}_2) + \text{ meas } (\mathcal{I}_3).$$

Let $\epsilon > 0$, by Theorem 3.2, $v_n$ is uniformly bounded sequence, thus there exists $t_\epsilon$, such that for $t \geq t_\epsilon$ we have

$$\text{meas } (\mathcal{I}_1) \leq \epsilon/3 \quad \text{and} \quad \text{meas } (\mathcal{I}_2) \leq \epsilon/3.$$

In the approximate problem $(\mathcal{P}_n)$, we take $T_t(v_n)$ as test function and following the outlines of Theorem 3.2, we get

$$\int_\Omega \frac{|\nabla T_t(v_n)|^{p(x)}}{(1+|v_n|)^{\theta(x)}}\, dx \leq \frac{\|g\|_1}{\alpha} t, \quad \text{for all } n \geq 0 \text{ and } t > 0,$$

which implies that for all $n \geq 0$ and $t > 0$,

$$\begin{aligned} \int_\Omega |\nabla T_t(v_n)|^{p(x)}\, dx &\leq \frac{\|g\|_1}{\alpha} t(1+t)^{\theta^+} \\ &\leq \frac{\|g\|_1}{\alpha}(1+t)^{\theta^+ + 1}. \end{aligned}$$

Sobolev embedding imply that there exists a subsequence still denoted by $(T_t(v_n))_n$ such that

$$\begin{array}{l} T_t(v_n) \rightharpoonup T_t(v) \text{ weakly in } W_0^{1,p(\cdot)}(\Omega), \\ T_t(v_n) \to T_t(v) \text{ strongly in } L^{m(\cdot)}(\Omega), \text{ for } 1 \leq m(\cdot) < p_*(\cdot), \\ T_t(v_n) \to T_t(v) \text{ a.e. in } \Omega, \end{array} \tag{7}$$

for all $t > 0$. Thus there exists $n_0(s, \epsilon) \in \mathbb{N}$ such that for all $n, m \geq n_0(s, \epsilon)$ we have

$$\begin{aligned} \text{meas } (\mathcal{I}_3) &= \int_{\{|T_t(v_n) - T_t(v_m)| > s\}} dx \\ &\leq \int_\Omega \left( \frac{|T_t(v_n) - T_t(v_m)|}{s} \right)^{m(x)} dx \\ &\leq \frac{1}{s^{m^\pm}} \int_\Omega |T_t(v_n) - T_t(v_m)|^{m(x)}\, dx \leq \epsilon/3. \end{aligned}$$

Finely for all $n, m \geq n_0(s, \epsilon)$ we have

$$\text{meas}\,\{|v_n - v_m| > s\} \leq \text{meas}\,(\mathcal{I}_1) + \text{meas}\,(\mathcal{I}_2) + \text{meas}\,(\mathcal{I}_3) \leq \epsilon,$$

which implies that $(v_n)_n$ is a Cauchy sequence in measure.

Following the standard argument as in [16], proving that $(\nabla v_n)_n$ is a Cauchy sequence in measure is an easy task.

In the second step we passing to the limit.

Let $v_n$ be a solution of approximate problem $(\mathcal{P}_n)$, for $w \in W_0^{1,p(\cdot)}(\Omega)$ we have

$$\int_\Omega b(x, T_n(v_n), \nabla v_n)\nabla w\, dx = \int_\Omega g_n(x) w\, dx.$$

Taking $w = T_t(v_n - \varphi)$ with $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ we get

$$\int_\Omega b(x, T_n(v_n), \nabla v_n)\nabla T_t(v_n - \varphi)\, dx = \int_\Omega g_n(x) T_t(v_n - \varphi)\, dx.$$

For the term in the right hand side, since $g_n$ converge strongly to $g$ in $L^1(\Omega)$ and $T_t(v_n - \varphi)$ converge weakly-* to $T_t(v - \varphi)$ in $L^\infty(\Omega)$, and a.e. in $\Omega$ we have

$$\int_\Omega g_n(x) T_t(v_n - \varphi)\, dx \longrightarrow \int_\Omega g(x) T_t(v - \varphi)\, dx.$$

For the left hand side we have

$$\begin{aligned}
\int_\Omega b(x, T_n(v_n), \nabla v_n) \cdot \nabla T_t(v_n - \varphi)\, dx &= \int_{\{|v_n - \varphi| \leq t\}} b(x, T_n(v_n), \nabla v_n) \cdot \nabla v_n\, dx \\
&\quad - \int_{\{|v_n - \varphi| \leq t\}} b(x, T_n(v_n), \nabla v_n) \cdot \nabla \varphi\, dx \\
&= \int_{\{|v_n - \varphi| \leq t\}} b(x, v_n, \nabla v_n) \cdot \nabla v_n\, dx \\
&\quad - \int_{\{|v_n - \varphi| \leq t\}} b(x, v_n, \nabla T_r(v_n)) \cdot \nabla \varphi\, dx
\end{aligned}$$

with $r = t + \|\varphi\|_\infty$.

By (2) and (7), we can prove that $b(x, v_n, \nabla T_r(v_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$, and converges weakly to $b(x, v, \nabla T_r(v))$ in $(L^{p'(\cdot)}(\Omega))^N$. Therefore we have

$$\int_{\{|v_n - \varphi| \leq t\}} b(x, v_n, \nabla T_r(v_n)) \cdot \nabla \varphi\, dx \longrightarrow \int_{\{|v - \varphi| \leq t\}} b(x, v, \nabla v) \cdot \nabla \varphi\, dx. \tag{8}$$

Since $b(x, v_n, \nabla v_n) \cdot \nabla v_n$ converge almost everywhere to $b(x, v, \nabla v) \cdot \nabla v$ in $\Omega$, by Fatou's lemma we have

$$\liminf_n \int_{\{|v_n - \varphi| \leq t\}} b(x, T_n(v_n), \nabla v_n) \cdot \nabla v_n\, dx \geq \int_{\{|v - \varphi| \leq t\}} b(x, v, \nabla v) \cdot \nabla v\, dx. \tag{9}$$

By (8) and (9), for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ we have

$$\int_\Omega b(x, v, \nabla v)\nabla T_t(v-\varphi)\,dx \le \int_\Omega g(x)T_t(v-\varphi)\,dx.$$

The results of regularity follow from Theorems 3.3 and 3.5.

### 3.3 Existence of weak solutions

In this section, first we find a sufficient conditions to have $m_0(\cdot) \gg 1$ and $m_1(\cdot) \gg 1$, and then we prove that the entropy solutions to the problem $(\mathcal{P})$ are also weak solutions under assumption $m_1(\cdot) \gg p(\cdot) - 1$.

**Lemma 3.7** *Let the exponents $m_0(\cdot)$ and $m_1(\cdot)$ as defined in Theorem 3.6, if $p(\cdot) \gg \theta^+ + 2 - \frac{1}{N}$. Then*

$$m_0(\cdot) \gg 1 \quad \text{and} \quad m_1(\cdot) \gg 1.$$

***Proof*** First, we prove that $m_0(\cdot) \gg 1$.

For all $x$ in $\Omega$, we have

$$\gamma = \left(\frac{p(x)}{p(x) - (\theta(x) + 1)}\right)^+ \le \left(\frac{p(x)}{p(x) - (\theta^+ + 1)}\right)^+ = \frac{p^-}{p^- - (\theta^+ + 1)}. \tag{10}$$

On the other hand, by a simple computations, we get

$$(p_*(\cdot))^- = \frac{Np^-}{N - p^-} > \frac{p^-}{p^- - (\theta^+ + 1)},$$

which gives according to (10) that $(p_*(\cdot))^- > \gamma$ and then $m_0(\cdot) \gg 1$.

Now we prove that $m_1(\cdot) \gg 1$.

By definition of $m_1(\cdot)$ we have for all $x \in \Omega$

$$\begin{aligned} m_1(x) &= \frac{Np^2(x)}{Np(x) + \gamma(\theta(x) + 1)(N - p(x))} \\ &\ge \frac{Np^2(x)}{Np(x) + \gamma(\theta^+ + 1)(N - p(x))}. \end{aligned}$$

Using that $\frac{Np^2(\cdot)}{Np(\cdot)+\gamma(\theta^+ + 1)(N - p(\cdot))}$ is increasing in $p(\cdot)$, $p(\cdot) \gg \theta^+ + 2 - \frac{1}{N}$ and (17) we have

$$
\begin{aligned}
\left(m_1(\cdot)\right)^- &\geq \left(\frac{Np^2(\cdot)}{Np(\cdot)+\gamma(\theta^++1)(N-p(\cdot))}\right)^- \\
&= \frac{Np^{-2}}{Np^-+\gamma(\theta^++1)(N-p^-)} \\
&\geq \frac{Np^-(p^--(\theta^++1))}{Np^--N(\theta^++1)+(\theta^++1)(N-p^-)} \\
&= \frac{N(p^--(\theta^++1))}{N-(\theta^++1)} > 1.
\end{aligned}
$$

□

**Theorem 3.8** *Assume that (1)–(2) hold true and $p(\cdot) \gg \theta^+ + 2 - \frac{1}{N}$, if $m_1(\cdot) \gg p(\cdot) - 1$ then the entropy solutions of problem $(\mathcal{P})$ are also weak solutions.*

***Proof*** Let $v_n$ be a solution to the approximate problem $(\mathcal{P}_n)$, we have

$$\int_\Omega b(x, T_n(v_n), \nabla v_n)\nabla\varphi\, dx = \int_\Omega g_n(x)\varphi\, dx, \quad \forall\, \varphi \in \mathcal{D}(\Omega). \tag{11}$$

Let $E$ be a subset of $\Omega$, by (2) we have

$$\int_E |b(x, T_n(v_n), \nabla v_n)|\, dx \leq \beta \int_E |\nabla v_n|^{p(x)-1}\, dx. \tag{12}$$

By Theorem 3.6, Lemma 3.7 and $m_1(\cdot) \gg p(\cdot) - 1$ the terms $|\nabla v_n|^{p(x)-1}$ is uniformly bounded in $L^{m(\cdot)}(\Omega)$, for some $m(\cdot) \gg 1$, so the terms in the right hand side of (12) goes to zero when a measure of $E$ is small enough.

Since $b(x, T_n(v_n), \nabla v_n)$ converge almost everywhere to $b(x, v, \nabla v)$, by Vitali theorem we conclude that $b(x, T_n(v_n), \nabla v_n)$ converge strongly to $b(x, v, \nabla v)$ in $L^1(\Omega)$. Now passing to the limit in (11), by using the previous results and $g_n$ converge strongly to $g$ in $L^1(\Omega)$ we obtain

$$\int_\Omega b(x, v, \nabla v)\nabla\varphi\, dx = \int_\Omega g(x)\varphi\, dx, \quad \forall\, \varphi \in \mathcal{D}(\Omega).$$

□

***Remark* 4** We can deal with the following degenerate elliptic problem and obtain the same results as above

$$\begin{cases} -\operatorname{div}\ b(x, v, \nabla v) + h(x, v) = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

with the lower order term $h(\cdot, v(\cdot))$ is a Carathéodory function that verifies a sign condition and with a natural growth.

## 4 Regularity results

In this section we assume that $g \in L^{q(\cdot)}(\Omega)$ with $q(x) \geq 1,\ \forall x \in \Omega$ and $q^- < \frac{p_*^-(p^- - \theta^-)}{p_*^-(p^- - \theta^-) - p^-}$. We prove some regularity of entropy solutions for a problem $(\mathcal{P})$.

***Remark* 5** For $\gamma > 0, v \in M^{\frac{p_*(\cdot)}{\gamma}}(\Omega)$ implies that $v \in M^{\frac{p_*^-}{\gamma}}(\Omega)$, which gives $\mu_v(t) \leq c_1 t^{-\frac{p_*^-}{\gamma}}$, with $c_1$ is a positive constant.

***Remark* 6** Since $g \in L^{m(\cdot)}(\Omega) \subset M^{m(\cdot)}(\Omega)$ we have $g \in M^{m^-}(\Omega)$, which gives $g_*(t) \leq c_2 t^{-\frac{1}{m^-}}$, with $c_2$ is a positive constant.

**Theorem 4.1** *Under assumptions* (1) *and* (2). *If $v$ is a solution in the sense of Definition* 3.1 *that belongs to* $M^{\frac{p_*(\cdot)}{\gamma}}(\Omega)$, *then there exists a positive constant* $C$, *depending only on* $p(\cdot), N,$ *and* $\Omega$, *such that*

$$\int_{\{|v| \leq t\}} \frac{|\nabla v|^{p(x)}}{(1+|v|)^{\theta(x)}}\, dx \leq C t^{1-\frac{(p_*)^-}{\gamma(q^-)'}},$$

*for all* $t > 0$ *and* $\gamma > \frac{(p_*)^-}{(q^-)'}$.

***Proof*** Since $v$ is an entropy solution to the problem $(\mathcal{P})$, we have

$$\int_\Omega a(x, v, \nabla v) \nabla T_t(v - \varphi)\, dx \leq \int_\Omega g(x) T_t(v - \varphi)\, dx$$

for all $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, by taking $\varphi = T_s(v)$ with $s \geq 1$, we have

$$\int_{\{s<|v|\leq t+s\}} a(x, v, \nabla v) \nabla v\, dx \leq t \int_{\{|v|>s\}} |g(x)|\, dx. \tag{13}$$

By Young inequality and dividing in the both sides by $t$, (1) gives

$$\frac{\alpha}{t} \int_{\{s<|v|\leq t+s\}} \frac{|\nabla v|^{p(x)}}{(1+|v|)^{\theta(x)}}\, dx \leq \int_{\{|v|>s\}} |g(x)|\, dx. \tag{14}$$

Passing to the limit in (14), for $t$ goes to zero we have

$$\alpha \frac{d}{ds} \int_{\{|v|\leq s\}} \frac{|\nabla v|^{p(x)}}{(1+|v|)^{\theta(x)}}\, dx \leq \int_0^{\mu_v(s)} g_*(\tau)\, d\tau.$$

Integrating between 0 and $t$, we get

$$\alpha \int_{\{|v|\leq t\}} \frac{|\nabla v|^{p(x)}}{(1+|v|)^{\theta(x)}}\, dx \leq \int_0^t \int_0^{\mu_v(s)} g_*(\tau)\, d\tau\, ds. \tag{15}$$

Now by using Remarks 5 and 6 the term in the right hand side of the previous inequality becomes

$$\begin{aligned}
\int_0^t \int_0^{\mu_v(s)} g_*(\tau)\, d\tau\, ds &\le c_2 \int_0^t \int_0^{\mu_v(s)} \tau^{-\frac{1}{q^-}}\, d\tau\, ds \\
&\le c_2 \int_0^t \mu_v(s)^{1-\frac{1}{q^-}}\, ds \\
&\le c_2 c_1^{1-\frac{1}{q^-}} \int_0^t s^{-\frac{(p_*)^-}{(q^-)'\gamma}}\, ds \\
&\le 2^{\theta^+} c_2 c_1^{1-\frac{1}{q^-}} \int_0^t s^{-\frac{(p_*)^-}{(q^-)'\gamma}}\, ds \\
&= \frac{2^{\theta^+} c_2 c_1^{\frac{1}{(q^-)'}}}{1-\frac{(p_*)^-}{(q^-)'\gamma}} \times t^{1-\frac{(p_*)^-}{(q^-)'\gamma}},
\end{aligned}$$

which implies that there exists a constant $c > 0$ such that

$$\int_\Omega \frac{|\nabla T_t(v)|^{p(x)}}{(1+|v|)^{\theta(x)}}\, dx \le c t^{1-\frac{(p_*)^-}{(q^-)'\gamma}}.$$

□

**Theorem 4.2** *Assume that the assumptions* (1) *and* (2) *hold true. If* $v$ *is an entropy solution of* $(\mathcal{P})$*, then there exists a positive constant* $c$*, depending only on* $p^\pm$*,* $N$*, and* $\Omega$*, such that*

(1) $\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}}\, dx \le c$*, for all* $t > 0$*, with* $\gamma_q = \left(\frac{q^- p(\cdot)-(p_*)^-(q^- -1)}{q^-(p(\cdot)-(\theta(\cdot)+1))}\right)^+$.

(2) *Let* $m_{0,q}(x) = \frac{p_*(x)}{\gamma_q}$*, for all* $m(\cdot)$ *such that* $0 \ll m(\cdot) \ll m_{0,q}(\cdot)$*, we have* $|v|^{m(\cdot)} \in L^1(\Omega)$*. Moreover there exists a constant positive* $c_0$ *such that* $\int_\Omega |v|^{m(x)}\, dx \le c_0$.

***Remark* 7** We remark that $\gamma_q > \frac{p_*^-}{(q^-)'}$. Ended by using $\theta^- \le \theta(x),\ \forall x \in \Omega$ we have

$$\gamma_q = \left(\frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta(x)+1))}\right)^+ \ge \left(\frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta^- + 1))}\right)^+,$$

and since $q^- < \frac{p_*^-(p^- - \theta^-)}{p_*^-(p^- - \theta^-) - p^-}$ we have

$$\frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta^- + 1))} \quad \text{is non-increasing in } p(\cdot).$$

Moreover we have

$$\gamma_q = \left( \frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta^- + 1))} \right)^+ = \frac{q^- p^- - (p_*)^-(q^- - 1)}{q^-(p^- - (\theta^- + 1))} > \frac{p_*^-}{(q^-)'}.$$

***Proof***

(1) We find $\gamma_q$ such that $v \in M^{\frac{p_*(x)}{\gamma_q}}$.
*Case 1:* $t \geq 1$, by using Proposition 2.2 and Sobolev embedding we have

$$\begin{aligned}
\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}}\, dx &= \int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}} \left|\frac{T_t(v)}{t}\right|^{p_*(x)} dx \\
&= \int_{\{|v|>t\}} |t^{(\frac{1}{\gamma_q}-1)} T_t(v)|^{p_*(x)}\, dx \\
&\leq \left\| t^{(\frac{1}{\gamma_q}-1)} T_t(v) \right\|_{p_*(x)}^{\alpha_1} \leq c \left\| \nabla (t^{(\frac{1}{\gamma_q}-1)} T_t(v)) \right\|_{p(\cdot)}^{\alpha_1} \\
&\leq c \left( \int_\Omega t^{(\frac{1}{\gamma_q}-1)} |\nabla T_t(v)|^{p(x)}\, dx \right)^{\alpha_1/\alpha_2}
\end{aligned} \tag{16}$$

by using Theorem 4.1 we obtain

$$\begin{aligned}
&\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}}\, dx \\
&\leq c \left( \int_\Omega t^{(\frac{1}{\gamma_q}-1)p(x)} \frac{|\nabla T_t(v)|^{p(x)}}{(1+|v|)^{\theta(x)} t^{1-\frac{(p_*)^-}{(q^-)'\gamma_q}}} t^{1-\frac{(p_*)^-}{(q^-)'\gamma_q}} (1+|v|)^{\theta(x)}\, dx \right)^{\alpha_1/\alpha_2} \\
&\leq 2^{\theta^+} c \left( \int_\Omega t^{(\frac{1}{\gamma_q}-1)p(x)} \frac{|\nabla T_t(v)|^{p(x)}}{(1+|v|)^{\theta(x)} t^{1-\frac{(p_*)^-}{(q^-)'\gamma_q}}} t^{1+\theta(x)-\frac{(p_*)^-}{(q^-)'\gamma_q}}\, dx \right)^{\alpha_1/\alpha_2} \\
&\leq 2^{\theta^+} c \left( \int_\Omega |t^{(\frac{1}{\gamma_q}-1)p(x)+1+\theta(x)-\frac{(p_*)^-}{(q^-)'\gamma_q}}\, dx \right)^{\alpha_1/\alpha_2},
\end{aligned} \tag{17}$$

if we choose $\gamma_q$ such that $(\frac{1}{\gamma_q} - 1)p(x) + 1 + \theta(x) - \frac{(p_*)^-}{(q^-)'\gamma_q} \leq 0$
i.e.

$$\gamma_q \geq \frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta(x) + 1))}, \quad \forall x \in \Omega,$$

and by taking $\gamma_q = \left( \frac{q^- p(x) - (p_*)^-(q^- - 1)}{q^-(p(x) - (\theta(x) + 1))} \right)^+$, (17) gives

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}}\, dx \leq c.$$

*Case 2:* $0 < t < 1$

$$\int_{\{|v|>t\}} t^{\frac{p_*(x)}{\gamma_q}} \leq |\Omega|.$$

Combining the estimates in both cases, the result follows.

(2) Let $0 \ll m(\cdot) \ll m_{0,q}(\cdot)$ and $\epsilon = (m_{0,q}(\cdot) - m(\cdot))^- > 0$. By Theorem 4.1, we have

$$\int_{\{|v|>t\}} t^{m_{0,q}(x)}\, dx \leq c, \quad \text{for all } t > 0.$$

From Lemma 2.7, we get

$$\int_\Omega |v|^{m(x)}\, dx \leq 2|\Omega| + c\Big(\frac{m_{0,q}(\cdot) - \epsilon}{\epsilon}\Big)^+, \text{ which gives the results .}$$

□

**Theorem 4.3** *Assume that the assumptions* (1) *and* (2) *hold true. Let $v$ be an entropy solution of* ($\mathcal{P}$). *If there exists a positive constant $c$ such that $\int_{\{|v|>t\}} t^{m_{0,q}(x)}\, dx \leq c$, for all $t > 0$, then $|\nabla v|^{\alpha(\cdot)} \in M^{m_{0,q}(\cdot)}(\Omega)$, where $\alpha(\cdot) = \frac{\gamma_q (q^-)' p(\cdot)}{\gamma_q (q^-)'[m_{0,q}(\cdot)+\theta(\cdot)+1] - p_*(\cdot)}$. Moreover there exists a positive constant $C'$ such that*

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_{0,q}(x)}\, dx \leq C', \quad \text{for all } t > 0.$$

***Proof*** Using Theorem 4.1, and the definition of $\alpha(\cdot)$, for $t > 1$ we have

$$\begin{aligned}
\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_{0,q}(x)}\, dx &\leq \int_{\{|\nabla v|^{\alpha(\cdot)}>t\}\cap\{|v|\leq t\}} t^{m_{0,q}(x)}\, dx + \int_{\{|\nabla v|^{\alpha(\cdot)}>t\}\cap\{|v|>t\}} t^{m_{0,q}(x)}\, dx \\
&\leq \int_{\{|v|\leq t\}} t^{m_{0,q}(x)} \left(\frac{|\nabla v|^{\alpha(x)}}{t}\right)^{p(x)/\alpha(x)} dx + c' \\
&= \int_{\{|v|\leq t\}} t^{m_{0,q}(x)+1-\frac{(p_*)^-}{\gamma_q(q^-)'}-\frac{p(x)}{\alpha(x)}} \frac{|\nabla T_t(v)|^{p(x)}(1+|v|)^{\theta(x)}}{t^{1-\frac{(p_*)^-}{\gamma_q(q^-)'}}(1+|v|)^{\theta(x)}}\, dx + c' \\
&\leq 2^{\theta^+} \int_{\{|v|\leq t\}} t^{m_{0,q}(x)+1+\theta(x)-\frac{(p_*)^-}{\gamma_q(q^-)'}-\frac{p(x)}{\alpha(x)}} \frac{|\nabla T_t(v)|^{p(x)}}{t^{1-\frac{(p_*)^-}{\gamma_q(q^-)'}}(1+|v|)^{\theta(x)}}\, dx + c' \\
&\leq 2^{\theta^+} \int_{\{|v|\leq t\}} \frac{|\nabla T_t(v)|^{p(x)}}{t^{1-\frac{(p_*)^-}{\gamma_q(q^-)'}}}\, dx + c' \\
&\leq C'.
\end{aligned}$$

where $c'$ and $C'$ are positive constants.

For $t \leq 1$ we have

$$\int_{\{|\nabla v|^{\alpha(\cdot)}>t\}} t^{m_{0,q}(x)}\, dx \leq |\Omega|.$$

□

**Theorem 4.4** *Assume that the assumptions* (1) *and* (2) *hold true. Let* $m_{0,q}(\cdot)$ *be defined in Theorem* 4.2 *and* $m_{1,q}(\cdot) = m_{0,q}(\cdot)\alpha(\cdot)$.

*If* $v$ *is an entropy solution of problem* $(\mathcal{P})$, *then* $|\nabla v|^{m(\cdot)} \in L^1(\Omega)$, *for all* $m(\cdot)$ *such that* $0 \ll m(\cdot) \ll m_{1,q}(\cdot)$. *Moreover there exists a constant C such that*

$$\int_\Omega |\nabla v|^{m(x)}\, dx \leq C.$$

***Proof*** By Theorem 4.3, we have

$$|\nabla v|^{\alpha(\cdot)} \in M^{m_{0,q}(\cdot)}(\Omega), \quad \text{with } \alpha(\cdot) = \frac{p(\cdot)}{m_{0,q}(\cdot) + \theta(\cdot) + 1 - \frac{(p_*)^-}{\gamma_q (q^-)'}}.$$

Let $0 \ll m(\cdot) \ll m_{1,q}(\cdot)$ and $r(\cdot) = m(\cdot)/\alpha(\cdot) \ll m_{0,q}(\cdot)$.

Using the Theorem 4.2 we obtain

$$\int_\Omega |\nabla v|^{m(x)}\, dx = \int_\Omega |\nabla v|^{\alpha(x) r(x)}\, dx \leq C.$$

□

***Remark* 8** If $q(\cdot) \equiv 1$, i.e., $g \in L^1(\Omega)$, we remark that $m_0(\cdot)$ coincide with $m_{0,q}(\cdot)$ and $m_1(\cdot)$ coincide with $m_{1,q}(\cdot)$, which implies that the regularity results obtained in the current section are a generalization of those obtained in Theorem 3.6 of Sect. 3.

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