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Nonlinear anisotropic degenerate parabolic equations with variable exponents and irregular data

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Abstract

In this paper, we prove the existence and the regularity of weak solutions for a class of nonlinear anisotropic parabolic equations with $p_i(\cdot)$ growth conditions, degenerate coercivity and $L^{m(\cdot)}$ data, with $m(\cdot) > 1$ being small. The functional setting involves Lebesgue-Sobolev spaces with variable exponents.

Keywords Anisotropic parabolic equations · Variable exponents · Degenerate coercivity · Regularity of weak solution · Irregular data

Mathematics Subject Classification 35K55 · 35K65

1 Introduction

In this paper, we are interested in the existence and regularity of solutions for some nonlinear parabolic equations with principal part having degenerate coercivity:

$$\begin{aligned} &\partial_t u - \sum_{i=1}^{N} D_i\big(b_i(t,x,u)a_i(t,x,Du)\big) + F(t,x,u) = f \quad \text{in } Q_T, \\ &u = 0 \quad \text{on } \Sigma_T, \\ &u(0,x) = u_0(x) \quad \text{in } \Omega. \end{aligned} \tag{1}$$

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where $Q_T \doteq (0,T)\times\Omega, \Sigma_T = (0,T)\times\partial\Omega, \Omega$ is a bounded open subset of $\mathbb{R}^N$, $(N \geq 2)$, $T > 0$ is a real number, $f \in L^{m(\cdot)}$, and $u_0 \in L^{(m(\cdot)-1)s_+(\cdot)+1}(\Omega)$, $s_+(\cdot) = \max_{1\leq i\leq N} s_i(\cdot)$. Here, we suppose that $b_i : Q_T \times \mathbb{R} \longrightarrow \mathbb{R}$, $a_i : Q_T \times \mathbb{R}^N \longrightarrow \mathbb{R}$, $F : Q_T \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and satisfying for a.e.$(t,x) \in Q_T$, for all $u \in \mathbb{R}$, for all $\xi, \xi' \in \mathbb{R}^N$, and for all $i = 1, \dots, N$ the following:

$$a_i(t,x,\xi)\cdot\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \tag{2}$$

$$|a_i(t,x,\xi)| \leq \left(g + \sum_{j=1}^{N} |\xi_j|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}, \tag{3}$$

$$\Big(a_i(t,x,\xi_i) - a_i(t,x,\xi_i')\Big)\cdot(\xi_i - \xi_i') > 0, \quad \xi_i \neq \xi_i', \tag{4}$$

$$\frac{C_2}{(1+|u|)^{\sigma(x)}} \leq b_i(t,x,u) \leq C_1, \tag{5}$$

where $\alpha > 0$, $C_1 > 0$, $C_2 > 0$, $m,\ \sigma \in C(\overline{\Omega})$, $m(\cdot) > \underline{1,}\ \sigma(\cdot) \geq 0$, $g \in L^1(Q_T)$ is non-negative function, and the variable exponents $p_i : \overline{\Omega} \longrightarrow (1,+\infty)$ are continuous functions.

$$\sup_{|u|\leq\lambda} |F(t,x,u)| \in L^1(Q_T), \qquad \text{for all } \lambda > 0, \tag{6}$$

$$F(t,x,u)\,\text{sign}(u) \geq \sum_{i=1}^{N} |u|^{s_i(x)},\ a.e.\ (t,x) \in Q_T,\ \text{for all } u \in \mathbb{R}, \tag{7}$$

where $s_i : \overline{\Omega} \longrightarrow (1,+\infty)$ are continuous functions on $\overline{\Omega}$.

Let us consider for example the operator

$$\begin{aligned} Au &= -\sum_{i=1}^{N} D_i\big(b_i(t,x,u)a_i(t,x,Du)\big) \\ &= -\sum_{i=1}^{N}\left(D_i\left(\frac{|D_iu|^{p_i(x)-2}D_iu}{\Big(\ln(e+|u|)\Big)^{\sigma(x)}}\right)\right). \end{aligned}$$

The main difficulties in studying (1) are the fact that, due to assumption (5), the differential operator Au is not coercive if u is very large, and the problem (1) has a more complicated nonlinearity than the classical case $p_i(\cdot) = p_i$ since it is nonhomogeneous. This shows that the classical methods for the constant case [7] can't be applied here. In the classical case $\sigma = 0$ and $p_i(\cdot) = p_i$ the existence and regularity solution have been treated in [7]. It is worth pointing out that the problem (1) has been studied in [5] in the particular case $p_i(\cdot) = 2$, $i \in \{1,2,\cdots,N\}$, $m(\cdot) = m$, and

$\sigma(\cdot) = \theta \in \left[0, 1 + \frac{2}{N}\right)$ with $u_0 = 0$, where the author have discussed the existence and regularity results based on (Lemma 2.2, [5]), but this technique do not work in the anisotropic case. In this paper, we assume that condition (7) holds true and we treat the regularity of u depending simultaneously on F and f. It is an open question to solve our problem without an additional control on u ($F = 0$). As in elliptic case [1, 9], we give also a better regularity result on Du when $s(\cdot)$ is large enough because if $\sigma(\cdot) = \sigma$, $m(\cdot) = m$, and $S_+(\cdot) > \frac{(N+1)\overline{p}-N(1+\sigma)}{N-(m-1)\overline{p}(\cdot)}$ we have

$$\frac{mp_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\sigma} > \frac{mp_i(\cdot)}{\overline{p}(\cdot)}\left(\frac{(N+1)\overline{p}(\cdot)-N(1+\sigma)}{N+1-(1+\sigma)(m-1)}\right), \quad \frac{1}{\overline{p}(\cdot)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(\cdot)}.$$

So, Theorem 4 improves Theorem 3 and (Theorem 1.4, [5]).

The proof of existence and regularity results under the assumptions (2)-(5), where $p_i(\cdot)$ is assumed to be merely a continuous function (is not as in (10)), is essentially based on the approximate problems (24) with some non degenerate coercivity and regular data. To describe briefly the tools we use, firstly we have the anisotropic Sobolev inequality to overcome the difficulties of getting the regularity in the Lemma 7, secondly we introduce the Lemma 9 to facilitate the control of the term ∂u_n of the regularized problem. Thirdly to prove Theorems 4, a key result about an $L^{m(\cdot)s_+(\cdot)}(Q_T)$ estimate for solution to (1) is proved. For the uniqueness of the weak solution, where f is irregular data, it is necessary to impose additional conditions on the data of the problem (1). Our regularity results are new and have not been proven before neither in the isotropic nor in the anisotropic case.

This paper is organized in the following way: In Sect. 2, we introduce the function spaces. The main Results are presented in Sect. 3. Theorems 3–5 are proved in Sect. 4.

2 Preliminaries

Let $p_i(\cdot) : \Omega \to (1, +\infty)$ be a continuous function for all $i = 1, \ldots, N$ and let $p_i^- = \min_{x\in\overline{\Omega}} p(x)$, $p_i^+ = \max_{x\in\overline{\Omega}} p(x)$. The appropriate Sobolev space to study problem (1) is the anisotropic spaces

$$W^{1,p_i(\cdot)}(\Omega) = \left\{u \in L^{p_i(\cdot)}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$
$$W_0^{1,p_i(\cdot)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$

which are Banach spaces under the norm

$$\|u\|_i = \|u\|_{L^{p_i(\cdot)}(\Omega)} + \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad i = 1, \ldots, N, \tag{8}$$

where

$$\|u\|_{L^{p_i(\cdot)}(\Omega)} = \inf\left\{ \lambda > 0 \mid \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p_i(x)} dx \le 1 \right\}.$$

The following inequality will be used later

$$\min\left\{ \|u\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}, \|u\|^{p_i^+}_{L^{p_i(\cdot)}(\Omega)} \right\} \le \int_\Omega |u(x)|^{p_i(x)} dx \le \max\left\{ \|u\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}, \|u\|^{p_i^+}_{L^{p_i(\cdot)}(\Omega)} \right\}. \tag{9}$$

The smooth functions are in general not dense in $\bigcap_{i=1}^{i=N} W_0^{1,p_i(\cdot)}(\Omega)$, but if the exponent variable $p_i(\cdot) > 1$ for each $i = 1, \dots, N$ satisfies the log-Hölder continuity condition (10), that is $\exists M > 0$:

$$|p_i(x) - p_i(y)| \le -\frac{M}{\ln(|x-y|)} \ \forall x \ne y \in \Omega \ \text{ such that } |x-y| \le \frac{1}{2}, \tag{10}$$

then the smooth functions are dense in $\bigcap_{i=1}^{i=N} W_0^{1,p_i(\cdot)}(\Omega)$. The Poincaré type inequality is not correct in the variable anisotropic case but we have the following

Theorem 1 ([4]) *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p_i(\cdot) > 1$ are continuous functions. Suppose that*

$$p_i(x) < \overline{p}^*(x), \tag{11}$$

where

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \overline{p}(x) < N \\ +\infty, & \text{if } \overline{p}(x) \ge N. \end{cases}$$

Then the following Poincaré-type inequality holds:

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \le C \sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad \forall u \in \bigcap_{i=1}^N W_0^{1,p_i(\cdot)}(\Omega),$$

where C is a positive constant independent of u. Thus, $\sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^N W_0^{1,p_i(\cdot)}(\Omega)$.

The following embedding result for the anisotropic constant exponent Sobolev space is well-known [11, 13].

Lemma 2 *Let Q be a cube of $\mathbb{R}^N$ with faces parallel to the coordinate planes. Suppose $p_i \ge 1, i = 1, ..., N$ and $u \in \bigcap_{i=1}^N W^{1,p_i}(Q)$. Then*

$$\|u\|_{L^s(Q)} \le K \prod_{i=1}^N \left(\|u\|_{L^{p_i}(Q)} + \|D_i u\|_{L^{p_i}(Q)} \right)^{\frac{1}{N}}, \tag{12}$$

where $s = \overline{p}^* = \frac{N\overline{p}}{N-\overline{p}}$ *if* $\overline{p} < N$ *with* $\overline{p}$ *given by* $\frac{1}{\overline{p}} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i}$. *The constant K depends on N and* p_i. *Furthermore, if* $\overline{p} \geq N$, *the inequality* (12) *is true for all* $s \geq 1$, *and K depends on s and* $|Q|$.

Remark 1 ([3]) Let Ω be a bounded subset of $\mathbb{R}^N$, and $p_i : \Omega \to (1, +\infty)$ be a continuous function. We have the following continuous dense embeddings

$$L^{p_i^+}(0,T;L^{p_i(\cdot)}(\Omega)) \hookrightarrow L^{p_i(\cdot)}(Q_T) \hookrightarrow L^{p_i^-}(0,T;L^{p_i(\cdot)}(\Omega)).$$

Throughout the paper we suppose that $p_i(\cdot) > 1$ are continuous functions satisfied the assumption (11).

3 Statements of results

Definition 1 A function u is a weak solution of problem (1) if:

$$u \in L^1(0,T;W_0^{1,1}(\Omega)) \cap \left(L^{s_+(\cdot)}(Q_T)\right),\ a_i \in L^1(0,T;L^1(\Omega)),\ F \in L^1(Q_T),$$

and

$$\begin{aligned} &-\int_0^T\int_\Omega u\partial_t\varphi\,dx\,dt + \sum_{i=1}^N\int_0^T\int_\Omega b_i(t,x,u)a_i(t,x,Du)D_i\varphi\,dx\,dt \\ &+\int_0^T\int_\Omega F(t,x,u)\varphi\,dx\,dt = \int_0^T\int_\Omega \varphi(t,x)f\,dx\,dt + \int_\Omega \varphi(0,x)u_0(x)\,dx, \end{aligned} \tag{13}$$

for all $\varphi \in C_c^1([0,T) \times \Omega)$, the C_c^1 functions with compact support.

Our main existence results for (1) are the following:

Theorem 3 *Let* $m(\cdot) = m$, $\sigma(\cdot) = \sigma$, $f \in L^m(Q_T)$ *with* $m > 1$, *such that*

$$\frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+1)\overline{p}(\cdot) - 2N(1+\sigma) + (N+\sigma+2)\overline{p}(\cdot)} < m < \frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+\sigma+2)\overline{p}(\cdot) - N(\sigma+1)}, \tag{14}$$

$$0 \leq \sigma < \min\left\{\overline{p}(\cdot) - 1 + \frac{\overline{p}(\cdot)}{N}, \overline{p}(\cdot) - 2 + \frac{m\overline{p}(\cdot)}{N}\right\},\ \overline{p}(\cdot) \geq 2. \tag{15}$$

Assume that $s_i(\cdot)$, $p_i(\cdot)$ *are continuous functions such that for all* $i = 1, \ldots, N$

$$\frac{\overline{p}(\cdot)\Big(N+1-(1+\sigma)(m-1)\Big)}{m\Big((N+1)\overline{p}(\cdot) - N(1+\sigma)\Big)} < p_i(\cdot) < \frac{\overline{p}(\cdot)\Big(N+1-(1+\sigma)(m-1)\Big)}{mN(1+\sigma) - (m-1)(N+\sigma+2)\overline{p}(\cdot)}, \tag{16}$$

and

$$s_i(\cdot) \geq p_i(\cdot). \tag{17}$$

Let a_i, F be Carathéodory functions, where a_i satisfying (2)-(4) and F satisfying (6)-(7). Then, the problem (1) has at least one weak solution

$$u \in \bigcap_{i=0}^{N} L^{q_i^-}\left(0, T; W_0^{1,q_i(\cdot)}(\Omega)\right),$$

where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying for all $i = 1, \dots, N$

$$1 \leq q_i(\cdot) < \frac{mp_i(\cdot)}{\overline{p}(\cdot)}\left(\frac{(N+1)\overline{p}(\cdot) - N(1+\sigma)}{N+1-(1+\sigma)(m-1)}\right). \tag{18}$$

Remark 2 The lower bound for m in (14) is due to the fact that $q_i(\cdot)$ must not be smaller than 1. The upper bound for m in (14) implies $q_i(\cdot) < p_i(\cdot)$. In the Theorem 3, we suppose that $m > 1$ because if $0 \leq \sigma < \frac{(N+1)\overline{p}(\cdot)-2N}{2N}$, then $\frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+1)\overline{p}(\cdot)-2N(1+\sigma)+(N+\sigma+2)\overline{p}(\cdot)} < 1$.

Remark 3 We note that (14) (resp. (16)) is well defined since we have (15) (resp. (14)).

Remark 4 We not that (14) implies

$$\overline{p}(\cdot) < \frac{N(N+\sigma+2)}{m(N+1)}. \tag{19}$$

Therefore, by (18) we have $\overline{q}(\cdot) < N$.

Remark 5 Under the assumption $0 \leq \sigma < \frac{(N+2)\overline{p}(\cdot)-2(N+1)}{1+N-\overline{p}(\cdot)}$ that is

$$\frac{(N+\sigma+2)\overline{p}(\cdot)}{(N+\sigma+2)\overline{p}(\cdot) - N(\sigma+1)} < \overline{p}(\cdot),$$

we can deduce that f is never in the dual space $\left(\bigcap_{i=1}^{N} L^{p_i^-}\left(0, T; W_0^{1,p_i(\cdot)}(\Omega)\right)\right)'$, so that the result of this paper deals with irregular data as in [2, 10]. If m tends to be 1, then $q_i(\cdot) < \frac{p_i(\cdot)}{\overline{p}(\cdot)}\left(\overline{p}(\cdot) - \frac{N(\sigma+1)}{N+1}\right)$, which is bound on $q_i(\cdot)$ obtained in [10]. Furthermore if $p_i(.) = 2$ the assumption (15) is equivalent to [(1.2), [5]], and then $q_i(.) = q = \frac{m(N(1-\sigma)+2)}{N+1-(1+\sigma)(m-1)}$, which is bound on q obtained in [(1.6), [5]].

Theorem 4 *Let $f \in L^{m(\cdot)}(Q_T)$, $1 < m(\cdot) < p_i'(\cdot)$ $p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot)-1}$, $s_i(\cdot) > 0$, $i = 1, \dots, N$ and $\sigma(\cdot) \geq 0$, such that*

$$\frac{1+\sigma(\cdot)}{m(\cdot)-1} > s_+(\cdot) > (1+\sigma(\cdot))\max\left\{\frac{(p_i(\cdot)-1)}{(m(\cdot)-1)p_i(\cdot)+1};(p_i(\cdot)-1)\right\} \tag{20}$$
$$\nabla s_+ \in L^\infty(Q_T),\ \nabla m \in L^\infty(Q_T)$$

$$p_i(\cdot) > \frac{1}{m(\cdot)}\Big(1+\frac{1+\sigma(\cdot)}{s_+(\cdot)}\Big). \tag{21}$$

Let a_i, F be Carathéodory functions, where a_i satisfying (2)-(4), and F satisfying (6)-(7). Then, the problem (1) has at least one weak solution

$$u \in \bigcap_{i=0}^{N} L^{q_i^-}\Big(0,T;W_0^{1,q_i(\cdot)}(\Omega)\Big),$$

where $q_i(\cdot)$ are continuous functions on $\overline{\Omega}$ satisfying for all $i=1,\dots,N$

$$q_i(\cdot) = \frac{m(\cdot)p_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\sigma(\cdot)}. \tag{22}$$

Theorem 5 *Let $f \in L^{m(\cdot)}(Q_T)$, $1<m(\cdot)<p_i'(\cdot)$, $\sigma(\cdot)\geq 0$, such that*

$$s_+(\cdot) \geq \frac{1+\sigma(\cdot)}{m(\cdot)-1}, \qquad \nabla\sigma \in L^\infty(\Omega). \tag{23}$$

Under the hypotheses (2)-(10) and (6)-(7), the problem (1) has at least one weak solution $u \in \cap_{i=1}^N L^{p_i^-}(0,T,W_0^{1,p_i(\cdot)}(\Omega)) \cap L^{1+s_+(\cdot)+\sigma(\cdot)}(Q_T)$.

Remark 6 The assumption $1<m(\cdot)<p_i'(\cdot)$ implies that (20) holds, otherwise (20) become empty. By (21) we have that $q_i(\cdot)>1$ in (22).

4 Proof of Theorems 3, 4, and 5

4.1 Approximation of (1)

Let $(f_n)_{n\in\mathbb{N}^\star} \subset C_c^\infty(Q_T)$ and $(u_{0,n})_{n\in\mathbb{N}^\star} \subset C_c^\infty(\Omega)$ be sequences of functions satisfying

$$|f_n| \leq n,\ |u_{0,n}| \leq n,\ \forall n \geq 1,$$
$$\|f_n\|_{L^{m(\cdot)}(Q_T)} \leq \|f\|_{L^{m(\cdot)}(Q_T)},\ \|u_{0,n}\|_{L^{(m(\cdot)-1)s_+(\cdot)+1}(\Omega)} \leq \|u_0\|_{L^{(m(\cdot)-1)s_+(\cdot)+1}(\Omega)}\ \forall n \geq 1.$$

Then, there exists at least one weak solution (see [2])

$$\begin{cases} u_n \in \bigcap_{i=1}^N L^{p_i^-}\Big(0,T;W_0^{1,p_i(\cdot)}(\Omega)\Big) \cap C([0,T];L^2(\Omega)),\\ \partial_t u_n \in \sum_{i=1}^N L^{p_i^{-\prime}}\Big(0,T;(W_0^{1,p_i(\cdot)}(\Omega))'\Big) + L^1(Q_T),\end{cases}$$

of problems

$$
\begin{aligned}
&\partial_t u_n - \sum_{i=1}^{N} D_i\big(b_i(t,x,T_n(u_n))a_i(t,x,Du_n)\big) + F(t,x,u_n) = f_n \quad \text{in } Q_T,\\
&u_n = 0 \quad \text{on } \Sigma_T,\\
&u_n(0,x) = u_{0,n}(x) \quad \text{in } \Omega,
\end{aligned} \tag{24}
$$

each of them satisfying the weak formulation

$$
\begin{aligned}
&\int_0^T < \partial_t u_n, \varphi > \, dt + \sum_{i=1}^{N} \int_0^T \int_\Omega b_i(t,x,T_n(u_n))a_i(t,x,Du_n).D_i\varphi \, dx\, dt\\
&+ \int_0^T \int_\Omega F(t,x,u_n).\varphi \, dx\, dt = \int_0^T \int_\Omega f_n \varphi \, dx\, dt,
\end{aligned} \tag{25}
$$

for all $\varphi \in \bigcap_{i=1}^{N} L^{p_i^-}\Big(0,T; W_0^{1,p_i(\cdot)}(\Omega)\Big) \cap L^\infty(Q_T)$. The truncation function T_k at height k, $k > 0$ is defined by $T_k(t) = \max\{-k, \min\{k,t\}\}$, $t \in \mathbb{R}$.

4.2 Uniform estimates

In this section, we state and prove uniform estimates for the solutions u_n of problem (24). In the remainder of this paper, we denote by C or $C_j, j \in \mathbb{N}^\star$, various positive constants depending only on the structure of a_i, F, $|\Omega|$, $\|f\|_{L^m(Q_T)}$, and T, never on n.

Lemma 6 ([10]) *There exists a constant* $C > 0$ (*independent of* n) *such that*

$$
\int_0^T \int_\Omega |F(t,x,u_n)| \, dx\, dt \le C.
$$

Lemma 7 *Let* m, $p_i(\cdot)$, σ, $s_i(\cdot)$ $i = 1,\dots,N$ *are restricted as in Theorem* 3. *Then, for all* $i = 1,\dots,N$, $(D_i u_n)$ *is bounded in* $L^{q_i^-}(0,T; L^{q_i(\cdot)}(\Omega))$, *furthermore* (u_n) *is bounded in* $L^{\overline{q}^-}(0,T, L^{\overline{q}^*(\cdot)}(\Omega))$, *where the exponents* $q_i(\cdot)$ *are defined as in* (18), $\overline{q}^*(\cdot) = \frac{N\overline{q}(\cdot)}{N-\overline{q}(\cdot)}$, $\overline{q}(\cdot) < N$.

Proof For all $\delta \in (0,1)$ and $\tau \in (0,T)$ using $\varphi_\delta(u_n) = \big((1+|u_n|)^{1-\delta} - 1\big)\text{sign}(u_n)\chi_{(0,\tau)}$, as a test function in (25), where $\chi_{(0,\tau)}$ denotes the characteristic function of $(0,\tau)$ in $(0, T]$, one gets

$$
\begin{aligned}
&\int_0^T < \partial_t u_n, \varphi_\delta(u_n) > \, dt + \sum_{i=1}^{N} \int_0^T \int_\Omega b_i(t,x,T_n(u_n))a_i(t,x,Du_n).D_i\varphi_\delta(u_n) \, dx\, dt\\
&+ \int_0^T \int_\Omega F(t,x,u_n).\varphi_\delta(u_n) \, dx\, dt = \int_0^T \int_\Omega f_n \varphi_\delta(u_n) \, dx\, dt.
\end{aligned}
$$

From (2) and (5), we obtain

$$\begin{aligned}&\int_\Omega dx\int_0^{u_n(\tau,x)}\varphi_\delta(r)\,dr+\alpha(1-\delta)C_2\sum_{i=1}^N\int_0^T\int_\Omega\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{(\delta+\sigma(x))}}\,dx\,dt\\&+\int_0^T\int_\Omega F(t,x,u_n)\varphi_\delta(u_n)\,dx\,dt\\&\le\int_0^T\int_\Omega\varphi_\delta(u_n)f_n\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}\varphi_\delta(r)\,dr.\end{aligned}\tag{26}$$

Observing that there exist two positive constants C_3 and C_4 such that

$$\forall r\in\mathbb{R},\qquad\int_0^r\varphi_\delta(t)dt\ge C_3|r|^{2-\delta}-C_4.$$

(26) and (6), yield

$$\begin{aligned}&\|u_n\|^{2-\delta}_{L^\infty(0,T;L^{2-\delta}(\Omega))}+\sum_{i=1}^N\int_0^T\int_\Omega\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{(\delta+\sigma(x))}}\,dx\,dt\\&\le C_5+C_6\left(\int_{Q_T}\Big(1+|u_n|\Big)^{(1-\delta)m'}\,dx\,dt\right)^{\frac{1}{m'}}.\end{aligned}\tag{27}$$

Now, let $q_i^+=\max_{x\in\overline{\Omega}}\{q_i(x)\}$, $i=1,...,N$ be a constant such that

$$q_i^+<\min_{x\in\overline{\Omega}}\left\{\frac{mp_i(x)}{\overline{p}(x)}\left(\frac{(N+1)\overline{p}(x)-N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right)\right\}=\frac{mp_i^-}{\overline{p}^-}\left(\frac{(N+1)\overline{p}^{-}-N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right)=\alpha_i.$$

By (15) we have

$$1<m<\frac{(N+\sigma+2)\overline{p}^{-}}{(N+\sigma+2)\overline{p}^{-}-N(\sigma+1)}<\frac{N}{\overline{p}^{-}}+1.\tag{28}$$

According to (14) we get

$$\frac{\alpha_i}{p_i^-}=\frac{\overline{\alpha}}{\overline{p}^-}=\theta\in(0,1).$$

Hölder's inequality and (27) imply that, for all $i=1,\dots,N$

$$\begin{aligned}y_{ni}&=\int_0^T\int_\Omega|D_iu_n|^{\alpha_i}\,dx\,dt=\int_0^T\int_\Omega\frac{|D_iu_n|^{\alpha_i}}{(1+|u_n|)^{(\delta+\sigma)\theta}}(1+|u_n|)^{(\delta+\sigma)\theta}\,dx\,dt\\&\le\left(C+C\left(\int_{Q_T}\Big(1+|u_n|\Big)^{(1-\delta)m'}\,dx\,dt\right)^{\frac{1}{m'}}\right)^{\theta}\left(\int_{Q_T}\Big(1+|u_n|\Big)^{(\delta+\sigma)\frac{\theta}{1-\theta}}\,dx\,dt\right)^{1-\theta}.\end{aligned}\tag{29}$$

Let $\delta=\frac{\overline{p}^-(1-m)(N+2)+mN(\sigma+1)-N\sigma}{\overline{p}^-(1-m)+N}$, then we have $(1-\delta)m'=\frac{(\delta+\sigma)\theta}{1-\theta}=\frac{N+2-\delta}{N}\overline{\alpha}$, and by (28)-(14) we deduce that $\delta\in(0,1)$. Putting $d=\frac{N+2-\delta}{N}\overline{\alpha}$, we obtain

$$d=\frac{m\Big((N+1)\overline{p}^{-}-N(\sigma+1)\Big)}{\overline{p}^{-}(1-m)+N}. \tag{30}$$

From (29), we deduce that

$$y_{ni}^{\frac{\overline{\alpha}}{N\alpha_i}}\leq C\left(1+\int_{Q_T}\Big(1+|u_n|\Big)^d\,dx\,dt\right)^{(1-\frac{\theta}{m})\frac{\overline{\alpha}}{N\alpha_i}},$$

hence

$$\prod_{i=1}^{N}y_{ni}^{\frac{\overline{\alpha}}{N\alpha_i}}\leq C^N\left(1+\int_{Q_T}\Big(1+|u_n|\Big)^d\,dx\,dt\right)^{(1-\frac{\theta}{m})}. \tag{31}$$

By (15) and (19), we have

$$2-\delta-d=\frac{N(2+\sigma)-(N+m)\overline{p}^{-}}{\overline{p}^{-}(1-m)+N}<0,$$

and

$$d-\overline{\alpha}^{\star}=\frac{m\Big((N+1)\overline{p}^{-}-N(\sigma+1)\Big)\Big(N(2+\sigma)-(N+m)\overline{p}^{-}\Big)}{(\overline{p}^{-}(1-m)+N)(N(N+2+\sigma)-m(N+1)\overline{p}^{-})}<0.$$

Hence $2-\delta<d<\overline{\alpha}^{\star}$. Using the interpolation inequality, we get

$$\|u_n(\cdot,t)\|_{L^d(\Omega)}\leq\|u_n(\cdot,t)\|_{L^{2-\delta}(\Omega)}^{1-\tau}\|u_n(\cdot,t)\|_{L^{\overline{\alpha}^*}(\Omega)}^{\tau},\qquad \tau=\frac{(2-\delta-d)\overline{\alpha}^{\star}}{(2-\delta-\overline{\alpha}^{\star})d}. \tag{32}$$

By virtue of (30) and (32), we obtain

$$\tau=\frac{N}{N+2-\delta},\ \text{and } d\tau=\overline{\alpha}.$$

Using (27) and (32), the result is

$$\begin{aligned}\int_0^T\|u_n\|_{L^d(\Omega)}^d\,dt&\leq\|u_n\|_{L^\infty(0,T;L^{2-\delta}(\Omega))}^{(1-\tau)d}\int_0^T\|u_n\|_{L^{\overline{\alpha}^\star}(\Omega)}^{d\tau}\,dt\\&\leq C\left(1+\int_{Q_T}\big(1+|u_n|\big)^d\,dx\,dt\right)^{\frac{(1-\tau)d}{(2-\delta)m'}}\int_0^T\|u_n\|_{L^{\overline{\alpha}^\star}(\Omega)}^{\overline{\alpha}}\,dt\\&=C\left(1+\int_{Q_T}\big(1+|u_n|\big)^d\,dx\,dt\right)^{\frac{\overline{\alpha}}{Nm'}}\int_0^T\|u_n\|_{L^{\overline{\alpha}^\star}(\Omega)}^{\overline{\alpha}}\,dt.\end{aligned} \tag{33}$$

From Lemma 2, we have

$$\int_0^T \|u_n\|_{L^{\overline{\alpha}^\star}(\Omega)}^{\overline{\alpha}} dt \le C \int_0^T \prod_{i=1}^N \left(\int_\Omega |D_i u_n|^{\alpha_i} dx \right)^{\frac{\overline{\alpha}}{N\alpha_i}} dt.$$

$\sum_{i=1}^N \frac{\overline{\alpha}}{N\alpha_i} = 1$, and the Hölder's inequality, yield

$$\int_0^T \|u_n\|_{L^{\overline{\alpha}^\star}(\Omega)}^{\overline{\alpha}} dt \le C \prod_{i=1}^N \left(\int_{Q_T} |D_i u_n|^{\alpha_i} dx\, dt \right)^{\frac{\overline{\alpha}}{N\alpha_i}}. \tag{34}$$

In view of (31), (33), and (34), we deduce

$$\begin{aligned} \int_{Q_T} |u_n|^d dx\, dt \le & C\left(1 + \int_{Q_T} |u_n|^d dx\, dt\right)^{1+\frac{\overline{\alpha}}{Nm'}-\frac{\theta}{m}} \\ = & C\left(1 + \int_{Q_T} |u_n|^d dx\, dt\right)^{1+\frac{\overline{\alpha}}{N}-\frac{\overline{\alpha}}{Nm}-\frac{\overline{\alpha}}{m\overline{p}^-}}. \end{aligned} \tag{35}$$

By (28) we have

$$1 + \frac{\overline{\alpha}}{N} - \frac{\overline{\alpha}}{Nm} - \frac{\overline{\alpha}}{m\overline{p}^-} < 1.$$

Therefore, (35) implies that the sequence (u_n) is bounded on $L^d(Q_T)$. Which then yields, by (29), a bound on the norm of $(D_i u_n)$ in L^{α_i}, also in $L^{q_i^+}$. The result of Lemma 7 follows from $q_i(\cdot) \le q_i^+$, Remark 1, and (34).

Now let us consider a continuous variable exponent $q_i(\cdot)$ on $\overline{\Omega}$ satisfying (18) such that

$$q_i^+ \ge \frac{mp_i^-}{\overline{p}^-}\left(\frac{(N+1)\overline{p}^- - N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right).$$

By the continuity of $q_i(\cdot)$ and $p_i(\cdot)$ on $\overline{\Omega}$, there exists a constant $\delta > 0$ such that for all $x \in \Omega$

$$\max_{z\in \overline{Q(x,\delta)\cap\Omega}} q_i(z) < \min_{z\in \overline{Q(x,\delta)\cap\Omega}} \left\{ \frac{mp_i(z)}{\overline{p}(z)}\left(\frac{(N+1)\overline{p}(z) - N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right)\right\}$$

where $Q(x,\delta)$ is a cube with center x and diameter δ. Observe that $\overline{\Omega}$ is compact and, therefore, we can cover it with a finite number of cubes $(Q_j')_{j=1,\dots,k}$ with edges parallel to the coordinate axes. We denote by q_{ij}^+ (resp. p_{ij}^-) the local maximum of $q_i(\cdot)$ on $\overline{(Q_j' \cap \Omega)}$ (resp. the local minimum of $p_i(\cdot)$ on $\overline{(Q_j' \cap \Omega)}$), such that

$$q_{ij}^+ < \frac{mp_{ij}^-}{\overline{p}_j^-}\left(\frac{(N+1)\overline{p}_j^- - N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right) = \alpha_{ij}, \quad \text{for all} \quad j = 1, \dots, k.$$

Observing that (7) and Lemma imply that (u_n) is bounded in $L^{s_+(\cdot)}(\Omega)$. So from (17) and (12), it is easy to check that, instead of the global estimate (34), we find

$$\int_0^T \|u_n\|^{\overline{\alpha}_j}_{L^{\overline{\alpha}_j^\star}(Q'_j\cap\Omega)}\,dt \le C\prod_{i=1}^N \left(1+y_{nij}\right)^{\frac{\overline{\alpha}_j}{N\alpha_{ij}}}, \tag{36}$$

where $y_{nij}=\int_{(0,T)\times(Q'_j\cap\Omega)}|D_iu_n|^{\alpha_{ij}}\,dx\,dt$, $\frac{1}{\overline{\alpha}_j}=\frac{1}{N}\sum_{i=1}^N\frac{1}{\alpha_{ij}}$. According to (31), we obtain

$$\prod_{i=1}^N\left(1+y_{nij}\right)^{\frac{\overline{\alpha}}{N\alpha_{ij}}}\le C^N\left(1+\int_{(0,T)\times(Q'_j\cap\Omega)}\left(1+|u_n|\right)^{d_j}dx\,dt\right)^{(1-\frac{\theta_j}{m})}, \tag{37}$$

where $\frac{\alpha_{ij}}{p^-_{ij}}=\frac{\overline{\alpha}_j}{\overline{p}^-_j}=\theta_j$, $d_j=\frac{N+2-\delta_j}{N}\overline{\alpha}_j$, $\delta_j=\frac{\overline{p}^-_j(1-m)(N+2)+mN(\sigma+1)-N\sigma}{\overline{p}^-_j(1-m)+N}$. Arguing locally as in (35), we obtain

$$\int_{(0,T)\times(Q'_j\cap\Omega)}|u_n|^{d_j}\,dx\,dt\le C\left(1+\int_{(0,T)\times(Q'_j\cap\Omega)}|u_n|^{d_j}\,dx\,dt\right)^{1+\frac{\overline{\alpha}_j}{N}-\frac{\overline{\alpha}_j}{Nm}-\frac{\overline{\alpha}_j}{m\overline{p}^-_j}}, \tag{38}$$

where $1+\frac{\overline{\alpha}_j}{N}-\frac{\overline{\alpha}_j}{Nm}-\frac{\overline{\alpha}_j}{m\overline{p}^-_j}<1$. Combining (36), (37), and (38), we obtain

$$\|u_n\|_{L^{\overline{\alpha}_j}(0,T,L^{\overline{\alpha}^*_j}((Q'_j\cap\Omega)))}\le C \quad\text{and}\quad \|D_iu_n\|_{L^{\alpha_{ij}}((0,T)\times(Q'_j\cap\Omega))}\le C.$$

Knowing that $q_i(x)\le q^+_{ij}\le\alpha_{i,j}$ and $\overline{q}^*(x)\le\overline{q}^{+*}_j\le\overline{\alpha}_j^{\,*}$ for all $x\in(Q'_j\cap\Omega)$, and all $j=1,\dots,k$, we conclude that (D_iu_n) is bounded in $L^{q_i(\cdot)}((0,T)\times\Omega)$. Consequently, by (36), (u_n) remains in a bounded set of $L^{\overline{q}^-}(0,T;L^{\overline{q}^*(\cdot)}(\Omega))$. This finishes the proof of the Lemma 7. □

Now we consider the following family of functions $(\phi_k)_{k>0}$:

- ϕ_k is a twice differentiable function, ϕ'_k, ϕ''_k are bounded on $\mathbb{R}$.
- $\phi_k(\sigma)=\sigma$ if $|\sigma|\le k$, and $\phi'_k(\sigma)=0$ if $|\sigma|\ge k+(1/k)$, $0<\phi'_k<1$ on the set $(k,k+(1/k))\cup(-(k+(1/k)),-k)$.

The construction of this family $(\phi_k)_{k>0}$ can be made explicitly (See [6]).

Lemma 8 [10] *There exists a constant C_k dependent of k such that*

$$\int_{Q_T}|D_i\phi_k(u_n)|^{p_i(x)}\,dx\,dt\le C_k,\quad i=1,\dots,N.$$

Next we show that (∂_tu_n) is in a bounded set of $L^r(0,T;W^{-1,r}(\Omega))+L^1(Q_T)$ for some $r>1$.

Lemma 9 *Let*

$$1<r<\min_i\min_{x\in\overline{\Omega}}\left\{\frac{m(N+1)p_i(x)}{(N+1-(1+\sigma)(m-1))(p_i(x)-1)\overline{p}(x)}\left(\overline{p}(x)-\frac{N(\sigma+1)}{N+1}\right)\right\}. \tag{39}$$

The sequence $(\partial_t u_n)$ remains in a bounded set of $L^r(0,T;W^{-1,r}(\Omega))+L^1(Q_T)$.

Proof It is similar to the proof of Lemma 2.11 of [6]. The existence of $r>1$ is by virtue of the upper bound in the assumption (16). Knowing that $(f_n - F(\cdot,\cdot,u_n))$ is in a bounded set of $L^1(Q_T)$, we have to show that

$$w_n = \sum_{i=1}^{N} D_i\big(b_i(t,x,T_n(u_n))a_i(t,x,Du_n)\big)$$

belongs to a bounded set of $L^r(0,T;W^{-1,r}(\Omega))$. In fact, setting for $t\in(0,T)$, $w_n(t)=w_n$. By (39), (3) and (5), we get

$$\begin{aligned}\|w_n\|_{W^{-1,r}(\Omega)} &= \sup_{\varphi\in W_0^{1,r'}(\Omega),\ \|\varphi\|_{W_0^{1,r'}(\Omega)}\le 1}\left|\int_\Omega \sum_{i=1}^{N}\big(b_i(t,x,T_n(u_n))a_i(t,x,Du_n)\big)D_i\varphi\ dx\right| \\ &\le C_1 \sup_{\|\varphi\|_{W_0^{1,r'}(\Omega)}\le 1}\sum_{i=1}^{N}\int_\Omega\left(g+\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)}\right)^{1-\frac{1}{p_i(\cdot)}}|D_i\varphi|\ dx.\end{aligned}$$

By the Hölder inequality, we get

$$\begin{aligned}&\|w_n\|_{W^{-1,r}(\Omega)} \\ &\le C_1 \sup_{\|\varphi\|_{W_0^{1,r'}(\Omega)}\le 1}\sum_{i=1}^{N}\left(\int_\Omega G_i(t,x)dxdt\right)^{\frac{1}{r}}\sum_{i=1}^{N}\|D_i\varphi\|_{L^{r'}(\Omega)}\end{aligned}$$

where

$$G_i(t,x)=\left(g+\sum_{j=1}^{N}|D_ju_n|^{p_j(\cdot)}\right)^{(1-\frac{1}{p_i(\cdot)})r}(t,x),\ \text{for all } i=1,\cdots,N.$$

Thus

$$\int_0^T \|w_n\|^r_{W^{-1,r}(\Omega)}\,dt \le C\sum_{i=1}^{N}\int_0^T\int_\Omega G_i(t,x)\,dx\,dt.$$

Thanks to (39) we have

$$\left(1-\frac{1}{p_i(\cdot)}\right)r < \frac{m(N+1)}{N+1-(1+\sigma)(m-1)}\left(1-\frac{N(\sigma+1)}{\overline{p}(x)(N+1)}\right), i=1,\dots,N.$$

There exist θ such that for all $i=1,\dots,N$

$$\left(1-\frac{1}{p_i(\cdot)}\right)r<\theta<\frac{m(N+1)}{N+1-(1+\sigma)(m-1)}\left(1-\frac{N(\sigma+1)}{\overline{p}(\cdot)(N+1)}\right),$$

from the upper bound in (14) we obtain that

$$\frac{m(N+1)}{N+1-(1+\sigma)(m-1)}\left(1-\frac{N(\sigma+1)}{\overline{p}(\cdot)(N+1)}\right)<1.$$

Therefore, $\theta \in (0,1)$ and

$$1\le \theta p_i(\cdot)<\frac{mp_i(\cdot)}{\overline{p}(\cdot)}\left(\frac{(N+1)\overline{p}(\cdot)-N(\sigma+1)}{N+1-(1+\sigma)(m-1)}\right),\ \left(1-\frac{1}{p_i(\cdot)}\right)\frac{r}{\theta}<1. \quad (40)$$

Writing $G_i = G_i^{\frac{\theta}{\theta}}$, by the Hölder inequality, we deduce

$$\int_0^T \|w_n\|^r_{W^{-1,r}_{(}\Omega)}\,dt \le C_5\sum_{i=1}^N\left(\int_0^T\int_\Omega\left(g^\theta+\sum_{j=1}^N |D_ju_n|^{\theta p_j(\cdot)}\right)dx\,dt\right)^{(1-\frac{1}{p_i(\cdot)})\frac{r}{\theta}}$$

$$\le C_5\sum_{i=1}^N\left(\int_0^T\int_\Omega\left(g^\theta+\sum_{j=1}^N |D_ju_n|^{\theta p_j(\cdot)}\right)dx\,dt\right)^{(1-\frac{1}{p_i^+})\frac{r}{\theta}}+C_5N.$$

By (9), Lemma 7, and (40) we get

$$\int_0^T\int_\Omega\sum_{j=1}^N |D_ju_n|^{\theta p_j(\cdot)}\,dx\,dt\le\sum_{j=1}^N\max\left\{\|D_ju_n\|^{\theta p_j^-}_{L^{\theta p_j(\cdot)}(Q_T)},\|D_ju_n\|^{\theta p_j^+}_{L^{\theta p_j(\cdot)}(Q_T)}\right\}\le C.$$

Since $g\in L^1(Q_T)$, we find

$$\int_0^T \|w_n\|^r_{W^{-1,r}(\Omega)}\,dt\le C_6.$$

This complete the proof of Lemma 9. □

Lemma 10 *Let p_i, s_i, σ, m $i=1,\dots,N$ are restricted as in Theorem 4. Then, there exists a constant $C>0$ independent of n, such that*

$$\int_0^T\int_\Omega\frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{1-(m(x)-1)s_+(x)+\sigma(x)}}\,dx\,dt+\int_0^T\int_\Omega|u_n|^{m(x)s_+(x)}\,dx\,dt\le C. \quad (41)$$

Proof As in elliptic case [9], taking

$$\varphi(u_n)=\left((1+|u_n|)^{(m(x)-1)s_+(x)}-1\right)\text{sign}(u_n),$$

as a test function in (25), by (2), (3), (5), (14), and the fact that for a.e. $(t,x)\in Q_T$

$$D_i\varphi(u_n)=(m(x)-1)(1+|u_n|)^{(m(x)-1)s_+(x)}\text{sign}(u_n)D_is_+(x)\ln(1+|u_n|)$$
$$+\frac{(m(x)-1)s_+(x)D_iu_n}{(1+|u_n|)^{1-(m(x)-1)s_+(x)}}+D_im(x)(1+|u_n|)^{(m(x)-1)s_+(x)}\text{sign}(u_n)s_+(x)\ln(1+|u_n|)$$

we obtain

$$\int_\Omega dx \int_0^{u_n(\tau,x)} \varphi(r)\,dr + \alpha(m^- - 1)s_+^- \sum_{i=1}^N \int_0^T \int_\Omega \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{1-(m(x)-1)s_+(x)+\sigma(x)}}\,dx\,dt$$
$$+ \sum_{i=1}^N \int_0^T \int_\Omega |u_n|^{s_+(x)}\left((1+|u_n|)^{(m(x)-1)s_+(x)} - 1\right) dx\,dt$$
$$\le \int_0^T \int_\Omega |f_n|\left((1+|u_n|)^{(m(x)-1)s_+(x)} - 1\right) dx\,dt$$
$$+ C_7 \sum_{i=1}^N \int_0^T \int_\Omega (1+|u_n|)^{(m(x)-1)s_+(x)} \ln(1+|u_n|).\left(g + \sum_{j=1}^N |D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} dx\,dt$$
$$+ \int_\Omega dx \int_0^{u_n(0,x)} \varphi(r)\,dr.$$

By dropping the positif term, the fact that $|u_n|^{s_+(\cdot)} \ge 2^{-s_+(\cdot)}(1+|u_n|)^{s_+(\cdot)} - 1$, (9), and Young inequality, we have

$$\sum_{i=1}^N \int_0^T \int_\Omega \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{1-(m(x)-1)s_+(x)+\sigma(x)}}\,dx\,dt + \frac{1}{2}\sum_{i=1}^N \int_0^T \int_\Omega (1+|u_n|)^{m(x)s_+(x)}\,dx\,dt$$
$$\le C_8 + C_8 \max\left(\|f_n\|^{m^+}_{L^{m(\cdot)}(Q_T)}, \|f_n\|^{m^-}_{L^{m(\cdot)}(Q_T)}\right)$$
$$+ C_8 \sum_{i=1}^N \int_0^T \int_\Omega (1+|u_n|)^{(m(x)-1)s_+(x)} \ln(1+|u_n|) \times \left(g + \sum_{j=1}^N |D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} dx\,dt. \tag{42}$$

We can estimate the last term in (42) by application of Young's inequality

$$(1+|u_n|)^{(m(x)-1)s_+(x)} \ln(1+|u_n|) \times \left(g + \sum_{j=1}^N |D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}$$
$$= (1+|u_n|)^{\sigma(x)+1-\frac{(1-(m(x)-1)s_+(x)+\sigma(x))}{p_i(x)}} \ln(1+|u_n|)$$
$$\times \left(g + \sum_{j=1}^N |D_j u_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}} (1+|u_n|)^{-\frac{(1-(m(x)-1)s_+(x)+\sigma(x))(p_i(x)-1)}{p_i(x)}} \tag{43}$$
$$\le C_9 (1+|u_n|)^{\sigma(x)p_i(x)+p_i(x)-(1-(m(x)-1)s_+(x)+\sigma(x))}(\ln(1+|u_n|))^{p_i(x)} + \frac{1}{4NC_8} g$$
$$+ \frac{1}{4NC_8} \sum_{i=1}^N \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{1-(m(x)-1)s_+(x)+\sigma(x)}}.$$

By (42) and (43), we obtain

$$\sum_{i=1}^{N}\int_0^T\int_\Omega \frac{|D_iu_n|^{p_i(x)}}{(1+|u_n|)^{1-(m(x)-1)s_+(x)+\sigma(x)}}\,dx\,dt+\sum_{i=1}^{N}\int_0^T\int_\Omega (1+|u_n|)^{m(x)s_+(x)}\,dx\,dt$$
$$\leq C_{10}+C_{11}\int_0^T\int_\Omega (1+|u_n|)^{\sigma(x)p_i(x)+p_i(x)-(1-(m(x)-1)s_+(x)+\sigma(x))}(\ln(1+|u_n|))^{p_i(x)}dxdt$$
$$=I. \tag{44}$$

We observe that

$$(\sigma(x)+1)(p_i(x)-1)-s_+(x)\leq((\sigma(x)+1)(p_i(x)-1)-s_+(x))^+=d_i<\frac{d_i}{2}<0,$$

due to the hypotheses (20), so $(1+|u_n|)^{(\sigma(x)+1)(p_i(x)-1)-s_+(x)-\frac{d_i}{2}}(\ln(1+|u_n|))^{p_i(x)}$ is bounded for all $x\in\overline{\Omega}$. We get by Young's inequality,

$$I=C_{10}+C_{11}\int_0^T\int_\Omega (1+|u_n|)^{m(x)s_i(x)+\frac{d_i}{2}}(1+|u_n|)^{(\sigma(x)+1)(p_i(x)-1)-s_i(x)-\frac{d_i}{2}}(\ln(1+|u_n|))^{p_i(x)}$$
$$\leq C_{12}+\frac{1}{2}\int_0^T\int_\Omega (1+|u_n|)^{m(x)s_i(x)}dxdt. \tag{45}$$

Therefore, (44) and (45) yield (41).

□

Lemma 11 *Let p_i, s_i, σ, m $i=1,\dots,N$ are restricted as in Theorem 5. Then, the approximate solution u_n is bounded in $\cap_{i=1}^{N}L^{p_i^-}(0,T,W_0^{1,p_i(\cdot)}(\Omega))\cap L^{1+s_+(\cdot)+\sigma(\cdot)}(Q_T)$.*

Proof Using $\varphi(u_n)=\big((1+|u_n|)^{1+\sigma(\cdot)}-1\big)\text{sign}(u_n)$ as test function in (25) and dropping the positif term, by (2) and (3), we obtain for all $\varepsilon>0$

$$\alpha\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}\,dx\,dt$$
$$+\sum_{i=1}^{N}\int_0^T\int_\Omega |u_n|^{s_+(x)}\big((1+|u_n|)^{(1+\sigma(x))}-1\big)\,dx\,dt$$
$$\leq C_{13}(\varepsilon)\int_0^T\int_\Omega |f_n|^{m(x)}dxdt+\varepsilon\int_0^T\int_\Omega (1+|u_n|)^{(1+\sigma(x))m'(x)}dxdt\,dx\,dt$$
$$+C_{14}\sum_{i=1}^{N}\int_0^T\int_\Omega (1+|u_n|)^{(1+\sigma(x))}\ln(1+|u_n|).\left(g+\sum_{j=1}^{N}|D_ju_n|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}dx\,dt$$
$$+\int_\Omega (1+|u_n(0,x)|)^{2+\sigma(x)}\,dx+C_{15}.$$

Since (23) we have $(1+\sigma)m'\leq 1+s_++\sigma$ and $2+\sigma\leq(m-1)s_++1$. It follows from the Young inequality that

$$\sum_{i=1}^{N}\int_0^T\int_\Omega |D_i u_n|^{p_i(x)}\,dx\,dt+\int_0^T\int_\Omega |u_n|^{1+\sigma(x)+s_+(x)}\,dx\,dt \\ \leq C_{16}\sum_{i=1}^{N}\int_0^T\int_\Omega (1+|u_n|)^{(1+\sigma(x))p_i(x)}\ln(1+|u_n|)^{p_i(x)}\,dx\,dt+C_{17}. \tag{46}$$

Let us write

$$(\sigma(\cdot)+1)p_i(\cdot)=(\sigma(\cdot)+1)(p_i(\cdot)-1)-s_+(\cdot)-\frac{\nu_i}{2}+\sigma(\cdot)+1+s_+(\cdot)+\frac{\nu_i}{2},$$

by (20), we get

$$\nu_i=\min_{x\in\overline{\Omega}}\{(\sigma+1)(p_i(x)-1)-s_+(x)\}<0.$$

Arguing as in (45) and using (46), we obtain

$$\sum_{i=1}^{N}\int_0^T\int_\Omega |D_i u_n|^{p_i(x)}\,dx\,dt+\int_0^T\int_\Omega |u_n|^{1+\sigma(x)+s_+(x)}\,dx\,dt\leq C_{18}.$$

This concludes the proof of the lemma. □

Lemma 12 *Let $p_i(\cdot)$, $\sigma(\cdot)$, $s_i(\cdot)$, $m(\cdot)$ $i=1,...,N$ are restricted as in Theorem 4. Then, every solution u_n of (25) satisfies the estimate*

$$\|D_i u_n\|_{L^{q_i(\cdot)}(Q_T)}\leq C,$$

where the $q_i(\cdot)$ defined as in (22).

Proof Observe that (22) implies that $q_i(\cdot)<p_i(\cdot)$ and

$$\frac{(1+s_+(\cdot)+\sigma(\cdot))q_i(\cdot)}{p_i(\cdot)}<m(\cdot)s_+(\cdot).$$

Then, by Young's inequality, we have

$$\int_0^T\int_\Omega |D_i u_n|^{q_i(x)}\,dx\,dt \\ =\int_0^T\int_\Omega \frac{|D_i u_n|^{q_i(x)}}{(1+|u_n|)^{\frac{(1-(m(x)-1)s_+(x)+\sigma(x))q_i(x)}{p_i(x)}}}(1+|u_n|)^{\frac{(1-(m(x)-1)s_+(x)+\sigma(x))q_i(x)}{p_i(x)}}\,dx\,dt, \\ \leq\int_0^T\int_\Omega\left(\frac{q_i(x)}{p_i(x)}\right)\frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{(1-(m(x)-1)s_+(x)+\sigma(x))}}\,dx\,dt \\ +\int_0^T\int_\Omega\left(1-\frac{q_i(x)}{p_i(x)}\right)(1+|u_n|)^{\frac{(1-(m(x)-1)s_+(x)+\sigma(x))q_i(x)}{p_i(x)-q_i(x)}}\,dx\,dt.$$

From (22), we deduce

$$\int_0^T \int_\Omega |D_i u_n|^{q_i(x)}\,dx\,dt$$
$$\leq C \int_0^T \int_\Omega \frac{|D_i u_n|^{p_i(x)}}{(1+|u_n|)^{(1-(m(x)-1)s_+(x)+\sigma(x))}}\,dx\,dt + C\int_0^T \int_\Omega (1+|u_n|)^{m(x)s_+(x)}\,dx\,dt. \tag{47}$$

Consequently, (47) and (41) imply the desert result. □

4.3 Passage to the limit and proof of Theorem 3

By Lemma 7, the sequence (u_n) remains in a bounded set of $\cap_{i=1}^N L^{q_i^-}(0,T;W_0^{1,q_i^-}(\Omega))$ where the $q_i(\cdot)$ defined as in (18) and from Lemma 9, the sequence $(\partial_t u_n)$ remains in a bounded set of the space

$$L^1(0,T;(W^{1,r'}(\Omega))') + L^1(Q_T) \hookrightarrow L^1(0,T;W^{-1,s}(\Omega)) + L^1(Q_T)$$

for all $s < \min\{N/(N-1), r\}$. Therefore, $(\partial_t u_n)$ is bounded in $L^1(0,T;W^{-1,s}(\Omega)) + L^1(Q_T)$.

Now, we can use Corollary 4 in [12], we obtain that

$$u_n \quad \text{is relatively compact in} \quad L^1(Q_T).$$

This implies that we can extract a subsequence (denote again by (u_n)) such that

$$u_n \to u \qquad \text{a.e on } Q_T. \tag{48}$$

Lemma 13 ([8]) *Let a_i be a function satisfying* (2)-(4) *and let F satisfy* (6)-(7). *Then*

$$F(t,x,u_n) \to F(t,x,u) \quad \text{strongly in} \quad L^1(0,T;L^1(\Omega)).$$

Now, using Lemma 8 and adapting the approach of [10], there exists a subsequence (still denoted u_n) such that

$$Du_n \to Du \quad \text{a.e on } Q_T. \tag{49}$$

From (48), (49), Lemma 7, and assumption (3), we get

$$b_i(t,x,T_n(u_n))a_i(t,x,Du_n) \to b_i(t,x,u)a_i(t,x,Du) \quad \text{strongly in } L^{\kappa_i(\cdot)}(Q_T), \tag{50}$$

for all continuous function κ_i on Q_T such that

$$1 < \kappa_i(\cdot) < \frac{mp_i(\cdot)}{(p_i(\cdot)-1)\overline{p}(\cdot)}\left(\frac{(N+1)\overline{p}(\cdot) - N(1+\sigma)}{N+1-(1+\sigma)(m-1)}\right).$$

This is possible because since we have the upper bound in (16). Using (48), Lemma (13), and (50), we can easily pass to the limit in (24). This proves Theorem (3).

4.4 Proof of Theorem 4

In order to prove this Theorem, we modify the proof of Theorem 3. It's sufficient to replace only (50) with the following

$$b_i(t,x,T_n(u_n))a_i(t,x,Du_n) \to b_i(t,x,u)a_i(t,x,Du) \quad \text{strongly in } L^{\tau_i(.)}(Q_T), \quad (51)$$

for all continuous function τ_i on Q_T such that

$$1 < \tau_i(\cdot) < \frac{m(\cdot)p_i(\cdot)s_+(\cdot)}{(1+s_+(\cdot)+\sigma(\cdot))(p_i(\cdot)-1)}.$$

This is possible because we have (20). Thus by (51) and Lemma 13, we can deduce that the limit function u is a weak solution of (1) possessing the regularity stated in (22). This proves Theorem 4.

4.5 Proof of Theorem 5

In the same way of the proof of Theorem 4 we have by (3) and Lemma 11 that

$$b_i(t,x,T_n(u_n))a_i(t,x,Du_n) \rightharpoonup b_i(t,x,u)a_i(t,x,Du) \quad \text{weakly in } L^{p_i'(.)}(Q_T),$$

therefore, we can easily passe to the limit in (24). So the theorem is proved.

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