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On nonlinear parabolic equations with singular lower order term

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Abstract
In this paper we study existence and regularity results for solution to a nonlinear and singular parabolic problem. The model is

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left((a(x,t) + |u|^q)\nabla u\right) = \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u(x,t) = 0 & \text{on} \quad \Gamma, \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega \times (0,T)$, $T > 0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $q > 0$, $\gamma > 0$, and f is non-negative function belonging to some Lebesgue space $L^m(Q)$, $m \geq 1$ and $u_0 \in L^\infty(\Omega)$ such that

$$\forall\, \omega \subset\subset \Omega,\ \exists\, D_\omega > 0 : \ u_0 \geq D_\omega \quad \text{in} \ \omega.$$

Keywords Singular problem · Nonlinear parabolic equations · Lower order term

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1 Introduction

In this work we prove existence and regularity results for a class of nonlinear singular parabolic equations. More precisely, we are interested in the following nonlinear problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left((a(x,t)+|u|^q)\nabla u\right) = \frac{f}{u^\gamma} & \text{in} \quad Q, \\ u(x,t)=0 & \text{on} \quad \Gamma, \\ u(x,0)=u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, Q is the cylinder $\Omega \times (0,T)$, $T>0$, Γ the lateral surface $\partial\Omega \times (0,T)$, $q>0$, $\gamma>0$, and f is non-negative function which belongs to some Lebesgue space $L^m(Q)$, $m \geq 1$, the data u_0 satisfies

$$u_0 \in L^\infty(\Omega) \;\text{ and }\; \forall\, \omega \subset\subset \Omega,\; \exists\, D_\omega > 0 \,:\; u_0 \geq D_\omega \;\text{ in }\; \omega. \tag{2}$$

Moreover $a(x,t)$ is a measurable function satisfying

$$0 < \alpha \leq a(x,t) \leq \beta \;\; a.e.\; Q; \tag{3}$$

where α, β are fixed real numbers.

The interest in problem as (1) started in [12] in connection with the study of thermo-conductivity (u^γ represented the resistivity the material), and later in the study of signal transmission and in the theory of non-Newtonian pseudo-plastic fluids, see [13, 20, 22].

If $\gamma = 0$ many works have appeared concerning the existence and regularity of elliptic equations. Boccardo In [5] has been studied the existence and regularity results of quasi linear elliptic problem

$$\begin{cases} -\operatorname{div}\left((a(x)+|u|^q)\nabla u\right) + b(x)u|u|^{p-2}|\nabla u|^2 = f(x) & \text{in} \quad \Omega, \\ u=0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $a(x)$, $b(x)$ are measurable bounded functions, $p, q \geq 0$ and $0 \leq f \in L^m(\Omega)$, $1 \leq m \leq \frac{N}{2}$, see also [19]. In the case parabolic the authors in [18] has been studied the existence and regularity results of nonlinear problems

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left((a(x,t)+|u|^q)\nabla u\right) + b(x,t)u|u|^{p-1}|\nabla u|^2 = f & \text{in} \quad Q, \\ u=0 & \text{on} \quad \partial\Omega, \\ u(t=0)=0 & \text{in} \quad \Omega, \end{cases}$$

where $a(x, t)$, $b(x, t)$ are measurable positive bounded functions, $p, q > 0$ and f belongs to $L^m(Q)$ for some $m \geq 1$. If $q=0$, then the operator $A(x,t,\xi) = b(x,t)\xi$ existing in [14] and [8]($p=2$) is linear coercive, monotone and satisfying the growth condition $|A(x,t,\xi)| \leq C(d(x,t)+|\xi|)$ with C a positive constant and $d \in L^2(Q)$, we highlight that our case ($q>0$) the required growth of $A(x,t,s,\xi) = (a(x,t)+s^q)\xi$ is more general, handling growths greater then linear case (see also [3, 10, 15, 28]).

In the elliptic framework and when $\gamma > 0$ a rich amount of research has been conducted to prove the existence of solution to singular problems, see [25, 26]. For example Boccardo and Orsina in [6] proved the existence and regularity results to problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $\gamma > 0$ and f is a nonnegative function belonging to $L^m(Q), m \geq 1$. In the same concept the authors in [23] proved the existence of solution to problem

$$\begin{cases} -\Delta u = \frac{f(x)}{u^\gamma} + \mu & \text{in} \quad \Omega, \\ u > 0 & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $\gamma > 0, f$ is a nonnegative function on Ω, and μ is a nonnegative bounded Radon measures on Ω. Hence Charkaoui and Alaa [7] established the existence of weak periodic solution to singular parabolic problems

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u = \frac{f(x)}{u^\gamma} & \text{in} \quad Q, \\ u = 0 & \text{on} \quad \Gamma, \\ u(.,0) = u(.,T) & \text{in} \quad \Omega, \end{cases}$$

with $\gamma > 0$ and f is a nonnegative integrable function periodic in time with period T. Let us observe that we refer to [8, 9, 11, 17, 24] for more details on singular parabolic problems.

If $\gamma = 0$ and $q = 0$, the problem (1) has been studied in [14]. When $q = 0$ and $\gamma > 0$, the existence and regularity results of problem (1) has been obtained in [8]. The aim of this paper to prove the existence and regularity of solutions of problem (1) depending on the summability of the datum f and the parameters $q, \gamma > 0$. As we will see, our growth assumption on the function $a(x,t) + |u|^q$ has a regularization effect on the solution u and its gradient ∇u, allowing in some cases to have finite energy solution (i.e $u \in L^2(0,T;H_0^1(\Omega))$ even if $f \in L^1(Q)$.

Notation. Hereafter, we will make use of two truncation functions T_k and G_k : for every $k \geq 0$ and $s \in \mathbb{R}$, let

$$T_k(s) = \min(k, \max(r, -k)), \quad G_k(s) = s - T_k(s).$$

We will denote with $\rho^* = \frac{\rho N}{N-\rho}$ the Sobolev conjugate of $1 \leq \rho < N$.

For the sake of simplicity we will use when referring to the integrals the following notation

$$\int_Q f = \int_0^T \int_\Omega f = \int_Q f dx dt.$$

Finally, throughout this paper, C will indicate any positive constant which depends only on the data and whose value change from line to line and, some times in the same line.

Our aim is to prove the existence of weak solutions to problem (1). Here is the definition of solution we will consider.

Definition 1 If $\gamma \leq 1$, a solution of (1) is a function $u \in L^1(0,T;W_0^{1,1}(\Omega))$ such that

$$\forall \omega \subset\subset \Omega \;\; \exists\, C_\omega > 0 : u \geq C_\omega \;\; in \;\; \omega \times (0,T), \tag{4}$$

$$(a(x,t) + u^q)\nabla u \in L^1(0,T;L^1_{loc}(\Omega)), \tag{5}$$

and

$$-\int_\Omega u_0(x)\varphi(x,0) - \int_0^T \int_\Omega u\frac{\partial\varphi}{\partial t} + \int_0^T \int_\Omega (a(x,t) + u^q)\nabla u \nabla\varphi = \int_0^T \int_\Omega \frac{f\varphi}{u^\gamma}, \tag{6}$$

$\forall \varphi \in C^1_c(\Omega \times [0,T))$.

If $\gamma > 1$, a solution of problem (1) is a function $u \in L^2(0,T;H^1_{loc}(\Omega))$, $u^r \in L^1(0,T;W_0^{1,1}(\Omega))$, for some $r > 1$ and u satisfying (4)–(6).

Now we give a consequence of the Gagliardo–Nirenberg inequality, see [21].

Lemma 1 *Let v be a function in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T;L^s(\Omega))$, with $p \geq 1, s \geq 1$. Then $v \in L^\sigma(Q)$ with $\sigma = p\frac{N+s}{N}$ and*

$$\int_Q |v|^\sigma \leq C||v||^{\frac{sp}{N}}_{L^\infty(0,T:L^s(\Omega))} \int_Q |\nabla v|^p.$$

2 The approximation scheme

Let f be a non-negative measurable function which belongs to some Lebesgue space, let $n \in \mathbb{N}$, $f_n = \frac{f}{1+\frac{1}{n}f}$, and let us consider the following approximation of problem (1)

$$\begin{cases} \dfrac{\partial u_n}{\partial t} - \operatorname{div}((a(x,t) + u_n^q)\nabla u_n) = \frac{f_n}{(u_n+\frac{1}{n})^\gamma} & \text{in } Q, \\ u_n(x,t) = 0 & \text{on } \Gamma, \\ u_n(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{7}$$

Lemma 2 *The problem* (7) *has a non-negative solution* $u_n \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$.

Proof Let $k, n \in \mathbb{N}$, be fixed $v \in L^2(Q)$ and define $w := S(v)$ to be the unique solution of (see [16])

$$\begin{cases} \dfrac{\partial w}{\partial t} - \operatorname{div}\left((a(x,t) + |T_k(v)|^q)\nabla w\right) = \frac{f_n}{(|v|+\frac{1}{n})^\gamma} & \text{in } Q, \\ w = 0 & \text{on } \Gamma, \\ w(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Using w as test function by (3) and dropping the non-negative terms, we have

$$\alpha \int_Q |\nabla w|^2 \leq n^{\gamma+1} \int_Q |w| + \frac{1}{2} \int_\Omega u_0^2,$$

an application of Poincaré inequality on the left hand side and Hölder inequality on the right hand side and the fact that $u_0 \in L^\infty(\Omega)$ yields

$$\int_Q |w|^2 \leq C n^{\gamma+1} \left(\int_Q |w|^2 \right)^{\frac{1}{2}} + \frac{1}{2} ||u_0||^2_{L^2(\Omega)},$$

this by Young inequality with ϵ, implies that

$$\int_Q |w|^2 \leq M,$$

where M is a positive constant independent of v. So that the ball of radius M is invariant under S.

• Now we prove that S is continuous.

Let us choose a sequence $v_n \to v$ strongly in $L^2(Q)$; then by Lebesgue convergence Theorem :

$$\frac{f_n}{(|v_n| + \frac{1}{n})^\gamma} \to \frac{f_n}{(|v| + \frac{1}{n})^\gamma} \quad \text{in } L^2(Q),$$

and the uniqueness of solution for linear problem yields that $w_r = S(v_r)$
$\to w = S(v)$ strongly in $L^2(Q)$. Therefore, we proved that S is continuous.

As we proved before, we have that:

$$\int_Q |\nabla S(v)|^2 \leq C(n, \gamma, ||u_0||_{L^2(\Omega)}), \quad \text{for every} \quad v \in L^2(Q).$$

Then, $S(v)$ is relatively compact in $L^2(Q)$, and by Shauder's fixed point Theorem, there exist $u_{n,k} \in L^2(0,T;H_0^1(\Omega))$ such that $S(u_{n,k}) = u_{n,k}$ for each n, k fixed. Moreover, $u_{n,k} \in L^\infty(Q)$, for all $k, n \in \mathbb{N}$. Indeed, for $h \geq 1$ fixed, using $G_h(u_{n,k})$ as test function, we obtain, since $u_{n,k} + \frac{1}{n} \geq h \geq 1$ on $\{u_{n,k} \geq h\}$

$$\frac{1}{2}\int_{\Omega}|G_h(u_{n,k})|^2+\alpha\int_Q|\nabla G_h(u_{n,k})|^2\leq\int_Q f_nG_h(u_{n,k})+\frac{1}{2}\int_{\Omega}u_0^2.$$

From now, we can follow the standard technique used for the non-singular case in [1] to get $u_{n,k}\in L^{\infty}(Q)$. Furthermore, the estimate of $u_{n,k}\in L^{\infty}(Q)$ is independent from $k\in\mathbb{N}$, then for k large enough and for n fixed, $u_n\in L^2(0,T;H_0^1(\Omega))\cap L^{\infty}(Q)$ is the solution of the following approximate problem

$$\begin{cases}\dfrac{\partial u_n}{\partial t}-\ \operatorname{div}\left((a(x,t)+u_n^q)\nabla u_n\right)=\frac{f_n}{(|u_n|+\frac{1}{n})^{\gamma}} & \text{in}\quad Q,\\ u_n(x,t)=0 & \text{on}\quad \Gamma,\\ u_n(x,0)=u_0(x) & \text{in}\quad \Omega.\end{cases}$$

Since $\frac{f_n}{(|u_n|+\frac{1}{n})^{\gamma}}\geq 0$. The maximum principle implies that $u_n\geq 0$, and this concludes the proof. □

Lemma 3 *Let u_n be a solution of (7). Then for every $\omega\subset\subset\Omega$ there exists $C_{\omega}>0$ independent on n such that $u_n\geq C_{\omega}$ in $\omega\times(0,T)$, $\forall n\in\mathbb{N}$.*

Proof Define for $s\geq 0$ the function

$$\psi_{\delta}(s)=\begin{cases}1 & \textit{if}\ \ 0\leq s\leq 1,\\ \frac{1}{\delta}(1+\delta-s) & \textit{if}\ \ 1\leq s\leq\delta+1,\\ 0 & \textit{if}\ \ s>\delta+1.\end{cases}$$

We choose $\psi_{\delta}(u_n)\varphi$ as test function in (7) with $\varphi\in L^2(0,T;H_0^1(\Omega))\cap L^{\infty}(\Omega)$, $\varphi\geq 0$ then we have

$$\begin{aligned}&\int_0^T\int_{\Omega}\frac{\partial u_n}{\partial t}\psi(u_n)\varphi+\int_Q(a(x,t)+u_n^q)\nabla u_n\nabla\varphi\psi_{\delta}(u_n)\\ &\quad=\frac{1}{\delta}\int_{\{1\leq u_n\leq\delta+1\}}(a(x,t)+u_n^q)|\nabla u_n|^2\varphi+\int_Q\frac{f_n}{(u_n+\frac{1}{n})^{\gamma}}\psi_{\delta}(u_n)\varphi,\end{aligned}$$

thus, dropping the non-negative term $\frac{1}{\delta}\int_{\{1\leq u_n\leq\delta+1\}}(a(x,t)+u_n^q)|\nabla u_n|^2\varphi$, and letting δ goes to zero, we obtain

$$\begin{aligned}&\int_0^T\int_{\Omega}\frac{\partial u_n}{\partial t}\chi_{\{0\leq u_n<1\}}\varphi+\int_Q(a(x,t)+u_n^q)\nabla u_n\cdot\nabla\varphi\chi_{\{0\leq u_n<1\}}\\ &\quad\geq\int_Q\frac{f_n}{(u_n+\frac{1}{n})^{\gamma}}\varphi\chi_{\{0\leq u_n<1\}}.\end{aligned}$$

Then for the last inequality we can write as follows

$$\int_0^T \int_\Omega \frac{\partial T_1(u_n)}{\partial t}\varphi + \int_Q (a(x,t) + T_1(u_n)^q)\nabla T_1(u_n)\nabla\varphi \geq \int_Q \frac{f}{2^\gamma(1+f)}\varphi\chi_{\{0\leq u_n<1\}},$$

for all $0 \leq \varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q)$. Since $\frac{f}{2^\gamma(1+f)}\chi_{\{0\leq u_n<1\}}$ not identically zero and $\alpha \leq a(x,t) + T_1(u_n)^q \leq \beta + 1$, then we have

$$\int_0^T \int_\Omega \frac{\partial T_1(u_n)}{\partial t}\varphi + (\beta+1)\int_Q \nabla T_1(u_n).\nabla\varphi \geq 0. \tag{8}$$

This yields that $v_n = T_1(u_n)$ is a weak solution of the variational inequality

$$\begin{cases} \frac{1}{\beta+1}\frac{\partial v_n}{\partial t} - \triangle v_n \geq 0 & \text{in} \quad Q, \\ v_n(x,t) = 0 & \text{on} \quad \Gamma, \\ v_n(x,0) = T_1(u_0(x)) & \text{in} \quad \Omega, \end{cases}$$

where $v_n = T_1(u_n)$. We are going to prove that

$$\forall\, \omega \subset\subset \Omega,\ \exists\, C_\omega > 0: \quad v_n(x,t) \geq C_\omega \ \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{9}$$

Let w_n be the solution of the following problem

$$\begin{cases} \frac{1}{\beta+1}\frac{\partial w_n}{\partial t} - \triangle w_n = 0 & \text{in} \quad Q, \\ w_n(x,t) = 0 & \text{on} \quad \Gamma, \\ w_n(x,0) = v_n(x,0) & \text{in} \quad \Omega. \end{cases} \tag{10}$$

From (8) v_n is a supersolution of (10), we have $v_n \geq w_n$, so that we only have to prove that

$$\forall\, \omega \subset\subset \Omega,\ \exists\, C_\omega > 0: \quad w_n(x,t) \geq C_\omega \ \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{11}$$

Since by (2)

$$\forall\, \omega \subset\subset \Omega,\ \exists\, d_\omega > 0: \quad w_n(x,0) = v_n(x,0) \geq d_\omega \ \text{ in } \ \omega \times (0,T),\ \forall n \in \mathbb{N}. \tag{12}$$

For the rest of the proof we can argue as Boccardo, Orsina and Porzio in [4] (see pp $414-416$), we deduce that there exists $C_\omega > 0$ such that $w_n \geq C_\omega$ in $\omega \times (0,T)$, $\forall \omega \subset\subset \Omega$, since $v_n \geq w_n$, then $T_1(u_n) = v_n \geq C_\omega$ in $\omega \times (0,T)$, $\forall \omega \subset\subset \Omega$. As $u_n \geq T_1(u_n) = v_n$, then we obtain

$$u_n \geq C_\omega \ \text{ in } \ \omega \times (0,T), \quad \forall \omega \subset\subset \Omega,\ \forall n \in \mathbb{N}.$$

□

3 A priori estimates and main results

3.1 Case $\gamma < 1$

Lemma 4 *Let u_n be a solution of (7), with $\gamma < 1$ and $q > 1-\gamma$. Assume that $f \in L^1(Q)$, then u_n is bounded in $L^2(0,T;H_0^1(\Omega))$.*

Proof For n fixed, we choose $\epsilon < \frac{1}{n}$ and using $\phi(u_n) = ((u_n+\epsilon)^\gamma - \epsilon^\gamma) \times (1-(1+u_n)^{1-(q+\gamma)})$ as test function, then we have

$$\begin{aligned} &\int_\Omega \Psi(u_n(x,t)) + \gamma \int_Q (u_n+\epsilon)^{\gamma-1}(1-(1+u_n)^{1-(q+\gamma)})(a(x,t)+u_n^q)|\nabla u_n|^2 \\ &+ (q+\gamma-1)\int_Q ((u_n+\epsilon)^\gamma - \epsilon^\gamma)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \\ &= \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-(q+\gamma)}) + \int_\Omega \Psi(u_0), \end{aligned} \tag{13}$$

where $\Psi(s) = \displaystyle\int_0^s \phi(\ell)d\ell$. Dropping the first and second non-negative terms in the left hand side of (13), since $u_0 \in L^\infty(\Omega)$ and using (3), $\epsilon < \frac{1}{n}$ we have

$$\begin{aligned} &(q+\gamma-1)\int_Q ((u_n+\epsilon)^\gamma - \epsilon^\gamma)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \\ &\leq \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\frac{1}{n})^\gamma - \epsilon^\gamma)(1-(1+u_n)^{1-(q+\gamma)}) \leq \int_Q f + C, \end{aligned} \tag{14}$$

and passing to the limit on ϵ, we get

$$\int_Q (\alpha u_n^\gamma + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \leq C\int_Q f + C. \tag{15}$$

By working in $\{u_n \geq 1\}$, we have

$$\int_{\{u_n\geq 1\}} (\alpha + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \leq \int_Q (\alpha u_n^\gamma + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}},$$

then it follows from (15) that

$$\frac{\min(\alpha,1)}{2^{q+\gamma-1}}\int_{\{u_n\geq 1\}} |\nabla u_n|^2 \leq \min(\alpha,1)\int_{\{u_n\geq 1\}} \frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2 \leq C\int_Q f + C.$$

we can deduce that

$$\int_{\{u_n \ge 1\}} |\nabla u_n|^2 \le C. \tag{16}$$

Now, we choose $(T_k(u_n)+\epsilon)^\gamma - \epsilon^\gamma$ as a test function with $\epsilon < \frac{1}{n}$ in (7), by (3) and dropping the nonnegative terms, we get

$$\begin{aligned}\alpha \int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}} &\le \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_k(u_n)+\epsilon)^\gamma - \epsilon^\gamma)\\ &\quad + \frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1} - \epsilon^\gamma \int_\Omega u_0 \le \int_Q f\\ &\quad + \frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1} - \epsilon^\gamma \int_\Omega u_0.\end{aligned}$$

Therefore

$$\begin{aligned}\int_Q |\nabla T_k(u_n)|^2 &= \int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}}(T_k(u_n)+\epsilon)^{1-\gamma}\\ &\le (k+\epsilon)^{1-\gamma}\int_Q \frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\epsilon)^{1-\gamma}}\\ &\le (k+\epsilon)^{1-\gamma}\left[\int_Q f + \frac{1}{\gamma+1}\int_\Omega (T_k(u_0)+\epsilon)^{\gamma+1} - \epsilon^\gamma \int_\Omega u_0\right].\end{aligned}$$

By the fact that $u_0 \in L^\infty(\Omega)$ and letting ϵ goes to zero, implies that

$$\int_Q |\nabla T_k(u_n)|^2 \le Ck^{1-\gamma}. \tag{17}$$

Combining (16) and (17) we obtain

$$\int_Q |\nabla u_n|^2 = \int_{\{u_n \ge 1\}} |\nabla u_n|^2 + \int_{\{u_n \le 1\}} |\nabla u_n|^2 \le C.$$

Hence by last inequality we deduce that u_n is bounded in $L^2(0,T;H_0^1(\Omega))$ with respect to n. □

Lemma 5 *Let u_n be a solution of problem (7), with $\gamma < 1$ and $q \le 1-\gamma$. Suppose that f belongs to $L^1(Q)$, then u_n is bounded in $L^r(0,T;W_0^{1,r}(\Omega))$; with $r = \frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$.*

Proof For n fixed, we choose $\epsilon < \frac{1}{n}$ and using $\psi(u_n) = (u_n+\epsilon)^\gamma - \epsilon^\gamma$ as test function in (7), we obtain

$$\int_\Omega \Psi(u_n(x,t)) + \gamma \int_Q (a(x,t) + u_n^q)(u_n+\epsilon)^{\gamma-1}|\nabla u_n|^2$$
$$= \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma - \epsilon^\gamma) + \int_\Omega \Psi(u_0),$$

where $\Psi(s) = \int_0^s \psi(\ell)d\ell$. By removing the first nonnegative terms and using (3), $u_0 \in L^\infty(\Omega)$, since $q \le 1-\gamma < 1, \epsilon < \frac{1}{n} < 1$ and by the fact that

$$\min(\alpha,1)(u_n+\epsilon)^q \le \min(\alpha,1)(u_n+1)^q \le \min(\alpha,1)(1+u_n^q) \le \alpha + u_n^q \le a(x,t) + u_n^q,$$

we have

$$\gamma \min(\alpha,1)\int_Q (u_n+\epsilon)^{q+\gamma-1}|\nabla u_n|^2 \le \gamma \int_Q (\alpha+u_n^q)(u_n+1)^{\gamma-1}|\nabla u_n|^2$$
$$\le \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\epsilon)^\gamma - \epsilon^\gamma) \le \int_Q f + C.$$

If $q = 1-\gamma$, then u_n is bounded in $L^2(0,T;H_0^1(\Omega))$ with respect to n.

If $q < 1-\gamma$, then applying Sobolev inequality, we have

$$\left(\int_Q ((u_n+\epsilon)^{\frac{q+\gamma+1}{2}} - \epsilon^{\frac{q+\gamma+1}{2}})^{2^*}\right)^{\frac{2}{2^*}} \le C\int_Q |\nabla(u_n+\epsilon)^{\frac{q+\gamma+1}{2}}|^2 \le C\int_Q f + C, \tag{18}$$

letting $\epsilon \to 0$, then (18) implies

$$\int_Q u_n^{\frac{2^*(q+\gamma+1)}{2}} \le C. \tag{19}$$

Therefore, u_n is bounded in $L^{\frac{N(q+1+\gamma)}{N-2}}(Q)$ with respect to n.

Now, if $r < 2$ as in the statement of Lemma 5, we have by the Hölder inequality

$$\int_Q |\nabla u_n|^r = \int_Q \frac{|\nabla u_n|^r}{(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2}}}(u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2}}$$
$$\le \left(\int_Q \frac{|\nabla u_n|^2}{(u_n+\epsilon)^{1-(q+\gamma)}}\right)^{\frac{r}{2}} \left(\int_Q (u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2-r}}\right)^{1-\frac{r}{2}}$$
$$\le C\left(\int_Q (u_n+\epsilon)^{(1-(q+\gamma))\frac{r}{2-r}}\right)^{1-\frac{r}{2}}.$$

Thanks to (19), the value of r is such that $\frac{(1-(q+\gamma))r}{2-r} = \frac{N(q+\gamma+1)}{N-2}$, so that the right hand side of the above inequality is bounded, and then

$$\int_Q |\nabla u_n|^r \le M, \tag{20}$$

where M is a positive constant independent of n. Then u_n is bounded in $L^r(0,T;W_0^{1,r}(\Omega))$ with respect to n, with $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$ as desired. $\square$

Remark 1 As consequence of both Lemmas 4 and 5, there exists a sub-sequence (not relabeled) and a function u such that u_n converge weakly to u in $L^r(0,T;W_0^{1,r}(\Omega))$ (with $r=\frac{N(q+1+\gamma)}{N-(1-(q+\gamma))}$) and almost everywhere in Q as $n\to\infty$.

In the next lemma we give an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for any $\rho<\frac{N}{N-1}$.

Lemma 6 *Let u_n be a solution of problem (7), with $\gamma<1$. Suppose that $f\in L^1(Q)$, then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(Q)$ for every $\rho<\frac{N}{N-1}$.*

Proof For n fixed, we choose $\epsilon<\frac{1}{n}$ and we take as test function $\psi(u_n)=((T_1(u_n)+\epsilon)^\gamma-\epsilon^\gamma)(1-(1+u_n)^{1-\lambda})$, with $\lambda>1$, we have

$$\begin{aligned}&\int_\Omega \Psi(u_n(x,t))+\gamma\int_Q (T_1(u_n)+\epsilon)^{\gamma-1}(1-(1+u_n)^{1-\lambda})(a(x,t)+u_n^q)|\nabla T_1(u_n)|^2\\&+(\lambda-1)\int_Q (T_1(u_n)+\epsilon)^\gamma-\epsilon^\gamma)(a(x,t)+u^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda}\\&=\int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_1(u_n)+\epsilon)^\gamma-\epsilon^\gamma)(1-(1+u_n)^{1-\lambda})+\int_\Omega \Psi(u_0),\end{aligned}\tag{21}$$

where $\Psi(s)=\displaystyle\int_0^s \psi(\sigma)d\sigma.$

In the following, we ignore the first and second non-negative terms in the left hand side of (21), using (3) and the fact that $\alpha+u_n^q\ge c_0(1+u_n)^q$ yield

$$\begin{aligned}&(\lambda-1)c_0\int_Q ((T_1(u_n)+\epsilon)^\gamma-\epsilon^\gamma)(1+u_n)^{q-\lambda}|\nabla u_n|^2\\&\le\int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_1(u_n)+\epsilon)^\gamma-\epsilon^\gamma)(1-(1+u_n)^{1-\lambda})+\int_\Omega \Psi(u_0).\end{aligned}\tag{22}$$

Letting ϵ goes to zero and using the fact that $u_0\in L^\infty(\Omega)$, then (22) becomes

$$\int_{\{u_n\ge1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\le\int_Q T_1^\gamma(u_n)(1+u_n)^{q-\lambda}|\nabla u_n|^2\le C\int_Q f+C.\tag{23}$$

Combining (17) and (23) lead to

$$\begin{aligned}\int_Q(1+u_n)^{q-\lambda}|\nabla u_n|^2&=\int_{\{u_n\ge1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\\&+\int_{\{u_n\le1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\le C.\end{aligned}$$

Now, let $\rho = \frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$ and using the previous result together with Hölder inequality, we have

$$\int_Q u_n^{q\rho}|\nabla u_n|^\rho \le \int_Q (1+u_n)^{\frac{\rho(q+\lambda)}{2}} \frac{|\nabla u_n|^\rho}{(1+u_n)^{\frac{\rho(\lambda-q)}{2}}} \le C\left(\int_Q (1+u_n)^{\frac{\rho(q+\lambda)}{2-\rho}}\right)^{\frac{2-\rho}{2}},$$

and by Sobolev inequality, we get

$$\left(\int_Q u_n^{\rho^*(q+1)}\right)^{\frac{\rho}{\rho^*}} \le C\left(\int_Q u_n^{\frac{\rho(q+\lambda)}{2-\rho}}\right)^{\frac{2-\rho}{2}},$$

the previous choice of ρ implies that $\rho^*(q+1) = \rho(q+\lambda)/(2-\rho)$, and since $\lambda > 1$, we obtain an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for every $\rho < N/(N-1)$, as desired. In order to pass to the limit in the approximate equations, the almost everywhere convergence of the ∇u_n to ∇u is required, this result will be proved following the same techniques as in [2] (see also [19]). □

Lemma 7 *The sequence* $\{\nabla u_n\}$ *converges to* ∇u *a.e. in* Q.

Proof Let $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$ independent of $t \in [0,T]$ $\varphi \equiv 1$ on $w = Supp\varphi \subset\subset \Omega$ and using $T_h(u_n - T_k(u))\varphi$ as a test function in (7)

$$\begin{aligned} &\int_0^T \int_\Omega \frac{\partial u_n}{\partial t} T_h(u_n - T_k(u))\varphi + \int_0^T \int_\Omega (a(x,t) + u_n^q)\varphi \nabla u_n \nabla T_h(u_n - T_k(u)) \\ &\quad + \int_0^T \int_\Omega (a(x,t) + u_n^q) T_h(u_n - T_k(u)) \nabla u_n \nabla \varphi \\ &= \int_0^T \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} T_h(u_n - T_k(u))\varphi. \end{aligned} \tag{24}$$

Since $w = Supp\varphi \subset\subset \Omega$ and by Lemma 3 we have $u_n \ge C_{Supp\varphi}$, then we the above equality becomes

$$\begin{aligned} &\frac{1}{2}\int_\Omega T_h^2(u_n - T_k(u))\varphi + \int_0^T \int_\Omega (a(x,t) + u_n^q)|\nabla T_h(u_n - T_k(u))|^2 \varphi \\ &\le Ch||\nabla\varphi||_{L^\infty} + h||\varphi||_{L^\infty} \frac{1}{C_{Supp\varphi}^\gamma} \int_0^T \int_{Supp\varphi} f + \frac{1}{2}\int_\Omega T_h^2(u_0 - T_k(u_0))\varphi \\ &\quad - \int_0^T \int_\Omega (a(x,t) + u_n^q)\nabla T_h(u) \nabla T_h(u_n - T_k(u))\varphi, \end{aligned} \tag{25}$$

by removing the first non-negative term, we obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega (a(x,t) + u_n^q)|\nabla T_h(u_n - T_k(u))|^2 \varphi \\
&\quad \le Ch||\nabla \varphi||_{L^\infty} + h||\varphi||_{L^\infty} \frac{1}{C^\gamma_{Supp\varphi}} \int_0^T \int_{Supp\varphi} f + \frac{1}{2} h^2 \text{ meas } (\Omega) \\
&\quad - \int_0^T \int_\Omega (a(x,t) + u_n^q)\nabla T_h(u)\nabla T_h(u_n - T_k(u))\varphi.
\end{aligned} \tag{26}
$$

Since $\nabla T_h(u_n - T_k(u)) \neq 0$ (which implies that $u_n \le h + k$), we can easily to pass the limit as n tends to ∞, thanks to Remark 1, in the right hand side of the above inequality, so that

$$
\alpha \limsup_{n\to\infty} \int_0^T \int_\Omega |\nabla T_h(u_n - T_k(u))|^2 \varphi \le Ch. \tag{27}
$$

Let now s be such that $s < r < 2$, where r is in the statement of Lemma 5

$$
\begin{aligned}
\int_0^T \int_w |\nabla u_n - \nabla u|^s &\le \int_0^T \int_\Omega |\nabla u_n - \nabla u|^s \varphi \\
&= \int_{\{|u_n - u| \le h, u \le k\}} |\nabla u_n - \nabla u|^s \varphi + \int_{\{|u_n - u| \le h, u > k\}} |\nabla u_n - \nabla u|^s \varphi \\
&\quad + \int_{\{|u_n - u| > h\}} |\nabla u_n - \nabla u|^s \varphi.
\end{aligned} \tag{28}
$$

From (20), we have

$$
\begin{aligned}
\int_0^T \int_\Omega |\nabla u_n - \nabla u|^s \varphi &\le \int_0^T \int_\Omega |\nabla T_h(u_n - T_k(u))|^s \varphi \\
&\quad + ||\varphi||_{L^\infty} \left(2^s M^s (meas\{u > k\})^{1-\frac{s}{r}} + 2^s M^s (meas\{|u_n - u| > h\})^{1-\frac{s}{r}} \right).
\end{aligned} \tag{29}
$$

Thus, combining (27) and (28), we obtain for every $h > 0$ and every $k > 0$

$$
\begin{aligned}
\limsup_{n\to\infty} \int_0^T \int_\Omega |\nabla u_n - \nabla u|^s \varphi &\le \left(\frac{2h}{\alpha} \int_0^T \int_\Omega \right)^{\frac{s}{2}} ||\varphi||_{L^\infty} meas(Q)^{1-\frac{s}{2}} \\
&\quad + ||\varphi||_{L^\infty} 2^s M^s (meas\{u > k\})^{1-\frac{s}{r}}.
\end{aligned} \tag{30}
$$

Letting h tends to zero and k tends to infinity, we finally that

$$
\limsup_{n\to\infty} \int_0^T \int_\Omega |\nabla u_n - \nabla u|^s \varphi = 0, \quad \forall s < 2.
$$

Therefore, up to sub sequence, $\{\nabla u_n\}$ converges to ∇u a.e., and Lemma 7 is completely proved. □

Now we are in position to prove our existence result given by

Theorem 1 *Let $\gamma < 1$ and f be nonnegative function in $L^1(Q)$, then there exists a non-negative solution u of problem (1) in the sense of Definition 1. Moreover, u belong to $L^2(0,T;H_0^1(\Omega))$ if $q > 1-\gamma$ and it belongs to $L^r(0,T;W_0^{1,r}(\Omega))$ (with r as in the statement of Lemma 5) if $q \leq 1-\gamma$.*

Proof As we have already said (see Remark 1), there exists a function $u \in L^r(0,T;W_0^{1,r}(\Omega))$, such that u_n converges weakly to u in $L^r(0,T;W_0^{1,r}(\Omega))$.

By Lemma 3, we have $\frac{f_n}{(u_n+\frac{1}{n})^\gamma}$ is bounded in $L^1(0,T;L^1_{loc}(\Omega))$ and Lemma 6 gives $(a(x,t)+u_n^q)|\nabla u_n|$ is bounded in $L^\rho(Q)$, $\rho < \frac{N}{N-1} < 2$ then $\operatorname{div}((a(x,t)+u_n^q)\nabla u_n)$ is bounded $L^{\rho'}(Q) \subset L^2(Q) \subset L^2(0,T;H^{-1}(\Omega))$, then we deduce $\{\frac{\partial u_n}{\partial t}\}_n$ is bounded in $L^1(0,T;L^1_{loc}(\Omega)) + L^2(0,T;H^{-1}(\Omega))$, using compactness argument in [27], we deduce that

$$u_n \longrightarrow u \quad \text{strongly in } L^1(Q). \tag{31}$$

On the other hand, Lemmas 6, 7 and Remark 1 imply that the sequence $u_n^q|\nabla u_n|$ converges weakly to $u^q|\nabla u|$ in $L^\rho(Q)$ for every $\rho < \frac{N}{N-1}$. Hence for every $\varphi \in C_c^1(\Omega \times [0,T))$

$$\lim_{n\to\infty} \int_Q (a(x,t)+u_n^q)\nabla u_n \cdot \nabla\varphi = \int_Q (a(x,t)+u^q)\nabla u \cdot \nabla\varphi. \tag{32}$$

For the limit of the right hand of (7). Let $w = \{\varphi \neq 0\}$, then by Lemma 3, one has, for every $\varphi \in C_c^1(\Omega \times [0,T))$

$$\left| \frac{f_n\varphi}{(u_n+\frac{1}{n})^\gamma} \right| \leq \frac{||\varphi||_{L^\infty}}{C_w^\gamma} f, \tag{33}$$

then by Remark 1, (33) and dominated convergence theorem, we get

$$\frac{f_n}{(u_n+\frac{1}{n})^\gamma} \longrightarrow \frac{f}{u^\gamma} \quad \text{strongly in } L^1_{loc}(Q). \tag{34}$$

Let $\varphi \in C_c^1(\Omega \times [0,T))$ as test function in (7), by (31), (32), (33), (34) and letting $n \to +\infty$, we obtain

$$-\int_\Omega u_0(x)\varphi(x,0) - \int_Q u\frac{\partial\varphi}{\partial t} + \int_Q (a(x,t)+u^q)\nabla u.\nabla\varphi = \int_Q \frac{f}{u^\gamma}\varphi. \tag{35}$$

Hence, we conclude that the solution u satisfies the conditions (4), (5) and (6) of Definition 1, so that the proof of Theorem 1 is now completed. □

3.2 Case $\gamma = 1$

Lemma 8 *Let u_n be a solution of problem (7), with $\gamma = 1$. Suppose that f belongs to $L^1(Q)$. Then u_n is bounded in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \cap L^{\frac{N(q+2)}{N-2}}(Q)$.*

Proof we use $u_n\chi_{(0,t)}$ as test function in (7) and by (3), we obtain

$$\frac{1}{2}\int_\Omega |u_n(x,t)|^2 + \alpha\int_0^t\int_\Omega |\nabla u_n|^2 + \int_0^t\int_\Omega u_n^q|\nabla u_n|^2 \le \int_0^t\int_\Omega f_n + \frac{1}{2}\int_\Omega u_0^2,$$

as $f_n \le f$ and $u_0 \in L^\infty(\Omega)$, passing to supremum for $t \in (0,T)$ in the above estimate, we get

$$\begin{aligned}\frac{1}{2}||u_n||_{L^\infty(0,T;L^2(\Omega))} + \alpha\int_Q |\nabla u_n|^2 + \int_Q u_n^q|\nabla u_n|^2 \\ \le \int_Q f + \frac{1}{2}||u_0||^2_{L^2(\Omega)} \le C.\end{aligned} \tag{36}$$

This implies that

$$||u_n||_{L^\infty(0,T;\,L^2(\Omega))} \le C \text{ and } ||u_n||_{L^2(0,T;\,H_0^1(\Omega))} \le C. \tag{37}$$

In the other hand by Sobolev embedding Theorem and from (36), we can get

$$\int_Q u_n^{\frac{(q+2)2^*}{2}} \le \frac{4S}{(q+2)^2}\int_Q |\nabla u_n^{\frac{q+2}{2}}|^2 \le \int_Q f + \frac{1}{2}||u_0||^2_{L^2(\Omega)} \le C.$$

where S the constant of Sobolev embedding, hence the above estimate implies that the boundedness of u_n in $L^{\frac{N(q+2)}{N-2}}(Q)$ with respect to n. Then the proof of Lemma 8 is completed. □

Lemma 9 *Let u_n be a solution of problem (7), with $\gamma = 1$. Suppose that $f \in L^1(Q)$, then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(Q)$ for every $\rho < N/(N-1)$.*

Proof We take $\varphi(u_n) = T_1(u_n)(1-(1+u_n)^{1-\lambda})$, with $\lambda > 1$, as test function in (7), we obtain

$$\begin{aligned}&\int_\Omega \psi(u_n) + \gamma\int_Q T_1(u_n)(1-(1+u_n)^{1-\lambda})(a(x,t)+u_n^q)|\nabla T_1(u_n)|^2 \\ &+ (\lambda-1)\int_Q T_1(u_n)(a(x,t)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda} \\ &= \int_Q \frac{f_n}{u_n+\frac{1}{n}}T_1(u_n)(1-(1+u_n)^{1-\lambda}) + \int_\Omega \psi(u_0),\end{aligned}$$

where $\psi(s)=\int_0^s\varphi(\ell)d\ell$. Dropping the non-negative terms, from (3) and by the fact that $u_0\in L^\infty(\Omega)$, $\alpha+u_n^q\geq c_0(1+u_n)^q$, we have

$$\int_Q T_1(u_n)(1+u_n)^{q-\lambda}|\nabla u_n|^2\leq C\int_Q f+C.$$

By working in the set $\{u_n\geq 1\}$ and using the above estimate, we get

$$\int_{\{u_n\geq 1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\leq\int_Q T_1(u_n)^\gamma(1+u_n)^{q-\lambda}|\nabla u_n|^2\leq C\int_Q f+C. \tag{38}$$

The inequality (37) with (38), yields

$$\begin{aligned}\int_Q(1+u_n)^{q-\lambda}|\nabla u_n|^2=&\int_{\{u_n\geq 1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\\&+\int_{\{u_n<1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\leq C.\end{aligned} \tag{39}$$

Now let us fix $\rho=\frac{N(2+q-\lambda)}{N(q+1)-(\lambda+q)}$, by Hölder's inequality and (39), we have

$$\int_Q u_n^{q\rho}|\nabla u_n|^\rho=\int_Q\frac{|\nabla u_n|^\rho}{(1+u_n)^{\frac{\rho(\lambda-q)}{2}}}(1+u_n)^{\frac{\rho(\lambda+q)}{2}}\leq C\Big(\int_Q(1+u_n)^{\frac{\rho(q+\lambda)}{2-\rho}}\Big)^{\frac{2-\rho}{2}},$$

applying Sobolev inequality and using the above estimate, we deduce

$$\Big(\int_Q u_n^{\rho^*(q+1)}\Big)^{\frac{\rho}{\rho^*}}\leq C\Big(\int_Q u_n^{\frac{\rho(q+\lambda)}{2-\rho}}\Big)^{\frac{2-\rho}{2}}.$$

The previous choice of ρ implies that $\rho^*(q+1)=\frac{\rho(q+\lambda)}{2-\rho}$, and since $\lambda>1$, we obtain an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(Q)$ for every $\rho<N/(N-1)$. □

Theorem 2 *Let $\gamma=1$ and f be a function in $L^1(Q)$. Then there exists a solution u in $L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H_0^1(\Omega))\cap L^{\frac{N(q+2)}{N-2}}(Q)$ of problem (1) in the sense of Definition 1.*

Proof By Lemmas 3, 7, 8 and 9, the proof of Theorem 2 is identical to the one of Theorem 1. □

3.3 The strongly singular case γ > 1

In this case we do not have an estimate on u_n in $L^2(0,T;H_0^1(\Omega))$, but we can prove that u_n is bounded in $\boldsymbol{L^2(0,T;H^1_{loc}(\Omega))}$ such that $\boldsymbol{u^{\frac{q+\gamma+1}{2}}\in L^2(0,T;H_0^1(\Omega))}$.

Lemma 10 *Let u_n be a solution of the problem (7), with $\gamma > 1$. Suppose that f belongs to $L^1(Q)$, then $u_n^{\frac{q+\gamma+1}{2}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$, and u_n is bounded in $L^2(0,T;H^1_{loc}(\Omega)) \cap L^\infty(0,T;L^{\gamma+1}(\Omega))$. Moreover if $q \le \gamma - 1$, then $u_n^q|\nabla u_n|$ is bounded in $L^2(w \times (0,T))$ for every $w \subset\subset \Omega$.*

Proof Choosing $u_n^\gamma \chi_{(0,t)}$, as test function in (7) with $(0 < t \le T)$. Since $0 \le \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \le 1$, recalling that (3), the fact that $0 \le f_n \le f$ and the dropping the non-negative term, we have;

$$\frac{1}{\gamma+1}\int_\Omega u_n(x,t)^{\gamma+1} + \gamma \int_0^t \int_\Omega u_n^{q+\gamma-1}|\nabla u_n|^2 \le \int_0^t \int_\Omega \frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} + \frac{1}{\gamma+1}\int_\Omega u_0^{\gamma+1} \le \int_0^t \int_\Omega f + \frac{1}{\gamma+1}\int_\Omega u_0^{\gamma+1}.$$

Since $u_0 \in L^\infty(\Omega)$ and passing to supremum in $t \in [0,T]$, we obtain

$$\frac{1}{\gamma+1}||u_n||_{L^\infty(0,T;L^{\gamma+1}(\Omega))} + \gamma \int_Q u_n^{q+\gamma-1}|\nabla u_n|^2 \le \int_Q f + \frac{1}{\gamma+1}||u_0||^{\gamma+1}_{L^{\gamma+1}(\Omega)}, \quad (40)$$

then we get

$$\frac{4}{(q+\gamma+1)^2}\int_Q |\nabla u_n^{\frac{q+\gamma+1}{2}}|^2 = \int_Q u_n^{q+\gamma-1}|\nabla u_n|^2 \le \int_Q f + \frac{1}{\gamma+1}||u_0||^{\gamma+1}_{L^{\gamma+1}(\Omega)},$$

hence

$$\int_Q |\nabla u_n^{\frac{q+\gamma+1}{2}}|^2 \le C.$$

The last inequality and (40), imply that $u_n^{\frac{q+\gamma+1}{2}}$ is bounded in $L^2(0,T;H_0^1(\Omega))$ and u_n is bounded in $L^\infty(0,T;L^{\gamma+1}(\Omega))$ with respect to n. We choose now

$\varphi(u_n) = u_n^\gamma(1-(1+u_n)^{1-(q+\gamma)})$ as test function, dropping the non-negative terms, from (3), we have

$$(q+\gamma-1)\int_Q u_n^\gamma(\alpha + u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \le \int_Q \frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} + \int_\Omega \Psi(u_0) \le \int_Q f + \int_\Omega \Psi(u_0),$$

where $\Psi(s) = \int_0^s \varphi(\ell)d\ell$. By working in the set $\{u_n \ge 1\}$ and the fact that $u_0 \in L^\infty(\Omega)$, we get

$$\int_{\{u_n\geq 1\}}(\alpha+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \leq \int_Q(\alpha u_n^\gamma + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(u_n+1)^{q+\gamma}}$$
$$\leq \int_Q f + C,$$

the above estimate implies

$$\frac{\min(\alpha,1)}{2^{q+\gamma-1}}\int_{\{u_n\geq 1\}}|\nabla u_n|^2 \leq \min(\alpha,1)\int_{\{u_n\geq 1\}}\frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2$$
$$\leq C\int_Q f + C,$$

then we get

$$\int_{\{u_n\geq 1\}}|\nabla u_n|^2 \leq C. \tag{41}$$

Now we take $(T_k(u_n))^\gamma$ as test function in (7), by (3), Lemma 3 and the fact that $\frac{T_k(u_n)^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq \frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1, u_0 \in L^\infty(\Omega)$ and dropping the nonnegative terms, we obtain

$$\alpha C_w^{\gamma-1}\int_0^T\int_w|\nabla T_k(u_n)|^2 \leq \alpha\int_Q T_k(u_n)^{\gamma-1}|\nabla T_k(u_n)|^2$$
$$\leq \int_Q f + \frac{1}{\gamma+1}\int_\Omega T_k(u_0)^{\gamma+1} \leq \int_Q f + \frac{1}{\gamma+1}||u_0||^{\gamma+1}_{L^{\gamma+1}(\Omega)},$$

then we get that

$$\int_0^T\int_w|\nabla T_k(u_n)|^2 \leq C \quad \forall w \subset\subset \Omega. \tag{42}$$

Combining (41) and (42), we can deduce that

$$\int_0^T\int_w|\nabla u_n|^2 \leq \int_0^T\int_{w\cap\{u_n\geq 1\}}|\nabla u_n|^2 + \int_0^T\int_w|\nabla T_1(u_n)|^2 \leq C \tag{43}$$

$\forall w \subset\subset \Omega$, so that u_n is bounded in $L^2(0,T,H^1_{loc}(\Omega))$ is achieved. Now going back to (40), we have

$$\int_{\{u_n\geq 1\}}u_n^{q+\gamma-1}|\nabla u_n|^2 \leq \int_Q u_n^{q+\gamma-1}|\nabla u_n|^2 \leq \frac{1}{\gamma}\int_Q f + \frac{1}{\gamma(\gamma+1)}||u_0||^{\gamma+1}_{L^{\gamma+1}(\Omega)}.$$

Then we obtain since $2q \leq q+\gamma-1$

$$\int_0^T \int_w u_n^{2q}|\nabla u_n|^2 \le \int_0^T \int_{w\cap\{u_n\ge 1\}} u_n^{q+\gamma-1}|\nabla u_n|^2 + \int_0^T \int_w |\nabla T_1(u_n)|^2 \le C, \quad \forall w \subset\subset \Omega, \tag{44}$$

then the last inequality implies that $u_n^q|\nabla u_n|$ is bounded in $L^2(w\times(0,T))$ for every $w \subset\subset \Omega$. □

Remark 2 We note that by virtue of Lemma 10 we easily deduce the almost everywhere convergence of ∇u_n to ∇u following exactly the same proofs as the one of Lemma 7.

Theorem 3 *Let $\gamma > 1, q \le \gamma - 1$ and f be a nonnegative function in $L^1(Q)$. Then there exists a nonnegative solution $u \in L^2(0,T;H^1_{loc}(\Omega))$ of problem (1) in the sense of Definition 1. Moreover $u^{\frac{q+\gamma+1}{2}} \in L^2(0,T;H^1_0(\Omega))$.*

Proof Thanks to Lemmas 3, 7, 10, the proof of Theorem 3 is identical to the one of Theorem 1. □

4 Regularity results

In this section we study the regularity results of solution of problem (1) depending on $q, \gamma > 0$ and the summability of f.

Theorem 4 *Let $\gamma < 1$, f be a nonnegative function in $L^m(Q)$, $1 < m < \frac{N}{2} + 1$. Then the solution found in Theorem 1, satisfies the following summabilities:*

(i) *If $\frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)} \le m < \frac{N}{2} + 1$, $q \le 1-\gamma$ then u belongs to $L^2(0,T;H^1_0(\Omega)) \cap L^\sigma(Q)$, where*

$$\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

(ii) *If $1 < m < \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, $q > 1-\gamma$ then u belongs to $L^r(0,T;W^{1,r}_0(\Omega)) \cap L^\sigma(Q)$, where*

$$r = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N+2-m(1-\gamma)+q(m-1)}, \quad \sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

Proof Let u_n be a solution of (7) given by Lemma 2, such that u_n converges to a solution of (1). We choose $\varphi(u_n) = ((u_n+1)^\lambda - 1)\chi_{(0,t)}$, $(\lambda > 0)$ as test function in (7), we have

$$\int_\Omega \Psi(u_n(x,t)) + \lambda \int_0^t \int_\Omega (1+u_n)^{\lambda-1}(a(x,t)+u_n^q)|\nabla u_n|^2 \\ \leq C \int_0^t \int_\Omega |f_n| u_n^{\lambda-\gamma} + \int_\Omega \Psi(u_0),$$

where $\Psi(s) = \int_0^s \varphi(\ell)d\ell$. From the condition (3) and the fact that $u_0 \in L^\infty(\Omega)$, $c_0(1+u_n)^q \leq \alpha + u_n^q$, and applying Hölder's inequality, we obtain

$$\int_\Omega \Psi(u_n(x,t)) + \lambda c_0 \int_0^t \int_\Omega (1+u_n)^{\lambda+q-1}|\nabla u_n|^2 \\ \leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \tag{45}$$

By the definition of $\Psi(s)$ and $\varphi(s)$, if $\gamma \leq 1-q \leq \lambda$, we can write

$$\Psi(s) \geq \frac{|s|^{\lambda+1}}{\lambda+1} \quad \forall s \in \mathbb{R}.$$

From the above estimate and some simplification the inequality (45), we can estimate as follows

$$\frac{1}{\lambda+1}\int_\Omega [|u_n(x,t)|^{\frac{\lambda+q+1}{2}}]^{\frac{2(\lambda+1)}{\lambda+q+1}} + \frac{4\lambda c_0}{(\lambda+q+1)^2}\int_0^t \int_\Omega |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ \leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C.$$

Now passing to supremum for $t \in [0,T]$, we get

$$\frac{1}{\lambda+1}|||u_n|^{\frac{\lambda+q+1}{2}}||^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} + \frac{4\lambda c_0}{(\lambda+q+1)^2}\int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ \leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C. \tag{46}$$

By Lemma 1 (where $v = u_n^{\frac{\lambda+q+1}{2}}$, $s = \frac{2(\lambda+1)}{\lambda+q+1}$, $p = 2$), (46), we have

$$\int_Q [|u_n|^{\frac{\lambda+q+1}{2}}]^{2\frac{N+\frac{2(\lambda+1)}{\lambda+q+1}}{N}} \leq \left(\| |u_n|^{\frac{\lambda+q+1}{2}} \|^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} \right)^{\frac{2}{N}} \int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2$$
$$\leq \left[C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C \right]^{\frac{2}{N}+1}$$
$$\leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C,$$

then, we can obtain

$$\int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} \leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{(\frac{2}{N}+1)\frac{1}{m'}} + C. \tag{47}$$

Now choosing λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-\gamma)m', \tag{48}$$

then implies that

$$\lambda = \frac{N(q+1)+2+N\gamma m'}{Nm'-N-2}, \quad \sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

By virtue of $m < \frac{N}{2}+1$, then $(\frac{2}{N}+1)\frac{1}{m'} < 1$, and combining (47) and (48) with Young's inequality, we obtain

$$\int_Q |u_n|^\sigma \leq C. \tag{49}$$

The condition $m \geq \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, ensure that $\lambda \geq 1-q \geq \gamma$ and going back to (45), from (48) and (49), we have

$$\begin{aligned} \int_Q |\nabla u_n|^2 &\leq \int_Q (1+u_n)^{\lambda+q-1}|\nabla u_n|^2 \\ &\leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}} + C \leq C\left(\int_Q u_n^\sigma\right)^{\frac{1}{m'}} + C \leq C. \end{aligned} \tag{50}$$

The estimate (49) and (50), implies that u_n is bounded in $L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$ with respect to n, so $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$. Hence the proof of (i) is desired.

Now we prove (ii)

If $\gamma \leq \lambda < 1-q$, by definition $\varphi(s)$, $\Psi(s)$, we can get

$$\Psi(s) \geq C|s|^{\lambda+1} - C,$$

from the last inequality and going back to (45), we have

$$C\int_{\Omega}|u_n(x,t)|^{\lambda+1}+\lambda c_0\int_0^t\int_{\Omega}\frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}}$$
$$\leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}}+\int_{\Omega}\Psi(u_0)+Cmeas(\Omega),$$

by the fact that $u_0\in L^\infty(\Omega)$ and passing to supremum for $t\in[0,T]$, then we get

$$C||u_n||^{\lambda+1}_{L^\infty(0,T;L^{\lambda+1}(\Omega))}+\lambda c_0\int_Q\frac{|\nabla u_n|^2}{(1+u_n)^{1-\lambda-q}} \leq C\left(\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}}+C. \tag{51}$$

Let $\delta\leq 2$, applying Hölder's inequality, we have

$$\begin{aligned}\int_Q|\nabla u_n|^\delta&=\int_Q\frac{|\nabla u_n|^\delta}{(1+u_n)^{\frac{\delta(1-\lambda-q)}{2}}}(1+u_n)^{\frac{\delta(1-\lambda-q)}{2}}\\&\leq\left(\int_Q\frac{|\nabla u_n|^2}{(u_n+1)^{1-\lambda-q}}\right)^{\frac{\delta}{2}}\left(\int_Q(1+u_n)^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}}\\&\leq C\left(1+\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{\delta}{2m'}}\left(1+\int_Q u_n^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}}.\end{aligned} \tag{52}$$

Applying Lemma 1 (where $v=u_n,\ s=\lambda+1,\ p=\delta$) we get

$$\begin{aligned}\int_Q u_n^{\frac{\delta(N+\lambda+1)}{N}}&\leq||u_n||^{\frac{\delta(\lambda+1)}{N}}_{L^\infty(0,T;L^{\lambda+1}(\Omega))}\int_Q|\nabla u_n|^\delta\\&\leq C\left(1+\int_Q u_n^{(\lambda-\gamma)m'}\right)^{\frac{\delta}{Nm'}+\frac{\delta}{2m'}}\left(1+\int_Q u_n^{\frac{\delta(1-\lambda-q)}{2-\delta}}\right)^{\frac{2-\delta}{2}}.\end{aligned} \tag{53}$$

Let choose λ such that

$$\sigma=\frac{\delta(N+\lambda+1)}{N}=(\lambda-\gamma)m'=\frac{\delta(1-\lambda-q)}{2-\delta}, \tag{54}$$

then we deduce

$$\lambda=\frac{N(q+1)+2+N\gamma m'}{Nm'-N-2},\quad \sigma=m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

$$r=m\frac{N(q+\gamma+1)+2(\gamma+1)}{N+2-m(1-\gamma)+q(m-1)}.$$

From (54), the inequality (53), becomes

$$\int_Q u_n^\sigma \le C\left(1+\int_Q u_n^\sigma\right)^{\frac{\delta}{Nm'}+\frac{\delta}{2m'}+\frac{2-\delta}{2}}.$$

By virtue of $m < \frac{2(N+2-q)}{N(q+\gamma+1)+2(2-q)}$, ensure that $\frac{\delta}{Nm'}+\frac{\delta}{2m'}+\frac{2-\delta}{2} < 1$, then applying Young's inequality we can deduce that

$$\int_Q u_n^\sigma \le C. \tag{55}$$

We combine (54) and (55) in (52), yields

$$\int_Q |\nabla u_n|^\delta \le C. \tag{56}$$

Two last inequalities proved that the sequence u_n is bounded in $L^\delta(0,T;W_0^{1,\delta}(\Omega))$ $\cap L^\sigma(Q)$, and so $u \in L^\delta(0,T;W_0^{1,\delta}(\Omega)) \cap L^\sigma(Q)$. □

Theorem 5 *Let $\gamma = 1$, f be a nonnegative function in $L^m(Q)$, $1 \le m < \frac{N}{2}+1$. Then the solution found in Theorem 2, satisfy the following summability $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\sigma(Q)$ with $\sigma = \frac{m(N(q+2)+4)}{N-2m+2}$.*

Proof Let u_n be a solution of (7) given by Lemma 2, such that u_n converges to a solution of (1). Choosing $u_n^\lambda \chi_{(0,t)}$ as test function, with $\lambda \ge 1$, using (3) and applying Hölder's inequality, we have

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1} + \lambda\int_0^t\int_\Omega (\alpha+u_n^q)u_n^{\lambda-1}|\nabla u_n|^2\\ &\le C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + \frac{1}{\lambda+1}\int_\Omega u_0^{\lambda+1},\end{aligned}$$

thanks to $u_0 \in L^\infty(\Omega)$ and dropping the nonnegative term, we get

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega |u_n(x,t)|^{\lambda+1} + \lambda\int_0^t\int_\Omega u_n^{\lambda+q-1}|\nabla u_n|^2\\ &\le C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + \frac{1}{\lambda+1}||u_0||_{L^{\lambda+1}(\Omega)}^{\lambda+1} \le C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + C,\end{aligned}$$

by simple simplification the above estimate becomes

$$\begin{aligned}&\frac{1}{\lambda+1}\int_\Omega [|u_n(x,t)|^{\frac{\lambda+q+1}{2}}]^{\frac{2(\lambda+1)}{\lambda+q+1}} + \frac{4\lambda}{(\lambda+q+1)^2}\int_0^t\int_\Omega |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2\\ &\le C\left(\int_Q u_n^{(\lambda-1)m'}\right)^{\frac{1}{m'}} + C.\end{aligned}$$

Passing to supremum in $t \in [0,T]$, then we obtain

$$\frac{1}{\lambda+1}||u_n^{\frac{\lambda+q+1}{2}}||^{\frac{2(\lambda+1)}{\lambda+q+1}}_{L^\infty(0,T;L^{\frac{2(\lambda+1)}{\lambda+q+1}}(\Omega))} + \frac{4\lambda}{(\lambda+q+1)^2}\int_Q |\nabla u_n^{\frac{\lambda+q+1}{2}}|^2 \\ \leq C\Big(\int_Q u_n^{(\lambda-1)m'}\Big)^{\frac{1}{m'}} + C. \tag{57}$$

By Lemma 1 (where $v = u_n^{\frac{\lambda+q+1}{2}}$, $s = \frac{2(\lambda+1)}{\lambda+q+1}$, $p = 2$), we use the same proof as before, we get

$$\int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} \leq \Big[C\Big(\int_Q u_n^{(\lambda-1)m'}\Big)^{\frac{1}{m'}} + C\Big]^{\frac{2}{N}+1} \\ \leq C\Big(\int_Q u_n^{(\lambda-1)m'}\Big)^{\frac{1}{m'}(\frac{2}{N}+1)} + C. \tag{58}$$

Choosing λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-1)m', \tag{59}$$

then

$$\lambda = \frac{N(q+1)+2+Nm'}{Nm'-N-2}, \quad \sigma = \frac{m(N(q+2)+4)}{N-2m+2}.$$

Thanks to (59) and (58), implies that

$$\int_Q |u_n|^\sigma \leq C\Big(\int_Q |u_n|^\sigma\Big)^{\frac{1}{m'}(\frac{2}{N}+1)} + C.$$

The condition $m < \frac{N}{2}+1$ ensure that $\frac{1}{m'}(\frac{2}{N}+1) < 1$ and $\lambda \geq 1$ implies that $m \geq 1$, and using Young's inequality in the above estimate gives

$$\int_Q |u_n|^\sigma \leq C, \tag{60}$$

then we deduce that u_n is bounded in $L^\sigma(Q)$ and so u belong to $L^\sigma(Q)$. □

Theorem 6 *Let $\gamma > 1, q > \gamma - 1$ and f be a nonnegative function in $L^m(Q), m > 1$. Then there exists a solution u of problem (1) such that if*

$\max(1, \frac{(N+2)(2q-\gamma+1)}{N(q+\gamma+1)+4(q+1)}) < m < \frac{N}{2}+1$, *then u belong to $L^\sigma(Q)$ with*

$$\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}.$$

Proof We will take $u_n^\lambda \chi_{(0,t)}(\lambda > 1)$ as test function in (7), as in the case $\gamma = 1$ we will follow the proof of Theorem 5, repeating the same passage in order to arrive to the inequality

$$\int_Q |u_n|^{\frac{N(\lambda+q+1)+2(\lambda+1)}{N}} \leq C\left(\int_Q |u_n|^{(\lambda-\gamma)m'}\right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C. \tag{61}$$

We now choose λ such that

$$\sigma = \frac{N(\lambda+q+1)+2(\lambda+1)}{N} = (\lambda-\gamma)m', \tag{62}$$

i.e $\lambda = \frac{N(q+1)+2+N\gamma m'}{Nm'-N-2}$, $\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}$. Combining (61) and (62), implies that

$$\int_Q |u_n|^\sigma \leq C\left(\int_Q |u_n|^\sigma\right)^{\frac{1}{m'}(\frac{2}{N}+1)} + C,$$

by virtue of $m < \frac{N}{2}+1$, then we have $\frac{1}{m'}(\frac{2}{N}+1) < 1$ and $\lambda > 1$ ensure that $m > 1$, then by Young's inequality, we get

$$\int_Q |u_n|^\sigma \leq C. \tag{63}$$

Hence from (63) it follows that u_n is bounded in $L^\sigma(Q)$ so that $u \in L^\sigma(Q)$. Next we testing (7) by $u_n^\gamma T_1(u_n - T_k(u_n))$, we have

$$\int_Q \frac{\partial u_n}{\partial t} u_n^\gamma T_1(u_n - T_k(u_n)) + \gamma \int_Q u_n^{\gamma-1}(a(x,t)+u_n^q)|\nabla u_n|^2 T_1(u_n - T_k(u_n))$$
$$+ \int_{Q\cap\{k\leq u_n\leq k+1\}} u_n^\gamma(a(x,t)+u_n^q)|\nabla u_n|^2 = \int_Q \frac{f_n}{(u_n+\frac{1}{n})^\gamma} u_n^\gamma T_1(u_n - T_k(u_n)). \tag{64}$$

Dropping the first and second nonnegative terms in the left hand side of (64) and using the assumption (3), we obtain

$$\int_{Q\cap\{u_n\geq k\}} u_n^\gamma |\nabla u_n|^2 \leq \frac{1}{\gamma\alpha}\int_{Q\cap\{u_n\geq k\}} f + C. \tag{65}$$

Thus, thanks to the estimate (65), implies that

$$\int_{Q\cap\{u_n>k\}} u_n^q|\nabla u_n| \leq \left(\int_{Q\cap\{u_n>k\}} u_n^{2q-\gamma+1}\right)^{\frac{1}{2}}\left(\int_{Q\cap\{u_n>k\}} u_n^{\gamma-1}|\nabla u_n|^2\right)^{\frac{1}{2}}$$
$$\leq \left(\int_{Q\cap\{u_n>k\}} u_n^{2q-\gamma+1}\right)^{\frac{1}{2}}\left(\frac{1}{\gamma\alpha}\int_{Q\cap\{u_n>k\}} f + C\right)^{\frac{1}{2}}.$$

Since u_n is bounded in $L^\sigma(Q)$, then $2q-\gamma+1 \le \sigma$ is equivalent to $m \ge \frac{(N+2)(2q-\gamma+1)}{N(q+\gamma+1)+4(q+1)}$, hence we get

$$\int_{Q\cap\{u_n>k\}} u_n^q|\nabla u_n| \le C\left(\frac{1}{\gamma\alpha}\int_{Q\cap\{u_n>k\}} f\right)^{\frac{1}{2}}. \tag{66}$$

Now let $\varphi \in C_c^1(\Omega\times[0,T))$, $\varphi \equiv 1$ on $w\times(0,T)$, $w \subset\subset \Omega$, and E be a measurable subset of Q, from (66) and Lemma 10, we can get

$$\begin{aligned}\int_{E\cap\{w\times(0,T)\}} u_n^q|\nabla u_n| &\le \int_E u_n^q|\nabla u_n|\varphi \le \int_{Q\cap\{u_n>k\}} u_n^q|\nabla u_n|\varphi + k^q\int_E|\nabla u_n|\varphi \\ &\le C||\varphi||_{L^\infty}\left(\int_{Q\cap\{u_n>k\}} f + C\right)^{\frac{1}{2}} + ||\varphi||_{L^\infty}k^q meas(E)^{\frac{1}{2}}\left(\int_{w\times(0,T)}|\nabla u_n|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Taking the limit as meas (E) tends to zero, k tend to infinity and since $u_n^q|\nabla u_n|$ converge to $u^q|\nabla u|$ almost everywhere, we easily verify thanks to Vitali's Theorem that

$$u_n^q|\nabla u_n| \to u^q|\nabla u| \qquad \text{strongly in} \quad L^1(0,T;L^1_{loc}(\Omega)). \tag{67}$$

Therefore, putting together (67), Lemma 3 and Lemma 10, we conclude the proof of Theorem 6. □

Theorem 7 *Let $\gamma > 1, q \le \gamma-1$ and f be a non-negative function in $L^m(Q), 1 < m < \frac{N}{2}+1$. Then the solution found in Theorem3, satisfies the following summability, $u \in L^\sigma(Q)$, with $\sigma = m\frac{N(q+\gamma+1)+2(\gamma+1)}{N-2m+2}$.*

Proof The proof of Theorem 7 is similar to proof of item (i) of Theorem 4. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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