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On the asymptotic behavior of inverse problems for parabolic equation

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Abstract
We study two inverse initial-boundary value problems for a linear parabolic equation. These equations arise in mathematical modeling of the viscous heat-conducting fluid motion with two or one free boundaries. The unknown function of time enters the right-hand side of the equation additively and is found from the additional condition of integral overdetermination. For both problems, a priori estimates of solutions in the uniform metric are obtained. Stationary solutions are found. Sufficient conditions for the input data, under which the solutions with increasing time tend to the stationary regime according to the exponential law, are established.

Keywords Parabolic equation · Inverse problem · A priori estimates · Asymptotic behavior

Mathematics Subject Classification 35A02 · 35A23

1 Introduction

We consider the initial-boundary value problem

$$u_t = \nu u_{xx} + f(t) + g(x,t), \quad x \in (0,l), \quad t \in [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \quad x \in (0,l), \tag{1.2}$$

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$$\alpha_1 u(0,t) - \beta_1 u_x(0,t) = q_1(t), \quad \alpha_2 u(l,t) + \beta_2 u_x(l,t) = q_2(t),$$
$$\int\limits_0^l u(x,t)dx = q_3(t), \quad t \in [0,T]. \tag{1.3}$$

In (1.1 – 1.3) the functions $g(x,\ t)$, $u_0(x)$, $q_i(t), (i = 1,2,3)$ and constants $\nu > 0$, $T > 0$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\beta_1 \geq 0$, $\beta_2 \geq 0$ are given and $u(x,\ t)$, $f(t)$ are unknown. Thus, the problem (1.1 – 1.3) is inverse. For its smooth solutions, it is necessary that the conditions of the agreement have been met

$$\alpha_1 u_0(0) - \beta_1 u_{0x}(0) = q_1(0), \quad \alpha_2 u_0(l) + \beta_2 u_{0x}(l) = q_2(0), \quad \int\limits_0^l u_0(x)dx = q_3(0). \tag{1.4}$$

For $\beta_1 = \beta_2 = 0$, the mathematical modeling of two-dimensional creeping motions of a special type of viscous fluid in a flat channel [1] and microconvection models [2] leads to the problem (1.1 – 1.3). The qualitative properties of this problem, including the asymptotic behavior of the solution as $t \to \infty$ in the uniform metric, were studied in [3]. Earlier, a similar result in the integral metric was obtained in [4].

It should be noted that inverse problems for parabolic equations with an integral overdetermination condition of a more general form than (1.3) have been considered in a fairly large number of papers, for example, [5–7] and others. A more complete overview is given in [8]. As a rule, in these works the existence and uniqueness of the solution is proved, the methods of constructing approximate solutions for the case when the unknown function $f(t)$ enters multiplicatively in the right-hand side of the equation (system) are considered and justified.

In this paper, two subproblems are considered:$\alpha_1 = \alpha_2 = 0$, $\beta_1 \neq 0$, $\beta_2 \neq 0$ and $\alpha_1 \neq 0, \alpha_2 = 0, \beta_1 = 0, \beta_2 \neq 0$. The first problem simulates the unidirectional motion of a flat layer with two free boundaries and a known temperature distribution. The second task simulates a similar motion in a layer with a moving bottom solid wall and an upper free boundary. In this case, the unknown function $f(t)$ is the longitudinal pressure gradient along the layer, and ν is the kinematic viscosity. Integral condition (1.3) is a given liquid flow rate through the cross section of the layer.

Using the specifics of these two subproblems (one-dimensionality), it is possible to obtain sufficient conditions on the input data under which the solutions converge as $t \to \infty$ to stationary ones in the uniform metric.

For $\alpha_1 > 0,\ \alpha_2 > 0,\ \beta_1 \geq 0,\ \beta_2 \geq 0,\ f(t) = 0$ and integral condition (1.3) is absent (direct initial-boundary value problem), the sufficient conditions in the case of multidimensional linear parabolic equations of general form were established in [9]. As simple examples show, there is no such convergence for boundary conditions of the second kind $\left(-\beta_1 u_x(0,t) = q_1(t), \beta_2 u_x(l,t) = q_2(t)\right)$ or mixed ones $\left(\alpha_1 u(0,t) = q_1(t), \beta_2 u_x(l,t) = q_2(t)\right)$.

Moreover, the stationary regime for boundary conditions of the second kind is not the only one. However, for the inverse problems formulated above, under certain conditions, such convergence takes place.

2 Inverse problem in the case of boundary conditions of the second kind

Here $\alpha_1 = \alpha_2 = 0$, $\beta_1 > 0$, $\beta_2 > 0$, so that

$$u_x(0,t) = -\beta_1^{-1} q_1(t), \quad u_x(l,t) = \beta_2^{-1} q_2(t). \tag{2.1}$$

Integrating equation (1.1) over x from 0 to l and using the integral condition (1.3) and (2.1), we find

$$f(t) = \frac{1}{l}\left[q_3'(t) - \nu\left(\beta_1^{-1} q_1(t) + \beta_2^{-1} q_2(t)\right) - \int_0^l g(x,t)dx \right]. \tag{2.2}$$

Thus, the unknown function $f(t)$ is immediately determined from the input data of the problem. Substitution of (2.2) in equation (1.1) leads to a direct problem for the function $u(x, t)$ with initial condition (1.2) and boundary conditions (2.1).

Remark 1 If in (1.1) on the right-hand side $f(t)$ is given by equality (2.2), then the redefinition condition (1.3) is satisfied automatically.

According to Remark 1 and formula (2.2), the function $u(x, t)$ is a solution of the classical second initial-boundary value problem with a known right-hand side. Its solution is given by the formula (see [10], p. 58)

$$u(x,t) = \int_0^l u_0(y)dy + \int_0^t \int_0^l F(y,\tau)G(x,y,t-\tau)dyd\tau$$
$$+ \nu\beta_1^{-1} \int_0^t q_1(\tau)G(x,0,t-\tau)d\tau + \nu\beta_2^{-1} \int_0^t q_2(\tau)G(x,l,t-\tau)d\tau,$$

where G is the Green's function

$$G(x,y,t) = \frac{1}{l} + \frac{2}{l}\sum_{n=1}^{\infty} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi y}{l}\right)\exp\left(-\frac{\nu n^2\pi^2 t}{l^2}\right),$$
$$F(x,t) = f(t) + g(x,t).$$

The estimate $|u(x,t)|, x \in [0,l], t \in [0,T]$ from this representation is a rather laborious problem with long calculations [11, 12]. Therefore, using the specifics of the problem, let us reduce it to an auxiliary classical first initial-boundary value problem.

Differentiating equation (1.1) with respect to the variable x, we obtain the first initial-boundary value problem for $w(x,t) = u_x(x,t)$:

$$
\begin{aligned}
w_t &= \nu w_{xx} + g_x(x,t), \quad x \in (0,l), \quad t \in [0,T], \\
w(x,0) &= w_0(x) = u_{0x}(x), \quad x \in [0,l], \\
w(0,t) &= -\beta_1^{-1} q_1(t), \quad w(l,t) = \beta_2^{-1} q_2(t), \quad t \in [0,T].
\end{aligned} \tag{2.3}
$$

The resulting direct problem has a stationary solution

$$
w^s(x) = -\beta_1^{-1} q_1^s + \frac{1}{l}\left[\beta_1^{-1} q_1^s + \beta_2^{-1} q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right] x - \frac{1}{\nu}\int_0^x g^s(y)dy, \tag{2.4}
$$

where q_1^s, q_2^s, $g^s(x)$ are given constants and function, respectively.

Suppose, that $q_j(t)$ are defined for all $t \geq 0$ and

$$
|q_j(t)| \leq N_j(1+\tau)^{-\alpha}, \quad j=1,2, \quad |g(x,t)| \leq N_3(1+\tau)^{-\alpha}, \quad |g_x(x,t)| \leq N_4(1+\tau)^{-\alpha} \tag{2.5}
$$

with some positive constants $N_1, ..., N_4, \alpha$ for any $x \in [0,l], \tau = \nu l^{-2} t$ is dimensionless time. Then [9]

$$
|w(x,t| \leq N_5(1+\tau)^{-\alpha} \tag{2.6}
$$

with new constant $N_5 > 0, x \in [0,l]$.

The stationary solution $u^s(x)$ is found by integrating (2.4):

$$
u^s(x) = -\beta_1^{-1} q_1^s x - \frac{1}{\nu}\left(f^s \frac{x^2}{2} + \int_0^x (x-y) g^s(y)dy\right) + C. \tag{2.7}
$$

Here

$$
f^s = -\frac{\nu}{l}\left(\beta_1^{-1} q_1^s + \beta_2^{-1} q_2^s + \frac{1}{\nu}\int_0^l g^s(y)dy\right), \tag{2.8}
$$

and the constant C is determined from the stationary integral condition (1.3)

$$
C = l^{-1} q_3^s + \frac{\beta_1^{-1} l}{3} q_1^s - \frac{\beta_2^{-1} l}{6} q_2^s + \frac{1}{\nu l}\left(\int_0^l \int_0^x (x-y) g^s(y)dydx - l^2 \int_0^l g^s(y)dy\right). \tag{2.9}
$$

Remark 2 If $q_3'(t) \to 0, q_j(t) \to q_j^s, g(x,t) \to g^s(x)$ at $t \to \infty, x \in [0,l]$, then $f(t) \to f^s$, $t \to \infty$ ($f(t)$ and f^s are determined from (2.2), (2.8), respectively).

Let us proceed to obtaining an a priori estimate for $|u(x, t)|$. From the integral mean value theorem there is a point $x_0 \in (0,l)$ such that $u(x_0,t) = l^{-1} q_3(t)$ (see (1.3)). Therefore, for any $x \in [0,l]$, $t \geq 0$ we have

$$u(x,t) = u(x_0,t) + \int_{x_0}^{x} u_y(y,t)dy = l^{-1}q_3(t) + \int_{x_0}^{x} w(y,t)dy.$$

Hence, using (2.6), we obtain

$$|u(x,t)| \le l^{-1}|q_3(t)| + \int_0^l |w(y,t|dy \le l^{-1}|q_3(t)| + N_5 l(1+\tau)^{-\alpha}. \tag{2.10}$$

Let inequalities are satisfied

$$\begin{aligned} &|q_j(t) - q_j^s| \le D_j(1+\tau)^{-\alpha}, \quad j = 1,2,3, \\ &|q_3'(t)| \to 0,\ t \to \infty, \quad |g(x,t) - g^s(x)| \le D_4(1+\tau)^{-\alpha}, \\ &|g_x(x,t) - g_x^s(x)| \le D_5(1+\tau)^{-\alpha} \end{aligned} \tag{2.11}$$

with constants $D_i > 0\,(i = 1,...,5)$, $\alpha > 0$ for any $x \in [0,l]$. Then

$$\begin{aligned} &|u(x,t) - u^s(x)| \le D_6(1+\tau)^{-\alpha}, \\ &|u_x(x,t) - u_x^s| \le D_7(1+\tau)^{-\alpha}, \\ &|f(t) - f^s| \le D_8(1+\tau)^{-\alpha}, \end{aligned} \tag{2.12}$$

where D_6, D_7, D_8 are positive constants. Estimates (2.12) follow sequentially from (2.10), (2.6), formulas (2.2), (2.8), and assumptions (2.11). To obtain estimates (2.12), it is sufficient in the original problem to make the replacement $\tilde{u}(x,t) = u(x,t) - u^s(x)$, $\tilde{w}(x,t) = w(x,t) - w^s(x)$, $\tilde{f}(t) = f - f^s$. In this case, the input data is replaced by $\tilde{q}_j(t) = q_j(t) - q_j^s$, $j = 1,2,3$, $\tilde{g}(x,t) = g(x,t) - g^s(x)$.

Remark 3 If the right-hand sides of inequalities (2.11) are bounded by the exponent ($\exp(-\alpha\tau)$, $\alpha > 0$), then the solution of the inverse initial-boundary value problem exponentially with increasing time tends to the stationary regime $u^s(x)$, f^s, determined by the formulas (2.7 – 2.9).

Remark 4 The obtained results can be interpreted as the stability of the stationary solution (2.7 – 2.9) if, for example, conditions (2.11) are satisfied.

3 Mixed inverse initial-boundary value problem

Here $\alpha_1 \ne 0$, $\alpha_2 = 0$, $\beta_1 = 0$, $\beta_2 \ne 0$, i. e. $u(0,t) = q_1(t)$, $u_x(l,t) = \beta_2^{-1} q_2(t)$. The replacement

$$
\begin{aligned}
u(x,t) &= v(x,t) + \frac{3}{2l}\left(\frac{1}{l}\,q_1(t) + \frac{\beta_2^{-1}}{2}\,q_2(t) - \frac{1}{l^2}\,q_3(t)\right)x^2 \\
&\quad + \left(\frac{3}{l^2}\,q_3(t) - \frac{3}{l^2}\,q_1(t) - \frac{\beta_2^{-1}}{2}\,q_2(t)\right)x + q_1(t)
\end{aligned}
\tag{3.1}
$$

leads to a problem with homogeneous boundary conditions

$$
v_t = \nu v_{xx} + f(t) + g_1(x,t), \quad x \in (0,l), \quad t \in [0,T], \tag{3.2}
$$

$$
v(x,0) = v_0(x), \quad x \in [0,l], \tag{3.3}
$$

$$
v(0,t) = 0, \quad v_x(l,t) = 0, \quad \int_0^l v(y,t)dy = 0, \quad t \in [0,T], \tag{3.4}
$$

where

$$
\begin{aligned}
g_1(x,t) &= g(x,t) + \frac{3\nu}{l}\left(\frac{1}{l}\,q_1(t) + \frac{\beta_2^{-1}}{2}\,q_2(t) - \frac{1}{l^2}\,q_3(t)\right) - q_1'(t) - \\
&\quad - \frac{3}{2l}\left(\frac{1}{l}\,q_1'(t) + \frac{\beta_2^{-1}}{2}\,q_2'(t) - \frac{1}{l^2}\,q_3'(t)\right)x^2 - \left(\frac{3}{l^2}\,q_3'(t) - \frac{3}{l^2}\,q_1'(t) - \frac{\beta_2^{-1}}{2}\,q_2'(t)\right)x, \\
v_0(x) &= u_0(x) - \frac{3}{2l}\left(\frac{1}{l}\,q_1(0) + \frac{\beta_2^{-1}}{2}\,q_2(0) - \frac{1}{l^2}\,q_3(0)\right)x^2 - \\
&\quad - \left(\frac{3}{l^2}\,q_3(0) - \frac{3}{l^2}\,q_1(0) - \frac{\beta_2^{-1}}{2}\,q_2(0)\right)x - q_1(0).
\end{aligned}
\tag{3.5}
$$

Let us proceed to obtaining a priori estimates for the $|v(x,t)|$ and $|f(t)|$ for all $x \in [0,l]$ and $t \in [0,T]$. Multiplication of equation (3.2) by $v(x, t)$ and integration, using boundary conditions (3.4), leads to the equality

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\int_0^l v^2(x,t)dx\right) + \nu\int_0^l v_x^2(x,t)dx = \int_0^l g_1(x,t)v(x,t)dx. \tag{3.6}
$$

Since in our case the Poincare inequality holds

$$
\int_0^l v^2(x,t)dx \le \frac{l^2}{2}\int_0^l v_x^2(x,t)dx,
$$

from (3.6) we find the estimate for $t \in [0,T]$

$$\int_0^l v^2(x,t)dx \le \left[\left(\int_0^l v_0^2(x)dx\right)^{1/2} + \int_0^t e^{2\nu\tau/l^2} k(\tau)d\tau\right]^2 e^{-4\nu t/l^2} \tag{3.7}$$

with function

$$k(t) = \left(\int_0^l g_1^2(x,t)dx\right)^{1/2}. \tag{3.8}$$

Along with (3.6), there is another integral identity

$$\int_0^l v_t^2 dx + \frac{\nu}{2}\frac{\partial}{\partial t}\int_0^l v_x^2 dx = \int_0^l g_1 v_t dx.$$

Using this inequality and taking into account the notation (3.8), we obtain

$$\int_0^l v_x^2 dx \le \int_0^l v_{0x}^2 dx + \frac{1}{\nu}\int_0^t k^2(\tau)d\tau, \quad t \in [0,T]. \tag{3.9}$$

Using the first boundary condition (3.4), inequalities (3.7), (3.9), we have

$$v^2(x,t) = 2\int_0^x v v_x dx \le 2\left(\int_0^l v^2 dx\right)^{1/2}\left(\int_0^l v_x^2 dx\right)^{1/2} \le \left[\left(\int_0^l v_0^2 dx\right)^{1/2} + \int_0^t e^{2\nu\tau/l^2} k(\tau)d\tau\right]\left[\left(\int_0^l v_{0x}^2 dx\right)^{1/2} + \nu^{-1}\int_0^t k^2(\tau)d\tau\right]^{1/2} e^{-2\nu t/l^2} \equiv \bar{k}^2(t)e^{-2\nu t/l^2}. \tag{3.10}$$

Therefore, for all $x \in [0,l]$, $t \in [0,T]$ we get

$$|v(x,t)| \le \bar{k}(t)e^{-\nu t/l^2}. \tag{3.11}$$

We multiply equation (3.2) by $2lx - x^2$ and integrate over x. Using conditions (3.4), we find

$$f(t) = \frac{3}{2l^3}\left[\int_0^l (2lx - x^2)(v_t - g_1)dx\right]. \tag{3.12}$$

Remark 5 From (3.7) and (3.12) implies the uniqueness of solution of the inverse problem.

To estimate $|f(t)|$, it is necessary, according to (3.12), to estimate $|v_t(x,t)|$. Differentiation of problem (3.2), (3.4) with respect to t leads to a similar problem for $v_t(x,t)$, where $f(t)$ must be replaced by $f_t(t)$, $g_1(x,t)$ by $g_{1t}(x,t)$, and the initial data (3.3) by

$$v_t(x,0) = \nu v_{0xx} + f(0) + g_1(x,0) \equiv v_1(x), \quad x \in [0,l], \tag{3.13}$$

where

$$f(0) = \frac{1}{l}\left[\nu v_{0x}(0) - \int_0^l g_1(y,0)dy\right].$$

To obtain the last equality, it is sufficient to integrate equation (3.2) over x and set $t = 0$. Therefore, for $v_t(x,t)$, estimates (3.7), (3.11) hold, where instead of $k(t)$ there will be another function

$$k_1(t) = \left(\int_0^l g_{1t}^2(y,t)dy\right)^{1/2}. \tag{3.14}$$

So,

$$|v_t(x,t)| \le \bar{k}_1(t)e^{-\nu t/l^2} \tag{3.15}$$

with a bounded function for $t \in [0,T]$

$$\bar{k}_1(t) = \left\{2\left[\left(\int_0^l v_1^2(y)dy\right)^{1/2} + \int_0^t e^{2\nu\tau/l^2}k(\tau)d\tau\right]\left[\int_0^l v_1^2(y)dy + \nu^{-1}\int_0^t k_1^2(\tau)d\tau\right]^{1/2}\right\}^{1/2}, \tag{3.16}$$

where $v_1(x)$ is defined in (3.13). Turning to representation (3.12), we arrive at the estimate for all $t \in [0,T]$

$$|f(t)| \le \left[\bar{k}_1(t)e^{-\nu t/l^2} + \max_{x\in[0,l]}|g_1(x,t)|\right], \tag{3.17}$$

where $k_1(t)$ is given by equality (3.16).

We have proven

Theorem 1 *Let* $v_0(x) \in C^2[0,l]$, $g_1(x,t)$, $g_{1t}(x,t) \in C([0,l]\times[0,T])$ *and a smooth solution to problem* (3.2 – 3.4) *exists, then it is unique and estimates* (3.9), (3.15), (3.17) *hold, which are uniform for all* $x \in [0,l]$ *and* $t \in [0,T]$.

Remark 6 From equation (3.2) and inequalities (3.15), (3.17) follows the estimate of the second derivative

$$|v_{xx}(x,t)| \le \frac{5}{2\nu}\left[\bar{k}_1(t)e^{-\nu t/l^2} + \max_{x\in[0,l]}|g_1(x,t)|\right], \tag{3.18}$$

and from the mean value theorem follows a similar estimate for $|v_x(x,t)|$.

Remark 7 The conditions of Theorem 1 will be satisfied, if $q_j(t) \in C^2[0,T]$, $(j = 1,2,3)$, $g(x,t), g_t(x,t) \in C([0,l]\times[0,T])$, $u_0(x) \in C^2[0,l]$.

The stationary solution $v^s(x)$, f^s of the inverse problem (3.2), (3.4) has the form

$$\begin{aligned} v^s(x) =& \frac{1}{\nu}\left[\left(lx - \frac{x^2}{2}\right)f^s + \left(\int_0^l g_1^s(y)dy\right)x - \int_0^x (x-y)g_1^s(y)dy\right], \\ f^s =& \frac{3}{l^3}\left[\int_0^l\int_0^x (x-y)g_1^s(y)dydx - \frac{l^2}{2}\int_0^l g_1^s(y)dy\right], \end{aligned} \tag{3.19}$$

where

$$g_1^s(x) = g^s(x) + \frac{3\nu}{l}\left(\frac{1}{l}q_1^s + \frac{\beta_2^{-1}}{2}q_2^s - \frac{1}{l^2}q_3^s\right), \tag{3.20}$$

and $g^s(x)$, q_j^s $(j = 1,2,3)$ are the given function and constants of the original problem. According to (3.1)

$$u^s(x) = v^s(x) + \frac{3}{2l}\left(\frac{1}{l}q_1^s + \frac{\beta_2^{-1}}{2}q_2^s - \frac{1}{l^2}q_3^s\right)x^2 + \frac{3}{l^2}q_3^s - \frac{3}{l^2}q_1^s - \frac{\beta_2^{-1}}{2}q_2^s. \tag{3.21}$$

If $V(x,t) = v(x,t) - v^s(x)$, $F(t) = f(t) - f^s$, $G_1(x,t) = g_1(x,t) - g_1^s(x)$, $x \in [0,l]$, $t \in [0,T]$, then for $V(x,t)$, $F(t)$ the estimates (3.11), (3.15), (3.17), (3.18). Suppose, that the functions $q_j(t)$ $(j = 1,2,3)$, $g(x,t)$ are defined for all $t \ge 0$, $x \in [0,l]$. Then

Theorem 2 *Under the condition of the integral convergences*

$$\int_0^\infty K(\tau)e^{2\nu\tau/l^2}d\tau, \quad \int_0^\infty K_1(\tau)e^{2\nu\tau/l^2}d\tau, \tag{3.22}$$

where

$$K(t) = \left(\int_0^l G_1^2(x,t)dx \right)^{1/2}, \quad K_1(t) = \left(\int_0^l G_{1t}^2(x,t)dx \right)^{1/2}, \tag{3.23}$$

the solution of the inverse problem (3.2 – 3.4) *tends to the stationary regime* (3.19), (3.20) *exponentially.*

Proof Note that from convergence of the first integral (3.22) and formula (3.23) for $K(t)$ follows, that $G_1(x,t) \sim e^{-2\nu t/l^2}$ for large t. And then (3.17) implies the exponential convergence of $f(t)$ to f^s. A similar convergence of $v(x, t)$ to $v^s(x)$ (or $u(x, t)$ to $u^s(x)$ from (3.21)) follows from estimate (3.11) for $V(x, t)$. □

Remark 8 Under the conditions of Theorem 2, there is an exponential stability of the stationary solution (3.21). In terms of the initial data, for the integrals (3.22) to be bounded, it is sufficient to require the convergence of the integrals.

$$\int_0^\infty \left[\int_0^l (g(x,\tau) - g^s(x))^2 dx \right] e^{2\nu\tau/l^2} d\tau; \quad \int_0^\infty \int_0^l g_\tau(x,\tau) dx e^{2\nu\tau/l^2} d\tau;$$
$$\int_0^\infty (q_j(\tau) - q_j^s)^2 e^{2\nu\tau/l^2} d\tau, \quad j = 1,2,3; \quad \int_0^\infty \left(q^{(n)}(\tau) \right)^2 e^{2\nu\tau/l^2} d\tau, \quad n = 0,1,2.$$

Remark 9 For a more detailed description of the behavior of the solution to the problem considered here, it is convenient to use the Laplace transform method. In this case, the conditions on the input data can be weakened (they can have discontinuities of the first kind in time [12]). In addition, under fairly general assumptions, one can prove the convergence for $t \to \infty$ of solution $u(x, t)$, $f(t)$ to stationary regime $u^s(x)$, f^s.

Let us prove the existence of a solution to the inverse problem by constructing it in the form of a series on a special basis. Integrating (3.2) over x from 0 to l, we find

$$f(t) = \frac{1}{l} \left[\nu v_x(0,t) - \int_0^l g_1(y,t) dy \right]. \tag{3.24}$$

Put $w(x,t) = v_x(x,t)$, then

$$w_t = \nu w_{xx} + g_{1x}(x,t), \quad x \in (0,l), \quad t \in [0,T], \tag{3.25}$$

$$w(x,0) = v_{0x}(x) \equiv w_0(x). \tag{3.26}$$

Since

$$v(x,t) = \int_0^x w(z,t)dz, \tag{3.27}$$

then, taking into account the redefinition (1.3), we have the boundary conditions

$$w(l,t) = 0, \quad \int_0^l (l-y)w(y,t)dy = 0. \tag{3.28}$$

The problem (3.25), (3.26), (3.28) is direct initial-boundary value problem with nonlocal integral condition.

We will consider the spectral problem

$$\begin{aligned} &W_{xx} + \lambda W = 0, \quad x \in (0,l); \\ &W(l) = 0, \quad \int_0^l (l-y)W(y)dy = 0. \end{aligned} \tag{3.29}$$

It has a countable number of orthonormal eigenfunctions

$$W_n(x) = \frac{1}{\mu_n}\sqrt{\frac{2}{l}\left(1+\mu_n^2\right)}\sin\left[\mu_n\left(1-\frac{x}{l}\right)\right], \quad n = 1, 2, ..., \tag{3.30}$$

at that $\lambda_n = \mu_n^2 l^{-2}$, and μ_n are positive roots of the equation $\operatorname{tg}\mu_n = \mu_n$, $\mu_n = \xi - \xi^{-1} - 2\xi^{-3}/3 + O\left(\xi^{-5}\right)$ at $\xi \gg 1$, $\xi = \pi(n+1/2)$ [13]. It is clear that the system of functions $W_n(x)$ is complete in $L_2(0,l)$.

The solution to problem (3.25), (3.26), (3.28) has the form

$$w(x,t) = \sum_{n=1}^{\infty} w_n(t)W_n(x), \tag{3.31}$$

$$w_n(t) = \left[w_{0n}e^{-\lambda_n \nu t} + \int_0^t e^{-\lambda_n \nu(t-\tau)}g_{1n}(\tau)d\tau\right], \tag{3.32}$$

$$w_{0n} = \int_0^l w_0(x)W_n(x)dx, \quad g_{1n}(t) = \int_0^l g_{1x}(x,t)W_n(x)dx. \tag{3.33}$$

It is easy to prove (see, for example, [13]), that series (3.31) is a classical solution of the problem (3.25), (3.26), (3.28) at $t \ge \varepsilon > 0$ and $w_0(x) \in C[0,l]$, $g(x,t) \in C^1((0,l) \times [0,T])$.

From (3.24) and (3.31) we find

$$f(t) = \frac{1}{l}\left[\nu \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} w_n(t) - \int_0^l g_1(y,t)dy \right], \tag{3.34}$$

because $\sin \mu_n = \mu_n \left(1 + \mu_n^2\right)^{-1/2}$. From (3.27), (3.31) the function v will have the form

$$v(x,t) = \sqrt{2l} \sum_{n=1}^{\infty} \frac{\sqrt{1+\mu_n^2}}{\mu_n^2} w_n(t) \left\{ \cos\left[\mu_n\left(1 - \frac{x}{l}\right)\right] - \cos \mu_n \right\}. \tag{3.35}$$

The required function $u(x, t)$ is determined from the replacement (3.1). Formulas (3.34), (3.35) give the unique classical solution of the inverse problem.

4 Properties of the solution of the initial-boundary value problem for the loaded equation

We seek the solution of the initial-boundary value problem for the loaded equation (3.2) in the form of the Fourier series ($f(t)$ is defined by the formula (3.24))

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} v_n(t) \sin(\xi_n x), \quad \xi_n = \frac{\pi(2n+1)}{2l}, \\ v_n(t) &= v_{0n} e^{-\nu \xi_n^2 t} + \int_0^t \left[\frac{2}{l\xi_n} f(\tau) + g_{1n}(\tau) \right] e^{-\nu \xi_n^2 (t-\tau)} d\tau, \end{aligned} \tag{4.1}$$

wherein $v(0,t) = 0$, $v_x(l,t) = 0$ and the integral redefinition condition (3.4) was used to derive equality (3.24). The quantities v_{0n}, $g_{1n}(t)$ are the coefficients of the Fourier series of the functions $v_0(x)$ and $g_1(x,t)$:

$$v_{0n} = \frac{2}{l} \int_0^l v_0(x) \sin(\xi_n x) dx, \quad g_{1n}(t) = \frac{2}{l} \int_0^l g_1(x,t) \sin(\xi_n x) dx.$$

Using (4.1), we rewrite equality (3.24) as

$$\begin{aligned} f(t) &= \frac{1}{l}\left[\nu \sum_{n=0}^{\infty} \xi_n v_n(t) - \int_0^l g_1(y,t)dy \right], \\ f(0) &= \frac{1}{l}\left[\nu \sum_{n=0}^{\infty} \xi_n v_{0n} - \int_0^l g_1(y,0)dy \right]. \end{aligned} \tag{4.2}$$

Substitution of $v_n(t)$ from (4.1) into (4.2) leads to an integral equation of the second kind (Volterra equation) on $f(t)$:

$$f(t) = \frac{2\nu}{l^2} \int_0^t f(\tau) \sum_{n=0}^{\infty} e^{-\nu \xi_n^2 (t-\tau)} d\tau + m(t), \tag{4.3}$$

where

$$m(t) = \frac{\nu}{l} \sum_{n=0}^{\infty} \xi_n \left(v_{0n} + \int_0^t g_{1n}(\tau) e^{\nu \xi_n^2 \tau} d\tau \right) e^{-\nu \xi_n^2 t} - \frac{1}{l} \int_0^l g_1(y,t) dy, \quad m(0) = f(0). \tag{4.4}$$

The kernel of the integral equation is the sum of the Dirichlet series

$$K(z) = \frac{2\nu}{l^2} \sum_{n=0}^{\infty} e^{-\nu \xi_n^2 z}, \quad z = t - \tau. \tag{4.5}$$

We have two representations of the function $f(t)$ are (3.34) and (4.2).

Lemma *The function $f(t)$, defined by equality (3.34), is a solution of the integral equation (4.4).*

Proof By virtue of linearity, we carry out the proof for $w_0(x) = A\sqrt{2l^{-1}} \sin\left[\mu_k(1 - x/l)\right]$, where $A = \text{const}$, k is fixed and $g_1(x,t) = 0$. In the general case, the proof is carried out by similar calculations taking into account the equality

$$\int_0^l g_1(y,t) dy = \frac{2}{l} \sum_{n=0}^{\infty} \frac{g_{1n}(t)}{\xi_n}.$$

Because $w_{0k} = A \sin \mu_k$, then from (3.34) $f(t) = \nu\sqrt{2l^{-3}} A \exp\left(-\nu l^{-2} \mu_k^2 t\right) \sin \mu_k$, and from (3.35) we get

$$v(x,t) = A\sqrt{2l^3} \mu_k^{-1} \exp\left(-\nu l^{-2} \mu_k^2 t\right) \left\{ \cos\left[\mu_k \left(1 - \frac{x}{l}\right)\right] - \cos \mu_k \right\}.$$

Hence, the coefficients of the Fourier series of the initial function $v_0(x) = A\sqrt{2l^3} \mu_k^{-1} \left\{ \cos\left[\mu_k\left(1 - \frac{x}{l}\right)\right] - - \cos \mu_k \right\}$ with respect to system $\sin\left(\xi_n x\right)$, $\xi_n = \pi(2n+1)/2l$, are as follows

$$v_{0n} = -\frac{2A\sqrt{2l} \sin \mu_k}{\xi_n \left(\mu_k^2 - \xi_n^2 l^2\right)}. \tag{4.6}$$

The function $m(t)$ from (4.4) with the help of formula (4.6) can be rewritten as

$$m(t) = -\frac{2\nu A\sqrt{2l}}{l^2} \sin \mu_k \sum_{n=0}^{\infty} \frac{e^{-\nu \xi_n^2 t}}{\mu_k^2 - \xi_n^2 l^2}. \tag{4.7}$$

Let us calculate the first term on the right-hand side of equation (4.3)

$$\frac{2\nu^2}{l^3}\sqrt{\frac{2}{l}}A\sin\mu_k\int_0^t e^{-\nu l^{-2}\mu_k^2\tau}\sum_{n=0}^{\infty}e^{-\nu\xi_n^2(t-\tau)} = \\ = \frac{2\nu}{l}\sqrt{\frac{2}{l}}A\sin\mu_k\left[e^{-\nu l^{-2}\mu_k^2 t}\sum_{n=0}^{\infty}\frac{1}{\xi_n^2 l^2-\mu_k^2}+\sum_{n=0}^{\infty}\frac{e^{-\nu\xi_n^2 t}}{\mu_k^2-\xi_n^2 l^2}\right]. \tag{4.8}$$

Since $\xi_n l = \pi(2n+1)/2$, then the sum of the first row in square brackets is ([14], p. 688)

$$\frac{1}{\pi\bar{\mu}_k}\,\mathrm{tg}\left(\frac{\pi\bar{\mu}_k}{2}\right), \quad \bar{\mu}_k = \frac{2}{\pi}\mu_k.$$

This sum is equal to 1/2, because $\mathrm{tg}\mu_k = \mu_k$. Hence, the right-hand side of (4.8), taking into account (4.7), is $f(t) - m(t)$, which proves the lemma.

Thus, the function $f(t)$ from (3.34) is an exact solution to the Volterra equation (4.3). According to the reasoning in item 3, it is the only solution. □

5 The solution of the inverse problem in Laplace images

To obtain quantitative information about the behavior of solving linear problems, the Laplace transform method is often used. If $z(t)$ is original, $t \in [0, \infty)$, then its Laplace transform $\hat{z}(p)$ (image) is the integral

$$\hat{z}(p) = \int_0^{\infty} z(t)e^{-pt}dt. \tag{5.1}$$

Identification and properties of the Laplace transformation set out in many textbooks, for example in [15]. It is applicable for a wide class of functions $z(t)$, in particular, having a finite number of discontinuity points of the 1st kind. In specific tasks, the original may depend on other variables and parameters.

Application of the Laplace transform to problem (3.2 – 3.4) leads to a boundary value problem for $\hat{v}(x,p)$, $\hat{f}(p)$:

$$\hat{v}_{xx} - \frac{p}{\nu}\hat{v} = -\frac{1}{\nu}\left[\hat{f}(p) + \hat{g}_1(x,p) + v_0(x)\right], \quad 0 < x < l;$$

$$\hat{v}(0,p) = \hat{v}_x(l,p) = 0, \quad \int_0^l \hat{v}(y,p)dy = 0.$$

The last problem has a solution in quadratures:

$$\hat{v}(x,p) = \frac{1}{\sqrt{\nu p}}\left\{ \sqrt{\frac{\nu}{p}}\hat{f}(p)\left[\mathrm{th}\sqrt{\frac{p}{\nu}}\,l\,\mathrm{sh}\sqrt{\frac{p}{\nu}}\,x + 1 - \mathrm{ch}\sqrt{\frac{p}{\nu}}\,x\right] \right.$$
$$+ \frac{1}{\mathrm{ch}\sqrt{\frac{p}{\nu}}\,l}\int_0^l G(y,p)\mathrm{ch}\left[\sqrt{\frac{p}{\nu}}\,(l-y)\right]dy\,\mathrm{sh}\sqrt{\frac{p}{\nu}}\,x \tag{5.2}$$
$$\left. - \int_0^x G(y,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{\nu}}\,(x-y)\right]dy\right\}, \quad G(x,p) = \hat{g}_1(x,p) + v_0(x);$$

$$\hat{f}(s) = \frac{1}{l - \sqrt{\frac{\nu}{p}}\,\mathrm{th}\sqrt{\frac{p}{\nu}}\,l}\left\{\sqrt{\frac{p}{\nu}}\int_0^l\left(\int_0^x G(y,p)\,\mathrm{sh}\left[\sqrt{\frac{p}{\nu}}\,(x-y)\right]dy\right)dx\right.$$
$$\left. + \frac{1 - \mathrm{ch}\sqrt{\frac{p}{\nu}}\,l}{\mathrm{ch}\sqrt{\frac{p}{\nu}}\,l}\int_0^l G(y,p)\mathrm{ch}\left[\sqrt{\frac{p}{\nu}}\,(l-y)\right]dy\right\} \tag{5.3}$$

Let $\lim_{t\to\infty} g_1(x,t) = g_1^s(x)$ uniformly for all $x \in (0,l)$ and there exist $\hat{g}_1(x,p)$, $\hat{g}_{1t}(x,p)$. Then ([15]) $\lim_{p\to 0} p\,\hat{g}_1(x,p) = g_1^s(x)$. Using asymptotic formulas for hyperbolic functions (here it is enough to take into account that for $y \to 0$: $\mathrm{sh}\,y = y + O(y^3)$, $\mathrm{ch}\,y = 1 + y^2/2 + O(y^4)$, $\mathrm{th}\,y = y - y^3/3 + O(y^5)$), it is easy to derive from (5.2), (5.3) the equalities

$$\lim_{p\to 0} p\,\hat{v}(x,p) = v^s(x), \quad \lim_{p\to 0} p\hat{f}(p) = f^s,$$

where $v^s(x)$, f^s is defined by formulas (3.19). In other words, as $t \to \infty$ the non-stationary solution tends to the stationary one under less restrictive conditions on the original functions $g(x,\ t)$, $q_j(t)$, $j = 1,2,3$, $u_0(x)$. More precise information on the behavior of $v(x,\ t)$, $f(t)$ for finite values of t can be obtained only by numerically inverting the Laplace transform (5.1), where $\hat{z} = \hat{v}$, ($\hat{z} = \hat{f}$).

Equation (4.3) can also be solved by the Laplace transform method, extending the function $g_1(x,t)$ by zero outside the segment [0, T] and assuming that for $t = T$ it has a discontinuity of the first kind. Namely [16],

$$\hat{f}(p) = \hat{m}(p) + \frac{\hat{K}(p)\hat{m}(p)}{1 - \hat{K}(p)}, \tag{5.4}$$

where $\hat{K}(p)$ is a kernel image (4.5). So,

$$f(t) = m(t) + \int_0^t K(t-\tau)m(\tau)d\tau, \tag{5.5}$$

and function $K(t)$ is original of $\hat{K}(p)\left[1-\hat{K}(p)\right]^{-1}$. In our case, from (4.5), as $\widehat{\exp(-\nu\xi_n^2 t)} = \left(p+\nu\xi_n^2\right)^{-1}$, we find

$$\frac{\hat{K}(p)}{1-\hat{K}(p)} = \frac{2\nu}{l^2}\sum_{n=0}^{\infty}\frac{1}{p+\nu\xi_n^2}\left[1-\frac{2\nu}{l^2}\sum_{n=0}^{\infty}\frac{1}{p+\nu\xi_n^2}\right]^{-1} = \hat{K}(p).$$

Since [14]

$$\frac{2\nu}{l^2}\sum_{n=0}^{\infty}\frac{1}{p+\nu\xi_n^2} = \frac{1}{l}\sqrt{\frac{\nu}{p}}\,\mathrm{th}\left(\sqrt{\frac{p}{\nu}}\,l\right),$$

we get

$$\hat{K}(p) = \frac{\mathrm{th}\left(\sqrt{\frac{p}{\nu}}\,l\right)}{\sqrt{\frac{p}{\nu}}\,l - \mathrm{th}\left(\sqrt{\frac{p}{\nu}}\,l\right)}. \tag{5.6}$$

Formula (5.6) can be useful for numerically finding the original $f(t)$ from (5.4). In addition, using (4.4), (5.6), it is easy to show the coincidence of expressions (3.34) and (5.5) for $f(t)$.

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