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Critical fractional elliptic equations with exponential growth

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Abstract
In this paper we establish, using variational methods combined with the Moser–Trudinger inequality, existence and multiplicity of weak solutions for a class of critical fractional elliptic equations with exponential growth without a Ambrosetti–Rabinowitz-type condition. The interaction of the nonlinearities with the spectrum of the fractional operator will be used to study the existence and multiplicity of solutions. The main technical result proves that a local minimum in $C^0_s(\overline{\Omega})$ is also a local minimum in $W_0^{s,p}$ for exponentially growing nonlinearities.

Keywords Topological methods in PDEs · Variational methods · Fractional p-Laplacian · Critical and subcritical exponential growth in Trudinger–Moser sense

Mathematics Subject Classification 35A16 · 35A15 · 35R11 · 35B33

1 Introduction

In this paper we consider existence and multiplicity of solutions to the Dirichlet problem

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$$\begin{cases} (-\Delta)^s_p u = -\lambda |u|^{q-2}u + a|u|^{p-2}u + f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.1}$$

where $(-\Delta)^s_p$ is the fractional p-Laplacian, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda > 0$ and $a \in \mathbb{R}$ are parameters, $N = sp$, and $0 < s < 1 < q < 2 \leq p$. Here

$$(-\Delta)^s_p u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^N \backslash B(x,\epsilon)} \frac{-|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} \mathrm{d}y,$$

where u is a measurable function and $x \in \mathbb{R}^N$.

We suppose that the nonlinearity f has exponential growth, both critical and subcritical in the Trudinger–Moser sense.

Recently, non-local problems have been extensively studied in the literature and have attracted the attention of many mathematicians from different fields of research. They appear in the description of various phenomena in the applied sciences, such as optimization, finance, phase transitions, material science and water waves, image processing, etc. See the excellent book by Caffarelli on this subject [11], but also an elementary introduction to this topic by Di Nezza et al. [22].

In 1994, Ambrosetti et al. [2] established existence and multiplicity of solution for a local problem involving concave-convex nonlinearities and Sobolev critical exponent, namely, $2^* = \frac{2N}{N-2} (N \geq 3)$. This work caused a growing interest in the study of multiplicity of solutions for local problems of the type

$$-\Delta u = \mu |u|^{q-2}u + g(u) \quad \text{in} \quad \Omega,$$

when g is asymptotically linear and asymmetric, that is, g satisfies the Ambrosetti–Prodi-type condition given by (see [18]) $g_- = \lim\limits_{t \to -\infty} \dfrac{g(t)}{t} < \lambda_k < g_+ = \lim\limits_{t \to +\infty} \dfrac{g(t)}{t}$, where $\{\lambda_k\}_{k \geq 1}$ denotes the sequence of eigenvalues of $(-\Delta)$ considered in $H^1_0(\Omega)$. In Chabrowsky and Yang [12] a problem with Neumann boundary condition was considered, while in Motreanu et al. [38] a problem involving a local p-Laplacian was considered. In [20], de Paiva and Massa studied the local problem

$$\begin{cases} -\Delta u = -\lambda |u|^{q-2}u + au + g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $1 < q < 2$, $\lambda > 0$, $a \in [\lambda_k, \lambda_{k+1})$, and the nonlinearity g satisfying subcritical polynomial growth at infinity, among other conditions. The critical case was considered in de Paiva and Presoto [21], where three solutions for problem (1.2) were obtained: a positive, a negative and a sign-changing solution. The problem (1.2) with critical polynomial growth was handled by Miyagaki et al. [37] for the fractional Laplacian operator. To complete our references, we would like to cite some papers. For instance, [2, 3, 14, 42] for concave problems, [4, 6, 7, 9] for problems involving the fractional Laplacian and, for the fractional p-Laplacian, we cite [8, 13, 24, 35, 39]. See also references therein.

With respect to nonlinearities with exponential growth for a problem like (1.1), in the limit case $N = sp$, Bahrouni [5] proved a version of the Trudinger–Moser inequality for fractional spaces, which was improved by Takahashi [45], who obtained, among other things, optimality of the upper bound. With respect to local elliptic problems with exponential growth nonlinearity we would like to cite, e.g., [16, 17, 19, 32] and references therein.

The pioneering paper for fractional Laplacian, by Iannizzotto and Squassina [26], considered a nonlinearity with exponential growth, but it was proved by de Figueiredo et al. [16, 17, p.142] that the Ambrosetti–Rabinowitz (AR) condition was satisfied in [26]. Namely, the (AR) condition is fulfilled if there exist $\mu > p$ and $R > 0$ such that

$$0 < \mu F(t) \leq f(t)t, \quad \text{for all } |t| \geq R, \quad \text{where } F(t) = \int_0^t f(s)ds \tag{AR}$$

and in this situation,

$$\lim_{|t|\to+\infty} \frac{F(t)}{|t|^p} = +\infty$$

follows immediately from (AR). The main role of (AR) is to guarantee that Palais–Smale sequences are bounded. Many authors have been working to drop this condition in problems with polynomial growth, e.g., [15, 28, 31, 33, 34, 44] and references therein. With respect to exponential growth without the (AR) condition we cite, for instance, [29, 30]. Recently, Pei [40] proved a existence result for a superlinear p-fractional problem with exponential growth.

Motivated by [40] and [21], in this work we obtain results of existence and multiplicity of solutions for (1.1).

We look for solutions to (1.1) in the uniformly convex Sobolev space

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y < \infty \right\}.$$

Because solutions must be equal 0 outside Ω, it is natural to consider the space

$$X_p^s = \left\{ u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ on } \mathbb{R}^N \setminus \Omega \right\}.$$

Since $\Omega \subset \mathbb{R}^N$ is a bounded, smooth domain and $0 < s < 1 < p$, this space can be considered with the Gagliardo norm (see [27, p.4]) defined by

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y \right)^{1/p},$$

which will be denoted by $\|\cdot\|_{X_p^s}$. Also, consider $A : X_p^s \to (X_p^s)^*$ defined, for all $u, v \in X_p^s$, by

$$\langle A(u), v \rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y. \tag{1.3}$$

Finally, denote by $\varphi_1 > 0$, the (L^p-normalized) autofunction associated with the first eigenvalue

$$\lambda_1 = \inf\left\{ [u]^p_{W^{s,p}(\mathbb{R}^N)} \ : \ u \in X^s_p, \ \|u\|_{L^p(\Omega)} = 1 \right\}$$

of $(-\Delta)^s_p$ in the space X^s_p.

To cope with nonlinearities involving exponential growth, the main tool is the so called "Moser–Trudinger inequality". We will make use of the following version of this inequality, based on [5, Lema 2.5].

Proposition 1.1 *Suppose that* $0 < s < 1$, $p \geq 2$ *and* $N = sp$. *Then there exists* $\alpha^*_{s,N} = \alpha(s,N)$ *such that, for all* $0 \leq \alpha < \alpha^*_{s,N}$,

$$\int_\Omega \exp\left(\alpha |u|^{\frac{N}{N-s}}\right) \mathrm{d}x \leq H_\alpha,$$

for all $u \in X^s_p$ *such that* $\|u\|_{X^s_p} \leq 1$, *where* $H_\alpha > 0$ *is a constant.*

An adequate version of Proposition 1.1 in the special case $p = 2$, $s = 1/2$ and $N = 1$ is given in [45, Theorem 1] and [36, Proposition 1.1].

Considering (1.1) in the case of subcritical exponential growth in the Trudinger–Moser sense, we suppose that f satisfies

($f_{1,p}$) $f \in C(\mathbb{R}, \mathbb{R})$, $f(0) = 0$ and $F(t) \geq 0$ for all $t \in \mathbb{R}$, where $F(t) = \displaystyle\int_0^t f(s)\mathrm{d}s$;

($f_{2,p}$) $\displaystyle\lim_{|t|\to\infty} \frac{|f(t)|}{\exp(\alpha |t|^{\frac{N}{N-s}})} = 0$, for all $\alpha > 0$;

($f_{3,p}$) $\displaystyle\lim_{|t|\to 0} \frac{f(t)}{|t|^{p-2}t} = 0$;

($f_{4,p}$) $\displaystyle\lim_{|t|\to\infty} \frac{F(t)}{|t|^p} = +\infty$.

In the case of a critical exponential growth, we change ($f_{2,p}$) for

($f'_{2,p}$) there exists $\alpha_0 > 0$ such that

$$\lim_{|t|\to\infty} \frac{|f(t)|}{\exp(\alpha |t|^{\frac{N}{N-s}})} = \begin{cases} \infty, & \text{if} \quad 0 < \alpha < \alpha_0 \\ 0, & \text{if} \quad \alpha > \alpha_0. \end{cases}$$

Keeping up with the conditions ($f_{1,p}$) and ($f_{3,p}$), we suppose additionally that f satisfies

($f_{5,p}$) $\dfrac{f(t)}{|t|^{p-2}t}$ is increasing if $t > 0$, and decreasing if $t < 0$;

($f_{6,p}$) For all sequence $(u_n) \subset X^s_p$, if

$$u_n \rightharpoonup u, \text{in} \quad X_p^s,\ f(u_n) \to f(u), \text{in} \quad L^1(\Omega),$$

then $F(u_n) \to F(u)$ in $L^1(\Omega)$;

$(f_{7,p})$ There exist $r > p$ and $C_r > 0$ such that $F(t) \geq \frac{C_r}{r}|t|^r$, for all $t \in \mathbb{R}$, verifying $C_r > \left[2\frac{N}{s}\left(\frac{\alpha_0}{\alpha^*_{s,N}}\right)^{\frac{N-s}{s}}\frac{(r-p)}{pr}\right]^{\frac{r-p}{p}}\frac{1}{C}$, with $C = \inf_{u\in\mathbb{F}}\frac{\|u\|_{L^r}}{\|u\|_{X_p^s}}$, where $\alpha^*_{s,N}$ is the constant given in Proposition 1.1 and $\mathbb{F} = \operatorname{span}\{\varphi_1, \varphi\}$ for $\varphi \in W$.

Remark 1.2 Condition $(f_{6,p})$ was supposed by [29, 30] and [40] in the case $u = 0$. Observe that $(f_{7,p})$ implies $(f_{4,p})$.

Hypotheses $(f_{1,p})$–$(f_{4,p})$ are satisfied by $f(t) = |t|^{p-2}t\log(1 + |t|)$, a function that does not verify the (AR) condition.

On its turn, considering $0 < \sigma < 1$, the function

$$f(t) = \begin{cases} \sigma t^{r-1} + C_r t^{r-1}, & \text{if } 0 \leq t \leq (p-1)^{\frac{N-s}{N}}, \\ t^{\frac{N}{N-s}}\exp\left(t^{\frac{N}{N-s}} - (p-1)\right) + C_r t^{r-1} & \\ +\sigma(p-1)^{\frac{N-s}{N}(r-1)} - (p-1)^{\frac{s}{N}}, & \text{if } t > (p-1)^{\frac{N-s}{N}} \end{cases}$$

satisfies our hypotheses in the critical growth case, if $f(t) = -f(-t)$, for $t < 0$.

Our main result is the following. It will play an essential role in the sequence.

Theorem 1 *Let* $\Phi : X_p^s \to \mathbb{R}$ *be the* $C^1(X_p^s, \mathbb{R})$ *functional defined by*

$$\Phi(u) = \frac{1}{p}\|u\|^p_{X_p^s} - \int_\Omega G(u)\mathrm{d}x,$$

where $G(t) = \int_0^t g(s)\mathrm{d}s$.

Let us suppose that g *satisfies* $(f_{2,p})$ *or* $(f'_{2,p})$ *and that* 0 *is a local minimum of* Φ *in* $C_s^0(\overline{\Omega})$, *that is, there exists* $r_1 > 0$ *such that*

$$\Phi(0) \leq \Phi(z),\ \forall\, z \in X_p^s \cap C_s^0(\overline{\Omega}),\ \|z\|_{0,s} \leq r_1. \tag{1.4}$$

Then 0 *is a local minimum of* Φ *in* X_p^s, *that is, there exists* $r_2 > 0$ *such that*

$$\Phi(0) \leq \Phi(z),\ \forall\, z \in X_p^s,\ \|z\|_{X_p^s} \leq r_2.$$

(See definition of $C_s^0(\overline{\Omega})$ in Sect. 3.) Theorem 1 will play an essential role to obtain the next results.

In order to obtain the geometric conditions of the Linking Theorem, we define

$$\lambda^* = \inf\left\{ \|u\|^p_{X^s_p} \ : \ u \in W, \ \|u\|^p_{L^p(\Omega)} = 1 \right\},$$

where

$$W = \left\{ u \in X^s_p \ : \ \langle A(\varphi_1), u\rangle = 0 \right\}.$$

Theorem 2 (subcritical case) *If $\lambda_1 \le a < \lambda^*$ and if f satisfies conditions $(f_{1,p}) - (f_{4,p})$ then, for λ small enough, problem (1.1) has at least three nontrivial solutions. Additionally, if f is odd, then (1.1) has infinitely many solutions.*

Theorem 3 (Critical case) *If f satisfies conditions $(f_{1,p})$, $(f'_{2,p})$, $(f_{3,p})$ and $(f_{5,p}) - (f_{7,p})$ then, for λ small enough, problem (1.1) has at least three nontrivial solutions in the case $\lambda_1 \le a < \lambda^*$.*

Remark 1.3 Analogous results are valid in the particular case $N = 1$, $p = 2$, $s = 1/2$ and $\Omega = (0, 1)$. Considering the eigenvalue sequence $\{\lambda_j\}_{j\ge 1}$ of $(-\Delta)^{1/2}$ in $X^{1/2}_2$, Theorems 2 and 3 are valid for any $\lambda_k \le a < \lambda_{k+1}$, if $\lambda > 0$ is small enough.

The main achievement of this paper is the minimization result that will be presented in Sect. 3 (see notation there): we prove that a local minimum in $C^0_s(\overline{\Omega})$ is also a local minimum in $W^{s,p}_0$ for nonlinearities with exponential growth. They are the counterpart of the result obtained by de Paiva and Massa [20] (also de Paiva and Presoto [21]) and their proofs are obtained by applying ideas developed by Barrios et al. [6], Giacomoni, Prashanth and Sreenadh [23] and Iannizzoto, Mosconi and Squassina [25]. We would like to emphasize that with exception of [23], which deals with local N-Laplacian case with exponential growth, other references treated local or non-local Laplacian with polynomial growth.

2 Preliminaries

Definition 2.1 We say that $u \in X^s_p$ is a weak solution to (1.1) if

$$\langle A(u), v\rangle = -\lambda \int_\Omega |u|^{q-2}uv\mathrm{d}x + a\int_\Omega |u|^{p-2}uv\mathrm{d}x + \int_\Omega f(u)v\mathrm{d}x,$$

for all $v \in X^s_p$, with $A : X^s_p \to (X^s_p)^*$ being defined by (1.3).

We recall that X^s_p is compactly immersed in $L^r(\Omega)$ for all $1 \le r < \infty$, the immersion being continuous in the case $r = \infty$ (see [22, Teorema 6.5, 7.1]).

We define the functional $I_{\lambda,p} : X^s_p \to \mathbb{R}$ by

$$I_{\lambda,p}(u) = \frac{1}{p}\|u\|^p_{X^s_p} + \frac{\lambda}{q}\int_\Omega |u|^q\mathrm{d}x - \frac{a}{p}\int_\Omega |u|^p\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x.$$

The next result is a direct consequence of [41, Proposição 1.3.].

Lemma 2.2 *If $u_n \rightharpoonup u$ in X_p^s and $\langle A(u_n), u_n - u\rangle \to 0$, then $u_n \to u$ in X_p^s.*

Let us consider the Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded, smooth domain, $s \in (0, 1)$, $p > 1$ and $f \in L^\infty(\Omega)$.

The next two results can be found in Iannizzotto et al. [25], Theorems 1.1 and 4.4, respectively.

Proposition 2.3 *There exist $\alpha \in (0, s]$ and $C_\Omega > 0$ depending only on N, p, s, with C_Ω also depending on Ω, such that, for all weak solution $u \in X_p^s$ of (2.1), $u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \leq C_\Omega \|f\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}.$$

Proposition 2.4 *Let $u \in X_p^s$ satisfies $\left|(-\Delta)_p^s u\right| \leq K$ weakly in Ω for some $K > 0$. Then*

$$|u| \leq \left(C_\Omega K\right)^{\frac{1}{p-1}} \delta^s \quad a.e. \text{ in } \Omega,$$

for some $C_\Omega = C(N, p, s, \Omega)$.

By adapting arguments of Zhang and Shen [46, Lemma 2] we obtain the following result.

Lemma 2.5 (Critical and subcritical cases) *If f satisfies $(f_{1,p})$, $(f_{2,p})$ (or $(f_{2,p}')$) and $(f_{4,p})$, then any (PS)-sequence for $I_{\lambda,p}$ is bounded.*

In order to obtain a positive solution for problem (1.1), we define

$$I_{\lambda,p}^\pm : X_p^s \to \mathbb{R}$$
$$I_{\lambda,p}^\pm(u) = \frac{1}{p}\|u\|^p + \frac{\lambda}{q}\int_\Omega |u^\pm|^q \mathrm{d}x - \frac{a}{p}\int_\Omega |u^\pm|^p \mathrm{d}x - \int_\Omega F(u^\pm)\mathrm{d}x.$$

We have that $I_{\lambda,p}^\pm \in C^1(X_p^s, \mathbb{R})$ and

$$\langle (I_{\lambda,p}^\pm)'(u), h\rangle = \langle A(u), h\rangle + \lambda \int_\Omega |u^\pm|^{q-1} h\mathrm{d}x - a\int_\Omega |u^\pm|^{p-1} h\mathrm{d}x - \int_\Omega f(u^\pm) h\mathrm{d}x$$

for all $u, h \in X_p^s$. Observe that a critical point for $I_{\lambda,p}^\pm$ is a weak solution to the problem

$$\begin{cases} (-\Delta)_p^s u = -\lambda|u^\pm|^{q-1} + a|u^\pm|^{p-1} + f(u^\pm) & \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R} \setminus \Omega, \end{cases}$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$. It is not difficult to see that a critical point of $I^+_{\lambda,p}$ is a non-negative function.

3 Proof of Theorem 1

We start showing a regularization result that will be useful in the proof of our main result.

Lemma 3.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded, smooth domain and f a function satisfying $(f_{2,p})$ or $(f'_{2,p})$. Let $(v_\epsilon)_{\epsilon\in(0,1)} \subseteq X^s_p$ be a family of solution to the problem*

$$\begin{cases} (-\Delta)^s_p u = \left(\dfrac{1}{1-\xi_\epsilon}\right) f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\xi_\epsilon \le 0$ and $\|v_\epsilon\|_{X^s_p} \le 1$, for all $\epsilon \in (0,1)$. Then

$$\sup_{\epsilon\in(0,1)} \|v_\epsilon\|_{L^\infty(\Omega)} < \infty.$$

Proof We define, for $0 < k \in \mathbb{N}$,

$$T_k(s) = \begin{cases} s+k, & \text{if } \quad s \le -k, \\ 0, & \text{if } \quad -k < s < k, \\ s-k, & \text{if } \quad s \ge k \end{cases}$$

and

$$\Omega_k = \{x \in \Omega \ : \ |v_\epsilon(x)| \ge k\}.$$

Observe that $T_k(v_\epsilon) \in X^s_p$ and $\|T_k(v_\epsilon)\|^p_{X^s_p} \le C^p \|v_\epsilon\|^p_{X^s_p} < \infty$ for a constant $C > 0$. Taking $T_k(v_\epsilon)$ as a test-function, we obtain

$$\langle A(v_\epsilon), T_k(v_\epsilon)\rangle \le \int_\Omega |f(v_\epsilon)||T_k(v_\epsilon)|\mathrm{d}x.$$

We claim that

$$\langle A(v_\epsilon), T_k(v_\epsilon)\rangle_{X^s_p} \le C\left(\int_\Omega |T_k(v_\epsilon)|^r \mathrm{d}x\right)^{1/r} |\Omega_k|^{p/r}. \tag{3.1}$$

In fact, suppose that f satisfies $(f_{2,p})$. Then, for all $t \in \mathbb{R}$ and $\alpha > 0$ we have

$$|f(t)| \le C \exp(\alpha |t|^{\frac{N}{N-s}}) \in L^1(\Omega), \tag{3.2}$$

where $C > 0$ is a constant. If $0 < \alpha < \alpha^*_{s,N}$ (see Proposition 1.1), we can fix $\theta > 1$ so that $0 < \theta\alpha < \alpha^*_{s,N}$. Applying the (generalized) Hölder inequality and recalling $\|v_\epsilon\|_{X^s_p} \le 1$, it follows from Proposition 1.1 the proof of our claim. The proof in the case that f satisfies $(f'_{2,p})$ is analogous.

Denote

$$T(x,y) = \frac{|v_\epsilon(x) - v_\epsilon(y)|^{p-2}(v_\epsilon(x) - v_\epsilon(y))(T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y))}{|x-y|^{N+sp}}.$$

Noting that the following inequality holds

$$|s-t|^{p-2}(s-t)(T_k(s) - T_k(t)) \ge |T_k(s) - T_k(t)|^p, \quad \text{for all} \quad s,t \in \mathbb{R},$$

since both $T_k(s)$ and $s - T_k(s)$ are non decreasing functions, we obtain

$$T(x,y) \ge \frac{|T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y)|^p}{|x-y|^{N+sp}}.$$

Therefore, we have the estimate

$$A(v_\epsilon) \cdot T_k(v_\epsilon) \ge \int\limits_{\mathbb{R}^{2N}} \frac{|T_k(v_\epsilon)(x) - T_k(v_\epsilon)(y)|^p}{|x-y|^{N+sp}} \mathrm{d}x\mathrm{d}y = \|T_k(v_\epsilon)\|^p_{X^s_p}.$$

The continuous immersion $X^s_p \hookrightarrow L^r(\Omega)$ yields (for a constant $C_1 > 0$)

$$C_1 \left(\int_\Omega |T_k(v_\epsilon)|^r \mathrm{d}x \right)^{p/r} \le \langle A(v_\epsilon), T_k(v_\epsilon) \rangle. \tag{3.3}$$

Thus, it follows from (3.1) and (3.3) the existence of $C > 0$ such that

$$\int_\Omega |T_k(v_\epsilon)|^r \mathrm{d}x \le C|\Omega_k|^{p/(p-1)}.$$

Since, for all $s \in \mathbb{R}$, we have $|T_k(s)| = (|s| - k)(1 - \chi_{[-k,k]}(s))$, we conclude that, if $0 < k < h \in \mathbb{N}$, then $\Omega_h \subset \Omega_k$. Thus,

$$\int_\Omega |T_k(v_\epsilon)|^r \mathrm{d}x = \int_{\Omega_k} (|v_\epsilon| - k)^r \ge \int_{\Omega_h} (|v_\epsilon| - k)^r \ge (h-k)^r |\Omega_h|.$$

Defining, for $0 < k \in \mathbb{N}$,

$$\phi(k) = |\Omega_k|,$$

we obtain

$$\phi(h) \le C(h-k)^{-r} \phi(k)^{p/(p-1)}, \quad 0 < k < h \in \mathbb{N}.$$

Considering the sequence (k_n) defined by $k_0 = 0$ and $k_n = k_{n-1} + d/2^n$, where $d = 2^p C^{1/r}|\Omega|^{1/(p-1)r}$, we have $0 \le \phi(k_n) \le \phi(0)/(2^{nr(p-1)})$ for all $n \in \mathbb{N}$. Thus $\lim_{n\to\infty}\phi(k_n) = 0$.

Since $\phi(k_n) \ge \phi(d)$ implies $\phi(d) = 0$, we have $|v_\epsilon(x)| \le d$ a.e. in Ω, for all $\epsilon \in (0,1)$. We are done. □

We recall the definitions of the spaces $C^0_\delta(\overline{\Omega})$ and $C^{0,\alpha}_\delta(\overline{\Omega})$. For this, we define $\delta : \overline{\Omega} \to \mathbb{R}^+$ by $\delta(x) = \mathrm{dist}(x, \mathbb{R}^N \setminus \Omega)$. Then, if $0 < \alpha < 1$,

$$C^0_s(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) \ : \ \frac{u}{\delta^s} \ \text{has a continuous extension to } \overline{\Omega} \right\}$$
$$C^{0,\alpha}_s(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) \ : \ \frac{u}{\delta^s} \ \text{has a } \alpha - \text{Hölder extension to } \overline{\Omega} \right\}$$

with the respective norms

$$\|u\|_{0,\delta} = \left\| \frac{u}{\delta^s} \right\|_{L^\infty(\Omega)} \text{ and } \|u\|_{\alpha,\delta} = \|u\|_{0,\delta} + \sup_{x,y\in\overline{\Omega},\, x\neq y.} \frac{|u(x)/\delta(x)^s - u(y)/\delta(y)^s|}{|x-y|^\alpha}.$$

Proof of Theorem 1 For $0 < \epsilon < 1$, let us denote $B_\epsilon = \{z \in X^s_p \ : \ \|z\|_{X^s_p} \le \epsilon\}$. By contradiction, suppose that for each $\epsilon > 0$, there exists $u_\epsilon \in B_\epsilon$ such that

$$\Phi(u_\epsilon) < \Phi(0). \tag{3.4}$$

It is not difficult to verify that $\Phi : B_\epsilon \to \mathbb{R}$ is weakly lower semicontinuous. Therefore, there exists $v_\epsilon \in B_\epsilon$ such that $\inf_{u\in B_\epsilon} \Phi(u) = \Phi(v_\epsilon)$. It follows from (3.4) that

$$\Phi(v_\epsilon) = \inf_{u\in B_\epsilon} \Phi(u) \le \Phi(u_\epsilon) < \Phi(0).$$

We will show that

$$v_\epsilon \to 0 \text{ in } C^0_s(\overline{\Omega}) \text{ as } \quad \epsilon \to 0,$$

since this implies that, for $r_1 > 0$, the existence of $z \in C^0_s(\overline{\Omega})$, such that $\|z\|_{0,s} < r_1$ and $\Phi(z) < \Phi(0)$, contradicting (1.4).

Since v_ϵ is a critical point of Φ in X^s_p, by Lagrange multipliers we obtain the existence of $\xi_\epsilon \le 0$ such that $\langle \Phi'(v_\epsilon), \phi\rangle = \xi_\epsilon \langle v_\epsilon, \phi\rangle$, for all $\phi \in X^s_p$.

Thus, v_ϵ satisfies

$$\begin{aligned} (-\Delta)^s_p v_\epsilon &= \left(\frac{1}{1-\xi_\epsilon}\right) g(v_\epsilon) =: g^\epsilon(v_\epsilon) \qquad \text{in } \Omega, \\ v_\epsilon &= 0 \qquad \text{in } \mathbb{R}\backslash\Omega, \end{aligned}$$

If $\|v_\epsilon\|_{X^s_p} \le \epsilon < 1$, Proposition 3.1 show the existence of a constant $C_1 > 0$, not depending on ϵ, such that

$$\|v_\epsilon\|_{L^\infty(\Omega)} \le C_1. \tag{3.5}$$

Since $\xi_\epsilon \leq 0$, (3.2) and (3.5) show that $\|g^\epsilon(v_\epsilon)\|_{L^\infty(0,1)} \leq C_2$ for some constant $C_2 > 0$. Theorem 2.3 then yields $\|v_\epsilon\|_{C^{0,\beta}(\overline{\Omega})} \leq C_3$, for $0 < \beta \leq s$ and a constant C_3 not depending on ϵ.

It follows from Arzelà–Ascoli theorem the existence of a *sequence* (v_ϵ) such that $v_\epsilon \to 0$ uniformly as $\epsilon \to 0$. Passing to a subsequence, we can suppose that $v_\epsilon \to 0$ a. e. in Ω and, therefore, $v_\epsilon \to 0$, uniformly in $\overline{\Omega}$. But now follows from Proposition 2.4 that

$$\|v_\epsilon\|_{0,\delta} = \left\| \frac{v_\epsilon}{\delta^s} \right\|_{L^\infty(\Omega)} \leq C \sup_{x\in(0,1)} |g^\epsilon(v_\epsilon(x))|$$

for a constant $C > 0$. We are done. □

Remark 3.2 Observe that, if 0 a strict local minimum in $C^0_\delta(\overline{\Omega})$, then 0 is also a strict local minimum in X^s_p.

4 Proof of Theorem 2

In this section we deal with existence and multiplicity of solutions to the problem (1.1) when f has subcritical growth.

The proof of Theorem 2 will be given in 3 subsections. In the first subsection, we will obtain a positive solution by applying the Mountain Pass Theorem. Analogously, in the second subsection we will obtain a negative solution. In the last subsection, a third solution will be obtained by the Linking Theorem and we conclude the proof of Theorem 2.

4.1 Positive solution for the functional $I_{\lambda,p}$

Lemma 4.1 *Suppose that f satisfies $(f_{1,p})$, $(f_{2,p})$, $(f_{3,p})$ and $(f_{4,p})$. Then, for any $\lambda > 0$, the functional $I^+_{\lambda,p}$ satisfies the (PS) condition at any level.*

The same result is valid for the functional $I_{\lambda,p}$.

Proof Let $(u_n) \subset X^s_p$ be a (PS)-sequence for $I^+_{\lambda,p}$. By arguments similar to that used in the proof of Lemma 2.5, there exists $u_0 \in X^s_p$ such that $u_n \rightharpoonup u_0$ in X^s_p. We can also suppose that

$$u_n \to u_0 \quad \text{in } L^r(\Omega) \quad \text{for } r \geq 1 \text{ and } u_n(x) \to u_0(x) \text{ a.e. in } \Omega.$$

if $1 < q < 2 \leq p$, by applying Hölder's inequality we obtain

$$\int_\Omega |u_n^+|^{q-2} u_n^+ (u_n - u_0) \to 0 \quad \text{and} \quad \int_\Omega |u_n^+|^{p-2} u_n^+ (u_n - u_0) \to 0.$$

Observe that $u_n - u_0 \rightharpoonup 0$ implies $\langle (I^+_{\lambda,p})'(u_n), u_n - u_0 \rangle \to 0$. It follows that

$$\begin{aligned}\langle A(u_n), u_n - u_0\rangle &= \langle (I^+_{\lambda,p})'(u_n), u_n - u_0\rangle - \lambda \int_\Omega |u_n^+|^{q-2}u_n^+(u_n - u_0)\\ &\quad + a\int_\Omega |u_n^+|^{p-2}u_n^+(u_n - u_0) + \int_\Omega f(u_n^+)(u_n - u_0)\\ &= \int_\Omega f(u_n^+)(u_n - u_0) + o(1).\end{aligned}$$

Taking $0 < \alpha < \dfrac{\alpha^*_{s,N}}{rM^{\frac{N}{N-s}}}$, it follows from Hölder's inequality

$$\langle A(u_n), u_n - u_0\rangle \le CC_1\left(\int_\Omega |u_n - u_0|^{r/(r-1)}\right)^{(r-1)/r} + o(1),$$

for a positive constant C_1. Thus $\langle A(u_n), u_n - u_0\rangle \to 0$ and we conclude $u_n \to u_0$ in X^s_p as a consequence of Lemma 2.2.

The proof is analogous in the case of the functional $I_{\lambda,p}$. □

The next results will be useful when proving the geometric conditions of the Mountain Pass Theorem. We define

$$J_{\lambda,p}(u) := I_{\lambda,p}(u) - \frac{1}{p}\|u\|^p_{X^s_p} = \frac{\lambda}{q}\int_\Omega |u|^q\mathrm{d}x - \frac{a}{p}\int_\Omega |u|^p\mathrm{d}x - \int_\Omega F(u)\mathrm{d}x$$

and

$$J^+_{\lambda,p}(u) := I^+_{\lambda,p}(u) - \frac{1}{p}\|u\|^p_{X^s_p} = \frac{\lambda}{q}\int_\Omega |u^+|^q\mathrm{d}x - \frac{a}{p}\int_\Omega |u^+|^p\mathrm{d}x - \int_\Omega F(u^+)\mathrm{d}x.$$

Lemma 4.2 (Subcritical and critical cases) *Suppose that* $a > 0$ *and that* f *satisfies* $(f_{3,p})$. *Then, the trivial solution* $u = 0$ *is a strict local minimum of* $J^+_{\lambda,p}$ *for all* $\lambda > 0$.

Proof According to Theorem 1, it suffices to show that $u = 0$ is a strict local minimum for $J^+_{\lambda,p}$ in $C^0_\delta(\overline{\Omega})$. Condition $(f_{3,p})$ implies, for some $\omega > 0$,

$$\lim_{|t|\to 0}\frac{F(t)}{|t|^p} = 0 \quad \Rightarrow \quad |F(t)| < |t|^p, \ \text{ for all } 0 < |t| \le \omega.$$

Consider $u \in (C^0_\delta(\overline{\Omega}) \cap X^s_p) \setminus \{0\}$. Taking com $\|u\|_{0,\delta}$ small enough, we have $0 < |u^+| < \omega$, since $|u^+| \le M\|u\|_{0,\delta}$ for some $M > 0$. Thus,

$$J^+_{\lambda,p}(u) = \frac{\lambda}{q}\int_\Omega |u^+|^q\mathrm{d}x - \left(\frac{a}{p} + 1\right)\int_\Omega |u^+|^p\mathrm{d}x$$

For $1 < q < p$, we have $|u^+|^{p-q} \le (k_1)^{p-q}\|u\|^{p-q}_{0,\delta}$ for some constant $k_1 > 0$. Thus,

$$J^+_{\lambda,p}(u) \geq \left[\frac{\lambda}{q} - \left(\frac{a}{p}+1\right)(k_1)^{p-q}\|u\|^{p-q}_{0,\delta}\right]\int_\Omega |u^+|^q \mathrm{d}x,\ \frac{a}{p}+1>0.$$

Hence, there exists $R>0$, such that,

$$J^+_{\lambda,p}(u) > 0 = J^+_{\lambda,p}(0),\ \forall\, 0<\|u\|_{0,\delta}<R,$$

completing the proof. □

Remark 4.3 The same result holds for $J_{\lambda,p}$.

Lemma 4.4 (*Subcritical and critical cases*) *Suppose that* $a>0$ *and* f *satisfies* $(f_{4,p})$. *Then, for a fixed* $\Lambda>0$, *there exists* $t_0=t_0(\Lambda)$ *such that*

$$I^+_{\lambda,p}(t\varphi_1)<0,$$

for all $t\geq t_0$ *and* $0<\lambda<\Lambda$.

Proof It follows from $(f_{4,p})$ that, fixed $M>0$, there exists $C_M>0$ such that

$$F(t)\geq M|t|^p - C_M. \tag{4.1}$$

Thus, if $M>\dfrac{\lambda_1}{p}$, denoting by φ_1 the positive eigenfunction associated with the eigenvalue λ_1, with $\|\varphi_1\|_{L^p(\Omega)}=1$, we have

$$\begin{aligned}
I^+_{\lambda,p}(t\varphi_1) &\leq \frac{\lambda_1|t|^p}{p}\int_\Omega|\varphi_1|^p\mathrm{d}x + \frac{|t|^q\lambda}{q}\int_\Omega|\varphi_1|^q\mathrm{d}x - M|t|^p\int_\Omega|\varphi_1|^p\mathrm{d}x + C_M|\Omega|\\
&= t^p\left[\frac{\lambda}{q}\frac{1}{t^{p-q}}\int_\Omega|\varphi_1|^q\mathrm{d}x + \frac{1}{t^p}C_M|\Omega| - \left(M-\frac{\lambda_1}{p}\right)\right].
\end{aligned}$$

For a fixed $\Lambda>0$ we now choose $t_0=t_0(\Lambda)>0$ such that

$$\frac{\Lambda}{q}\frac{1}{t_0^{p-q}}\int_\Omega\varphi_1^q\mathrm{d}x + \frac{C_M}{t_0^p}|\Omega| - \left(M-\frac{\lambda_1}{p}\right)<0.$$

So, for $t\geq t_0$ and $\lambda<\Lambda$ we have

$$\begin{aligned}
\frac{\lambda}{q}\frac{1}{t^{p-q}}\int_\Omega\varphi_1^q\mathrm{d}x + \frac{C_M}{t^p}|\Omega| - \left(M-\frac{\lambda_1}{p}\right) &\leq \frac{\Lambda}{q}\frac{1}{t_0^{p-q}}\int_\Omega\varphi_1^q\mathrm{d}x + \frac{C_M}{t_0^p}|\Omega| - \left(M-\frac{\lambda_1}{p}\right)\\
&<0.
\end{aligned}$$

The result follows. □

Proposition 4.5 *Suppose that, for* $a>0$, f *satisfies* $(f_{1,p})$, $(f_{2,p})$, $(f_{3,p})$ $(f_{4,p})$. *Then, the subcritical problem* (1.1) *has at least one positive solution, for all* $0<\lambda<\Lambda$, *where* $\Lambda>0$ *is arbitrary.*

Proof It follows immediately from the Mountain Pass Theorem, as consequence of Lemmas 4.1, 4.2 and 4.4. □

4.2 Negative solution for the functional $I_{\lambda,p}$

The Palais–Smale condition is obtained by following the reasoning given in the proof of Lemma 4.1.

For $u \in X_p^s$, by defining

$$J^-_{\lambda,p}(u) := I^-_{\lambda,p}(u) - \frac{1}{p}\|u\|^p_{X_p^s} = \frac{\lambda}{q}\int_\Omega |u^-|^q \mathrm{d}x - \frac{a}{p}\int_\Omega |u^-|^p \mathrm{d}x - \int_\Omega F(u^-)\mathrm{d}x,$$

we obtain the result analogous to Lemma 4.2.

Mimicking the proof of Lemma 4.4, we obtain the second condition of the Mountain Pass Theorem. Thus, the negative solution follows, as before, from the Mountain Pass Theorem.

4.3 A third solution

In order to obtain the geometric conditions of the Linking Theorem, we define

$$\lambda^* = \inf\left\{ \|u\|^p_{X_p^s} \;:\; u \in W,\; \|u\|^p_{L^p(\Omega)} = 1 \right\},$$

where

$$W = \left\{ u \in X_p^s \;:\; \langle A(\varphi_1), u\rangle = 0 \right\},$$

with φ_1 the first autofunction, positive and normalized, of $(-\Delta)_p^s$.

The proof of the next result is simple.

Proposition 4.6 $X_p^s = W \oplus \mathrm{span}\{\varphi_1\}$.

Following ideas of Alves et al. [1] and Capozzi et al. [10], we obtain the next result.

Proposition 4.7 $\lambda_1 < \lambda^*$.

Proof Of course

$$\lambda_1 = \inf\left\{ \|u\|^p_{X_p^s} \;:\; \|u\|^p_{L^p(\Omega)} = 1 \right\} \le \inf\left\{ \|u\|^p_{X_p^s} \;:\; u \in W \text{ e } \|u\|^p_{L^p(\Omega)} = 1 \right\} = \lambda^*.$$

Suppose that $\lambda_1 = \lambda^*$. It follows the existence of a sequence $(u_n) \subset W$ such that

$$\|u_n\|^p_{L^p(\Omega)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \|u_n\|^p_{X_p^s} = \lambda^* = \lambda_1.$$

Since (u_n) is bounded in X_p^s, passing to a subsequence if necessary, there exists $u \in X_p^s$ such that

$$u_n \rightharpoonup u \text{ in } X_p^s, \quad u_n \to u \text{ in } L^q(\Omega),\ 1 \le q \le p, \quad u_n(x) \to u(x) \text{ a.e. in } \Omega.$$

Since

$$\lambda_1 \le \|u\|_{X_p^s}^p \le \liminf_{k\to\infty} \|u_k\|^p = \liminf_{k\to\infty} \lambda_k = \lambda_1.$$

we conclude that $u = t\varphi_1$ for some $t \neq 0$.

But $\langle A(\varphi_1), u_n\rangle \to \langle A(\varphi_1), u\rangle$. Since $(u_n) \subset W$, we have $\langle A(\varphi_1), u_n\rangle = 0$, thus implying $\langle A(\varphi_1), u\rangle = 0$. It follows $t\|\varphi_1\|_{X_p^s}^p = 0$ and $t = 0$, and we have reached a contradiction. □

Lemma 4.8 *If $a < \lambda^*$, then there exist $\beta, \rho > 0$ such that $I_{\lambda,p}(u) \ge \beta$ for all $u \in W$ such that $\|u\|_{X_p^s} = \rho$.*

Proof Take $\theta > p$ and $0 < \alpha < \alpha_{s,N}^*$ (see Proposition 1.1). It follows from $(f_{2,p})$ and $(f_{3,p})$ the existence of $0 \le \mu < \lambda^* - a$ and $C > 0$ such that

$$F(t) \le \frac{\mu}{p} t^P + C \exp(\alpha |t|^{\frac{N}{N-s}}) |t|^\theta, \quad \text{for all } t \in \mathbb{R}.$$

Thus, if $u \in W$ and $\|u\|_{X_p^s} \le 1$, then the definition of λ^* yields

$$I_{\lambda,p}(u) \ge \frac{\|u\|_{X_p^s}^p}{p} - (a+\mu)\frac{\|u\|_{X_p^s}^p}{p\lambda^*} - C\int_\Omega \exp(\alpha u^2)|u|^\theta \mathrm{d}x.$$

Take $r > 1$ so that $0 < r\alpha < \alpha_{s,N}^*$. Recalling that $\|u\|_{X_p^s} \le 1$, by combining the Hölder's inequality, the continuous immersion $X_p^s \hookrightarrow L^{r'q}(\Omega)$ and Proposition 1.1 guarantee the existence of $C_1 > 0$ such that

$$I_{\lambda,p}(u) \ge \frac{1}{p}\left(1 - \frac{a+\mu}{\lambda^*}\right)\|u\|_{X_p^s}^2 - C_1\|u\|_{X_p^s}^\theta.$$

Observe that $\theta > p$ and $1 - \dfrac{a+\mu}{\lambda^*} > 0$. Thus, for $\rho > 0$ small enough and $\|u\|_{X_p^s} = \rho$, we have

$$I_{\lambda,p}(u) \ge \rho^p \left\{ \frac{1}{p}\left(1 - \frac{a+\mu}{\lambda^*}\right) - C_1\rho^{\theta-p} \right\} := \beta > 0.$$

We are done. □

Lemma 4.9 *Suppose that f satisfies $(f_{4,p})$. If $Y \subset X_p^s$ is a subspace with* $\dim Y < \infty$, *then*

$$\lim_{u\in Y,\|u\|_{X_p^s}\to\infty} I_{\lambda,p}(u) = -\infty.$$

Proof Since all norms in Y are equivalent, there exist $C_1 > 0$ and $C_2 > 0$ such that, for all $y \in Y$ holds

$$\|u\|_{X_p^s}^p \le C_1\|u\|_{L^p(\Omega)}^p \quad \text{and} \quad \|u\|_{L^q(\Omega)}^q \le C_2\|u\|_{X_p^s}^q. \tag{4.2}$$

Now, by applying (4.1) for $M > C_1\left(\frac{1}{p} + \frac{\lambda C_2}{q}\right)$ and (4.2), we obtain

$$I_{\lambda,p}(u) \le \|u\|_{X_p^s}^p\left(\frac{1}{p} + \frac{\lambda C_2}{q} - \frac{M}{C_1}\right) + C_M|\Omega|.$$

The choice of M implies the result. □

Lemma 4.10 *If f satisfies $(f_{4,p})$ and if $\lambda_1 \le a < \lambda^*$, then there exists $\eta = \eta(\lambda) > 0$ such that*

$$I_{\lambda,p}(u) \le \eta(\lambda) \quad \text{and} \quad \lim_{\lambda\to 0}\eta(\lambda) = 0, \quad \text{for all } u \in \operatorname{span}\{\varphi_1\}.$$

Proof Since $u \in \operatorname{span}\{\varphi_1\}$, we have

$$I_{\lambda,p}(u) \le \left(\frac{\lambda_1 - a}{p}\right)\int_\Omega u^p\mathrm{d}x + \frac{\lambda}{q}\|u\|_{L^q(\Omega)}^q - \int_\Omega F(u)\mathrm{d}x.$$

But $\lambda_1 \le a < \lambda^*$, the continuous immersion and (4.1) imply that

$$I_{\lambda,p}(u) \le \frac{\lambda K_q}{q}\|u\|_{X_p^s}^q - MK_2\|u\|_{X_p^s}^p + C_M.$$

Since $1 < q < p$, we conclude that

$$\lim_{u\in\operatorname{span}\{\varphi_1\},\|u\|_{X_p^s}\to\infty} I_{\lambda,p}(u) = -\infty.$$

Thus, there exists $R > 0$ such that $I_{\lambda,p}(u) < 0$ for all $u \in \operatorname{span}\{\varphi_1\}$ satisfying $\|u\|_{X_p^s} > R$. If $u \in \operatorname{span}\{\varphi_1\}$ and $\|u\|_{X_p^s} \le R$, we have

$$0 \le I_{\lambda,p}(u) \le \frac{\lambda}{q}K_q\|u\|_{X_p^s}^q - \int_\Omega F(u)\mathrm{d}x \le \frac{\lambda}{q}K_qR^q - \int_\Omega F(u)\mathrm{d}x \le \frac{\lambda}{q}K_qR^q.$$

The result follows by defining $\eta(\lambda) = \frac{\lambda}{q}K_qR^q$. □

Proposition 4.11 *Suppose that f satisfies $(f_{1,p}) - (f_{4,p})$. If $\lambda_1 \le a < \lambda^*$, then problem* (1.1) *has at least a third solution, for all $\lambda > 0$ small enough.*

Proof We already know that $X_p^s = W \oplus \operatorname{span}\{\varphi_1\}$ and that the functional $I_{\lambda,p}$ satisfies the Palais-Smale condition at all levels, for any $\lambda > 0$. Therefore, the Linking theorem guarantees that $I_{\lambda,p}$ has a critical value $C \ge \beta$ given by

$$C = \inf_{\gamma\in\Gamma}\max_{u\in Q} I_{\lambda,p}(\gamma(u))$$

where $\Gamma = \{\gamma \in C(\bar{Q}, E)\ ;\ \gamma = I_d \text{ in } \partial Q\}$.

Taking into account Lemma 4.8, to conclude our result from the Linking Theorem, it suffices to show the existence of $e \in (\partial B_1)\cap W$, constants $R > \rho$ and $\alpha > 0$ such that $I_{\lambda,p}\big|_{\partial Q} < \alpha < \beta$, where $Q = (B_R \cap \operatorname{span}\{\varphi_1\}) \oplus (0, Re)$.

So, take $\varphi \in W$ with $\|\varphi\|_{X_p^s} = 1$. Lemma 4.9 guarantees the existence of $\bar{R} > 0$ such that

$$I_{\lambda,p}(u) < 0 \quad \text{for all } u \in \operatorname{span}\{\varphi_1, \varphi\},\ \|u\|_{X_p^s} \geq \bar{R}. \tag{4.3}$$

By applying Lemma 4.8 for $\rho\varphi \in \operatorname{span}\{\varphi_1, \varphi\}$, we obtain $I_{\lambda,p}(\rho\varphi) \geq \beta > 0$, proving that $\bar{R} > \rho$. We now consider

$$Q = \{u = w + t\varphi,\ w \in \operatorname{span}\{\varphi_1\} \cap B_{\bar{R}},\ 0 \leq t \leq \bar{R}\}$$

and the border $\partial Q = \bigcup_{i=1}^{3}\Gamma_i$ with

(1) $\Gamma_1 = \overline{B}_{\bar{R}}(0) \cap \operatorname{span}\{\varphi_1\}$,
(2) $\Gamma_2 = \{u \in X_p^s\ :\ u = w + \bar{R}\varphi, w \in B_{\bar{R}}(0) \cap \operatorname{span}\{\varphi_1\}\}$,
(3) $\Gamma_3 = \{u \in X_p^s\ :\ u = w + r\varphi,\ w \in \operatorname{span}\{\varphi_1\},\ \|w\|_{X_p^s} = \bar{R},\ 0 \leq r \leq \bar{R}\}$.

We have $I_{\lambda,p}\big|_{\Gamma_i} \leq \eta(\lambda)$, for $i = 1, 2, 3$. In fact, this follows from Lemma 4.10 if $u \in \Gamma_1 \subset \operatorname{span}\{\varphi_1\}$. However, if $u \in \Gamma_2$ or $u \in \Gamma_3$, then it is a consequence of (4.3).

By the Linking theorem, there exists a weak solution $u_\lambda \in X_p^s$ of the problem (1.1) such that

$$0 < \eta(\lambda) < \beta \leq I_{\lambda,p}(u_\lambda) = C_\lambda.$$

Observe that $u_\lambda \neq 0$, since $I_{\lambda,p}(0) = 0$.

In order to show that this third solution is different from the positive and negative solutions obtained before, consider $g_0^+ : [0,1] \to X_p^s$ given by $g_0^+(t) = t(t_0\varphi_1)$, with t_0 defined in Lemma 4.4. We have

$$g_0^+ \in \{g \in C([0,1], X_p^s)\ :\ g(0) = 0,\ g(1) = t_0\varphi_1\}.$$

It follows from Lemma 4.10 that

$$I_{\lambda,p}^+(g_0^+(t)) = I_{\lambda,p}(g_0^+(t)) \leq \eta(\lambda), \quad \text{for all } t \in [0,1]. \tag{4.4}$$

The result now follows by applying a result analogous to Lemma 4.4, valid for solutions with negative energy and defining $g_0^- \in \Gamma^-$, satisfying an estimate analogous to (4.4). □

Proof of Theorem 2 To conclude its proof we observe that, if f is odd, then $I_{\lambda,p}$ is even. Now, the existence of infinite many solutions follows by applying the symmetric version of the Mountain Pass Theorem, see [43, Theorem 9,12]. □

5 Proof of Theorem 3

5.1 Positive and negative solutions for the functional $I_{\lambda,p}$

Lemma 5.1 *Suppose that f satisfies the hypotheses of the critical exponential growth case. Then the functional* $I_{\lambda,p}$ *satisfies the* (PS)*-condition at any level* $c < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}$.

Proof For $c < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}$, let (u_n) be a $(PS)_c$ sequence in X^s_p. Lemma 2.5 guarantees that (u_n) is bounded. Therefore, passing to a subsequence, we can suppose that

$$u_n \rightharpoonup u \text{ in } X^s_p, \quad u_n \to u \text{ in } L^q(\Omega) \text{ for all } q \geq 1, \quad u_n(x) \to u(x) \text{ a.e. in } \Omega.$$

Since $\lambda > 0$, $(\|I'_{\lambda,p}(u_n)\|_{(X^s_p)^*})$ and $(I_{\lambda,p}(u_n))$ are bounded sequences in $\mathbb{R}$.

Therefore, since $(f_{5,p})$ implies $pF(t) \leq tf(t)$ for all $t \neq 0$, there exists $C > 0$ such that

$$\max\left\{\|u_n\|^2_{X^s_p}, \int_\Omega f(u_n)u_n \mathrm{d}x, \int_\Omega F(u_n)\mathrm{d}x\right\} \leq C.$$

It follows from (3.2) that $f(u_n), f(u) \in L^1(\Omega)$. Since $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega f(u_n)u_n \mathrm{d}x \leq C$, we conclude that

$$f(u_n) \to f(u) \text{ in } L^1(\Omega) \tag{5.1}$$

by applying [16, 17, Lema 2.1]. Thus, (5.1) and $(f_{6,p})$ allow us to conclude that

$$F(u_n) \to F(u) \text{ in } L^1(\Omega) \tag{5.2}$$

and

$$\frac{\|u_n\|^p_{X^s_p}}{p} \to c - \frac{\lambda}{q}\int_\Omega |u|^q \mathrm{d}x + \frac{a}{p}\int_\Omega |u|^p \mathrm{d}x + \int_\Omega F(u)\mathrm{d}x.$$

Since $I'_\lambda(u_n) \to 0$ in $(X^s_p)^*$, it follows that

$$\int_\Omega f(u_n)u_n \mathrm{d}x \to pc + \lambda\left(1 - \frac{p}{q}\right)\int_\Omega |u|^q \mathrm{d}x + p\int_\Omega F(u)\mathrm{d}x. \tag{5.3}$$

A new application of the inequality $pF(t) \leq tf(t)$ yields

$$\int_\Omega f(u_n)u_n \mathrm{d}x - p\int_\Omega F(u_n)\mathrm{d}x \geq 0.$$

We conclude from (5.3) that

$$pc \geq \lambda\left(\frac{p}{q}-1\right)\int_\Omega |u|^q,$$

thus showing that $c \geq 0$. Now, standard arguments show that $\langle I'_{\lambda,p}(u), v\rangle = 0$ for all $v \in X^s_p$.

Thus, $pF(t) \leq tf(t)$ yields

$$I_{\lambda,p}(u) > \frac{1}{p}\left(\|u\|^p_{X^s_p} + \lambda\int_\Omega |u|^q - a\int_\Omega |u|^2\mathrm{d}x - \int_\Omega f(u)u\mathrm{d}x\right) = \frac{1}{p}\langle I'_{\lambda,p}(u), u\rangle = 0$$

proving that $I_{\lambda,p}(u) > 0$, since $I_{\lambda,p}(0) = 0$.

To prove that $u_n \to u$ in X^s_p, it suffices to show that $I_{\lambda,p}(u) = c$, since this yields $\|u_n\|_{X^s_p} \to \|u\|_{X^s_p}$. In fact, it follows from (5.2) that

$$\begin{aligned} I_{\lambda,p}(u) &\leq \liminf_{n\to\infty}\left(\frac{1}{p}\|u_n\|^p_{X^s_p} + \frac{\lambda}{q}\int_\Omega |u_n|^q - \frac{a}{p}\int_\Omega |u_n|^p\mathrm{d}x - \int_\Omega F(u_n)\mathrm{d}x\right) \\ &= \liminf_{n\to\infty} I_{\lambda,p}(u_n) = c. \end{aligned}$$

If $I_{\lambda,p}(u) < c$, then we would have

$$\begin{aligned} \lim_{n\to\infty}\|u_n\|^p_{X^s_p} &> p\left(I_{\lambda,p}(u) - \frac{\lambda}{q}\int_\Omega |u|^q + \frac{a}{p}\int_\Omega |u|^2\mathrm{d}x + \int_\Omega F(u)\mathrm{d}x\right) \\ &= \|u\|^p_{X^s_p}. \end{aligned} \tag{5.4}$$

By defining $v_n = u_n/\|u_n\|_{X^s_p}$ and $v = u/c_0$, where

$$c_0 = \left(pc - \frac{p\lambda}{q}\int_\Omega |u|^q + a\int_\Omega |u|^p\mathrm{d}x + p\int_\Omega F(u)\mathrm{d}x\right)^{-1/p} > 0,$$

(5.4) would then imply that

$$\|v\|_{X^s_p} = \frac{\|u\|_{X^s_p}}{c_0} < \frac{\|u\|_{X^s_p}}{\|u\|_{X^s_p}} = 1.$$

We can conclude that $v_n \rightharpoonup v$ in X^s_p, by choosing $\alpha > \alpha_0$ so that

$$pr^{\frac{N-s}{s}}\alpha^{\frac{N-s}{s}} < \frac{\left(\alpha^*_{s,N}\right)^{\frac{N-s}{s}}}{c}.$$

Thus, $I_{\lambda,p}(u) > 0$ and (5.4) would then imply

$$\lim_{n\to\infty} r^{\frac{N-s}{s}}\alpha^{\frac{N-s}{s}}\|u_n\|^p_{X^s_p} = r^{\frac{N-s}{s}}\alpha^{\frac{N-s}{s}}c_0^p < \left(\alpha^*_{s,N}\right)^{\frac{N-s}{s}}\left(\frac{c_0^p}{p(c - I_{\lambda,p}(u))}\right).$$

It is not difficult to show that $c_0^p\big(p(c - I_\lambda)(u)\big)^{-1} = \left(1 - \|v\|^p_{X^s_p}\right)^{-1}$.

Thus,

$$b = \lim_{n\to\infty} r^{\frac{N-s}{s}} \alpha^{\frac{N-s}{s}} \|u_n\|^p_{X^s_p} < \frac{\left(\alpha^*_{s,N}\right)^{\frac{N-s}{s}}}{1 - \|v\|^p_{X^s_p}}.$$

So, for $\epsilon > 0$ small enough and $n \in \mathbb{N}$ large enough, we have

$$r\alpha\|u_n\|^{\frac{N}{N-s}}_{X^s_p} < \epsilon + b < \frac{\alpha^*_{s,N}}{(1 - \|v\|^p_{X^s_p})^{\frac{s}{N-s}}}.$$

Thus, there exist $1 < \mu < \frac{1}{(1-\|v\|^p_{X^s_p})^{\frac{s}{N-s}}}$ and $0 < \gamma < \alpha^*_{s,N}$ such that

$$r\alpha\|u_n\|^{\frac{N}{N-s}}_{X^s_p} < \gamma\mu < \frac{\alpha^*_{s,N}}{(1 - \|v\|^p_{X^s_p})^{\frac{s}{N-s}}}.$$

But (3.2) implies

$$\int_\Omega |f(u_n)|^r \mathrm{d}x \le C \int_\Omega \exp(r\alpha|u_n|^{\frac{N}{N-s}})\mathrm{d}x \le C \int_\Omega \exp\left(\gamma\mu|v_n|^{\frac{N}{N-s}}\right)\mathrm{d}x.$$

Our choice of μ and γ guarantees that the sequence $\left(\exp\left(\gamma|v_n|^{\frac{N}{N-s}}\right)\right)$ is bounded in $L^\mu(\Omega)$. Therefore, $(f(u_n))$ is bounded in $L^r(\Omega)$ for some $r > 1$.

By applying the Brezis-Lieb lemma, we conclude that $f(u_n) \rightharpoonup f(u)$ in $L^r(\Omega)$ and, since $u_n \to u$ in $L^{r'}(\Omega)$, we conclude that

$$\lim_{n\to\infty} \int_\Omega f(u_n)u_n \mathrm{d}x = \int_\Omega f(u)u\mathrm{d}x.$$

Thus,

$$\lim_{n\to\infty} \|u_n\|^p_{X^s_p} = \|u\|^p_{X^s_p} - \langle I'_{\lambda,p}(u), u\rangle = \|u\|^p_{X^s_p}.$$

we have reached a contradiction. Therefore, $I_{\lambda,p}(u) = c$. □

Remark 5.2 The same result is valid for the functionals $I^+_{\lambda,p}$ and $I^-_{\lambda,p}$.

Proposition 5.3 *Suppose that $a \ge \lambda_1$ and f satisfies $(f_{1,p})$, $(f'_{2,p})$, $(f_{3,p})$ and $(f_{5,p}) - (f_{7,p})$. Then, in the case of critical exponential growth, problem (1.1) has at least one positive solution for all $\lambda > 0$ small enough.*

Proof As in the subcritical growth case, the functional $I^+_{\lambda,p}$ satisfies the geometric hypotheses of the Mountain Pass Theorem.

We will show that $I^+_{\lambda,p}$ satisfies the (PS) condition at level C^+_λ, given by

$$C_\lambda^+ = \inf_{g\in\Gamma^+} \max_{u\in g([0,1])} I_{\lambda,p}^+(u),$$

where $\Gamma^+ = \{g \in C([0,1], X_p^s) \,:\, g(0) = 0,\ g(1) = t_0\varphi_1\}$, with t_0 given in Lemma 4.4. Observe that

$$\max_{t\in[0,1]} I_{\lambda,p}^+(g(t)) \geq I_{\lambda,p}^+(g(0)) = I_{\lambda,p}^+(0) = 0,\ \forall\, g \in \Gamma^+,$$

thus implying that $\max_{t\in[0,1]} I_{\lambda,p}^+(g(t)) \geq 0,\ \forall\, g \in \Gamma^+$.

It follows that

$$0 \leq \inf_{g\in\Gamma^+} \max_{u\in g([0,1])} I_{\lambda,p}^+(u) = C_\lambda^+ < \infty.$$

As in the proof of Lemma 5.1, we obtain that $I_{\lambda,p}^+$ satisfies the $(PS)_c$ condition for all $\lambda > 0$, where $c < \frac{s}{N}\left(\frac{\alpha_{s,N}^*}{\alpha_0}\right)^{\frac{N-s}{s}}$. We will show that $C_\lambda^+ < \frac{s}{N}\left(\frac{\alpha_{s,N}^*}{\alpha_0}\right)^{\frac{N-s}{s}}$, if $\lambda > 0$ is small enough.

In fact, by defining $g_0^+ : [0,1] \to X_p^s$ by $g_0^+(t) = t(t_0\varphi_1)$, the result follows by applying Lemma 4.10:

$$C_\lambda^+ \leq I_{\lambda,p}^+(g_0^+) < \eta(\lambda) < \frac{s}{N}\left(\frac{\alpha_{s,N}^*}{\alpha_0}\right)^{\frac{N-s}{s}},$$

if $\lambda > 0$ is small enough. □

The proof of existence of a negative solution is analogous to that of Proposition 5.3.

Proposition 5.4 *Suppose that $a \geq \lambda_1$ and that f satisfies $(f_{1,p}), (f'_{2,p}), (f_{3,p})$ and $(f_{5,p}) - (f_{7,p})$. Then, in the case of critical exponential growth, problem (1.1) has at least one negative solution for all $\lambda > 0$ small enough.*

5.2 A third solution

Proposition 5.5 *Suppose that f satisfies $(f_{1,p}), (f'_{2,p}), (f_{3,p})$, and $(f_{5,p}) - (f_{7,p})$. If $\lambda_1 \leq a < \lambda^*$ then, for all $\lambda > 0$ small enough, problem (1.1) has at least a third solution in the case of critical exponential growth.*

Proof According to Lemmas 4.8 and 4.9, the functional $I_{\lambda,p}$ satisfies the geometry of the Linking Theorem.

We maintain the notation introduced in Sect. 4, with $X_p^s = W \oplus \operatorname{span}\{\varphi_1\}$ and $Q = (B_R \cap \operatorname{span}\{\varphi_1\}) \oplus ([0, R\varphi])$ for $\varphi \in W$. So, it suffices to prove that

(iii) $\sup_{u\in Q} I_{\lambda,p}(u) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}$.

We claim that $(f_{7,p})$ implies, for $\lambda > 0$ small enough,

$$\max_{u\in\bar{Q}} I_{\lambda,p}(u) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}. \tag{5.5}$$

So, we write $I_{\lambda,p}$ in the form

$$I_{\lambda,p}(u) = J(u) + \frac{\lambda}{q}\int_\Omega |u|^q \mathrm{d}x,$$

with $J(u) = \frac{1}{p}\|u\|^p_{X^s_p} - \frac{a}{p}\int_\Omega |u|^p \mathrm{d}x - \int_\Omega F(u)\mathrm{d}x$.

In order to prove (iii), it is enough to verify that

$$\sup_{u\in\bar{Q}} J(u) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}} \tag{5.6}$$

or, what is the same, that for $\lambda > 0$ small enough, we have

$$\sup_{u\in\bar{Q}} I_{\lambda,p}(u) \le \sup_{u\in\bar{Q}} J(u) + \frac{\lambda}{q}\sup_{u\in\bar{Q}} |u|^q_{L^q(\Omega)} < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}},$$

thus showing (iii).

In order to prove (5.6), we will show that

$$\sup_{u\in\bar{Q}} J(u) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}.$$

Consider $\mathbb{F} = \mathrm{span}\{\varphi_1, \varphi\}$. We have

$$\sup_{u\in\bar{Q}} J(u) \le \max_{u\in\mathbb{F}} J(u) = \max_{u\in\mathbb{F}, t\neq 0} J\left(|t|\frac{u}{|t|}\right) = \max_{u\in\mathbb{F}, t>0} J(tu) \le \max_{u\in\mathbb{F}, t\ge 0} J(tu).$$

But

$$J(tu) = \frac{t^p}{p}\|u\|^p_{X^s_p} - \frac{a}{p}t^p\int_\Omega |u|^p\mathrm{d}x - \int_\Omega F(tu)\mathrm{d}x \le \frac{t^p}{p}\|u\|^p_{X^s_p} - \int_\Omega F(tu)\mathrm{d}x.$$

Define $\eta : [0, +\infty) \to \mathbb{R}$ by

$$\eta(t) = \frac{t^p}{p}\|u\|^p_{X^s_p} - \int_\Omega F(tu)\mathrm{d}x.$$

Since all norms in $\mathbb{F}$ are equivalent, it follows from $(f_{7,p})$ the existence of $C>0$ such that

$$\int_\Omega F(tu)\mathrm{d}x \ge \frac{C_r}{r}\int_\Omega t^r|u|^r\mathrm{d}x = \frac{C_r}{r}t^r\|u\|^r_{L^r(\Omega)} \ge C\frac{C_r}{r}t^r\|u\|^r_{X^s_p}.$$

Thus,

$$\eta(t) \le \frac{t^p}{p}\|u\|^p_{X^s_p} - C\frac{C_r}{r}t^r\|u\|^r_{X^s_p} \le \max_{t\ge 0}\left(\frac{t^p}{p}\|u\|^p_{X^s_p} - C\frac{C_r}{r}t^r\|u\|^r_{X^s_p}\right). \tag{5.7}$$

Therefore, (5.7) yields $\eta(t) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}$ and

$$\max_{u\in\mathbb{F},\, t\ge 0} J(tu) \le \max_{t\ge 0}\eta(t) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}},$$

and the proof of our claim is complete.

For $\lambda>0$ small enough, the functional $I_{\lambda,p}$ satisfies the (PS)-condition at the level $C_\lambda = \inf_{h\in\Gamma}\sup_{u\in\bar{Q}} I_{\lambda,p}(h(u))$, where $\Gamma = \{h\in C(\bar{Q}, X^s_p)\,;\, h = id \text{ in } \partial Q\}$. In fact, (5.5) implies that, for $\lambda>0$ small enough, we have

$$\inf_{h\in\Gamma}\sup_{u\in\bar{Q}} I_{\lambda,p}(h(u)) \le \sup_{u\in\bar{Q}} I_{\lambda,p}(u) < \frac{s}{N}\left(\frac{\alpha^*_{s,N}}{\alpha_0}\right)^{\frac{N-s}{s}}$$

and the $(PS)_{C_\lambda}$-condition is consequence of Lemma 5.1.

It follows from the Linking Theorem that $C_\lambda = \inf_{h\in\Gamma}\sup_{u\in\bar{Q}} I_{\lambda,p}(h(u))$ is a critical value for $I_{\lambda,p}$, with $C_\lambda\ge\beta$. Therefore, there exists $u_\lambda\in X^s_p$ weak solution of (1.1) satisfying $0<\beta\le I_{\lambda,p}(u_\lambda)$, what implies that $u_\lambda\ne 0$.

As in the proof of Proposition 4.11, we prove that this solution is different from the positive and negative solutions already obtained. □

Observe that we also conclude the proof of Theorem 3 by the same reasoning given in the proof of Theorem 2.

Remark 5.6 The proof of the analogous results in case $N=1$, $p=2$, $s=1/2$ and $\Omega=(0,1)$ are completely similar; in order to find a third solution by applying the Linking Theorem we consider the decomposition

$$X = V_k \oplus W_k,$$

where $V_k = \mathrm{span}\{\varphi_1,\dots,\varphi_k\}$ is the subspace generated by the autofunctions of $(-\Delta)^{1/2}$ corresponding to the eigenvalues $\lambda_1,\dots,\lambda_k$, e $W_k = V_k^\perp$.

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