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The $W^{1,2}_{(p,q)}$-solvability for a class of fully nonlinear parabolic equations

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Abstract

The solvability in the Sobolev-Lorentz spaces $W^{1,2}_{(p,q)}(\Omega_T)$ with $p > d + 1$ and $q > 0$ is proved for a class of fully nonlinear parabolic equations with small BMO nonlinearities in (x, t)-variables over a bounded parabolic domain with $C^{1,1}$-smooth lateral boundary. Here, we make use of the unified approach based on the Fefferman-Stein theorem in accordance with almost all pointwise estimate of the sharp functions to establish the estimates of D^2u and D_tu in the framework of Lorentz spaces.

Keywords Fully nonlinear parabolic equations · BMO nonlinearities · The Fefferman-Stein theorem · Sharp functions · Lorentz spaces

Mathematics Subject Classification Primary: 35B45, 35K61, 35R05 · Secondary: 46E30

1 Introduction

Nonlinear Calderón-Zygmund theory for elliptic and parabolic partial differential equations have been extensively studied in recent decades. In particular, an interior $W^{2,p}$-estimate with $p > d$ for a class of fully nonlinear elliptic equations, was first obtained by Caffarelli in [6] via the Aleksandrov-Bakel'man-Pucci estimate, the covering argument and the Harnack inequality by using an improvement of Krylov-Safonov's technique. A similar interior estimate was extended by Wang in [26] to fully nonlinear parabolic equations by adapting the Aleksandrov-Bakel'man-Pucci-Krylov-Tso maximum principle and the compactness method. Later, Escauriaza in

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[11] further sharpened Caffarelli's result in [6] to the range $p > d - \varepsilon$ with a small $\varepsilon > 0$ depending on the ellipticity constants and d by considering the reverse Hölder inequality. Meanwhile, we also notice that a solvability in $W^{2,1}_{p,loc}(\Omega_T) \cap C(\overline{\Omega_T})$ for the initial-boundary problem of fully nonlinear parabolic equations was obtained by Crandall, Kocan and Świech in [7]. Winter in [27] further made use of the Alexandroff maximum principle and the weak Harnack inequality on the boundary setting to establish the boundary estimate as well as the $W^{2,p}$-solvability for the Dirichlet boundary problem associated nonlinear elliptic equations. Recently, an interesting work from Krylov [14] provided a unified approach to show the L^p-solvability for nondivergence elliptic and parabolic equations under the regular assumption of VMO_x principle coefficients. This approach mainly relies on the pointwise estimates of sharp functions for the spatial derivatives of solutions due to the Fefferman-Stein theorem. Furthermore, such an approach was developed by Dong, Kim and Krylov to attain the $W^{2,p}$-solvability for fully nonlinear elliptic and parabolic equations with the nonlinearities being VMO in independent variables. For examples, Krylov [15] showed the $W^{2,p}$-solvability with $p > d$ to the Bellman's equations with the VMO nonlinearity by his unified argument. Dong-Krylov-Li [8] demonstrated an interior $W^{2,p}$-solvability for $p > d$ and $W^{2,1}_p$-solvability for $p > d + 1$, respectively, to fully nonlinear elliptic and parabolic equations. More generally, Krylov [16] studied the existence and uniqueness in $W^{2,p}$ for $p > d$ to the strong solution of fully nonlinear elliptic equations $H(u, Du, D^2u, x) = 0$. Dong and Krylov in [9] proved the solvability in $W^{2,1}_p$ for $p > d + 1$ to the strong solutions of fully nonlinear parabolic equations $\partial_t u(t, x) + H(u, Du, D^2u; t, x) = 0$ under some relaxed convexity assumptions instead of requiring H to be convex or concave with respect to D^2u. Very recently, Dong and Krylov [10] also derived the regularity in the mixed-norm Sobolev spaces to fully nonlinear elliptic and parabolic equations by improving the Fefferman-Stein theorem in the mixed-norm weighted Lebesgue spaces.

This paper is actually a continuation of Dong-Krylov-Li's work in [8], which extends it in two folds: the Fefferman-Stein theorem in the Lorentz spaces and Lorentz regularity of fully nonlinear parabolic equations. Here, the Fefferman-Stein inequality in the Lorentz spaces is also inspired by Dong and Krylov's paper [10] involving that in the mixed-norm form. Our main aim is to attain a global estimate in the Sobolev-Lorentz spaces by using a unified approach to fully nonlinear parabolic equations with the nonlinearities being small BMO oscillation in independent variable in a bounded $C^{1,1}$ parabolic domain. To state our problem under consideration more precisely, let us recall some related notations. Let Ω be a bounded domain of $\mathbb{R}^d$ for $d \geq 2$ with $C^{1,1}$-smooth boundary, and $\Omega_T := \Omega \times (0, T)$ for $T > 0$ with its usual parabolic boundary $\partial\Omega_T := \Big(\partial\Omega \times (0, T]\Big) \cup \Big(\Omega \times \{0\}\Big)$. We consider the following zero Cauchy-Dirichlet problem of a fully nonlinear parabolic equation:

$$\begin{cases} \partial_t u(x,t) + H(D^2u, Du, u, x, t) = 0 & \text{in } \Omega_T, \\ u = 0 & \text{on } \partial\Omega_T, \end{cases} \tag{1.1}$$

where $D^2u = \Big(D_{ij}u\Big)_{d\times d}$ denotes the Hessian matrix of u, and $Du = \Big(D_i u\Big)_{d\times 1}$ denotes its gradient.

As usual, we use $\mathring{W}^{1,2}_2(\Omega_T)$ to denote the set of all functions $u \in L^2(\Omega_T)$ that their weak derivatives Du, D^2u, D_tu belong to L^2-spaces, and u vanishes on the parabolic boundary $\partial\Omega_T$ in the trace sense. Therefore, a strong solution of (1.1) which is treated throughout this paper is a Sobolev function $u \in \mathring{W}^{1,2}_2(\Omega_T)$ satisfying (1.1) almost everywhere in Ω_T, for more details to see [18, Chapter VII]. In this context, we suppose that the nonlinearity $H(D^2u, Du, u, x, t)$ can be decomposed into following two parts:

$$H(D^2u, Du, u, x, t) = F(D^2u, x, t) + G(D^2u, Du, u, x, t).$$

Let $\mathcal{S}$ be the set of all symmetric $d \times d$-matrices, and they satisfy the following structural assumptions:

H1. We suppose that $F(u'', x, t)$ is convex and positive homogeneous of degree one with respect to $u'' \in \mathcal{S}$, and there exist two constants $0 < \lambda \leq \Lambda < \infty$ such that

$$\lambda\,|\xi|^2 \leq F(u'' + \xi\xi^*, x, t) - F(u'', x, t) \leq \Lambda\,|\xi|^2$$

for any $u'' \in \mathcal{S}$, $\xi \in \mathbb{R}^d$ and each $(x, t) \in \Omega_T$.

H2. For any fixed $u'' \in \mathcal{S}$, $u' \in \mathbb{R}^d$ and $(x, t) \in \Omega_T$, we let $G(u'', u', u, x, t)$ be a monotone nonincreasing function in $u \in \mathbb{R}$. Moreover, there exists a positive constant K such that

$$\left|G(u'', u', u, x, t) - G(u'', v', v, x, t)\right| \leq K\left(|u' - v'| + |u - v|\right)$$

and

$$\left|G(u'', u', u, x, t)\right| \leq \psi\left(|u''|\right)|u''| + K\left(|u'| + |u|\right) + g(x, t)$$

for any $u'' \in \mathcal{S}$, $u', v' \in \mathbb{R}^d$, $u, v \in \mathbb{R}$ and each $(x, t) \in \Omega_T$. In the above, the function $\psi(\rho) : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded, monotonically decreasing and $\psi(\rho) \to 0$ as $\rho \to 0$, while $g(x, t)$ is a given function specified later.

Before stating our main result, it is necessary to introduce some notations. For any $r > 0$ and $(x, t) \in \mathbb{R}^{d+1}$, we denote

$$B_r(x) = \left\{y \in \mathbb{R}^d \,:\, |x - y| < r\right\} \quad \text{and} \quad Q_r(x, t) = B_r(x) \times (t,\, t + r^2).$$

For convenience, we write $B_r = B_r(0)$ and $Q_r = Q_r(0, 0)$ in the following context. An average of $f(x, t)$ over Q_r is denoted by

$$\bar{f}_{Q_r} = \fint_{Q_r} f(x, t)\,dxdt = \frac{1}{|Q_r|}\int_{Q_r} f(x, t)\,dxdt,$$

where $|Q_r|$ is $(d + 1)$-dimensional Lebesgue measure of Q_r. To our aim, we need to impose the following (δ, R_0)-vanishing assumption on the leading term in (1.1) for some δ and $R_0 > 0$.

Assumption 1.1 For any $u'' \in S$ with $|u''| = 1$, we say that $F(u'', x, t)$ is (δ, R_0)-vanishing of (x, t). If there exists $R_0 > 0$ and a function $\widetilde{F}(u'')$ (independent of $z = (x, t)$) satisfying (H1), such that for any $r \in (0, R_0]$ and $(s, y) \in \overline{\Omega}_T$, we have

$$\sup_{u'' \in S,\ |u''|=1} \fint_{Q_r(s,y)} \left| F(u'', x, t) - \widetilde{F}(u'') \right| dxdt \leq \delta,$$

where $F(u'', x, t)$ is zero extended from $Q_r(s, y) \cap \Omega_T$ to $Q_r(s, y)$ for (x, t) close to the parabolic boundary, and the parameter $\delta > 0$ will be specified later.

We refer to [8] for the same assumptions as H1, H2 and Assumption 1.1, where Dong, Krylov and Li established the L^p estimate with $p > d + 1$ for the zero Cauchy-Dirichlet problems (1.1) in a bounded $C^{1,1}$-smooth domain. In the article, our aim is to extend it to a regularity in the scale of Lorentz space which is extremely useful when dealing with borderline existence as a refined version of Lebesgue spaces. So, it is necessary to recall the Lorentz space and Sobolev-Lorentz space since our Sobolev-Lorentz regularity for solutions is taken into consideration. The Lorentz space is a two-parameter scale of spaces which make the classical Lebesgue spaces refined in some sense, and the Sobolev-Lorentz space is similarly done from Sobolev space. More precisely, we have

Definition 1.2

(i) Let U be an open subset in $\mathbb{R}^{d+1}$. For $p \in [1, \infty)$ and $q \in (0, +\infty)$, the Lorentz space $L^{(p,q)}(U)$ is the set of all measurable functions $f : U \to \mathbb{R}^{d+1}$ such that

$$\|f\|_{L^{(p,q)}(U)} := \left(p \int_0^\infty \left(\kappa^p |\{(x,t) \in U : |f(x,t)| > \kappa\}| \right)^{\frac{q}{p}} \frac{d\kappa}{\kappa} \right)^{\frac{1}{q}} < +\infty. \quad (1.2)$$

For $q = \infty$, the space $L^{(p,\infty)}(U)$ is set to be the usual Marcinkiewicz space with quasinorm

$$\|f\|_{L^{(p,\infty)}(U)} := \sup_{\kappa > 0} \left(\kappa^p |\{(x,t) \in U : |f(x,t)| > \kappa\}| \right)^{\frac{1}{p}} < +\infty. \quad (1.3)$$

(ii) The Sobolev-Lorentz space $W^{1,2}_{(p,q)}(U)$ is defined by

$$W^{1,2}_{(p,q)}(U) = \left\{ f : f, \partial_t f, Df, D^2 f \in L^{(p,q)}(U) \right\},$$

where the quasinorm of f is

$$\|f\|_{W^{1,2}_{(p,q)}(U)} := \|f\|_{L^{(p,q)}(U)} + \left\|\partial_t f\right\|_{L^{(p,q)}(U)} + \|Df\|_{L^{(p,q)}(U)} + \left\|D^2 f\right\|_{L^{(p,q)}(U)}.$$

The regularity in the scale of Lorentz spaces concerning partial differential equations was originated from Talenti's work [23] based on the symmetrization argument. In 2010, Mingione [21] developed some sharp estimates in the Lorentz spaces for nonlinear elliptic equations in divergence form by using the duality exponent. Using the geometric argument first introduced by Byun and Wang [4], Mengesha and Phuc [20] recently obtained a global weighted Lorentz estimate to quasilinear elliptic equations with small BMO coefficients in Reifenberg flat domains. In addition, Baroni [2] showed a local Lorentz estimate for parabolic systems of p-Laplacian type by improving the classic Hardy inequality and the reverse Hölder inequality, and a similar result was also obtained in [3] for parabolic obstacle problems of p-Laplacian type by using the same technique. A global extension in the weighted Lorentz spaces was established by Tian and Zheng in [24] to the zero Cauchy-Dirichlet problems of nonlinear parabolic equations with small BMO nonlinearities in Reifenberg domains. Recently, Bui and Duong [5] got a global weighted Lorentz estimate to the weak solutions of divergence nonlinear parabolic equations with nonstandard growth over Reifenberg domains. Very recently, Tian and Zheng [25] also proved a global estimate of weak solutions in variable Lorentz spaces to parabolic obstacle problems with nonstandard growth over a quasiconvex domain based on the large-M-inequality principle originated from Acerbi and Mingione's work [1]. Zhang and Zheng in [28] also showed local Lorentz estimates for a class of fully nonlinear parabolic and elliptic equations by using the large-M-inequality principle.

We now summarize a global Sobolev-Lorentz estimate for fully nonlinear parabolic equation (1.1) as follows.

Theorem 1.3 *Let Ω be a bounded $C^{1,1}$-smooth domain and $T > 0$. For the Cauchy-Dirichlet problems (1.1) with assumptions H1 and H2, then there exists a small constant $\delta = \delta(d, \lambda, \Lambda, p, q, \partial\Omega) > 0$ such that if the $F(u'', x, t)$ is (δ, R_0)-vanishing shown as Assumption 1.1 for some $R_0 > 0$, then the following is true:*

(i) *For $u \in W^{1,2}_{(p,q)}(\Omega_T)$ satisfying (1.1) with $g \in L^{(p,q)}(\Omega_T)$ for $p \in (d+1, +\infty)$ and $q \in (0, +\infty]$, we have*

$$\|u\|_{W^{1,2}_{(p,q)}(\Omega_T)} \le N\big(\|g\|_{L^{(p,q)}(\Omega_T)} + \bar{\rho}\|\psi\|_{L^\infty}\big) \tag{1.4}$$

for some $\bar{\rho} > 0$, where N is a positive constant depending only on $d, p, q, \lambda, \Lambda, \delta, R_0, K, T, |\Omega|$ and $\|\partial\Omega\|_{C^{1,1}}$ while $q < \infty$, and the constant is independent of q while $q = \infty$.

(ii) *For any $g \in L^{(p,q)}(\Omega_T)$ with $p \in (d+1, +\infty)$ and $q \in (0, +\infty]$, there is a unique solution $u \in W^{1,2}_{(p,q)}(\Omega_T)$ of (1.1). In particular, if $\psi(0) = 0$ in (H2), then there is a unique solution $u \in W^{1,2}_{(p,q)}(\Omega_T)$ of (1.1) with the estimate*

$$\|u\|_{W^{1,2}_{(p,q)}(\Omega_T)} \le N\|g\|_{L^{(p,q)}(\Omega_T)}. \tag{1.5}$$

It is also worth noting that a solvability theorem in the space $W^{1,2}_{p,loc}(\Omega_T) \cap C(\overline{\Omega}_T)$ can be found in [7] for the initial boundary value problem for a class of fully nonlinear parabolic equations. The solvability in the Sobolev spaces $W^{1,2}_p(\overline{\Omega}_T)$ for $p > d+1$ of the terminal boundary value problem is proved by Dong, Krylov and

Li in [8] for such fully nonlinear parabolic equations (1.1), in bounded cylindrical domains, in the case of VMO "coefficients". Here, we further prove the solvability theorem 1.3 in the scale of Sobolev-Lorentz spaces under the assumptions of H1 and H2 along this line from Dong-Krylov's argument. A key ingredient of our main proof is that we need to improve the Fefferman-Stein type theorem of sharp functions in the Lorentz spaces. We note that Krylov in [15] gave a modified Fefferman-Stein theorem in Lebesgue spaces by a filtration of partitions, which leads to that the L^p-solvability was obtained for Bellman's equations with VMO coefficients in the whole space. Furthermore, Dong and Krylov [10] improved the Fefferman-Stein theorem in the mixed-norm weighted Lebesgue spaces, which yields a mixed-norm regularity for fully nonlinear elliptic and parabolic equations. Inspired by their works, we are going to establish a variant of the Fefferman-Stein theorem in the Lorentz spaces on the basis of the Calderón-Zygmund decomposition in the parabolic setting. Another important point is that the partitions of unity do not work for global estimates of the solutions in the Lorentz-Sobolev spaces, since the commuting transformation of integral does not hold in the Lorentz spaces. To this end, we make use of the idea of local L^p-estimate for nondivergence parabolic equations to handle the mean oscillation estimates of Hessian introduced by Krylov's paper [15], which originated from Lin's paper [19] for nondivergence elliptic equations.

Let us now consider the following Dirichlet problem of fully nonlinear elliptic equations:

$$\begin{cases} H(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

As a direct consequence of main Theorem , we immediately have the following solvability regarding the elliptic setting.

Corollary 1.4 *Let Ω be a bounded $C^{1,1}$-smooth domain. For the Dirichlet problem (1.6) with assumptions H1 and H2, then there exists a small constant $\delta = \delta(d, \lambda, \Lambda, p, q, \partial\Omega) > 0$ such that if the $F(u'', x)$ is (δ, R_0)-vanishing for some $R_0 > 0$, then the following is true*:

(i) *For $u \in W^2_{(p,q)}(\Omega)$ satisfying (1.6) with $g \in L^{(p,q)}(\Omega)$ with $p \in (d, +\infty)$ and $q \in (0, +\infty]$, we have*

$$\|u\|_{W^2_{(p,q)}(\Omega)} \leq N\big(\|g\|_{L^{(p,q)}(\Omega)} + \bar{\rho}\|\psi\|_{L^\infty}\big) \tag{1.7}$$

for some $\bar{\rho} > 0$, where N is a positive constant depending only on $d, p, q, \lambda, \Lambda, \delta, R_0, K, |\Omega|$ and $\|\partial\Omega\|_{C^{1,1}}$ if *$q < \infty$, while the constant is not dependent of q if $q = \infty$. Specifically, if $\psi(0) = 0$ in (H2), then we have*

$$\|u\|_{W^2_{(p,q)}(\Omega)} \leq N\|g\|_{L^{(p,q)}(\Omega)}. \tag{1.8}$$

(ii) *For any* $g \in L^{(p,q)}(\Omega)$ *with* $p \in (d, +\infty)$ *and* $q \in (0, +\infty]$, *there is a unique solution* $u \in W^2_{(p,q)}(\Omega)$ *of* (1.6).

The remainder of this paper is organized as follows. In Sect. 2, we present the Fefferman-Stein type theorem in the Lorentz spaces, and some necessary preliminary results concerning the Gagliardo-Nirenberg interpolation inequality and Alexandrov's estimate in the Lorentz spaces. In Sect. 3 we complete the proof of Theorem 1.3. Moreover, as a corollary of main Theorem 1.3, a global Sobolev-Lorentz estimate of the solution is proved for parabolic Bellman's equations with small BMO coefficients in a bounded $C^{1,1}$-smooth domain.

2 Preliminary results

This section is devoted to some related notations and basic facts which will be useful in the main proof. Throughout this paper, we denote $N(d, p, q, \lambda, \Lambda, \cdots)$ or $N_i(d, p, q, \lambda, \Lambda, \cdots)$ for $i = 0, 1, 2, \cdots$, a universal constant depending only on prescribed quantities and possibly varying from line to line. First, we give a generalized version of the Fefferman-Stein theorem of sharp functions in the Lorentz spaces. Let $\mathfrak{C}$ be a set of all $\mathcal{C}_{(R^2,R)}(x, t)$ for any $R > 0$ with the fixed point $(x, t) \in \mathbb{R}^{d+1}$. When f is locally integrable on $\mathbb{R}^{d+1}$, we define the parabolic maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x,t) = \sup_{\mathcal{C}\in\mathfrak{C}} \fint_{\mathcal{C}} |f(s,y)|\, dyds. \tag{2.1}$$

For $\gamma \in (0, 1]$, we set

$$f^{\sharp}_{\gamma}(x,t) = \sup_{\mathcal{C}\in\mathfrak{C}} \left(\fint_{\mathcal{C}} \fint_{\mathcal{C}} |f(s,y) - f(\tau,z)|^{\gamma}\ dydsdzd\tau \right)^{\frac{1}{\gamma}}. \tag{2.2}$$

Remark 2.1 While $\gamma = 1$, we observe that $f \in BMO$ is equivalent to $f^{\sharp}_1 \in L^{\infty}$. We also note that $BMO \cong \mathcal{L}^{p,d}$(the Campanato spaces) for any $1 \le p < \infty$ (cf. [22, Chapter IV]), which leads to this fact that the norms $\|f^{\sharp}_{\gamma}\|_{L^{\infty}}$ and $\|f^{\sharp}_1\|_{L^{\infty}}$ are equivalent for any $\gamma \in (0, 1]$.

The Hardy-Littlewood maximal function theorem states that if $f \in L^p$ for $1 < p \le \infty$, then $\mathcal{M}f \in L^p$ and $\|\mathcal{M}f\|_{L^p} \le N(p, d)\,\|f\|_{L^p}$, for instance, see [22, Chapter I]. By interpolating argument, Mengesha and Phuc in [20, Lemma 3.11] extended the above boundedness of the Hardy-Littlewood maximal function from the Lebesgue spaces to the weighted Lorentz spaces. Similarly, if $f^{\sharp}_{\gamma} \in L^p$ for $1 < p < \infty$, then $\mathcal{M}f \in L^p$ and $\|\mathcal{M}f\|_{L^p} \le N(d, p, \gamma)\,\|f^{\sharp}_{\gamma}\|_{L^p}$ as given by Krylov in [15, Theorem 5.3]. Therefore, the interpolation method is also applied to the Fefferman-Stein sharp function in the Lorentz spaces just as what Mengesha and Phuc did. Indeed, for any fixed $1 < p < \infty$, we take $p_1 = \frac{p+1}{2}$ to obtain

$$\|\mathcal{M}f\|_{L^{p_1}(\mathbb{R}^{d+1})} \le N_1 \|f_\gamma^\sharp\|_{L^{p_1}(\mathbb{R}^{d+1})}$$

for any $f_\gamma^\sharp \in L^{p_1}(\mathbb{R}^{d+1})$; while we take $p_2 = p+1$ to show

$$\|\mathcal{M}f\|_{L^{p_2}(\mathbb{R}^{d+1})} \le N_2 \|f_\gamma^\sharp\|_{L^{p_2}(\mathbb{R}^{d+1})}$$

for any $f_\gamma^\sharp \in L^{p_2}(\mathbb{R}^{d+1})$. Note that for any $1 < p < \infty$, $1 < p_1 < p < p_2 < +\infty$. By taking $\vartheta = 1 - \frac{1}{p}$ such that $\frac{1}{p} = \frac{1-\vartheta}{p_1} + \frac{\vartheta}{p_2}$, we use the interpolation theorem for Lorentz spaces (cf. [12, Theorem 1.4.19]) to obtain the following conclusion

$$\| \mathcal{M}f \|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le N_0 N_1^{1-\vartheta} N_2^{\vartheta} \| f_\gamma^\sharp \|_{L^{(p,q)}(\mathbb{R}^{d+1})}.$$

In summary, we have the following conclusion.

Lemma 2.2 *Let* $1 < p < \infty, 0 < q \le \infty$ *and* $\gamma \in (0,1]$. *Then, for any* $f_\gamma^\sharp \in L^{(p,q)}(\mathbb{R}^{d+1})$ *there exists a constant N depending only on d, p, q and γ such that*

$$\| \mathcal{M}f \|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le N \| f_\gamma^\sharp \|_{L^{(p,q)}(\mathbb{R}^{d+1})}.$$

Furthermore, we can also show the improved Fefferman-Stein Theorem (Lemma 2.2) in a half space via the odd or even extensions, see Lemma 5 in [8]. To this end, we write

$$\mathbb{R}^{d+1}_+ = \left\{ (x,t) = (x, t^1, \cdots, x^d) \in \mathbb{R}^{d+1} \ : \ x^1 > 0 \right\}.$$

Lemma 2.3 *Let* $1 < p < \infty, 0 < q \le \infty$ *and* $\gamma \in (0,1]$. *Then, for any* $f_\gamma^\sharp \in L^{(p,q)}(\mathbb{R}^{d+1}_+)$ *there exists a constant N depending only on d, p, q and γ such that*

$$\| \mathcal{M}f \|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)} \le N \| f_\gamma^\sharp \|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)}.$$

Next let us extend the Gagliardo–Nirenberg interpolation inequality from the Lebesgue spaces to the Lorentz spaces. We first recall the conclusion by Dong and Krylov in [10, Lemma 6.5 (ii)]. For a fix $\rho > 0$ and $f(x,t) \in L^1_{loc}(\mathbb{R}^{d+1})$, we introduce the confined maximal function and the confined sharp function:

$$\mathbb{M}_\rho f(x,t) = \sup_{\left\{\begin{array}{l} \mathcal{C} \in \mathfrak{C} \ : \\ R \in [\rho, \infty) \end{array}\right\}} \fint_{\mathcal{C}} |f(s,y)|\, dyds,$$

$$f^\sharp_{1,\rho}(x,t) = \sup_{\left\{\begin{array}{l} \mathcal{C} \in \mathfrak{C} \ : \\ R \in (0, \rho] \end{array}\right\}} \fint_{\mathcal{C}} \fint_{\mathcal{C}} |f(s,y) - f(\tau, z)|\, dydsdzd\tau.$$

Lemma 2.4 *For any $\rho > 0$, $p \in [1, \infty)$, and $f \in W^{1,2}_{p,\mathrm{loc}}(\mathbb{R}^{d+1})$, we have*

$$\mathbb{M}_\rho(|Df|^p) \le N\,\mathbb{M}_\rho^{\frac{1}{2}}\left(|D^2 f|^p\right)\mathbb{M}_\rho^{\frac{1}{2}}(|f|^p) + N\rho^{-p}\,\mathbb{M}_\rho(|f|^p),$$

where $N = N(d,p)$ is a positive constant.

Indeed, Lemma 2.4 yields the following interpolation inequality in the Lorentz space.

Lemma 2.5 *Let $1 < p < \infty$ and $0 < q \le \infty$. Then for any $f \in W^{1,2}_{(p,q),\mathrm{loc}}(\mathbb{R}^{d+1})$, we have*

$$\| Df \|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le N\left(\varepsilon\left\| D^2 f \right\|_{L^{(p,q)}(\mathbb{R}^{d+1})} + \varepsilon^{-1}\| f \|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right) \tag{2.3}$$

for any $\varepsilon > 0$, where $N = N(p,q,d)$ is a positive constant.

Proof Indeed, we note that for any $\rho > 0$,

$$(Df)^\sharp_1(x,t) \le (Df)^\sharp_{1,\rho}(x,t) + 2\mathbb{M}_\rho(|Df|)(x,t) \quad \text{in} \quad \mathbb{R}^{d+1}. \tag{2.4}$$

By the Poincaré inequality we get

$$(Df)^\sharp_{1,\rho}(x,t) \le N\rho\mathcal{M}(|D^2 f|)(x,t).$$

Following Lemma 2.4 with $p = 1$ it yields

$$\mathbb{M}_\rho(|Df|)(x,t) \le N\rho\mathcal{M}(|D^2 f|)(x,t) + N\rho^{-1}\,\mathcal{M}(|f|)(x,t).$$

Putting the above two estimates into (2.4) deduces

$$(Df)^\sharp_1(x,t) \le N\rho\mathcal{M}(|D^2 f|)(x,t) + N\rho^{-1}\,\mathcal{M}(|f|)(x,t)$$

for any $(x,t) \in \mathbb{R}^{d+1}$.

Now, we make use of Lemma 2.2 and the Hardy-Littlewood maximal function theorem in the Lorentz spaces to obtain the required result by taking $\varepsilon = \rho$. This completes the proof. □

In what follows, we consider the Cauchy-Dirichlet problem of linear parabolic equations in nondivergence form:

$$\begin{cases} \partial_t u + Lu := \partial_t u + a^{ij}(x,t)D_{ij}u + b^i(x,t)D_i u - c(x,t)u = \varphi(x,t) & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \partial\Omega_T. \end{cases} \tag{2.5}$$

Here, we suppose that all coefficients $a^{ij}(x,t)$, $b^i(x,t)$ and $c(x,t)$ are bounded measurable functions defined on Ω_T and the symmetric coefficient matrix $A(x,t) = \left(a^{ij}(x,t)\right)$

satisfies an uniformly ellipticity, which means that there exist positive constants $0 < \lambda \le \Lambda < \infty$ such that

$$|b^i(x,t)| \le \Lambda, \quad i = 1, \cdots, d; \quad |c(x,t)| \le \Lambda \tag{2.6}$$

and

$$\lambda|\xi|^2 \le A(x,t)\,\xi\xi^* \le \Lambda|\xi|^2 \tag{2.7}$$

for any $\xi \in \mathbb{R}^d$ and almost all $(x,t) \in \Omega_T$.

In the following we recall the classical Aleksandrov estimate to the Cauchy-Dirichlet problem (2.5), see [13, Lemma 3].

Lemma 2.6 *Let $p \ge d+1$. Assume that $u \in \mathring{W}_p^{1,2}(\Omega_T)$ is a solution of (2.5) with (2.6) and (2.7). If $\varphi(x,t) \in L^p(\Omega_T)$ and $c(x,t) \ge 0$ in Ω_T, then*

$$\operatorname{ess\,sup}_{(x,t)\in\Omega_T} |u(x,t)| \le N \left\| |c|^{\frac{d+1-p}{p}} |\det A|^{\frac{1}{p}} \varphi \right\|_{L^p(\Omega_T)},$$

where $N = N(d, p, \lambda, \Lambda, |\Omega|)$.

Combining Lemma 2.6 with the imbedding inequality in the Lorentz spaces, it yields the following $L^{(p,q)}$-estimate to the problem (2.5).

Lemma 2.7 *For $d+1 < p < \infty$ and $0 < q \le \infty$, let $u \in \mathring{W}_{(p,q)}^{1,2}(\Omega_T)$ be the solution of (2.5) with (2.6) and (2.7). If $\varphi(x,t) \in L^{(p,q)}(\Omega_T)$ and $c(x,t) \ge 0$ in Ω_T, then*

$$\| u \|_{L^{(p,q)}(\Omega_T)} \le N \| \varphi \|_{L^{(p,q)}(\Omega_T)}, \tag{2.8}$$

where $N = N(d, p, q, \lambda, \Lambda, T, |\Omega|)$.

Proof We first use Lemma 2.6 with $p = d+1$, and the assumptions (2.6, 2.7) to obtain

$$|u(x,t)| \le N \|\varphi\|_{L^{d+1}(\Omega_T)}$$

for almost all $(x,t) \in \Omega_T$. Furthermore, by the imbedding inequality in the Lorentz spaces (cf. [12]), for any $p > d+1$ it yields

$$\|\varphi\|_{L^{d+1}(\Omega_T)} \le N\, |\Omega_T|^{\frac{1}{d+1} - \frac{1}{p}} \|\varphi\|_{L^{(p,q)}(\Omega_T)},$$

where $N = N(p, q, d)$. Putting the above two estimates together leads to the required result. This completes the proof. □

3 Proof of Theorem 1.3

This section is devoted to our main proof. We first consider the following parabolic equation

$$\partial_t u(x,t) + F(D^2u, x, t) = h(x,t) \qquad \text{in} \quad \mathbb{R}^{d+1}. \tag{3.1}$$

Hereafter, we suppose that Assumption 1.1 holds with $\mathbb{R}^{d+1}$ instead of Ω_T. To obtain the Sobolev-Lorentz estimate of (3.1) in the whole space, for any fixed point $(t_0, x_0) \in \mathbb{R}^{d+1}$, we begin with establishing the following mean oscillation estimate on $Q_r(t_0, x_0)$ for $r > 0$.

Lemma 3.1 *For $\kappa \geq 2, r > 0, \beta \in (1,\infty)$ and $\alpha \in (0,1)$, let $u \in W^{1,2}_{\beta(d+1)}(\mathbb{R}^{d+1})$ be the solution of (3.1). Then, there exists a small constant $\delta_0 = \delta_0(d, \lambda, \Lambda, \alpha, \beta, \gamma, R_0) > 0$ with some $R_0 > 0$ such that the following holds: if Assumption1.1 holds for any $\delta \in (0, \delta_0)$, then we have*

$$\begin{aligned}
&\fint_{Q_r(t_0,x_0)} \fint_{Q_r(t_0,x_0)} \left| D^2u(x,t) - D^2u(s,y) \right|^{\gamma} dxdtdyds \\
&\leq N\kappa^{d+2} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |h|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\
&\quad + N\kappa^{d+2} \delta^{\frac{\gamma}{(d+1)\beta'}} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |D^2u|^{(d+1)\beta}\, dxdt \right)^{\frac{\gamma}{(d+1)\beta}} \\
&\quad + N\kappa^{-\alpha\gamma} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |D^2u|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\
&\quad + N\kappa^{d+2} R_0^{-2\gamma} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |u|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\
&\quad + N\kappa^{d+2} R_0^{-\gamma} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |Du|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}}
\end{aligned} \tag{3.2}$$

for some $\gamma \in (0,1]$, where $\beta' = \beta/(\beta-1)$ and $N = N(d, \delta, \lambda, \Lambda, \beta)$.

Proof The proof is divided into two folds.

Case 1. If $0 < \kappa r \leq R_0$, in the Assumption 1.1 we take

$$\widetilde{F}(D^2u) = \bar{F}_{Q_{\kappa r}(t_0,x_0)}(D^2u).$$

Consider

$$\partial_t u(x,t) + \widetilde{F}(D^2u) = \tilde{h}(x,t) \qquad \text{in} \quad \mathbb{R}^{d+1}, \tag{3.3}$$

where

$$\tilde{h}(x,t) = h(x,t) + \widetilde{F}(D^2u) - F(D^2u,x,t).$$

Let us take $\hat{u}'' = \dfrac{D^2u}{|D^2u|}$, then $\hat{u}'' \in S$ and $|\hat{u}''| = 1$. With (H1) we get

$$\begin{aligned}
&\fint_{Q_{\kappa r}(t_0,x_0)} \left|F(D^2u,x,t) - \widetilde{F}(D^2u)\right|^{d+1} dxdt \\
&\le \sup_{u''\in S,\ |u''|=1} \fint_{Q_{\kappa r}(t_0,x_0)} \left|F(u'',x,t) - \widetilde{F}(u'')\right|^{d+1} |D^2u|^{d+1}\, dxdt \\
&\le N\delta^{\frac{1}{\beta'}} \left(\fint_{Q_{\kappa r}(t_0,x_0)} |D^2u|^{(d+1)\beta}\, dxdt\right)^{\frac{1}{\beta}},
\end{aligned}$$

where δ is a small positive constant shown as in Assumption 1.1 and $N = N(d,\lambda,\Lambda)$. It combined with Lemma 5.6 in [8] yields

$$\begin{aligned}
&\fint_{Q_r(t_0,x_0)} \fint_{Q_r(t_0,x_0)} \left|D^2u(x,t) - D^2u(s,y)\right|^{\gamma} dxdtdyds \\
&\le N\kappa^{d+2}\left(\fint_{Q_{\kappa r}(t_0,x_0)} |h|^{d+1}\, dxdt\right)^{\frac{\gamma}{d+1}} \\
&\quad + N\kappa^{d+2}\delta^{\frac{\gamma}{(d+1)\beta'}}\left(\fint_{Q_{\kappa r}(t_0,x_0)} |D^2u|^{(d+1)\,\beta}\, dxdt\right)^{\frac{\gamma}{(d+1)\beta}} \\
&\quad + N\kappa^{-\alpha\gamma}\left(\fint_{Q_{\kappa r}(t_0,x_0)} |D^2u|^{d+1}\, dxdt\right)^{\frac{\gamma}{d+1}}
\end{aligned} \tag{3.4}$$

for some $\gamma \in (0,1]$.

Case 2. If $\kappa r > R_0$, we remark that

$$\begin{aligned}
&\fint_{Q_r(t_0,x_0)} \fint_{Q_r(t_0,x_0)} \left|D^2u(x,t) - D^2u(s,y)\right|^{\gamma} dxdtdyds \\
&\le 2\fint_{Q_r(t_0,x_0)} |D^2u(x,t)|^{\gamma}\, dxdt.
\end{aligned} \tag{3.5}$$

It is a well-known fact that condition (H1) implies that there is an operator $L_0 = a^{ij}(x,t)D_{ij}$ with assumption (2.7) such that $\partial_t u(x,t) + F(D^2u,x,t) = \partial_t u(x,t) + L_0u(x,t)$. Therefore, we use Corollary 4.2 in [15] with dilations and the standard cut-off argument to obtain

$$\fint_{Q_{\kappa r/2}(t_0,x_0)} |D^2u(x,t)|^\gamma \, dxdt \le N\Bigg(\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |h|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}} + \frac{1}{(\kappa r)^{2\gamma}}\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |u|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}} + \frac{1}{(\kappa r)^{\gamma}}\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |Du|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}}\Bigg)$$

for some $\gamma \in (0,1)$. With (3.5) and $\kappa \ge 2$ it leads to that

$$\begin{aligned}&\fint_{Q_r(t_0,x_0)}\fint_{Q_r(t_0,x_0)} \left|D^2u(x,t) - D^2u(s,y)\right|^\gamma dxdtdyds \\ &\le N\kappa^{d+2}\Bigg(\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |h|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}} \\ &+ R_0^{-2\gamma}\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |u|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}} \\ &+ R_0^{-\gamma}\Big(\fint_{Q_{\kappa r}(t_0,x_0)} |Du|^{d+1}\, dxdt\Big)^{\frac{\gamma}{d+1}}\Bigg).\end{aligned} \tag{3.6}$$

Putting the above two cases together completes the proof of the lemma. □

Using Lemma 3.1 and the modified Fefferman-Stein Lemma 2.2, we get the following theorem.

Theorem 3.2 *For $p \in (d+1, +\infty)$ and $q \in (0, +\infty]$, let $u \in W^{1,2}_{(p,q)}(\mathbb{R}^{d+1})$ satisfying (3.1) with $h \in L^{(p,q)}(\mathbb{R}^{d+1})$. Then there exists a small constant $\delta_0 = \delta_0(d, \lambda, \Lambda, R_0) > 0$ with some $R_0 > 0$ such that if Assumption 1.1 holds for any $\delta \in (0, \delta_0)$, then we have*

$$\begin{aligned}&\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})} + \|\partial_t u\|_{L^{(p,q)}(\mathbb{R}^{d+1})} \\ &\le N\big(\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1})} + \|u\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\big),\end{aligned} \tag{3.7}$$

where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0)$.

Proof By Lemma 3.1 with $\alpha \in (0,1)$, we obtain

$$
\begin{aligned}
&\left(D^2u\right)^{\sharp}_{\gamma}(t_0,x_0)\\
&\quad\le N\kappa^{\frac{d+2}{\gamma}}\mathcal{M}^{\frac{1}{d+1}}\left(|h|^{d+1}\right)(t_0,x_0)\\
&\qquad+N\kappa^{\frac{d+2}{\gamma}}\delta^{\frac{1}{(d+1)\beta'}}\mathcal{M}^{\frac{1}{(d+1)\beta}}\left(|D^2u|^{(d+1)\beta}\right)(t_0,x_0)\\
&\qquad+N\kappa^{-\alpha}\mathcal{M}^{\frac{1}{d+1}}\left(|D^2u|^{d+1}\right)(t_0,x_0)\\
&\qquad+N\kappa^{\frac{d+2}{\gamma}}R_0^{-2}\mathcal{M}^{\frac{1}{d+1}}\left(|u|^{d+1}\right)(t_0,x_0)\\
&\qquad+N\kappa^{\frac{d+2}{\gamma}}R_0^{-1}\mathcal{M}^{\frac{1}{d+1}}\left(|Du|^{d+1}\right)(t_0,x_0)
\end{aligned}
$$

for some $\gamma \in (0,1]$. Therefore, with the modified Fefferman-Stein Lemma 2.2 we get

$$
\begin{aligned}
\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le & N\left(\kappa^{\frac{d+2}{\gamma}}\delta^{\frac{1}{(d+1)\beta'}}+\kappa^{-\alpha}\right)\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\\
&+N\kappa^{\frac{d+2}{\gamma}}\left(\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right.\\
&\left.+\ \|u\|_{L^{(p,q)}(\mathbb{R}^{d+1})}+\|Du\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right),
\end{aligned}
\tag{3.8}
$$

where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0)$. Now, let us first take $\kappa > 0$ being big number, and then take small $\delta > 0$ such that

$$
N\left(\kappa^{\frac{d+2}{\gamma}}\delta^{\frac{1}{(d+1)\beta'}}+\kappa^{-\alpha}\right)\le\frac{1}{2}.
$$

Therefore, it follows from the Gagliardo-Nirenberg interpolation inequality (2.3) that

$$
\begin{aligned}
\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le & N\left(\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right.\\
&\left.+\|u\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right).
\end{aligned}
\tag{3.9}
$$

On the other hand, we notice that

$$
\partial_t u(x,t) = -F(D^2u,x,t) + h(x,t) \qquad \text{in} \quad \mathbb{R}^{d+1}.
$$

By (H1) we have

$$
\begin{aligned}
\left\|\partial_t u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})} \le & N\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1})}+\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\\
\le & N\left(\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1})}+\|u\|_{L^{(p,q)}(\mathbb{R}^{d+1})}\right).
\end{aligned}
\tag{3.10}
$$

Combining (3.9) with (3.10) yields (3.7). □

In the following we consider the initial-boundary problem of (3.1) over a half space. We let

$$
\partial\,\mathbb{R}^{d+1}_{+} = \left\{\,(x,t) = (x, t^1, \cdots, x^d) \in \mathbb{R}^{d+1}\ :\ x^1 = 0\,\right\}
$$

and consider

$$\begin{cases} \partial_t u(x,t) + F(D^2u, x, t) = h(x,t) & \text{in} \quad \mathbb{R}^{d+1}_+, \\ u = 0 & \text{on} \quad \partial\, \mathbb{R}^{d+1}_+. \end{cases} \tag{3.11}$$

Here, we suppose that there exists a small constant $\delta > 0$ such that Assumption 1.1 holds with $\mathbb{R}^{d+1}_+$ instead of Ω_T. For any $r > 0$ and $(x,t) \in \mathbb{R}^{d+1}_+$, we denote

$$Q^+_r(x,t) = Q_r(x,t) \cap \mathbb{R}^{d+1}_+, \qquad Q^+_r = Q^+_r(0,0), \qquad Q^+_r(x^1) = Q^+_r(0,x^1,0).$$

Let us fix $(t_0, x_0) \in \overline{\mathbb{R}^{d+1}_+}$. To attain the mean oscillation estimate on $Q^+_r(t_0,x_0)$ for any $r > 0$ we mainly make use of Lemma 7.4 in [8] in terms of the odd extension technique.

Lemma 3.3 *For* $\kappa \geq 2$, $r > 0$, $\beta \in (1, \infty)$ *and* $\alpha \in (0,1)$, *let* $u \in \mathring{W}^{1,2}_{d+1}(\mathbb{R}^{d+1}_+)$ *be the solution of* (3.11) *and* $(t_0,x_0) \in \mathbb{R}^{d+1}_+$. *Then there exists a small constant* $\delta_0 = \delta_0(d, \lambda, \Lambda, \beta) > 0$ *such that the following holds: if Assumption* 1.1 *holds for any* $\delta \in (0, \delta_0)$, *then we have*

$$\begin{aligned} &\fint_{Q^+_r(t_0,x_0)} \fint_{Q^+_r(t_0,x_0)} \left| D^2u(x,t) - D^2u(s,y) \right|^\gamma dxdtdyds \\ &\leq N\kappa^{d+2} \left(\fint_{Q^+_{\kappa r}(t_0,x_0)} |h|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\ &\quad + N\kappa^{d+2} \delta^{\frac{\gamma}{(d+1)\beta'}} \left(\fint_{Q^+_{\kappa r}(t_0,x_0)} |D^2u|^{(d+1)\beta}\, dxdt \right)^{\frac{\gamma}{(d+1)\beta}} \\ &\quad + N\kappa^{-\alpha\gamma} \left(\fint_{Q^+_{\kappa r}(t_0,x_0)} |D^2u|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\ &\quad + N\kappa^{d+2} R_0^{-2\gamma} \left(\fint_{Q^+_{\kappa r}(t_0,x_0)} |u|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \\ &\quad + N\kappa^{d+2} R_0^{-\gamma} \left(\fint_{Q^+_{\kappa r}(t_0,x_0)} |Du|^{d+1}\, dxdt \right)^{\frac{\gamma}{d+1}} \end{aligned}$$

with some $\gamma \in (0,1]$, *where* $\beta' = \beta/(\beta - 1)$ *and* $N = N(d, \delta, \lambda, \Lambda, \beta)$.

If f is measurable function defined in $\mathbb{R}^{d+1}_+$, then for any $(x,t) \in \overline{\mathbb{R}^{d+1}_+}$ and $r > 0$,

$$\fint_{Q^+_r(x,t)} |f(s,y)|\, dyds \leq 2 \fint_{Q_r(x,t)} |f(s,y) I_{\mathbb{R}^{d+1}_+}|\, dyds \leq N(d)\, \mathcal{M}\,(f\, I_{\mathbb{R}^{d+1}_+})\,(x,t),$$

where $I_{\mathbb{R}^{d+1}_+}$ is the indicator function of $\mathbb{R}^{d+1}_+$. On the other hand, for the smallest $r > 0$ such that $\mathcal{C} \subset Q^+_r$, we have

$$\left(\fint_{\mathcal{C}}\fint_{\mathcal{C}}|f(x,t)-f(s,y)|^{\gamma}\,dxdtdyds\right)^{\frac{1}{\gamma}}$$
$$\leq N(d,\gamma)\left(\fint_{Q_r^+}\fint_{Q_r^+}|f(x,t)-f(s,y)|^{\gamma}\,dxdtdyds\right)^{\frac{1}{\gamma}}.$$

With Lemma 3.3 we obtain the following result.

Corollary 3.4 *Under the same hypotheses as in Lemma 3.3, we have*

$$\begin{aligned}
&\left(D^2u\right)^{\sharp}_{\gamma}(t_0,x_0)\\
&\leq N\kappa^{\frac{d+2}{\gamma}}\mathcal{M}^{\frac{1}{d+1}}\left(|h|^{d+1}I_{\mathbb{R}^{d+1}_+}\right)(t_0,x_0)\\
&\quad+N\kappa^{\frac{d+2}{\gamma}}\delta^{\frac{1}{(d+1)\beta'}}\mathcal{M}^{\frac{1}{(d+1)\beta}}\left(|D^2u|^{(d+1)\beta}I_{\mathbb{R}^{d+1}_+}\right)(t_0,x_0)\\
&\quad+N\kappa^{-\alpha}\mathcal{M}^{\frac{1}{d+1}}\left(|D^2u|^{d+1}I_{\mathbb{R}^{d+1}_+}\right)(t_0,x_0)\\
&\quad+N\kappa^{\frac{d+2}{\gamma}}R_0^{-2}\mathcal{M}^{\frac{1}{d+1}}\left(|u|^{d+1}I_{\mathbb{R}^{d+1}_+}\right)(t_0,x_0)\\
&\quad+N\kappa^{\frac{d+2}{\gamma}}R_0^{-1}\mathcal{M}^{\frac{1}{d+1}}\left(|Du|^{d+1}I_{\mathbb{R}^{d+1}_+}\right)(t_0,x_0)
\end{aligned}$$

for any $(t_0,x_0)\in\overline{\mathbb{R}^{d+1}_+}$, *where* $N=N(d,\delta,\lambda,\Lambda,\beta)$.

Hence, the following regularity theorem in $\mathring{W}^{1,2}_{(p,q)}(\mathbb{R}^{d+1}_+)$ can be deduced from Corollary 3.4 and Lemma 2.3 with the same way as the proof of Theorem 3.2.

Theorem 3.5 *For* $p\in(d+1,+\infty)$ *and* $q\in(0,+\infty]$, *let* $u\in\mathring{W}^{1,2}_{(p,q)}(\mathbb{R}^{d+1}_+)$ *satisfying* (3.11) *with* $h\in L^{(p,q)}(\mathbb{R}^{d+1}_+)$. *Then there exists a small constant* $\delta_0=\delta_0(d,\lambda,\Lambda,R_0)>0$ *for some* $R_0>0$ *such that if Assumption* 1.1 *holds for any* $\delta\in(0,\delta_0)$ *in* $\mathbb{R}^{d+1}_+$, *then we have*

$$\begin{aligned}
&\left\|D^2u\right\|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)}+\|\partial_t u\|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)}\\
&\leq N\left(\|h\|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)}+\|u\|_{L^{(p,q)}(\mathbb{R}^{d+1}_+)}\right),
\end{aligned}\tag{3.12}$$

where $N=N(d,p,q,\lambda,\Lambda,\delta,R_0)$.

On the basis of Theorems 3.2 and 3.5, we now prove the main Theorem 1.3 to the Cauchy-Dirichlet problems (1.1) in Ω_T. Here Ω is a bounded $C^{1,1}$ domain and there exists a small constant $\delta>0$ such that Assumption 1.1 holds in Ω_T.

Proof of Theorem 1.3 (i) Let us define

$$h(x,t)=-G(D^2u,Du,u,x,t).\tag{3.13}$$

We employ the technique of flattening the boundary and Theorems 3.2 and 3.5 which is a similar way shown as in [17]. Then, we obtain

$$\left\|D^2u\right\|_{L^{(p,q)}(\Omega_T)} + \|\partial_t u\|_{L^{(p,q)}(\Omega_T)} \le N\big(\|h\|_{L^{(p,q)}(\Omega_T)} + \|u\|_{L^{(p,q)}(\Omega_T)}\big), \tag{3.14}$$

where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0, T, C^{1,1}$ norm of $\partial\Omega)$. The definition of h together with Assumption (H2) leads to that

$$\begin{aligned}\|h\|_{L^{(p,q)}(\Omega_T)} =& \left\|G(D^2u, Du, u, x, t)\right\|_{L^{(p,q)}(\Omega_T)}\\ \le& \left\|\psi\left(|D^2u|\right)|D^2u|\right\|_{L^{(p,q)}(\Omega_T)} + K\left(\|Du\|_{L^{(p,q)}(\Omega_T)}\right.\\ &\left.+\|u\|_{L^{(p,q)}(\Omega_T)}\right) + \|g\|_{L^{(p,q)}(\Omega_T)}.\end{aligned} \tag{3.15}$$

Recall that the function $\psi(\rho) : \bar{\mathbb{R}}^+ \to \bar{\mathbb{R}}^+$ is bounded, monotonically decreasing and $\psi(\rho) \to 0$ as $\rho \to 0$. Therefore, for some $\bar{\rho} > 0$ we obtain

$$\begin{aligned}&\left\|\psi\left(|D^2u|\right)|D^2u|\right\|_{L^{(p,q)}(\Omega_T)}\\ &\le \psi(\bar{\rho})\left\|D^2u\right\|_{L^{(p,q)}(\Omega_T)} + N(p,q)\,\bar{\rho}\|\psi\|_{L^\infty}|\Omega_T|^{\frac{1}{p}}.\end{aligned} \tag{3.16}$$

This together with (3.15) and (3.14) yield

$$\begin{aligned}&\left\|D^2u\right\|_{L^{(p,q)}(\Omega_T)} + \|\partial_t u\|_{L^{(p,q)}(\Omega_T)}\\ &\le N\,\psi(\bar{\rho})\left\|D^2u\right\|_{L^{(p,q)}(\Omega_T)} + N\,\bar{\rho}\|\psi\|_{L^\infty}|\Omega_T|^{\frac{1}{p}}\\ &\quad + N\left(\|Du\|_{L^{(p,q)}(\Omega_T)} + \|u\|_{L^{(p,q)}(\Omega_T)} + \|g\|_{L^{(p,q)}(\Omega_T)}\right),\end{aligned}$$

where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0, T, K, C^{1,1}$ norm of $\partial\Omega)$. Taking $\bar{\rho}$ large enough that $N\,\psi(\bar{\rho}) \le \frac{1}{2}$ and using again the Gagliardo-Nirenberg interpolation inequality (2.3) in the Lorentz spaces, we obtain

$$\begin{aligned}&\left\|D^2u\right\|_{L^{(p,q)}(\Omega_T)} + \|\partial_t u\|_{L^{(p,q)}(\Omega_T)}\\ &\le N\left(\|u\|_{L^{(p,q)}(\Omega_T)} + \|g\|_{L^{(p,q)}(\Omega_T)} + \bar{\rho}\|\psi\|_{L^\infty}\right),\end{aligned} \tag{3.17}$$

where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0, T, K, |\Omega|, C^{1,1}$ norm of $\partial\Omega)$. Finally, to estimate the Lorentz norm of u, we rewrite the Cauchy-Dirichlet problem (1.1) as

$$\begin{cases}\partial_t u(x,t) + F(D^2u, x, t) + G(D^2u, Du, u, x, t) - G(D^2u, 0, 0, x, t) = -G(D^2u, 0, 0, x, t) & \text{in} \quad \Omega_T,\\ u = 0 & \text{on} \quad \partial\Omega_T.\end{cases}$$

Conditions (H1) and (H2) imply that there exists an operator $L\cdot = a^{ij}D_{ij}\,\cdot + b^iD_i\,\cdot - c\,\cdot$ with (2.6) and (2.7), such that

$$\begin{cases} \partial_t u(x,t) + Lu = -G(D^2u, 0, 0, x, t) & \text{in} \quad \Omega_T, \\ u = 0 & \text{on} \quad \partial\Omega_T. \end{cases}$$

Since G is a monotonically nonincreasing function in u, by the Alexandrov estimate in the Lorentz spaces (cf. Lemma 2.7) we obtain

$$\| u \|_{L^{(p,q)}(\Omega_T)} \leq N \left\| G(D^2u, 0, 0, x, t) \right\|_{L^{(p,q)}(\Omega_T)}, \tag{3.18}$$

where $N = N(d, p, q, \lambda, \Lambda, T, |\Omega|)$. This combined with Assumption (H2) yields

$$\| u \|_{L^{(p,q)}(\Omega_T)} \leq N \left(\left\| \psi\left(|D^2u|\right) |D^2u| \right\|_{L^{(p,q)}(\Omega_T)} + \|g\|_{L^{(p,q)}(\Omega_T)} \right).$$

Therefore, with the above estimate and (3.17), we use the same argument as in the proof of (3.16) to obtain

$$\left\| D^2u \right\|_{L^{(p,q)}(\Omega_T)} + \|\partial_t u\|_{L^{(p,q)}(\Omega_T)} \leq N \left(\|g\|_{L^{(p,q)}(\Omega_T)} + \bar{\rho} \|\psi\|_{L^\infty} \right) \tag{3.19}$$

for some $\bar{\rho} > 0$, where $N = N(d, p, q, \lambda, \Lambda, \delta, R_0, T, K, |\Omega|, \|\partial\Omega\|_{C^{1,1}})$.

(ii) With the *a prior* estimate (3.19) in hand, we make use of an approximating argument to the nonlinear function H and the domain Ω, which is just as the same proof of [8, Theorem 1.2], to lead to the existence and uniqueness claims of (1.1) in the Sobolev-Lorentz spaces $W^{1,2}_{(p,q)}(\Omega_T)$. It completes the proof. □

Notice that parabolic Bellman's equation is a typical prototype as fully nonlinear parabolic equations. Therefore, as a consequence of Theorem 1.3, it yields the following Sobolev-Lorentz estimate of strong solutions to parabolic Bellman's equations under the assumption of small BMO coefficients in a bounded $C^{1,1}$ domain. More precisely, we consider the initial-boundary problem of parabolic Bellman's equation

$$\begin{cases} \partial_t u(x,t) + \sup\limits_{\omega \in \mathcal{A}} \Big[a^{ij}(\omega, x, t) D_{ij} u(x,t) + b^i(\omega, x, t) D_i u(x,t) \\ \qquad\qquad -c(\omega, x, t) u(x,t) + g(\omega, x, t) \Big] = 0 & \text{in } \Omega_T, \\ u(x,t) = 0 & \text{on } \partial\Omega_T, \end{cases} \tag{3.20}$$

where the set $\mathcal{A}$ is a separable metric space, $a(\omega, x, t) = \left(a^{ij}(\omega, x, t)\right)$, $b(\omega, x, t) = \left(b^i(\omega, x, t)\right)$, $c(\omega, x, t) \geq 0$ and $g(\omega, x, t)$ are given functions. Here, we assume that these functions are measurable in (x, t) for each ω, continuous in ω for each (x, t) and there exist positive constants $0 < \lambda \leq \Lambda < \infty$ such that

$$|b(\omega, x, t)| \leq \Lambda, \qquad c(\omega, x, t) \leq \Lambda, \qquad \hat{g}(x,t) := \sup_{\omega \in \mathcal{A}} \left[g(\omega, x, t) \right] < \infty \tag{3.21}$$

and

$$\lambda|\xi|^2 \le a^{ij}(\omega,x,t)\,\xi_i\xi_j \le \Lambda|\xi|^2 \tag{3.22}$$

for any $\xi \in \mathbb{R}^d$, $\omega \in \mathcal{A}$ and almostly all $(x,t) \in \Omega_T$. Moreover, $a^{ij}(\omega,x,t)$ is (δ, R_0) -vanishing of (x, t) which means that there exists $R_0 > 0$ such that for any $r \in (0, R_0]$ and $(s,y) \in \overline{\Omega}_T$,

$$\sup_{i,j} \fint_{Q_r(s,y)} \sup_{\omega\in\mathcal{A}} \left| a^{ij}(\omega,x,t) - \bar{a}^{ij}_{Q_r(s,y)}(\omega) \right| dxdt \le \delta, \tag{3.23}$$

where the parameter $\delta > 0$ will be specified later.

One can check that **H1**,**H2** and Assumptions 1.1 hold in Theorem 1.3 by taking

$$\begin{aligned} F =& \sup_{\omega\in\mathcal{A}} a^{ij}(\omega,x,t)D_{ij}u(x,t) \quad \text{and} \\ G =& \sup_{\omega\in\mathcal{A}} \left[b^i(\omega,x,t)D_iu(x,t) - c(\omega,x,t)u(x,t) + g(\omega,x,t) \right], \end{aligned}$$

which immediately leads to the following conclusion.

Corollary 3.6 *For $p \in (d+1, +\infty)$, $q \in (0, +\infty]$, $\Omega \in C^{1,1}$, let $u \in \mathring{W}^{1,2}_{(p,q)}(\Omega_T)$ with $T > 0$ be the solution of the Cauchy-Dirichlet problems (3.20) with assumptions (3.21, 3.22) and $\hat{g} \in L^{(p,q)}(\Omega_T)$. Then, there exists a small constant $\delta = \delta(d,p,q,\lambda,\Lambda,R_0) > 0$ for some $R_0 > 0$ such that if the condition (3.23) holds, then we have*

$$\|u\|_{W^{1,2}_{(p,q)}(\Omega_T)} \le N\|\hat{g}\|_{L^{(p,q)}(\Omega_T)}. \tag{3.24}$$

Here, N is a positive constant depending only on $d,p,q,\lambda,\Lambda,\delta,R_0,T,|\Omega|$ and $\|\partial\Omega\|_{C^{1,1}}$.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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