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Maximal regularity for the damped wave equations

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Abstract
We consider the problem of maximal regularity for non-autonomous second order Cauchy problems

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) \quad t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}$$

Here, the time dependent operator $\mathcal{A}(t)$ is bounded from the Hilbert space $\mathcal{V}$ to its dual space $\mathcal{V}'$ and $\mathcal{B}(t)$ is associated with a sesquilinear form $\mathfrak{b}(t,\cdot,\cdot)$ with domain $\mathcal{V}$. We prove maximal L^p-regularity results and other regularity properties for the solutions of the above equation under minimal regularity assumptions on the operators. Our result is motivated by boundary value problems.

Keywords Damped wave equation · Maximal regularity · Non-autonomous evolution equations

Mathematics Subject Classification 35K90 · 35K45 · 47D06

1 Introduction

The aim of this article is to study non-autonomous second order evolution equations governed by forms.

Let $(\mathcal{H}, (\cdot,\cdot), \|\cdot\|)$ be a separable Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider another separable Hilbert space $\mathcal{V}$ which is densely and continuously embedded into $\mathcal{H}$. We denote by $\mathcal{V}'$ the (anti-) dual space of $\mathcal{V}$ so that

$$\mathcal{V} \hookrightarrow_d \mathcal{H} \hookrightarrow_d \mathcal{V}'.$$

Hence there exists a constant $C > 0$ such that

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$$\|u\| \le C\|u\|_{\mathcal{V}} \ \ (u \in \mathcal{V}),$$

where $\|\cdot\|_{\mathcal{V}}$ denotes the norm of $\mathcal{V}$. Similarly,

$$\|\psi\|_{\mathcal{V}'} \le C\|\psi\| \ \ (\psi \in \mathcal{H}).$$

We denote by $\langle , \rangle$ the duality $\mathcal{V}'$–$\mathcal{V}$ and note that $\langle \psi, v\rangle = (\psi, v)$ if $\psi, v \in \mathcal{H}$.

In this paper we consider maximal regularity for second order Cauchy problems. We focus on the damped wave equation.

We consider a family of sesquilinear forms

$$\mathfrak{b} : [0,\tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C},$$

such that

[H1] $D(\mathfrak{b}(t)) = \mathcal{V}$ (constant form domain),
[H2] $|\mathfrak{b}(t,u,v)| \le M\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ ($\mathcal{V}$-uniform boundedness),
[H3] $\operatorname{Re} \mathfrak{b}(t,u,u) + \nu\|u\|^2 \ge \delta\|u\|^2_{\mathcal{V}}$ $(\forall u \in \mathcal{V})$ for some $\delta > 0$ and some $\nu \in \mathbb{R}$ (uniform quasi-coercivity).

We denote by $B(t), \mathcal{B}(t)$ the usual operators associated with $\mathfrak{b}(t)$(as operators on $\mathcal{H}$ and $\mathcal{V}'$). Recall that $u \in \mathcal{H}$ is in the domain $D(B(t))$ if there exists $h \in \mathcal{H}$ such that for all $v \in \mathcal{V}$: $\mathfrak{b}(t,u,v) = (h,v)$. We then set $B(t)u := h$. The operator $\mathcal{B}(t)$ is a bounded operator from $\mathcal{V}$ into $\mathcal{V}'$ such that $\mathcal{B}(t)u = \mathfrak{b}(t,u,\cdot)$. The operator $B(t)$ is the part of $\mathcal{B}(t)$ on $\mathcal{H}$.

It is a classical fact that $-B(t)$ and $-\mathcal{B}(t)$ are both generators of holomorphic semigroups $(e^{-rB(t)})_{r\ge 0}$ and $(e^{-r\mathcal{B}(t)})_{r\ge 0}$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. The semigroup $e^{-rB(t)}$ is the restriction of $e^{-r\mathcal{B}(t)}$ to $\mathcal{H}$. In addition, $e^{-rB(t)}$ induces a holomorphic semigroup on $\mathcal{V}$ (see, e.g., Ouhabaz [20, Chapter 1]). Let $\mathcal{A}(t) \in \mathcal{L}(\mathcal{V},\mathcal{V}')$ for all $t \in [0,\tau]$ and a function $h : [0,\tau] \to [0,\infty)$ such that $\int_0^\tau t^p h(t)^p \,\frac{dt}{t} < \infty$ and

$$\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le h(t), \quad \text{for almost every } t \in [0,\tau].$$

We denote by $A(t)$ the part of $\mathcal{A}(t)$ on $\mathcal{H}$, defined by

$$\begin{aligned} D(A(t)) &:= \{u \in \mathcal{V} : \mathcal{A}(t)u \in \mathcal{H}\} \\ A(t)u &:= \mathcal{A}(t)u. \end{aligned}$$

Given a function f defined on $[0,\tau]$ with values either in $\mathcal{H}$ or in $\mathcal{V}'$ we consider the second order evolution equation

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{1}$$

This is an abstract damped non-autonomous wave equation and our aim is to prove well-posedness and maximal L^p-regularity for $p \in (1,\infty)$ in $\mathcal{V}'$ and in $\mathcal{H}$.

Definition 1.1 Let $X = \mathcal{H}$ or $\mathcal{V}'$. We say that Problem (1) has maximal L^p-regularity in X, if for all $f \in L^p(0,\tau;X)$ and all (u_0,u_1) in the trace space (see Sects. 2 and 3 for more details) there exists a unique $u \in W^{2,p}(0,\tau;X) \cap W^{1,p}(0,\tau;\mathcal{V})$ which satisfies (1) in the L^p-sense.

The maximal L^2-regularity in $\mathcal{V}'$ was first considered by Lions [16] (p. 151). He assumes that $\mathcal{A}(t)$ is associated with a sesquilinear form $\mathfrak{a}(t)$ which satisfies the same properties as $\mathfrak{b}(t)$ together with an additional regularity assumption on the forms $t \to \mathfrak{a}(t,u,v)$ and $t \to \mathfrak{b}(t,u,v)$ for every fixed $u,v \in \mathcal{V}$. Dautray–Lions [10, p. 667] proved maximal L^2-regularity in $\mathcal{V}'$ without the regularity assumption by taking $f \in L^2(0,\tau;\mathcal{H})$ and considering mainly symmetric forms. Recently, Batty et al. [9] proved maximal L^p-regularity for general forms $\mathcal{B}(.)$ and $\mathcal{A}(.)$ for the case $u_0 = u_1 = 0$ and $h \in L^p(0,\tau)$ by reducing the problem to a first order non-autonomous Cauchy problem. Dier–Ouhabaz [11] proved maximal L^2-regularity in $\mathcal{V}'$ for $u_0 \in \mathcal{V}, u_1 \in \mathcal{H}$ and $\mathcal{A}(t)$ is also associated with a $\mathcal{V}$-bounded quasi-coercive non-autonomous form $\mathfrak{a}(t)$. We improve the result in [9] by proving maximal L^p-regularity in $\mathcal{V}'$ for u_0 and u_1 not necessarily 0 and $t \to t^{1-\frac{1}{p}}h(t) \in L^p(0,\tau)$. Our proof is based on the result of the first order problem as in [9], but the main difference being that we use a fixed point argument.

More interesting is the question of second order maximal regularity in $\mathcal{H}$, i.e. whether the solution u of (1) is in $H^2(0,\tau;\mathcal{H})$ provided that $f \in L^2(0,\tau;\mathcal{H})$. A first answer to this question was giving by Batty et al. [9] in the particular case $\mathcal{B}(.) = k\mathcal{A}(.)$ for some constant k and that $\mathcal{A}(.)$ has the maximal regularity in $\mathcal{H}$. By using the form method, Dier and Ouhabaz [11], proved maximal L^2-regularity in $\mathcal{H}$ without the rather strong assumption $\mathcal{B}(.) = k\mathcal{A}(.)$, but $\mathcal{A}(t)$ is also associated with $\mathcal{V}$-bounded quasi-coercive form $\mathfrak{a}(t)$ and $t \to \mathfrak{a}(t,u,v), \mathfrak{b}(t,u,v)$ are symmetric and Lipschitz continuous for all $u,v \in \mathcal{V}$. We extend the results in [11] in three directions. The first one is to consider general forms which may not be symmetric. The second direction is to deal with maximal L^p-regularity, for all $p \in (1,\infty)$. The third direction, which is our main motivation, is to assume less regularity on the operators $\mathcal{A}(t), \mathcal{B}(t)$ with respect to t.

Our main results can be summarized as follows (see Theorems 3.7 and 3.10 for more general and precise statements).

For $p \in (1,\infty)$ we assume the following

- $|\mathfrak{b}(t,u,v) - \mathfrak{b}(s,u,v)| \le w(|t-s|)\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$, for all $u,v \in \mathcal{V}$,
- $\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le w(|t-s|)$,

such that

- $\int_0^\tau \frac{w(t)}{t^{\frac{3}{2}}}\, dt < \infty$.
- For $p \ne 2$ or $p = 2$ and $D(B(0)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$, we assume

$$\int_0^\tau \frac{w(t)^p}{t^{\frac{\max\{p,2\}}{2}}}\, dt < \infty. \tag{2}$$

- In the case $p = 2$ but $D(B(0)^{\frac{1}{2}}) \not\hookrightarrow \mathcal{V}$, we assume

$$\int_0^\tau \frac{w(t)^2}{t^{1+\epsilon}}\, dt < \infty, \tag{3}$$

for some $\epsilon > 0$.
Here $w : [0, \tau] \to [0, \infty)$ is a non-decreasing function.
Let $f \in L^p(0, \tau; \mathcal{H})$ and one of the following conditions holds

1. for $p \geq 2$, u_0 is in the real-interpolation space $(\mathcal{V}, D(A(0)))_{1-\frac{1}{p},p}$ and $u_1 \in (\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p}$,
2. for $p < 2, u_0 \in \mathcal{V}$ and $u_1 \in (\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p}$.

Then (1) has maximal L^p-regularity in $\mathcal{H}$. Assume in addition that $D(B(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0, \tau]$ and $w(t) \leq Ct^\epsilon$ for some $\epsilon > 0$, then for all $f \in L^2(0, \tau; \mathcal{H})$ and $u_0, u_1 \in \mathcal{V}$, we prove that the solution $u \in H^2(0, \tau; \mathcal{H}) \cap C^1(0, \tau; \mathcal{V})$.

By induction, our approach allows to consider Cauchy problems of order N for any $N \geq 3$.

By using similar ideas as in [5, 12], we give examples for which the maximal regularity fails.

We illustrate our abstract results by two applications in the final section. One of them concerns the Laplacian with time dependent Robin boundary conditions on a bounded Lipschitz domain Ω.

Notation We denote by $\mathcal{L}(E, F)$ (or $\mathcal{L}(E)$) the space of bounded linear operators from E to F (from E to E). The spaces $L^p(a, b; E)$ and $W^{k,p}(a, b; E)$ or $H^k(a, b; E)$ if $p = 2$ denote respectively the Lebesgue and usual Sobolev spaces of order k of function on (a, b) with values in E. For $u \in W^{1,p}(a, b; E)$ we denote the first weak derivative by u' and for $u \in W^{2,p}(a, b; E)$ the second derivative by u''. Recall that the norms of $\mathcal{H}$ and $\mathcal{V}$ are denoted by $\| \cdot \|$ and $\| \cdot \|_{\mathcal{V}}$. The scalar product of $\mathcal{H}$ is $(\cdot, \cdot)$ and the duality $\mathcal{V}'$–$\mathcal{V}$ is $\langle, \rangle$. We denote by $m!$ the factorial of m.

Finally, we denote by C, C' or $C_0, C_1, c, \ldots$ all inessential constants. Their values may change from line to line.

2 Maximal regularity for the damped wave equation in $\mathcal{V}'$

In this section we prove maximal regularity in $\mathcal{V}'$ for the Problem (1).

We start by recalling a well-known result for the first order non autonomous problem.

Following [6], we introduce the following definition

Definition 2.1 Let $(\mathfrak{b}(t))_{t\in[0,\tau]}$ be a family of $\mathcal{V}$-bounded, sesquilinear forms. A function $t \to \mathfrak{b}(t)$ is called relatively continuous if for each $t \in [0, \tau]$ and all $\epsilon > 0$ there exists $\alpha > 0,\ \beta \geq 0$ such that for all $u, v \in \mathcal{V},\ s \in [0, \tau]$ and $|t - s| \leq \alpha$ implies that

$$|\mathfrak{b}(t, u, v) - \mathfrak{b}(s, u, v)| \leq (\epsilon \|u\|_{\mathcal{V}} + \beta \|u\|_{\mathcal{V}'}) \|v\|_{\mathcal{V}}.$$

Theorem 2.2 *Let* $(\mathfrak{b}(t))_{t\in[0,\tau]}$ *be a family of* $\mathcal{V}$*-bounded, sesquilinear forms and* $p \in (1,\infty)$.

We assume one of the following conditions

- *for* $p = 2, t \to \mathfrak{b}(t)$ *is measurable,*
- *for* $p \neq 2, t \to \mathfrak{b}(t)$ *is piecewise relatively continuous.*

Then for all $u_1 \in (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$ *and* $f \in L^p(0,\tau;\mathcal{V}')$ *there exists a unique solution* $v \in MR^p(\mathcal{V}, \mathcal{V}') = W^{1,p}(0,\tau;\mathcal{V}') \cap L^p(0,\tau;\mathcal{V})$ *to the problem*

$$\begin{cases} v'(t) + \mathcal{B}(t)v(t) = f(t) \quad t\text{-a.e.} \\ v(0) = u_1. \end{cases} \tag{4}$$

In addition, there exists a positive constant C such that

$$\|v\|_{MR^p(\mathcal{V},\mathcal{V}')} \le C\Big[\|u_1\|_{(\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}} + \|f\|_{L^p(0,\tau;\mathcal{V}')}\Big].$$

Here, $\|v\|_{MR^p(\mathcal{V},\mathcal{V}')} := \|v\|_{W^{1,p}(0,\tau;\mathcal{V}')} + \|v\|_{L^p(0,\tau;\mathcal{V})}$.

Proof For the case $p = 2$ the result is due to Lions [16]. Since $\mathcal{B}(s) + \nu$ is the generator of an analytic semigroup in $\mathcal{V}'$ for all $s \in [0,\tau]$, then for all $u_1 \in (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$ and $f \in L^p(0,\tau;\mathcal{V}'), p \in (1,\infty)$ there exists a unique solution $v \in MR^p(\mathcal{V}, \mathcal{V}')$ to the autonomous problem

$$\begin{cases} w'(t) + \mathcal{B}(s)w(t) = f(t) \quad t\text{-a.e.} \\ w(0) = u_1. \end{cases} \tag{5}$$

Now, we apply [6, Theorem 2.7] to get the desired result for $p \neq 2$. □

From [3, Theorem III 4.10.2] we have the following lemma

Lemma 2.3 *Let* E_1, E_2 *be two Banach spaces such that* $E_2 \subseteq E_1$*. Then*

$$W^{1,p}(0,\tau;E_1) \cap L^p(0,\tau,E_2) \hookrightarrow C([0,\tau];(E_1,E_2)_{1-\frac{1}{p},p}).$$

We introduce the maximal regularity space

$$MR^p(\mathcal{V},\mathcal{V},\mathcal{V}') := W^{2,p}(0,\tau;\mathcal{V}') \cap W^{1,p}(0,\tau;\mathcal{V}).$$

It is a Banach space for the norm

$$\|u\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} := \|u''\|_{L^p(0,\tau;\mathcal{V}')} + \|u\|_{W^{1,p}(0,\tau;\mathcal{V})}.$$

Let $v \in MR^p(\mathcal{V}, \mathcal{V}')$ be the solution of (4) for a giving $u_1 \in (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$ and $f \in L^p(0,\tau;\mathcal{V}')$. For $u_0 \in \mathcal{V}$ and $t \in [0,\tau]$ we set $w(t) = u_0 + \int_0^t v(s)\,ds$. Then $w'(t) = v(t)$ and

$$\begin{cases} w''(t) + \mathcal{B}(t)w'(t) = f(t) & t\text{-a.e.} \\ w(0) = u_0, \quad w'(0) = u_1. \end{cases} \tag{6}$$

Moreover, we have the following estimate

$$\|w\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} \leq C\Big(\|u_0\|_{\mathcal{V}} + \|u_1\|_{(\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}} + \|f\|_{L^p(0,\tau;\mathcal{V}')} \Big). \tag{7}$$

We note also that the solution of the Problem (6) is unique. Indeed, we suppose there are two solutions w_1, w_2, then $w = w_1 - w_2 \in MR^2(\mathcal{V}, \mathcal{V}, \mathcal{V}')$ is a solution to the following problem

$$\begin{cases} w''(t) + \mathcal{B}(t)w'(t) = 0 & t\text{-a.e.} \\ w(0) = 0, \quad w'(0) = 0. \end{cases} \tag{8}$$

Therefore for $t \in [0, \tau]$

$$2\mathrm{Re} \int_0^t \langle w''(s), w'(s)\rangle ds + 2\mathrm{Re} \int_0^t \langle \mathcal{B}(s)w'(s), w'(s)\rangle \, ds = 0.$$

Recall that $w' \in W^{1,2}(0, \tau;\mathcal{V}') \cap L^2(0, \tau;\mathcal{V})$. Then by using [10, Theorem 2, p. 477] we obtain

$$\begin{aligned} 2\mathrm{Re} \int_0^t \langle w''(s), w'(s)\rangle \, ds &= \int_0^t \frac{d}{ds}\|w'(s)\|^2 \, ds \\ &= \|w'(t)\|^2 - \|w'(0)\|^2 \\ &= \|w'(t)\|^2. \end{aligned}$$

The uniform quasi-coercivity of the forms $(\mathfrak{b}(t))_{t\in[0,\tau]}$ gives

$$\|w'(t)\|^2 + 2\delta \int_0^t \|w'(s)\|^2_{\mathcal{V}} \, ds \leq 2\nu \int_0^t \|w'(s)\|^2 \, ds.$$

We conclude by Gronwall's lemma that $w'(t) = 0$ for all $t \in [0, \tau]$, hence $w(t) = 0$ and consequently $w_1(t) = w_2(t)$ for all $t \in [0, \tau]$.

Using Lemma 2.3 and the Sobolev embedding we have

$$MR^p(\mathcal{V}, \mathcal{V}, \mathcal{V}') \hookrightarrow C^1([0, \tau];(\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}) \cap C^{1-\frac{1}{p}}([0, \tau];\mathcal{V}). \tag{9}$$

We define the associated trace space to $MR^p(\mathcal{V}, \mathcal{V}, \mathcal{V}')$ by

$$TR^p(\mathcal{V}, \mathcal{V}') := \{(u(0), u'(0)) \, : \, u \in MR^p(\mathcal{V}, \mathcal{V}, \mathcal{V}')\},$$

endowed with norm

$$\|(u(0), u'(0))\|_{TR^p(\mathcal{V},\mathcal{V}')} := \inf\{ \|v\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} \, : \, v(0) = u(0), v'(0) = u'(0)\}.$$

Note that $\Big(TR^p(\mathcal{V}, \mathcal{V}'), \| \cdot \|_{TR^p(\mathcal{V},\mathcal{V}')}\Big)$ is a Banach space.

Proposition 2.4 *For all $p \in (1,\infty)$ we have*

$$TR^p(\mathcal{V},\mathcal{V}') = \mathcal{V} \times (\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p} \quad \text{with equivalent norms.}$$

Proof The first injection $TR^p(\mathcal{V},\mathcal{V}') \hookrightarrow \mathcal{V} \times (\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}$ is obtained by (9). For the second injection "$\hookleftarrow$" let us take $u_1 \in (\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}$. Then by [17, Corollary 1.14] there exists $w \in MR^p(\mathcal{V},\mathcal{V}')$ such that $w(0) = u_1$. We set now $u(t) = u_0 + \int_0^t w(s)ds$, where $u_0 \in \mathcal{V}$. Then $u \in MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ and

$$\|u\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} \le C'\Big(\|u_0\|_{\mathcal{V}} + \|w\|_{MR^p(\mathcal{V},\mathcal{V}')}\Big).$$

We note that the trace space associated to $MR^p(\mathcal{V},\mathcal{V}')$ is isomorphic to the real interpolation space $(\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}$ (see [18, Chapter 1]). Then

$$\begin{aligned} &\inf\{\|u\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} : u(0) = u_0, u'(0) = u_1\} \\ &\le C'\Big(\|u_0\|_{\mathcal{V}} + \inf\{\|w\|_{MR^p(\mathcal{V},\mathcal{V}')} : w(0) = u_1\}\Big) \\ &\le C\Big(\|u_0\|_{\mathcal{V}} + \|u_1\|_{(\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p}}\Big). \end{aligned}$$

Thus $\mathcal{V} \times (\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p} \hookrightarrow TR^p(\mathcal{V},\mathcal{V}')$. □

Remark 2.5 From the previous proposition and (9), we can deduce that the operator

$$\begin{aligned} MR^p(\mathcal{V},\mathcal{V},\mathcal{V}') &\to C([0,\tau];TR^p(\mathcal{V},\mathcal{V}')) \\ u &\to (u,u') \end{aligned}$$

is well defined and bounded.

Our main result on maximal L^p-regularity in $\mathcal{V}'$ is the following.

Theorem 2.6 *Let $p \in (1,\infty)$. We assume one of the following conditions*

- *for $p = 2$, $t \to \mathfrak{b}(t)$ is measurable.*
- *for $p \neq 2$, $t \to \mathfrak{b}(t)$ is piecewise relatively continuous.*

Let $\mathcal{A}(t) \in \mathcal{L}(\mathcal{V},\mathcal{V}')$ for all $t \in [0,\tau]$ such that $\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le h(t)$ for almost every $t \in [0,\tau]$ and $\int_0^\tau t^p h(t)^p \frac{dt}{t} < \infty$. Then for all $f \in L^p(0,\tau;\mathcal{V}')$ and $(u_0,u_1) \in TR^p(\mathcal{V},\mathcal{V}')$, there exists a unique solution $u \in MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ to the problem

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{10}$$

Moreover, there exists a positive constant C independent of u_0, u_1 and f such that the following estimate holds

$$\|u\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} \leq C\Big[\|(u_0,u_1)\|_{TR^p(\mathcal{V},\mathcal{V}')} + \|g\|_{L^p(0,\tau;\mathcal{V}')}\Big]. \tag{11}$$

As mentioned in the introduction, this theorem was proved by Batty, Chill and Srivastava [9] but they consider only the case $u_0 = u_1 = 0$ and suppose that $t \to \|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \in L^p(0,\tau)$.

Proof We introduce the subspace

$$MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}') := \{u \in MR^p(\mathcal{V},\mathcal{V},\mathcal{V}') : u(0) = 0, u'(0) = 0\}.$$

We equip this subspace with the norm $u \to \|u''\|_{L^p(0,\tau;\mathcal{V}')} + \|u'\|_{L^p(0,\tau;\mathcal{V})}$. To prove existence and uniqueness of the solution we use the contraction fixed point theorem and the existence of a solution in $MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ for Problem (6). Indeed, let $z \in MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ and $v \in MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ be the solution of the problem

$$\begin{cases} v''(t) + \mathcal{B}(t)v'(t) = g(t) - \mathcal{A}(t)z(t) & t\text{-a.e.} \\ v(0) = 0, \quad v'(0) = 0 \end{cases} \tag{12}$$

for a given $g \in L^p(0,\tau;\mathcal{V}')$.

We consider the operator $F : z \to v$. It follows by (7) that $F : MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}') \to MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ is a bounded operator. Now, let $z_1, z_2 \in MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ and $v_1 = Fz_1, v_2 = Fz_2$. We set $v = v_1 - v_2, w = z_1 - z_2$. Obviously, v satisfies

$$\begin{cases} v''(t) + \mathcal{B}(t)v'(t) = -\mathcal{A}(t)w(t) & t\text{-a.e.} \\ v(0) = 0, \quad v'(0) = 0. \end{cases} \tag{13}$$

Therefore, by (7) we have

$$\begin{aligned} \|Fz_1 - Fz_2\|^p_{MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')} &\leq C\|\mathcal{A}(.)w\|^p_{L^p(0,\tau;\mathcal{V}')} \\ &\leq C\int_0^\tau h(t)^p \|w(t)\|^p_{\mathcal{V}}\, dt \\ &\leq C\int_0^\tau h(t)^p t^{p-1}\, dt \|w\|^p_{C^{1-\frac{1}{p}}([0,\tau];\mathcal{V})} \\ &\leq C\int_0^\tau h(t)^p t^{p-1}\, dt \|z_1 - z_2\|^p_{MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')}. \end{aligned}$$

We choose τ small enough such that $C\int_0^\tau h(t)^p t^{p-1} dt < 1$. Thus, F is a contraction on the Banach space $MR_0^p(\mathcal{V},\mathcal{V},\mathcal{V}')$. So by the contraction fixed point theorem, there exists a unique solution to the problem

$$\begin{cases} v''(t) + \mathcal{B}(t)v'(t) + \mathcal{A}(t)v(t) = g(t) & t\text{-a.e.} \\ v(0) = 0, \quad v'(0) = 0 \end{cases} \tag{14}$$

for all $g \in L^p(0,\tau;\mathcal{V}')$ and $\tau > 0$ small enough. In addition, we have from (7)

$$\|v\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} \le C\|g\|_{L^p(0,\tau;\mathcal{V}')}.$$

Now, let $u_0, u_1 \in \mathcal{V} \times (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$. Since by Proposition 2.4, $TR^p(\mathcal{V},\mathcal{V}') = \mathcal{V} \times (\mathcal{V}', \mathcal{V})_{1-\frac{1}{p},p}$, then there exists $z \in MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ (with minimal norm) such that $z(0) = u_0$ and $z'(0) = u_1$. We set now $u = z + v$. Thus, u belongs to $MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')$ and satisfies

$$\begin{cases} u''(t) + \mathcal{B}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) \quad t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases} \tag{15}$$

with $f = g + z'' + \mathcal{B}(.)z' + \mathcal{A}(.)z \in L^p(0,\tau;\mathcal{V}')$.

Therefore

$$\begin{aligned} \|u\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} &\le C\Big(\|v\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')} + \|z\|_{MR^p(\mathcal{V},\mathcal{V},\mathcal{V}')}\Big) \\ &\le C'\Big(\|g\|_{L^p(0,\tau;\mathcal{V}')} + \|(u_0,u_1)\|_{TR^p(\mathcal{V},\mathcal{V}')}\Big) \\ &\le c\Big(\|f\|_{L^p(0,\tau;\mathcal{V}')} + \|(u_0,u_1)\|_{TR^p(\mathcal{V},\mathcal{V}')}\Big). \end{aligned}$$

This proves the desired a priori estimate and completes the proof when τ is sufficiently small. We note that by Remark 2.5, $(u,u') \in C([0,\tau];TR^p(\mathcal{V},\mathcal{V}'))$. For arbitrary $\tau > 0$, we split $[0,\tau]$ into a finite number of subintervals with small sizes and proceed exactly as in the previous proof. Finally, we stick the solutions to get the desired result. □

3 Maximal regularity for the damped wave equation in $\mathcal{H}$

Let $\mathcal{A}(t)$ and $\mathcal{B}(t)$ be as before. In this section we assume moreover that $\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le M$, for all $t \in [0,\tau]$.

Let us define the spaces

$$\begin{aligned} MR(p,\mathcal{H}) &:= \{u \in W^{2,p}(0,\tau;\mathcal{H}) \cap W^{1,p}(0,\tau;\mathcal{V}) \,:\, \mathcal{B}(.)u' + \mathcal{A}(.)u \in L^p(0,\tau;\mathcal{H})\}. \\ Tr(p,\mathcal{H}) &:= \{(u(0),u'(0)) \;:\, u \in MR(p,\mathcal{H})\}, \end{aligned}$$

endowed with norms

$$\begin{aligned} \|u\|_{MR(p,\mathcal{H})} &:= \|u''\|_{L^p(0,\tau;\mathcal{H})} + \|u\|_{W^{1,p}(0,\tau;\mathcal{V})} \\ &\quad + \|\mathcal{B}(.)u'(.) + \mathcal{A}(.)u(.)\|_{L^p(0,\tau;\mathcal{H})}. \\ \|(u(0),u'(0))\|_{Tr(p,\mathcal{H})} &:= \inf\{\|v\|_{MR(p,\mathcal{H})} \;: \\ &\qquad v \in MR(p,\mathcal{H}), v(0) = u(0), v'(0) = u'(0)\}. \end{aligned}$$

Maximal L^p-regularity in $\mathcal{H}$ for Problem (10) consists of proving existence and uniqueness of a solution $u \in MR(p,\mathcal{H})$ provided $f \in L^p(0,\tau;\mathcal{H})$ and $(u_0,u_1) \in Tr(p,\mathcal{H})$.

3.1 Preparatory lemmas

In this subsection we prove several estimates and most of the main arguments which will play an important role in proofs of our main results.

Proposition 3.1 *Maximal L^p-regularity in $\mathcal{H}$ for the problem*

$$\begin{cases} v''(t) + \mathcal{B}(t)v'(t) + \mathcal{A}(t)v(t) = f(t) & t\text{-a.e.} \\ v(0) = u_0, \quad v'(0) = u_1 + \gamma u_0 \end{cases} \tag{16}$$

is equivalent to maximal L^p-regularity for the problem

$$\begin{cases} u''(t) + \Big(\mathcal{B}(t) + 2\gamma\Big)I)u'(t) + \Big(\mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I\Big)u(t) = e^{-\gamma t}f(t) & t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases} \tag{17}$$

for all $\gamma \in \mathbb{C}$.

Proof Let v be the solution of (16) and $\gamma \in \mathbb{C}$. We set $u(t) = e^{-\gamma t}v(t)$. By a simple computation we obtain that u satisfies (17). In addition, $f \in L^p(0,\tau;\mathcal{H})$ if and only if $t \to e^{-\gamma t}f(t) \in L^p(0,\tau;\mathcal{H})$ and it is clear that $v \in W^{2,p}(0,\tau;\mathcal{H}) \cap W^{1,p}(0,\tau;\mathcal{V})$ if and only if $u \in W^{2,p}(0,\tau;\mathcal{H}) \cap W^{1,p}(0,\tau;\mathcal{V})$. □

We deduce that we may replace $\mathcal{B}(t)$ by $\mathcal{B}(t) + \gamma$. Therefore, we may suppose without loss of generality that [H3] holds with $\nu = 0$. In particular, we may suppose that $B(t)$ and $\mathcal{B}(t)$ are invertible. We will do so in the sequel without mentioning it.

We note that for $\gamma > 0$ big enough ($\gamma > \max\{\frac{M}{\delta},\nu\}$) and $t \in [0,\tau]$, we have that $\mathcal{C}(t) = \mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I$ is associated with a $\mathcal{V}$-bounded coercive form $\mathfrak{c}(t)$ (i.e., it satisfies [H3] with $\nu = 0$). In fact, let $u \in \mathcal{V}$. We get

$$\begin{aligned} \text{Re}\ \mathfrak{c}(t,u,u) &= \text{Re}\ \langle \mathcal{A}(t)u,u\rangle + \gamma\text{Re}\ \mathfrak{b}(t,u,u) + \gamma^2\|u\|^2 \\ &\geq -\|\mathcal{A}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|u\|_{\mathcal{V}}^2 + \gamma\delta\|u\|_{\mathcal{V}}^2 + (\gamma^2 - \gamma\nu)\|u\|^2 \\ &= -M\|u\|_{\mathcal{V}}^2 + \gamma\delta\|u\|_{\mathcal{V}}^2 + (\gamma^2 - \gamma\nu)\|u\|^2 \\ &\geq (\gamma\delta - M)\|u\|_{\mathcal{V}}^2. \end{aligned}$$

We denote by S_θ the open sector $S_\theta = \{z \in \mathbb{C}^* : |arg(z)| < \theta\}$ with vertex 0.

Lemma 3.2 *For any $t \in [0,\tau]$, the operators $-B(t)$ and $-\mathcal{B}(t)$ generate strongly continuous analytic semigroups of angle $\gamma = \frac{\pi}{2} - arctan(\frac{M}{\alpha})$ on $\mathcal{H}$ and $\mathcal{V}'$, respectively. In addition, there exist constants C and C_θ, independent of t, such that*

1. $\|e^{-zB(t)}\|_{\mathcal{L}(\mathcal{H})} \leq 1$ *and* $\|e^{-z\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}')} \leq C$ *for all* $z \in S_\gamma$.
2. $\|B(t)e^{-sB(t)}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{s}$ *and* $\|\mathcal{B}(t)e^{-s\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}')} \leq \frac{C}{s}$ *for all* $s \in \mathbb{R}$.
3. $\|e^{-sB(t)}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \leq \frac{C}{\sqrt{s}}$.
4. $\|(zI - B(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \leq \frac{C_\theta}{\sqrt{1+|z|}}$ *and* $\|(zI - \mathcal{B}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \leq \frac{C_\theta}{\sqrt{1+|z|}}$ *for all* $z \notin S_\theta$ *with fixed* $\theta > \gamma$.
5. $\|(zI - B(t))^{-1}\|_{\mathcal{L}((\mathcal{H},\mathcal{V})_{\beta,p};\mathcal{V})} \leq \frac{C_{\theta,\beta}}{(1+|z|)^{\frac{1+\beta}{2}}}$ *for all* $\beta \in [0,1], z \notin S_\theta$ *and* $p \in (1,\infty)$.
6. *All the previous estimates hold for* $\mathcal{B}(t) + \alpha$ *with constants independent of* α *for* $\alpha > 0$.

Proof For assertions 1–3 and 4, 6 we refer to [14, Proposition 2.1]. For assertion 5, observe that $\|(zI - B(t))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \leq \frac{C_\theta}{\sqrt{1+|z|}}$ and $\|(zI - B(t))^{-1}\|_{\mathcal{L}(\mathcal{V})} \leq \frac{C_\theta}{1+|z|}$ (see e.g. [7, p. 3]) for all $z \notin S_\theta$ with fixed $\theta > \gamma$. Then the claim follows immediately by interpolation. □

For $p \in (1,\infty)$ and $f \in L^p(0,\tau;\mathcal{H})$ and for almost every $t \in [0,\tau]$ we define the operator L by

$$L(f)(t) := B(t)\int_0^t e^{-(t-s)B(t)} f(s)\,ds.$$

The following result is Lemmas 2.5 and 2.6 in [14].

Lemma 3.3 *Let* $p \in (1,\infty)$. *Suppose that* $\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \leq w(|t-s|)$, *where* $w : [0,\tau] \to [0,\infty)$ *is a non-decreasing function such that*

$$\int_0^\tau \frac{w(t)}{t}\,dt < \infty.$$

Then the operator L is bounded on $L^p(0,\tau;\mathcal{H})$.

Let $p \in (1,\infty)$. We introduce the following assumptions

- for $p \neq 2 : t \to \mathfrak{b}(t)$ is relatively continuous and for $p = 2 : t \to \mathfrak{b}(t)$ is measurable.
- $|\mathfrak{b}(t,u,v) - \mathfrak{b}(s,u,v)| \leq w(|t-s|)\|u\|_\mathcal{V}\|v\|_\mathcal{V}$ for all $u, v \in \mathcal{V}$.
- $\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \leq w(|t-s|)$,

where $w : [0,\tau] \to [0,\infty)$ is a non-decreasing function such that

$$\int_0^\tau \frac{w(t)}{t^{\frac{3}{2}}}\,dt < \infty. \tag{18}$$

We assume in addition that

- For $p \neq 2$ (or $p = 2$ with $D(B(0)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$)

$$\int_0^\tau \frac{w(t)^p}{t^{\frac{\max\{p,2\}}{2}}}\, dt < \infty. \tag{19}$$

- In the case where $p = 2$, but $D(B(0)^{\frac{1}{2}}) \not\hookrightarrow \mathcal{V}$

$$\int_0^\tau \frac{w(t)^2}{t^{1+\varepsilon}}\, dt < \infty, \tag{20}$$

for arbitrary small $\varepsilon > 0$.

Let $\gamma > 0$ be sufficiently large such that $\mathcal{C}(t) = \mathcal{A}(t) + \gamma\mathcal{B}(t) + \gamma^2 I$ is associated with a $\mathcal{V}$-bounded coercive form $\mathfrak{c}(t)$. We denote by $C(t)$ the part of $\mathcal{C}(t)$ on $\mathcal{H}$.

It is clear that $|\mathfrak{c}(t,u,v) - \mathfrak{c}(s,u,v)| \le (1+\gamma)w(|t-s|)\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ for all $u, v \in \mathcal{V}$.

In the following we set $B = B(0)$, $\mathcal{B} = \mathcal{B}(0)$.

Next we define the operator $\mathcal{B}^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{V}')$ by

$$\mathcal{B}^{-\frac{1}{2}}x := \frac{1}{\pi}\int_0^\infty \lambda^{-\frac{1}{2}}(\lambda + \mathcal{B})^{-1}x\, d\lambda\, (x \in \mathcal{V}'),$$

see [4, (3.52)] or [21, (Sec. 2.6 p. 69)]. Then $(\mathcal{B}^{-\frac{1}{2}})^2 = \mathcal{B}^{-1}$. Moreover, $\mathcal{B}^{-\frac{1}{2}}$ is injective. One defines $\mathcal{B}^{\frac{1}{2}}$ by $D(\mathcal{B}^{\frac{1}{2}}) = R(\mathcal{B}^{-\frac{1}{2}})$ and $\mathcal{B}^{\frac{1}{2}} = (\mathcal{B}^{-\frac{1}{2}})^{-1}$, where $R(\mathcal{B}^{-\frac{1}{2}})$ is the range of $\mathcal{B}^{-\frac{1}{2}}$. Then $-\mathcal{B}^{\frac{1}{2}}$ is a closed operator on $\mathcal{V}'$ (in fact, the generator of a analytic semigroup). We have $\mathcal{B}^{-\frac{1}{2}}x = B^{-\frac{1}{2}}x$ for all $x \in \mathcal{H}$ and $B^{-\frac{1}{2}}$ is injective and $D(B^{\frac{1}{2}}) = R(B^{-\frac{1}{2}})$, $B^{\frac{1}{2}} = (B^{-\frac{1}{2}})^{-1}$. It can happen that $R(B^{-\frac{1}{2}}) \ne \mathcal{V}$. The following is easy to see using that $(\mathcal{B}^{-\frac{1}{2}})^2 = \mathcal{B}^{-1}$ is an isomorphism from $\mathcal{V}'$ onto $\mathcal{V}$. For more details and references, see [20, Chapter 8].

Lemma 3.4 *We claim that*

(1) $\mathcal{V} \hookrightarrow D(B^{\frac{1}{2}})$ *if and only if* $D(B^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$.
(2) *If* $B = B^*$, *we have* $D(B^{\frac{1}{2}}) = D(B^{*\frac{1}{2}}) = \mathcal{V}$ *and*

$$\sqrt{\delta}\|u\|_{\mathcal{V}} \le \|B^{\frac{1}{2}}u\| \le \sqrt{M}\|u\|_{\mathcal{V}}.$$

(3) $D(B^\alpha) = [\mathcal{H}, \mathcal{V}]_{2\alpha}$ *for all* $0 \le \alpha < \frac{1}{2}$.
(4) $D(B^{1-\alpha}) \hookrightarrow \mathcal{V}$ *for all* $0 \le \alpha < \frac{1}{2}$.

Proof Let $u \in D(B^*)$. If $\mathcal{V} \hookrightarrow D(B^{\frac{1}{2}})$ we have

$$\begin{aligned}\|u\|_{\mathcal{V}}^2 &\le \frac{1}{\delta}\mathrm{Re}\,(B^{\frac{1}{2}}u, B^{*\frac{1}{2}}u)\\ &\le \frac{1}{\delta}\|B^{\frac{1}{2}}u\|\|B^{*\frac{1}{2}}u\|\\ &\le C\|u\|_{\mathcal{V}}\|B^{*\frac{1}{2}}u\|.\end{aligned}$$

Then by the density of $D(B^*)$ in $D(B^{*\frac{1}{2}})$ we obtain

$$\|u\|_{\mathcal{V}} \le C\|B^{*\frac{1}{2}}u\|$$

for all $u \in D(B^{*\frac{1}{2}})$. Then $D(B^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$.

Now, we assume that $D(B^{*\frac{1}{2}}) \hookrightarrow \mathcal{V}$. It follows that $B^{*-\frac{1}{2}} \in \mathcal{L}(\mathcal{H}, \mathcal{V})$.

Let $x \in \mathcal{H}$ and write $\mathcal{B}^{*\frac{1}{2}}x = \mathcal{B}^{*}B^{*-\frac{1}{2}}x$. We obtain

$$\|\mathcal{B}^{*\frac{1}{2}}x\|_{\mathcal{V}'} \le \|\mathcal{B}^{*}\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|B^{*-\frac{1}{2}}x\|_{\mathcal{V}} \le M\|B^{*-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\|x\|.$$

The boundedness of norm implies $\mathcal{B}^{*\frac{1}{2}} \in \mathcal{L}(\mathcal{H}, \mathcal{V}')$ and by duality we have $B^{\frac{1}{2}} \in \mathcal{L}(\mathcal{V}, \mathcal{H})$. Then $\mathcal{V} \subseteq D(B^{\frac{1}{2}})$ and we get for all $x \in \mathcal{V}$

$$\begin{aligned}\|x\|^2_{D(B^{\frac{1}{2}})} &= \|x\|^2 + \|B^{\frac{1}{2}}x\|^2_{\mathcal{H}}\\ &\le (C^2_{\mathcal{H}} + \|B^{\frac{1}{2}}\|^2_{\mathcal{L}(\mathcal{V},\mathcal{H})})\|x\|^2_{\mathcal{V}}.\end{aligned}$$

Thus, $\mathcal{V} \hookrightarrow D(B^{\frac{1}{2}})$. This shows (1).

We assume now that $B = B^{*}$. Because of the density of $D(B)$ in $\mathcal{V}$ and $D(B^{\frac{1}{2}})$, we get for all $u \in \mathcal{V}$

$$\begin{aligned}\delta\|u\|^2_{\mathcal{V}} &\le \operatorname{Re} \mathfrak{b}(0, u, u)\\ &= \|B^{\frac{1}{2}}u\|^2\\ &\le M\|u\|^2_{\mathcal{V}}.\end{aligned}$$

This shows (2).

For (3), we refer to [15, Theorem 3.1].

Let $0 \le \alpha < \frac{1}{2}$ and $u \in D(B)$. We have

$$\begin{aligned}\|u\|^2_{\mathcal{V}} &\le \frac{1}{\delta}\|B^{1-\alpha}u\|\|B^{*\alpha}u\|\\ &\le \frac{1}{\delta}\|B^{1-\alpha}u\|\|u\|_{[\mathcal{H},\mathcal{V}]_{2\alpha}}\\ &\le \frac{C(\alpha)}{\delta}\|B^{1-\alpha}u\|\|u\|^{2\alpha}_{\mathcal{V}}\|u\|^{1-2\alpha},\end{aligned}$$

where $C(\alpha) > 0$ depending on α. Thus, for all $u \in D(B^{1-\alpha})$

$$\|u\|_{\mathcal{V}} \le \frac{C_{\mathcal{H}}^{1-2\alpha}C(\alpha)}{\delta}\|B^{1-\alpha}u\|.$$

This shows (4). □

Next we set $X^p = \mathcal{V}$ for all $p \in (1, 2[$ and $X^p = (\mathcal{V}, D(C(0)))_{1-\frac{1}{p},p}$ for $p \ge 2$.

Lemma 3.5 *Let $u_1 \in (\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p}$ and $u_0 \in X^p$, then the operators*

$$R_1 u_1(t) = B(t)e^{-tB(t)}u_1$$
$$R_2 u_0(t) = e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0$$

are bounded from $(\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p}$ *and* X^p *into* $L^p(0,\tau;\mathcal{H})$, *respectively.*

Remark 3.6 We note that the operator R_1 is already studied in [1, Theorem 2.2]. Here we assume less regularity on the operators $\mathcal{B}(t)$ with respect to t compared with [1, Theorem 2.2].

Proof Firstly, we note that in the case $p < 2$ we have $(\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p} = (\mathcal{H}, \mathcal{V})_{2(1-\frac{1}{p}),p}$ (see [13, (p. 5)]). Then by Lemma 3.2

$$\|(\lambda I - \mathcal{B}(0))^{-1}\|_{\mathcal{L}((\mathcal{H},D(B(0)))_{1-\frac{1}{p},p};\mathcal{V})} \leq \frac{C}{|\lambda|^{\frac{3}{2}-\frac{1}{p}}}.$$

In the case $p > 2$, the embedding $(\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p} \hookrightarrow \mathcal{V}$ holds. In fact, we use the inclusion properties of the real interpolation spaces [17, Proposition 1.1.4] to obtain

$$\begin{aligned}(\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p} &= (\mathcal{H}, D(B(0)))_{\frac{1}{2}+[\frac{1}{2}-\frac{1}{p}],p} \\ &\hookrightarrow (\mathcal{H}, D(B(0)))_{\frac{1}{2}+[\frac{1}{2}-\frac{1}{p}]-\varepsilon,2} \\ &= D(B(0)^{1-(\frac{1}{p}+\varepsilon)}),\end{aligned}$$

with $\varepsilon < \frac{1}{2} - \frac{1}{p}$.

The embedding $D(B(0)^{1-(\frac{1}{p}+\epsilon)}) \hookrightarrow \mathcal{V}$ (see Proposition 3.4) gives

$$\|(\lambda I - \mathcal{B}(0))^{-1}\|_{\mathcal{L}((\mathcal{H},D(B(0)))_{1-\frac{1}{p},p};\mathcal{V})} \leq \frac{C}{|\lambda|}.$$

We consider now the case $D(B(0)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$. One has

$$\begin{aligned}&\|(\lambda I - \mathcal{B}(0))^{-1}u_1\|_{\mathcal{V}} \\ &\leq C_0\|(\lambda I - \mathcal{B}(0))^{-1}u_1\|_{D(B(0)^{\frac{1}{2}})} \\ &\leq C_1\|B(0)^{-\frac{1}{2}}\|_{\mathcal{L}(\mathcal{H},D(B(0)^{\frac{1}{2}}))}\|(\lambda I - \mathcal{B}(0))^{-1}B(0)^{\frac{1}{2}}u_1\| \\ &\leq C\frac{1}{|\lambda|}\|u_1\|_{D(B(0)^{\frac{1}{2}})}.\end{aligned}$$

For the other case $(D(B(0)^{\frac{1}{2}}) \not\hookrightarrow \mathcal{V})$, since $D(B(0)^{\frac{1+\varepsilon}{2}}) \hookrightarrow \mathcal{V}$ for all $\varepsilon > 0$ (see Proposition 3.4) then

$$\begin{aligned}
&\|(\lambda I - \mathcal{B}(0))^{-1}u_1\|_{\mathcal{V}}\\
&\quad\le C_1\|B(0)^{-\frac{1+\varepsilon}{2}}\|_{\mathcal{L}(\mathcal{H},D(B(0)^{\frac{1+\varepsilon}{2}}))}\|B(0)^{\frac{\varepsilon}{2}}(\lambda I - \mathcal{B}(0))^{-1}B(0)^{\frac{1}{2}}u_1\|\\
&\quad\le C\frac{1}{|\lambda|^{1-\frac{\varepsilon}{2}}}\|u_1\|_{D(B(0)^{\frac{1}{2}})}.
\end{aligned}$$

We write

$$R_1u_1(t) = \Big(B(t)e^{-tB(t)} - B(0)e^{-tB(0)}\Big)u_1 + B(0)e^{-tB(0)}u_1.$$

Choose a contour Γ in the positive half-plane and write by the holomorphic functional calculus for the sectorial operators $B(t)$, $B(0)$

$$B(t)e^{-tB(t)} - B(0)e^{-tB(0)} = \frac{1}{2\pi i}\int_\Gamma \lambda e^{-t\lambda}(\lambda I - B(t))^{-1}\Big(\mathcal{B}(t) - \mathcal{B}(0)\Big)(\lambda I - B(0))^{-1}\,d\lambda.$$

Therefore

$$\begin{aligned}
&\|B(t)e^{-tB(t)}u_1 - B(0)e^{-tB(0)}u_1\|\\
&\quad\le C\int_0^\infty |\lambda|e^{-t|\cos\gamma||\lambda|}\|(\lambda I - \mathcal{B}(0))^{-1}\|_{\mathcal{L}((\mathcal{H},D(B(0)))_{1-\frac{1}{p},p};\mathcal{V})}\\
&\qquad\times\|(\lambda I - \mathcal{B}(t))^{-1}\|_{\mathcal{L}(\mathcal{V}';\mathcal{H})}\,d|\lambda|\,\|\mathcal{B}(t) - \mathcal{B}(0)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\\
&\qquad\times\|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}}.
\end{aligned}$$

Then

- for $p \neq 2$ or $p = 2$ with $D(B(0)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$, we have

$$\|(B(t)e^{-tB(t)} - B(0)e^{-tB(0)})u_1\| \le C\frac{w(t)}{t^{\max(\frac{1}{2},\frac{1}{p})}}\|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}}.$$

- for $p = 2$ and $D(B(0)^{\frac{1}{2}}) \not\hookrightarrow \mathcal{V}$, we get

$$\|(B(t)e^{-tB(t)} - B(0)e^{-tB(0)})u_1\| \le C\frac{w(t)}{t^{\frac{1}{2}+\varepsilon}}\|u_1\|_{D(B(0)^{\frac{1}{2}})}.$$

On the other hand, since $B(0)$ is invertible, it is well-known that $t \to B(0)e^{-tB(0)}u_1 \in L^p(0,\tau;\mathcal{H})$ if and only if $u_1 \in (\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}$ (see e.g. [17, Proposition 5.1.1]) and we have

$$\int_0^\infty \|B(0)e^{-tB(0)} - B(0)u_1\|^p\,dt \le \|u_1\|^p_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}}.$$

Then

- for $p \neq 2$ (or $p = 2$ with $D(B(0)^{\frac{1}{2}}) \hookrightarrow \mathcal{V}$), we have

$$\|R_1 u_1\|_{L^p(0,\tau;\mathcal{H})} \leq C\Big[\Big(\int_0^\tau \frac{w(t)^p}{t^{\max(\frac{p}{2},1)}} dt\Big)^{\frac{1}{p}} + 1\Big]\|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}} < \infty.$$

- if $p = 2$ and $D(B(0)^{\frac{1}{2}}) \not\hookrightarrow \mathcal{V}$, we have

$$\|R_1 u_1\|_{L^2(0,\tau;\mathcal{H})} \leq C\Big[\Big(\int_0^\tau \frac{w(t)^2}{t^{1+\varepsilon}} dt\Big)^{\frac{1}{2}} + 1\Big]\|u_1\|_{D(B(0)^{\frac{1}{2}})}.$$

This proves that R_1 is bounded from $(\mathcal{H}, D(B(0)))_{1-\frac{1}{p},p}$ into $L^p(0,\tau;\mathcal{H})$.

Now, we consider the operator R_2 with $p < 2$. Clearly

$$\begin{aligned}\|e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0\| &\leq \|e^{-t\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})}\|\mathcal{C}(t)u_0\|_{\mathcal{V}'}\\ &\leq \frac{C}{\sqrt{t}}\ \|u_0\|_{\mathcal{V}}.\end{aligned}$$

Therefore $\|R_2 u_0\|_{L^p(0,\tau;\mathcal{H})} \leq C\|u_0\|_{\mathcal{V}}$.

Now for $p \geq 2$, we write

$$\begin{aligned}R_2 u_0(t) &= B(t)e^{-tB(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u_0\\ &= B(t)e^{-tB(t)}\mathcal{B}(t)^{-1}(\mathcal{C}(t) - \mathcal{C}(0))u_0\\ &\quad + B(t)e^{-tB(t)}(\mathcal{B}(t)^{-1} - \mathcal{B}(0)^{-1})\mathcal{C}(0)u_0\\ &\quad + B(t)e^{-tB(t)}\mathcal{B}(0)^{-1}\mathcal{C}(0)u_0\\ &:= I_1(t) + I_2(t) + I_3(t).\end{aligned}$$

For $i = 1$ or $i = 2$, we have the following estimate

$$\begin{aligned}\|I_i(t)\| &\leq Cw(t)\|e^{-t\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})}\|u_0\|_{\mathcal{V}}\\ &\leq C'\frac{w(t)}{\sqrt{t}}\ \|u_0\|_{\mathcal{V}}.\end{aligned}$$

Then

$$\|I_i\|_{L^p(0,\tau;\mathcal{H})} \leq C'\Big[\int_0^\tau \frac{w(t)^p}{t^{\frac{p}{2}}} dt\Big]^{\frac{1}{p}}\ \|u_0\|_{\mathcal{V}}.$$

We note that

$$\mathcal{C}(0) : \begin{cases} D(C(0)) & \to \mathcal{H}\\ \qquad\mathcal{V} & \to \mathcal{V}'.\end{cases}$$

Therefore

$$D(\mathcal{C}(0)\mid_{(\mathcal{V}',\mathcal{H})_{\alpha,p}}) = (\mathcal{V}, D(C(0)))_{\alpha,p},$$

where $\mathcal{C}(0)\mid_{(\mathcal{V}',\mathcal{H})_{\alpha,p}}$ is the part of $\mathcal{C}(0)$ on $(\mathcal{V}',\mathcal{H})_{\alpha,p}$.

Thus,

$$\mathcal{B}(0)^{-1}\mathcal{C}(0)\in\mathcal{L}\Big((\mathcal{V},D(C(0)))_{1-\frac{1}{p},p};(\mathcal{V},D(B(0))_{1-\frac{1}{p},p})\Big).$$

Observing that for $t\in[0,\tau]$, $I_3(t)=R_1\mathcal{B}(0)^{-1}\mathcal{C}(0)u_0(t)$. Using the first part of the proposition, we obtain

$$\begin{aligned}\|I_3\|_{L^p(0,\tau;\mathcal{H})}&\le C_1\|\mathcal{B}(0)^{-1}\mathcal{C}(0)u_0\|_{(\mathcal{H},D(C(0)))_{1-\frac{1}{p},p}}\\&\le C\|u_0\|_{(\mathcal{V},D(C(0)))_{1-\frac{1}{p},p}}.\end{aligned}$$

This shows that $t\to R_2u_0(t)\in L^p(0,\tau;\mathcal{H})$ and then the lemma is proved. □

3.2 The main result

Our aim in this subsection is to prove maximal L^p-regularity in $\mathcal{H}$ for the second order Cauchy problem (16).

Our main result is the following.

Theorem 3.7 *For all $f\in L^p(0,\tau;\mathcal{H})$ and $u_1\in(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}$, $u_0\in X^p$, with $p\in(1,\infty)$, there exists a unique solution $v\in MR(p,\mathcal{H})$ to the problem*

$$\begin{cases}v''(t)+\mathcal{B}(t)v'(t)+\mathcal{A}(t)v(t)=f(t) & t\text{-a.e.}\\ v(0)=u_0,\quad v'(0)=u_1+\gamma u_0.\end{cases}\tag{21}$$

In addition, there exists a positive constant C such that

$$\|v\|_{MR(p,\mathcal{H})}\le C\Big(\|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}}+\|u_0\|_{X^p}+\|f\|_{L^p(0,\tau;\mathcal{H})}\Big).$$

Proof Let $f\in L^p(0,\tau;\mathcal{H})$ and $(u_0,u_1)\in(X^p\times(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p})\subseteq(\mathcal{V}\times(\mathcal{V}',\mathcal{V})_{1-\frac{1}{p},p})$. Recall that by Theorem 2.6 there exists a unique $v\in W^{2,p}(0,\tau;\mathcal{V}')\cap W^{1,p}(0,\tau;\mathcal{V})$ solution to Problem (21). Then by Proposition 3.1, there exists a unique $u\in W^{2,p}(0,\tau;\mathcal{V}')\cap W^{1,p}(0,\tau;\mathcal{V})$ solution to Problem (17).

For simplicity of notation, we write $\mathcal{B}(t)$ instead of $\mathcal{B}(t)+\gamma$.

Fix $0\le t\le\tau$. We get from the equation in (17)

$$\begin{aligned}&\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}u''(s)\,ds+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{B}(s)u'(s)\,ds\\&\quad+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(s)u(s)\,ds=\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}g(s)\,ds.\end{aligned}\tag{22}$$

Here $g(s)=e^{-\gamma s}f(s)$ for almost every $s\in[0,\tau]$.

Hence,

$$\begin{aligned}&\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}u''(s)\,ds+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{B}(s)u'(s)\,ds\\&\quad+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{C}(s)-\mathcal{C}(t)\Big)u(s)\,ds\\&\quad+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u(s)\,ds\\&=\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}g(s)\,ds.\end{aligned}\tag{23}$$

Integrating by parts, we obtain

$$\begin{aligned}\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}u''(s)\,ds&=\mathcal{B}(t)u'(t)-B(t)e^{-tB(t)}u'(0)\\&\quad-\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{B}(t)u'(s)\,ds\end{aligned}\tag{24}$$

and

$$\begin{aligned}\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u(s)\,ds&=\mathcal{C}(t)u(t)-e^{-t\mathcal{B}(t)}\mathcal{C}(t)u(0)\\&\quad-\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u'(s)\,ds.\end{aligned}\tag{25}$$

Combining (24) with (25) and (23), we have

$$\begin{aligned}&\mathcal{B}(t)u'(t)-B(t)e^{-tB(t)}u_1-\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{B}(t)-\mathcal{B}(s)\Big)u'(s)\,ds\\&\quad+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{C}(s)-\mathcal{C}(t)\Big)u(s)\,ds\\&\quad+\mathcal{C}(t)u(t)-e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0-\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u'(s)\,ds\\&=B(t)\int_0^t e^{-(t-s)B(t)}g(s)\,ds.\end{aligned}\tag{26}$$

Therefore

$$\begin{aligned}\mathcal{B}(t)u'(t)+\mathcal{C}(t)u(t) &= B(t)e^{-tB(t)}u_1+e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0\\
&+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))\mathcal{B}(s)^{-1}(\mathcal{B}(s)u'(s)+\mathcal{C}(s)u(s))\,ds\\
&-\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))\mathcal{B}(s)^{-1}\mathcal{C}(s)u(s)\,ds+\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u'(s)\,ds\\
&+\mathcal{B}(t)\int_0^t e^{-(t-s)B(t)}(\mathcal{C}(t)-\mathcal{C}(s))u(s)\,ds\\
&+B(t)\int_0^t e^{-(t-s)B(t)}g(s)\,ds.\end{aligned}$$

This allows us to write

$$\begin{aligned}&(I-Q)(\mathcal{B}(.)u'+\mathcal{C}(.)u)(t)\\
&\quad=B(t)e^{-tB(t)}u_1+e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0\\
&\quad-\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{B}(t)-\mathcal{B}(s)\Big)\mathcal{B}(s)^{-1}\mathcal{C}(s)u(s)\,ds\\
&\quad+\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{C}(t)-\mathcal{C}(s)\Big)u(s)\,ds+\int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(s)u'(s)\,ds\\
&\quad+B(t)\int_0^t e^{-(t-s)B(t)}g(s)\,ds,\end{aligned}\tag{27}$$

where for almost every $t\in[0,\tau]$

$$(Qh)(t):=\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))B(s)^{-1}h(s)\,ds.$$

Then, if $I-Q$ is invertible on $L^p(0,\tau;\mathcal{H})$ we obtain

$$\begin{aligned}&\mathcal{B}(t)u'(t)+\mathcal{C}(t)u(t)\\
&\quad=(I-Q)^{-1}\Big[R_1u_1+R_2u_0+W_1(u)+L(f)+L_2(u)+W_2(u)\Big](t),\end{aligned}\tag{28}$$

here R_1 and R_2 are as in Proposition 3.5, L as in Proposition 3.3 and

$$\begin{aligned}W_1(u)(t) &:= -\mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))\mathcal{B}(s)^{-1}(\mathcal{C}(s)u(s))\,ds,\\
W_2(u)(t) &:= \mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{C}(t)-\mathcal{C}(s))u(s)\,ds,\\
L_2u(t) &:= \int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u'(s)\,ds.\end{aligned}$$

Now, we prove the boundedness of Q, W_1, W_1, L_1 on $L^p(0,\tau;\mathcal{H})$ for $p\in(1,\infty)$. Let $h\in L^p(0,\tau;\mathcal{H})$. We have

$$\begin{aligned}\|(Qh)(t)\| \leq & \int_0^t \|B(t)e^{-\frac{(t-s)}{2}B(t)}\|_{\mathcal{L}(\mathcal{H})}\|e^{-\frac{(t-s)}{2}\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{H})} \\ & \times \|\mathcal{B}(s)-\mathcal{B}(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|B(s)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\|h(s)\|\, ds \\ \leq & \, C\int_0^t \frac{w(t-s)}{(t-s)^{\frac{3}{2}}}\|B(s)^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})}\|h(s)\|\, ds.\end{aligned} \tag{29}$$

Now, once we replace $B(s)$ by $\alpha + B(s)$, (29) is valid with a constant independent of $\alpha > 0$ by Proposition 3.1). using the estimate

$$\|(\alpha I + B(s))^{-1}\|_{\mathcal{L}(\mathcal{H},\mathcal{V})} \leq \frac{c}{\sqrt{\alpha}}$$

in (29) for $\alpha + B(s)$, we see that

$$\|(Qh)(t)\| \leq C\alpha^{-\frac{1}{2}}\int_0^t \frac{w(t-s)}{(t-s)^{\frac{3}{2}}}\|h(s)\|\, ds.$$

Therefore, by using Young's inequality we obtain

$$\|Q\|_{\mathcal{L}(L^p(0,\tau;\mathcal{H}))} \leq C\alpha^{-\frac{1}{2}}\int_0^\tau \frac{w(s)}{s^{\frac{3}{2}}}\, ds.$$

Using the assumption on w and taking α large enough makes Q strictly contractive, so that $(I-Q)^{-1}$ is bounded on $L^p(0,\tau;\mathcal{H})$ by the Neumann series.

For W_i, with $i = 1, 2$ we have

$$\|W_i(u)\|_{L^p(0,\tau;\mathcal{H})} \leq C_1\int_0^\tau \frac{w(s)}{s^{\frac{3}{2}}}\, ds\ \|u\|_{L^\infty(0,\tau;\mathcal{V})}.$$

The Sobolev embedding gives

$$\|W_i(u)\|_{L^p(0,\tau;\mathcal{H})} \leq C\int_0^\tau \frac{w(s)}{s^{\frac{3}{2}}}\, ds\ \|u\|_{W^{1,p}(0,\tau;\mathcal{V})}.$$

The following estimate holds for L_2

$$\|L_2 u\|_{L^p(0,\tau;\mathcal{H})} \leq C_2\sqrt{\tau}\|u'\|_{L^p(0,\tau;\mathcal{V})}.$$

As a result, we obtain from (28), (11) and the previous estimates

$$\begin{aligned}&\|\mathcal{C}(.)u + \mathcal{B}(.)u'\|_{L^p(0,\tau;\mathcal{H})} \\ &\quad\leq C\Big(\|u\|_{W^{1,p}(0,\tau;\mathcal{V})} + \|g\|_{L^p(0,\tau;\mathcal{H})} \\ &\qquad + \|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}} + \|u_0\|_{X^p}) \\ &\quad\leq C_1\Big(\|f\|_{L^p(0,\tau;\mathcal{H})} + \|u_1\|_{(\mathcal{H},D(B(0)))_{1-\frac{1}{p},p}} + \|u_0\|_{X^p}\Big).\end{aligned}$$

Then $u'' = g - \mathcal{C}(.)u - \mathcal{B}(.)u' \in L^p(0,\tau;\mathcal{H})$ and consequently $u \in MR(p,\mathcal{H})$ which implies that the Problems (17) and (21) have maximal L^p-regularity in $\mathcal{H}$. This finishes the proof of the theorem. □

Following [2, Definition 3.4], we introduce the following definition

Definition 3.8 We say that $(B(t))_{t\in[0,\tau]}$ satisfies the uniform Kato square root property if $D(B(t)^{\frac{1}{2}}) = \mathcal{V}$ for all $t \in [0,\tau]$ and there are $c_1, c^1 > 0$ such that for all $v \in \mathcal{V}$

$$c_1\|v\|_{\mathcal{V}} \le \|B(t)^{\frac{1}{2}}v\| \le c^1\|v\|_{\mathcal{V}}. \tag{30}$$

The uniform Kato square root property is obviously satisfied for symmetric forms (see Lemma 3.4(2)). It is also satisfied for uniformly elliptic operators (not necessarily symmetric)

$$B(t) = -\sum_{k,l=1}^{d} \partial_k(a_{kl}(t,x)\partial_l)$$

on $L^2(\mathbb{R}^d)$ since $\|\nabla u\|_2$ is equivalent to $\|B(t)^{\frac{1}{2}}u\|_2$ with constants depending only on the dimension and the ellipticity constants, see [8].

From [1, Lemma 4.1] we have the following lemma

Lemma 3.9 *Suppose* (30). *Then for all* $f \in L_2(0,\tau;\mathcal{H}), 0 \le s \le t \le \tau$,

$$\|\int_s^t e^{-(t-r)B(t)}f(r)dr\|_{\mathcal{V}} \le C\|f\|_{L_2(s,t;\mathcal{H})}.$$

In the next result we prove maximal L^2-regularity where we improve the assumption on u_0 and prove that the solution belongs to $C^1([0,\tau],\mathcal{V})$. More precisely

Theorem 3.10 *We assume the uniform Kato property* (30) *and the following two conditions that for all* $s,t \in [0,\tau]$

1. $|\mathfrak{b}(t,u,v) - \mathfrak{b}(s,u,v)| \le w(|t-s|)\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$, *with* $w : [0,\tau] \to [0,\infty)$ *is a non-decreasing function such that*

$$\int_0^\tau \frac{w(t)}{t^{\frac{3}{2}}}\,dt < \infty, \quad w(t) \le ct^{\varepsilon}, \tag{31}$$

for an arbitrary $\varepsilon > 0$.

2. $\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le w_0(|t-s|)$, *with* $w_0 : [0,\tau] \to [0,\infty)$ *is a non-decreasing function continuous at* 0 *and satisfies*

$$\int_0^\tau \frac{w_0(t)}{t^{\frac{3}{2}}}\,dt < \infty, \quad \int_0^\tau \frac{w_0^2(t)}{t}\,dt < \infty. \tag{32}$$

Then for all $f \in L^2(0,\tau;\mathcal{H})$ *and* $(u_0,u_1) \in \mathcal{V}\times\mathcal{V}$, *there exists a unique* $u \in MR(2,\mathcal{H})$ *be the solution to the Problem* (21). *Moreover*, $u \in C^1([0,\tau];\mathcal{V})$.

Remark 3.11 We note that if $w(t) \le Ct^{\frac{1}{2}+\varepsilon}$ and $w_0(t) \le C't^{\frac{1}{2}+\varepsilon}$ for some $\varepsilon > 0$, then the assumptions (31), (32) are satisfied.

Proof Firstly, we set $w_1(t) = w_0(t) + \gamma w(t)$. It is clear that $|\mathfrak{c}(t,u,v) - \mathfrak{c}(s,u,v)| \le w_1(|t-s|)\|u\|_{\mathcal{V}}\|v\|_{\mathcal{V}}$ and w_1 satisfies the same conditions with w_0.

From (27) we have

$$\begin{aligned}(I-Q)\Big(\mathcal{B}(.)u' + \mathcal{C}(.)u\Big)(t) &= B(t)e^{-tB(t)}u_1 + e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0\\ &\quad - \mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{B}(t)-\mathcal{B}(s)\Big)\mathcal{B}(s)^{-1}\mathcal{C}(s)u(s)\,ds\\ &\quad + \mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}\Big(\mathcal{C}(s)-\mathcal{C}(t)\Big)u(s)\,ds\\ &\quad + \int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{C}(t)u'(s)\,ds\\ &\quad + B(t)\int_0^t e^{-(t-s)B(t)}g(s)\,ds,\end{aligned}$$

such that

$$(Qh)(t) = \mathcal{B}(t)\int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))B(s)^{-1}h(s)\,ds,\ t \in (0,\tau).$$

To prove maximal L^2-regularity we follow the same proof as in Theorem 3.7. The main difference is to prove that $t \to R_2u_0(t) = e^{-tB(t)}\mathcal{C}(t)u_0 \in L^2(0,\tau;\mathcal{H})$.

Observing that

$$\begin{aligned}R_2u_0(t) &= e^{-t\mathcal{B}(t)}\mathcal{C}(t)u_0\\ &= e^{-t\mathcal{B}(t)}(\mathcal{C}(t)-\mathcal{C}(0))u_0 + e^{-tB(t)}\mathcal{C}(0)u_0.\end{aligned} \tag{33}$$

For the first term in the RHS of (33), we have

$$\begin{aligned}&\int_0^\tau \|e^{-t\mathcal{B}(t)}(\mathcal{C}(t)-\mathcal{C}(0))u_0\|^2\,dt\\ &\le \int_0^\tau \|e^{-t\mathcal{B}(t)}\|^2_{\mathcal{L}(\mathcal{V}',\mathcal{H})}\|\mathcal{C}(t)-\mathcal{C}(0)\|^2_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|u_0\|^2_{\mathcal{V}}\,dt\\ &\le C\int_0^\tau \frac{w_1(t)^2}{t}\,dt\ \|u_0\|^2_{\mathcal{V}}.\end{aligned}$$

We write

$$e^{-t\mathcal{B}(t)}\mathcal{C}(0)u_0 = e^{-t\mathcal{B}(t)}\mathcal{C}(0)u_0 - e^{-tB(0)}\mathcal{C}(0)u_0 + e^{-t\mathcal{B}(0)}\mathcal{C}(0)u_0.$$

The functional calculus for the sectorial operators $B(t)$, $B(0)$ gives

$$\|e^{-t\mathcal{B}(t)}\mathcal{C}(0)u_0 - e^{-tB(0)}\mathcal{C}(0)u_0\| \le c\frac{w(t)}{t^{\frac{1}{2}}}\|u_0\|_{\mathcal{V}}.$$

Clearly,

$$e^{-t\mathcal{B}(0)}\mathcal{C}(0)u_0 = B(0)^{\frac{1}{2}}e^{-tB(0)}\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{C}(0)u_0.$$

We note that by [2, Lemma 3.5] we have the quadratic estimate, namely

$$\int_0^\tau \|B(0)^{\frac{1}{2}}e^{-tB(0)}x\|^2 dt \le c\|x\|^2 \tag{34}$$

for all $x \in \mathcal{H}$.

Hence,

$$\begin{aligned}
&\int_0^\tau \|e^{-t\mathcal{B}(0)}\mathcal{C}(0)u_0\|^2 dt \\
&= \int_0^\tau \|B(0)^{\frac{1}{2}}e^{-tB(0)}\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{C}(0)u_0\|^2 dt \\
&\le c\|\mathcal{B}(0)^{-\frac{1}{2}}\mathcal{C}(0)u_0\|^2.
\end{aligned}$$

Since $D(B(0)^{\frac{1}{2}}) = \mathcal{V}$, we have $D(\mathcal{B}(0)^{\frac{1}{2}}) = \mathcal{H}$ and $\mathcal{B}(0)^{-\frac{1}{2}} \in \mathcal{L}(\mathcal{V}', \mathcal{H})$ (see Lemma 3.4).

Therefore

$$\int_0^\tau \|e^{-t\mathcal{B}(0)}\mathcal{C}(0)u_0\|^2\, dt \le C\|u_0\|^2_{\mathcal{V}}.$$

Then $\|R_2u_0\|_{L^2(0,\tau;\mathcal{H})} \le C\|u_0\|_{\mathcal{V}}$ and maximal L^2-regularity holds in $\mathcal{H}$. Thus, $u \in MR(2, \mathcal{H})$.

We proceed to show that $u' \in L^\infty(0, \tau; \mathcal{V})$.

Fix $0 \le t \le \tau$ and use (26), we obtain

$$\begin{aligned}
u'(t) = &-\mathcal{B}(t)^{-1}\mathcal{C}(t)u(t) + e^{-tB(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u_0 \\
&+ e^{-tB(t)}u_1 + \int_0^t e^{-(t-s)B(t)}(\mathcal{B}(t) - \mathcal{B}(s))u'(s)\, ds \\
&+ \int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{C}(t) - \mathcal{C}(s))u(s)\, ds \\
&+ \int_0^t e^{-(t-s)\mathcal{B}(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u'(s)\, ds + \int_0^t e^{-(t-s)B(t)}g(s)\, ds.
\end{aligned} \tag{35}$$

Now, we define the operator K in $L^\infty(0, \tau; \mathcal{V})$ by

$$K(h)(t) := \int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(s))h(s)\,ds,$$

where $t \in [0,\tau]$. Taking the norm in $\mathcal{V}$, it follows that

$$\begin{aligned}\|K(h)(t)\|_{\mathcal{V}} &\le \int_0^t \|e^{-(t-s)\mathcal{B}(t)}\|_{\mathcal{L}(\mathcal{V}',\mathcal{V})}\|\mathcal{B}(t)-\mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|h(s)\|_{\mathcal{V}}\,ds\\ &\le C\int_0^t \frac{w(t-s)}{t-s}\,ds\,\|h\|_{L^\infty(0,t;\mathcal{V})}\\ &\le C\sqrt{t}\int_0^t \frac{w(s)}{s^{\frac{3}{2}}}\,ds\,\|h\|_{L^\infty(0,t;\mathcal{V})},\end{aligned}$$

where C is a positive constant independent of t. Now, we take τ small enough such that $C\sqrt{\tau}\int_0^\tau \frac{w(s)}{s^{\frac{3}{2}}}\,ds < c$, with $0<c<1$.

We conclude that $\|K(h)\|_{L^\infty(0,\tau;\mathcal{V})} \le c\|h\|_{L^\infty(0,\tau;\mathcal{V})}$ and hence $I-K$ is invertible on $L^\infty(0,\tau;\mathcal{V})$.

Using (35), we get

$$\begin{aligned}(I-K)(u')(t) &= -\mathcal{B}(t)^{-1}\mathcal{C}(t)u(t) + e^{-tB(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u_0\\ &\quad + e^{-tB(t)}u_1 + \int_0^t e^{-(t-s)\mathcal{B}(t)}(\mathcal{C}(t)-\mathcal{C}(s))u(s)\,ds\\ &\quad + \int_0^t e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u'(s)\,ds + \int_0^t e^{-(t-s)B(t)}g(s)\,ds.\end{aligned}$$

Therefore for τ small enough and by using Lemma 3.9 we obtain

$$\begin{aligned}\|u'\|_{L^\infty(0,\tau;\mathcal{V})} &\le C\Big(\|u\|_{L^\infty(0,\tau;\mathcal{V})} + \|u_0\|_{\mathcal{V}} + \|u_1\|_{V} + \|g\|_{L^2(0,\tau;\mathcal{H})}\\ &\quad + \|u'\|_{L^2(0,\tau;\mathcal{V})} + \int_0^t \frac{w_1(l)}{l}\,dl\,\|u\|_{L^\infty(0,\tau;\mathcal{V})}\Big).\end{aligned}$$

Since $u' \in L^\infty(0,\tau;\mathcal{V}) \cap C([0,\tau];\mathcal{H})$, we have by [2, Lemma 3.7] that $u'(t) \in \mathcal{V}$ for all $t \in [0,\tau]$. Now, for τ arbitrary we split $(0,\tau)$ into small intervals and proceed exactly as in the previous proof. In order to obtain a solution $u \in W^{1,\infty}(0,\tau;\mathcal{V})$, we glue the solutions on each-interval.

We fix s and t in $[0,\tau]$ such that $s<t$. We get from the equation in (17)

$$\begin{aligned}&\int_s^t e^{-(t-l)\mathcal{B}(t)}u''(l)\,dl + \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{B}(l)u'(l)\,dl\\ &\quad + \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{C}(l)u(l)\,dl = \int_s^t e^{-(t-l)\mathcal{B}(t)}g(l)\,dl.\end{aligned} \tag{36}$$

Hence,

$$\begin{aligned}\int_s^t e^{-(t-l)\mathcal{B}(t)}u''(l)\,dl &+ \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{B}(l)u'(l)\,dl \\ &+ \int_s^t e^{-(t-l)\mathcal{B}(t)}\Big(\mathcal{C}(l)-\mathcal{C}(t)\Big)u(l)\,dl \\ &+ \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{C}(t)u(l)\,dl \\ &= \int_s^t e^{-(t-l)\mathcal{B}(t)}g(l)\,dl.\end{aligned} \tag{37}$$

By performing an integration by parts we have

$$\begin{aligned}\int_s^t e^{-(t-l)\mathcal{B}(t)}u''(l)\,dl = u'(t) - e^{-(t-s)\mathcal{B}(t)}u'(s) \\ - \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{B}(t)u'(l)\,dl\end{aligned} \tag{38}$$

and

$$\begin{aligned}\int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{C}(t)u(l)\,dl = \mathcal{B}(t)^{-1}\mathcal{C}(t)u(t) - e^{-(t-s)\mathcal{B}(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u(s) \\ - \int_s^t e^{-(t-l)\mathcal{B}(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u'(l)\,dl.\end{aligned} \tag{39}$$

Combining (39) with (38) and (37), we obtain

$$\begin{aligned}&u'(t) - u'(s) \\ &\quad = \big(-\mathcal{B}(t)^{-1}\mathcal{C}(t)u(t) + e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u(s)\big) \\ &\qquad + \big(e^{-(t-s)B(t)}u'(s) - u'(s)\big) + \int_s^t e^{-(t-l)\mathcal{B}(t)}(\mathcal{B}(t)-\mathcal{B}(l))u'(l)\,dl \\ &\qquad + \int_s^t e^{-(t-l)\mathcal{B}(t)}(\mathcal{C}(t)-\mathcal{C}(l))u(l)\,dl \\ &\qquad + \int_s^t e^{-(t-l)B(t)}\mathcal{B}(t)^{-1}\mathcal{C}(t)u'(l)\,dl + \int_s^t e^{-(t-l)B(t)}g(l)\,dl \\ &\quad := K_1(t,s) + K_2(t,s) + K_3(t,s) + K_4(t,s) + K_5(t,s) + K_6(t,s).\end{aligned}$$

Observing that

$$
\begin{aligned}
& e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s) - \mathcal{B}(t)^{-1}\mathcal{C}(t)u(t) \\
&\quad = \left[e^{-(t-s)B(t)} - e^{-(t-s)B(s)}\right]\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s) \\
&\qquad + e^{-(t-s)B(s)}(\mathcal{B}(t)^{-1} - \mathcal{B}(s)^{-1})\mathcal{C}(s)u(s) \\
&\qquad - (\mathcal{B}(t)^{-1} - \mathcal{B}(s)^{-1})\mathcal{C}(t)u(t) \\
&\qquad + \mathcal{B}(s)^{-1}(\mathcal{C}(s)u(s) - \mathcal{C}(t)u(t)) \\
&\qquad + \mathcal{B}(s)^{-1}(e^{-(t-s)B(s)} - I)\mathcal{C}(s)u(s) \\
&\quad := I_1(s,t) + I_2(s,t) + I_3(s,t) + I_4(s,t) + I_5(s,t).
\end{aligned}
\tag{40}
$$

We write by the functional calculus for the sectorial operators $B(t)$, $B(s)$

$$
e^{-(t-s)B(t)} - e^{-(t-s)B(s)} = \frac{1}{2\pi i}\int_\Gamma e^{-(t-s)\lambda}(\lambda - \mathcal{B}(t))^{-1}(\mathcal{B}(t) - \mathcal{B}(s))(\lambda - B(s))^{-1}\, d\lambda,
$$

where $\Gamma = \partial S_\theta$ is the boundary for an appropriate sector S_θ, with $\theta \in (0, \frac{\pi}{2})$. Then

$$
\begin{aligned}
& \|I_1(s,t)\|_{\mathcal{V}} \\
&\quad = \|\left(e^{-(t-s)B(t)} - e^{-(t-s)B(s)}\right)\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s)\|_{\mathcal{V}} \\
&\quad = \frac{1}{2\pi}\|\int_\Gamma e^{-(t-s)\lambda}(\lambda - B(t))^{-1}(\mathcal{B}(t) - \mathcal{B}(s))(\lambda - B(s))^{-1}\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s)\, d\lambda\|_{\mathcal{V}} \\
&\quad \le \int_\Gamma e^{-(t-s)|\lambda|\cos\theta}\|(\lambda - B(t))^{-1}\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|(\lambda - B(s))^{-1}\|_{\mathcal{L}(\mathcal{V})}\, d|\lambda| \\
&\qquad \times \|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s)\|_{\mathcal{V}} \\
&\quad \le cw(t-s)\int_\Gamma e^{-(t-s)|\lambda|\cos\theta}(1 + |\lambda|)^{-1}\, d|\lambda|\, \|u(s)\|_{\mathcal{V}} \\
&\quad \le c_\varepsilon \frac{w(t-s)}{(t-s)^\varepsilon}\, \|u(s)\|_{\mathcal{V}},
\end{aligned}
$$

where ε is an appropriate small positive constant and $c_\varepsilon > 0$ depending on ε.

Now the fact $w(t) \le ct^\varepsilon$ for some $\varepsilon > 0$ imply that $\|I_1(s,t)\|_{\mathcal{V}} \to 0$ as $t \to s$.

For $i = 2, 3$, we have immediately

$$
\|I_i(s,t)\|_{\mathcal{V}} \le C\|\mathcal{B}(t) - \mathcal{B}(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}\|u\|_{L^\infty(s,t;\mathcal{V})}.
$$

For the last terms in (40) we get

$$
\begin{aligned}
\|I_4(s,t)\|_{\mathcal{V}} &\le C_1\|\mathcal{C}(s)u(s) - \mathcal{C}(t)u(t)\|_{\mathcal{V}'}. \\
\|I_5(s,t)\|_{\mathcal{V}} &\le C_2\|\left(e^{-(t-s)B(s)} - I\right)\mathcal{C}(s)u(s)\|_{\mathcal{V}'}.
\end{aligned}
$$

Since $t \to \mathcal{C}(t)u(t) \in C([0,\tau];\mathcal{V}')$ and by the strong continuity of the semigroup $r \to e^{-rB(s)}$ on $\mathcal{V}$ one has

$$
\|e^{-(t-s)B(t)}\mathcal{B}(t)^{-1}\mathcal{C}(s)u(s) - \mathcal{B}(t)^{-1}\mathcal{C}(t)u(t)\|_{\mathcal{V}} \to 0
$$

as $t \to s$. Therefore $\|K_1(t,s)\|_{\mathcal{V}} \to 0$ as $t \to s$.

We write

$$\|K_2(t,s)\|_{\mathcal{V}} = \|[e^{-(t-s)B(t)} - I]u'(s)\|_{\mathcal{V}}$$
$$\leq \|[e^{-(t-s)B(t)} - e^{-(t-s)B(s)}]u'(s)\|_{\mathcal{V}} + \|[e^{-(t-s)B(s)} - I]u'(s)\|_{\mathcal{V}}.$$

We estimate the first term on the RHS. By using again the functional calculus for the operators $B(t)$, $B(s)$ we have

$$\|[e^{-(t-s)B(t)} - e^{-(t-s)B(s)}]u'(s)\|_{\mathcal{V}} \leq c' \frac{w(t-s)}{(t-s)^{\varepsilon'}} \|u'(s)\|_{\mathcal{V}},$$

for appropriate small $\varepsilon' > 0$.

By the strong continuity of the semigroup $r \to e^{-rB(s)}$ on $\mathcal{V}$ one has

$$\|[e^{-(t-s)B(s)} - I]u'(s)\|_{\mathcal{V}} \to 0 \quad \text{as } t \to s.$$

Therefore $\|K_2(t,s)\|_{\mathcal{V}} \to 0$ as $t \to s$.

Taking the norm in $\mathcal{V}$, we obtain

$$\|K_3(t,s)\|_{\mathcal{V}} \leq C_1 \int_s^t \frac{\|\mathcal{B}(t) - \mathcal{B}(l)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-l}\, dl\, \|u'\|_{L^\infty(s,t;\mathcal{V})}$$
$$\leq C_1 \int_s^t \frac{w(t-l)}{t-l}\, dl\, \|u'\|_{L^\infty(s,t;\mathcal{V})}.$$
$$\|K_4(t,s)\|_{\mathcal{V}} \leq C_2 \int_s^t \frac{\|\mathcal{C}(t) - \mathcal{C}(l)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')}}{t-l}\, dl\, \|u\|_{L^\infty(s,t;\mathcal{V})}$$
$$\leq C_2 \int_s^t \frac{w_1(t-l)}{t-l}\, dl\, \|u\|_{L^\infty(s,t;\mathcal{V})}.$$
$$\|K_5(t,s)\|_{\mathcal{V}} \leq C_3(t-s)\, \|u'\|_{L^\infty(s,t;\mathcal{V})},$$

where C_1, C_2, C_3 are a positive constants independents of s and t.

By using Lemma 3.9 we get

$$\|K_6(t,s)\|_{\mathcal{V}} \leq C\|g\|_{L^2(s,t;\mathcal{V})}.$$

Hence, $\|u'(t) - u'(s)\|_{\mathcal{V}} \to 0$ as $t \to s$. This proves that u' is right continuous for the norm of $\mathcal{V}$.

It remains to prove left continuity of u'.

Fix $0 \leq s \leq t \leq \tau$. We integrate the Eq. (17) from s to t to obtain

$$\int_s^t e^{-(l-s)B(s)} u''(l)\, dl + \int_s^t e^{-(l-s)B(s)} \mathcal{B}(l) u'(l)\, dl$$
$$+ \int_s^t e^{-(l-s)B(s)} \mathcal{C}(l) u(l)\, dl = \int_s^t e^{-(l-s)B(s)} g(l)\, dl.$$

Now, by integration by parts we get

$$u'(s) = e^{-(t-s)B(s)}u'(t) - \mathcal{B}(s)^{-1}e^{-(t-s)B(s)}\mathcal{C}(t)u(t) + \mathcal{B}(s)^{-1}\mathcal{C}(t)u(s)$$
$$+ 2\int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{B}(l)u'(l) + \mathcal{C}(l)u(l))\, dl$$
$$+ \int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{B}(s) - \mathcal{B}(l))u'(l)\, dl$$
$$- \int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{C}(l) - \mathcal{C}(t))u(l)\, dl$$
$$- \int_s^t e^{-(l-s)B(s)}\mathcal{B}(s)^{-1}\mathcal{C}(t)u'(l)\, dl - \int_s^t e^{-(l-s)B(s)}g(l)\, dl.$$

Hence,

$$u'(s) - u'(t) = e^{-(t-s)B(s)}u'(t) - u'(t)$$
$$- \mathcal{B}(s)^{-1}e^{-(t-s)B(s)}\mathcal{C}(t)u(t) + \mathcal{B}(s)^{-1}\mathcal{C}(t)u(s)$$
$$+ 2\int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{B}(l)u'(l) + \mathcal{C}(l)u(l))\, dl$$
$$+ \int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{B}(s) - \mathcal{B}(l))u'(l)\, dl$$
$$- \int_s^t e^{-(l-s)\mathcal{B}(s)}(\mathcal{C}(l) - \mathcal{C}(t))u(l)\, dl$$
$$- \int_s^t e^{-(l-s)B(s)}\mathcal{B}(s)^{-1}\mathcal{C}(t)u'(l)\, dl - \int_s^t e^{-(l-s)B(s)}g(l)\, dl.$$

We now proceed analogously as the proof of the right continuity of u', we obtain that

$$\|u'(t) - u'(s)\|_{\mathcal{V}} \to 0 \quad \textit{as } s \to t.$$

We have proved that u' is left continuous in $\mathcal{V}$ and finally $u \in C^1([0,\tau];\mathcal{V})$. Therefore $v \in C^1([0,\tau];\mathcal{V})$. This finishes the proof of the theorem. □

For higher order equations we have

Theorem 3.12 *Let $(\mathcal{A}_i(t))_{t\in[0,\tau], i\in[1,N]}$, $N \in \mathbb{N}^*$ such that $\mathcal{A}_i(t) \in \mathcal{L}(\mathcal{V}, \mathcal{V}')$ for all $i \in [1,N]$ and $\|\mathcal{A}_i(t)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le M$. We suppose that $(\mathcal{A}_N(t))_{t\in[0,\tau]}$ is associated with $\mathcal{V}$ -bounded quasi-coercive forms and for all $i \in [1,N]$*

$$\|\mathcal{A}_i(t) - \mathcal{A}_i(s)\|_{\mathcal{L}(\mathcal{V},\mathcal{V}')} \le K|t-s|^{\alpha}$$

for some $K > 0$ and $\alpha > \frac{1}{2}$. We assume in addition that $(\mathcal{A}_N(t) + \nu)_{t\in[0,\tau]}$ satisfies the uniform Kato property (30). Then for all $f \in L^2(0,\tau;\mathcal{H})$ and $u_0, \dots, u_{N-1} \in \mathcal{V}$ there exists a unique $u \in W^{N,2}(0,\tau;\mathcal{H}) \cap C^{N-1}(0,\tau;\mathcal{V})$ be the solution to the problem

$$\begin{cases} u^{(N)}(t) + \mathcal{A}_N(t)u^{(N-1)}(t) + \mathcal{A}_{N-1}(t)u^{(N-2)}(t) + \cdots + \mathcal{A}_1(t)u(t) = f(t) \quad t\text{-a.e.} \\ u^{(N-1)}(0) = u_{N-1}, \ldots, u(0) = u_0. \end{cases} \tag{41}$$

In addition, there exists a positive constant C independent of $u_0, \ldots, u_{N-1}$ *and f such that*

$$\|u\|_{W^{N,2}(0,\tau;\mathcal{H})\cap C^{N-1}(0,\tau;\mathcal{V})} \le C\left(\sum_{i=0}^{N-1} \|u_i\|_{\mathcal{V}} + \|f\|_{L^2(0,\tau;\mathcal{H})}\right).$$

Proof We give only the main ideas of the proof. We prove the theorem by induction. In case $N = 1$ the result follows from [1, Theorem 4.2]. The theorem holds for $N = 2$ by Theorem 3.10. Now, we assume that the theorem is true at order $N - 1$ where N is an arbitrary positive integer. By integration and following the same strategy of proof as in Theorem 2.6 we prove maximal L^2-regularity in $\mathcal{V}'$ for the Cauchy problem (41) and we have $u \in W^{N,2}(0, \tau;\mathcal{V}') \cap W^{N-1,2}(0, \tau;\mathcal{V})$.

Let $\gamma > 0$ and set $v(t) = e^{-\gamma t}u(t)$. By Leibniz's rule and using the Eq. (41) we get that v is the solution to the problem

$$\begin{cases} v^{(N)}(t) + (\mathcal{A}_N(t) + N\gamma I)v^{(N-1)}(t) + \sum_{j=0}^{N-2} \mathcal{C}_j(t)v^{(j)}(t) = e^{-\gamma t}f(t) \quad t\text{-a.e.} \\ v^{(k)}(0) = v_k = \sum_{j=0}^{k} C_j^k(-\gamma)^{k-j}u_j, \quad k \in [0, N-1], \end{cases} \tag{42}$$

where $\mathcal{C}_j(t) = \left(\sum_{m=j}^{N-1}(-1)^{N+1-m}C_m^N C_j^m\right)\gamma^{N-j}I + \sum_{m=j}^{N-1} C_j^m \gamma^{m-j}\mathcal{A}_{m+1}(t)$, for all $j \in [0, N-1]$ and $C_j^m = \frac{m!}{j!(m-j)!}$. Here $v^{(j)}$ is the derivative of order j.

We assume now that $\gamma > \frac{|\nu|}{N}$, then $\mathcal{C}_{N-1}(t) = \mathcal{A}_N(t) + N\gamma I$ is associated with $\mathcal{V}$-bounded coercive form for all $t \in [0, \tau]$.

By performing an integration by parts as in (24), (25) we obtain

$$\begin{aligned} &\mathcal{C}_{N-1}(t)v^{(N-1)}(t) + \mathcal{C}_{N-2}(t)v^{(N-2)}(t) + \cdots + \mathcal{C}_0(t)v(t) \\ &\quad = \mathcal{C}_{N-1}(t)e^{-t\mathcal{C}_{N-1}(t)}v_{N-1} + e^{-t\mathcal{C}_{N-1}(t)}\Big(\mathcal{C}_1(t)v_0 + \cdots + \mathcal{C}_{N-2}(t)v_{N-2}\Big) \\ &\qquad + \mathcal{C}_{N-1}(t)\int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)}\Big(\mathcal{C}_{N-1}(t) - \mathcal{C}_{N-1}(s)\Big)v^{(N-1)}(s)\,ds \\ &\qquad + \mathcal{C}_{N-1}(t)\int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)}\Big(\mathcal{C}_{N-2}(t) - \mathcal{C}_{N-2}(s)\Big)v^{(N-2)}(s)\,ds \\ &\qquad + \cdots + \mathcal{C}_{N-1}(t)\int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)}\Big(\mathcal{C}_0(t) - \mathcal{C}_0(s)\Big)v(s)\,ds \\ &\qquad + \int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)}\Big(\mathcal{C}_0(t)v'(s) + \cdots + \mathcal{C}_{N-2}(t)v^{(N-1)}(s)\Big)\,ds \\ &\qquad + \mathcal{C}_{N-1}(t)\int_0^t e^{-(t-s)\mathcal{C}_{N-1}(t)}e^{-\gamma s}f(s)\,ds. \end{aligned}$$

We now proceed analogously to the proof of Theorem 3.10 to prove maximal L^2-regularity in $\mathcal{H}$. The details are left to the reader. □

4 Counter-examples

In this section we give some examples where the maximal regularity fails.

Let

$$\mathfrak{c} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

be bounded coercive form and let $\mathcal{C}$ is the operator associated to $\mathfrak{c}$ in $\mathcal{V}'$ and $C = \mathcal{C} \mid_{\mathcal{H}}$.

We introduce the following space

$$L^2(0,\tau;D(C)) := \{u \in L^2(0,\tau;\mathcal{H}) \ : \ Cu \in L^2(0,\tau;\mathcal{H})\},$$

with norm

$$\|u\|_{L^2(0,\tau;D(C))} := \|Cu\|_{L^2(0,\tau;\mathcal{H})}.$$

Let us consider the space

$$MR_C(2,\mathcal{H}) := \{u \in W^{2,2}(0,\tau;\mathcal{H}) \cap W^{1,2}(0,\tau;\mathcal{V}) : \ C(u+u') \in L^2(0,\tau;\mathcal{H})\}$$

endowed with norm

$$\|u\|_{MR_C(2,\mathcal{H})} := \|u''\|_{L^2(0,\tau;\mathcal{H})} + \|C(u+u')\|_{L^2(0,\tau;\mathcal{H})}.$$

We define the associated trace space by

$$TR_{\mathfrak{c}}(2,\mathcal{H}) = \{(u(0),u'(0)) : \ u \in MR_C(2,\mathcal{H})\},$$

with norm

$$\|(u_0,u_1)\|_{TR_{\mathfrak{c}}(2,\mathcal{H})} = \inf\{\|u\|_{MR_C(2,\mathcal{H})} : u_0 = u(0), u'(0) = u_1\}.$$

Proposition 4.1 *We have*

$$TR_{\mathfrak{c}}(2,\mathcal{H}) = D = \{(u_0,u_1) : u_0 \in \mathcal{V},\ u_0 + u_1 \in (\mathcal{H},D(C))_{\frac{1}{2},2}\}.$$

Proof Let us first prove that $TR_{\mathfrak{c}}(2,\mathcal{H}) \subseteq D$. Indeed, let $u \in MR_C(2,\mathcal{H})$ and we set $v = u + u'$. Then

$$v \in L^2(0,\tau;D(C)) \cap W^{1,2}(0,\tau;\mathcal{H})$$

and by Lemma 2.3 one has $v \in C([0,\tau];(\mathcal{H},D(C))_{\frac{1}{2},2})$.

Moreover, $v(0) \in (\mathcal{H},D(C))_{\frac{1}{2},2}$, $u(0)+u'(0) \in (\mathcal{H},D(C))_{\frac{1}{2},2}$ and $u(0) \in \mathcal{V}$.

Thus, $TR_{\mathfrak{c}}(2,\mathcal{H}) \subseteq D$.

We next prove $D \subseteq TR_{\mathfrak{c}}(2,\mathcal{H})$. Indeed, let $(u_0,u_1) \in D$. By [17, Corollary 1.14], there exist $v \in L^2(0,\tau;D(C)) \cap W^{1,2}(0,\tau;\mathcal{H})$ such that $v(0) = u_0 + u_1$.

Now, we set

$$u(t) = e^{-t}u_0 + \int_0^t v(s)\,ds, \qquad t \in [0, \tau].$$

We conclude that $u(0) = u_0$, $u'(t) = -e^{-t}u_0 + v(t)$ and $u'(0) = u_1$. Furthermore

$$u \in W^{2,2}(0, \tau;\mathcal{H}) \cap W^{1,2}(0, \tau;\mathcal{V}) \quad \text{and} \quad u + u' \in L^2(0, \tau;D(C)).$$

Thus $D \subseteq TR_{\mathfrak{c}}(2, \mathcal{H})$. Then the claim follows immediately. □

4.1 Dier's counter-example

The example below is inspired by [5] who considered the first order Cauchy problem.

According to [19], there exist Hilbert spaces $\mathcal{V}, \mathcal{H}$ with $\mathcal{V} \hookrightarrow_d \mathcal{H}$ and a $\mathcal{V}$-bounded coercive form $\mathfrak{b} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that $D(B^{\frac{1}{2}}) \neq D(B^{*\frac{1}{2}})$, where $\mathcal{B}$ is the associated operator with $\mathfrak{b}$ on $\mathcal{V}'$ and B is the restriction of $\mathcal{B}$ to $\mathcal{H}$.

We define the symmetric form $\mathfrak{c} : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ by

$$\mathfrak{c}(v, w) = \frac{1}{2}\left[\mathfrak{b}(v, w) + \overline{\mathfrak{b}(w, v)}\right].$$

Let $\mathcal{C}$ be the associated operator with the form $\mathfrak{c}$. We set $C = \mathcal{C}\,|_{\mathcal{H}}$ is the part of $\mathcal{C}$ in $\mathcal{H}$.

It follows that

$$D(C^{\frac{1}{2}}) = D(C^{*\frac{1}{2}}) = \mathcal{V}.$$

Let $\phi \in MR_B(2, \mathcal{H})$ such that $(\phi(0), \phi'(0)) = (0, u_1)$ and $u_1 \in D(B^{\frac{1}{2}}) \setminus \mathcal{V}$ $(D(B^{*\frac{1}{2}}) \setminus \mathcal{V})$. Note such u_1 exists since from Lemma 3.4 or [15, Theorem 1] either $D(B^{\frac{1}{2}}) \setminus \mathcal{V}$ or $D(B^{*\frac{1}{2}}) \setminus \mathcal{V}$ is not empty.

We set $v(t) = -t^2\phi(1 - t)$ with $t \in [0, 1]$, one has $v'(t) = -2t\phi(1 - t) + t^2\phi'(1 - t)$ with $v'(0) = 0$, $v(0) = 0$ and $v(1) = 0$, $v'(1) = u_1$.

Remark 4.2 Note that by Proposition 4.1 we have $(0, u_1) \in TR_{\mathfrak{b}}(2, \mathcal{H})$ but $(0, u_1) \notin TR_{\mathfrak{c}}(2, \mathcal{H})$.

We define the non-autonomous forms

$$\mathfrak{a}(t;\cdot, \cdot) = \mathbb{1}_{[0,1]}(t)\mathfrak{b}(\cdot, \cdot) + \mathbb{1}_{[1,2]}(t)\mathfrak{c}(\cdot, \cdot),$$

where $\mathbb{1}$ is the indicator function and we denote by $\mathcal{A}(t)$ the associated operator to $\mathfrak{a}(t)$ in $\mathcal{V}'$.

Set

$$f(t) = \begin{cases} v''(t) + \mathcal{A}(t)v'(t) + \mathcal{A}(t)v(t), & t \in (0, 1) \\ 0 \quad \text{otherwise} . \end{cases} \tag{43}$$

Let u be the solution to the problem

$$\begin{cases} u''(t) + \mathcal{A}(t)u'(t) + \mathcal{A}(t)u(t) = f(t) & t\text{-a.e.} \\ u(0) = 0, \quad u'(0) = 0, \end{cases} \tag{44}$$

with $t \in [0, 2]$. Then by Theorem 2.6 we get $u \in W^{2,2}(0, 2; \mathcal{V}') \cap W^{1,2}(0, 2; \mathcal{V})$.

Moreover, $u\,|_{[0,1]} = v$, $u'(1) = u_1$ and $u(1) = 0$. We put now $w(t) = u|_{[1,2]}(t-1)$, one has $w(0) = 0$ and $w'(0) = u_1$.

Since $(0, u_1) \notin TR_c(2, \mathcal{H})$, we obtain that $w \notin MR_C(2, \mathcal{H})$ and w is the solution to the following problem on $L^2(0, 1; \mathcal{V}')$

$$\begin{cases} w''(t) + C(w'(t) + w(t)) = 0 & t\text{-a.e.} \\ w(0) = 0, \quad w'(0) = u_1. \end{cases} \tag{45}$$

Thus, $w \notin W^{2,2}(0, 1; \mathcal{H})$ and $u \notin W^{2,2}(0, 2; \mathcal{H})$.

4.2 Fackler's counter-example

Proposition 4.3 *For all $\tau \in (0, \infty)$ there exist a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ and a $\mathcal{V}$-bounded coercive, symmetric, non-autonomous forms*

$$\mathfrak{a}, \mathfrak{b} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C},$$

with $t \to \mathfrak{a}(t, u, v), \mathfrak{b}(t, u, v) \in C^{\frac{1}{2}}([0, \tau])$, for all $u, v \in \mathcal{V}$ such that the second order Cauchy problem (21) does not have maximal L^2-regularity in $\mathcal{H}$.

Proof According to [13, Theorem 5.1], there exist a Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}'$ and a bounded coercive, symmetric, non-autonomous form

$$\mathfrak{a} : [0, \tau] \times \mathcal{V} \times \mathcal{V} \to \mathbb{C}$$

with $t \to \mathfrak{a}(t, u, v) \in C^{\frac{1}{2}}([0, \tau])$ for all $u, v \in \mathcal{V}$, such that the first order Cauchy problem

$$\begin{cases} u'(t) + \mathcal{A}(t)u(t) = f(t) & t\text{-a.e.} \\ u(0) = 0 \end{cases} \tag{46}$$

does not have maximal L^2-regularity in $\mathcal{H}$, or equivalently, there exists $f \in L^2(0, \tau; \mathcal{H})$ such that $u \in W^{1,2}(0, \tau; \mathcal{V}') \cap L^2(0, \tau; \mathcal{V})$ but $u \notin W^{1,2}(0, \tau; \mathcal{H})$. Now, we take $\mathfrak{b}(t) = \mathfrak{a}(t) + I$ and we set $v(t) = \int_0^t e^{-(t-s)} u(s)\, ds$. Consequently, $v(t) + v'(t) = u(t)$ and so $v'(t) + v''(t) = u'(t)$.

We get by Theorem 2.6 that $v \in W^{2,2}(0, \tau; \mathcal{V}') \cap W^{1,2}(0, \tau; \mathcal{V})$ is the unique solution to the problem

$$\begin{cases} v''(t) + (\mathcal{A}(t) + I)v'(t) + \mathcal{A}(t)v(t) = f(t) & t\text{-a.e.} \\ v(0) = 0, \quad v'(0) = 0. \end{cases} \tag{47}$$

Note that $u \in W^{1,2}(0, \tau; \mathcal{H})$ if and only if $v \in W^{2,2}(0, \tau; \mathcal{H})$.

Let the form $\mathfrak{a}(\cdot)$ and $\mathcal{V}, \mathcal{H}$ be as in [13, Theorem 5.1]. Then there exists $f \in L^2(0, \tau; \mathcal{H})$ such that $u \notin W^{1,2}(0, \tau; \mathcal{H})$, where u is the solution to Problem (46).

It follows that $v'' \notin W^{2,2}(0,\tau;\mathcal{H})$ and so Problem (47) does not have maximal L^2 -regularity in $\mathcal{H}$. □

5 Applications

This section is devoted to some applications of the results given in the previous sections. We give examples illustrating the theory without seeking for generality.

5.1 Laplacian with time dependent Robin boundary conditions

Let Ω be a bounded domain of $\mathbb{R}^d$, with Lipschitz boundary Γ. Denote by σ the $(d-1)$-dimensional Hausdorff measure on Γ. Let

$$\beta_1,\ \beta_2\ :\ [0,\tau]\times\Gamma\to\mathbb{R}$$

be bounded measurable functions which are $(\frac{1}{2}+\varepsilon)$-Hölder continuous w.r.t. the first variable with $\varepsilon>0$, i.e.,

$$|\beta_i(t,\sigma)-\beta_i(s,\sigma)|\le K|t-s|^{\frac{1}{2}+\varepsilon}$$

for $i=1,2$, with $t,s\in[0,\tau]$, $\sigma\in\Gamma$ and $K>0$.

We consider the forms $\mathfrak{a},\mathfrak{b}$

$$\mathfrak{a},\mathfrak{b}\ :\ [0,\tau]\times H^1(\Omega)\times H^1(\Omega)\to\mathbb{R}$$

defined by

$$\mathfrak{a}(t,u,v)=\int_\Omega \nabla u\nabla v\,dx+\int_\Gamma \beta_1(t,.)uv\,d\sigma$$

and

$$\mathfrak{b}(t,u,v)=\int_\Omega \nabla u\nabla v\,dx+\int_\Gamma \beta_2(t,.)uv\,d\sigma.$$

The forms $\mathfrak{a},\mathfrak{b}$ are $H^1(\Omega)$-bounded, quasi-coercive and symmetric. The first statement follows readily from the continuity of the trace operator and the boundedness of β_i, $i=1,2$. The second one is a consequence of the inequality

$$\int_\Gamma |u|^2 d\sigma\le\varepsilon\|u\|^2_{H^1(\Omega)}+C_\varepsilon\|u\|^2_{L^2(\Omega)}$$

which is valid for all $\varepsilon>0$ (C_ε is a constant depending on ε). Note that this is a consequence of compactness of the trace as an operator from $H^1(\Omega)$ into $L^2(\Gamma,d\sigma)$.

Let $\mathcal{A}(t)$ be the operator associated with $\mathfrak{a}(t,\cdot,\cdot)$ and $\mathcal{B}(t)$ the operator associated with $\mathfrak{b}(t,\cdot,\cdot)$. Note that the part $A(t)$ in $H:=L^2(\Omega)$ of $\mathcal{A}(t)$ is interpreted as (minus) the Laplacian with time dependent Robin boundary conditions:

$$\partial_\nu v + \beta_1(t,.)v = 0 \quad \text{on } \Gamma.$$

Here we use the following weak definition of the normal derivative. Let $v \in H^1(\Omega)$ such that $\Delta v \in L^2(\Omega)$. Let $h \in L^2(\Gamma, d\sigma)$. Then $\partial_\nu v = h$ by definition if $\int_\Omega \nabla v \nabla w \, dx + \int_\Omega \Delta v \nabla w \, dx = \int_\Gamma hw \, d\sigma$ for all $w \in H^1(\Omega)$. Based on this definition, the domain of $A(t)$ is the set

$$D(A(t)) := \{u \in H^1(\Omega) : \Delta u \in L^2(\Omega),\ \partial_\nu u + \beta_1(t,.)u = 0\}$$

and for $u \in D(A(t))$ the operator is given by $A(t)u := -\Delta u$. The same definition for the operator $B(t)$.

In the next proposition we suppose that $w_1 \in (L^2(\Omega), D(B(0)))_{1-\frac{1}{p},p}, w_0 \in H^1(\Omega)$ for $p \le 2$ and $w_1 \in (L^2(\Omega), D(B(0)))_{1-\frac{1}{p},p},\ w_0 \in (H^1(\Omega), D(A(0)))_{1-\frac{1}{p},p}$ for $p > 2$.

We note that for $p < 2$

$$(L^2(\Omega), D(B(0)))_{1-\frac{1}{p},p} = (L^2(\Omega), H^1(\Omega))_{2(1-\frac{1}{p}),p} = B_p^{2(1-\frac{1}{p}),2}(\Omega),$$

where $B_p^{2(1-\frac{1}{p}),2}(\Omega)$ is the classical Besov space.

Proposition 5.1 *For all $f \in L^p(0,\tau;H)$, there exists a unique solution of the problem*

$$\begin{cases} w''(t) - \Delta w'(t) - \Delta w(t) = f(t) \quad t\text{-a.e.} \\ w(0) = w_0, \quad w'(0) = w_1, \\ \partial_\nu(w'(t) + w(t)) + \beta_2(t,.)w'(t) + \beta_1(t,.)w(t) = 0 \quad \text{on } \Gamma, \end{cases} \tag{48}$$

where $w \in W^{2,p}(0,\tau;L^2(\Omega)) \cap W^{1,p}(0,\tau;H^1(\Omega))$ and for $p \ge 2$, we have $w \in C^1([0,\tau];H^1(\Omega))$.

The proposition follows from Theorems 3.7 and 3.10.

Maximal L^p-regularity for the Laplacian with time dependent Robin boundary condition with $\beta_1 = \beta_2$ and $w_0 = w_1 = 0$ was previously proved in [9] and maximal L^2-regularity with $t \to \beta_1(t,\cdot), \beta_2(t,\cdot) \in C^1$ was proved in [11].

5.2 Elliptic operators on $\mathbb{R}^d$

Let $\mathcal{H} = L^2(\mathbb{R}^d)$ and $\mathcal{V} = H^1(\mathbb{R}^d)$. Suppose that $a_{jk}^l \in L^\infty(I \times \mathbb{R}^d)$, where $I = [0,\tau]$ and $j,k \in (1,\dots,d)$, $l \in (1,2)$ and there exists a constant $\alpha > 0$, such that

$$\sum_{j,k=1}^{d} a_{jk}^l(t,x)\xi_i \overline{\xi_j} \ge \alpha\, |\xi|^2 \left(t \in I,\ x \in \mathbb{R}^d,\ \xi \in \mathbb{C}^d\right).$$

We define the forms

$$\mathfrak{a}^l(t,u,v) = \sum_{j,k=1}^{d} \int_{\mathbb{R}^d} a^l_{jk}(t,x)\partial_j u\, \overline{\partial_k v}\, dx$$

with domain $\mathcal{V} = H^1(\mathbb{R}^d)$. For each t, the corresponding operator is formally given by

$$A(t)^l = -\sum_{j,k=1}^{d} \partial_j(a^l_{jk}(t,x)\partial_k) = -\mathrm{div}\Big((a^l_{jk}(t,\cdot))_{jk}\nabla\Big).$$

Proposition 5.2 *Let $u_0, u_1 \in H^1(\mathbb{R}^d)$ and $f \in L^2(0,\tau;L^2(\mathbb{R}^d))$ and if $a^l_{jk} \in C^{\frac{1}{2}+\varepsilon}(I;L^\infty(\mathbb{R}^d))$ for some $\varepsilon > 0$, there exists a unique $u \in H^2(I;L^2(\mathbb{R}^d)) \cap C^1(I;H^1(\mathbb{R}^d))$ such that*

$$\begin{cases} u''(t) - \mathrm{div}\Big((a^1_{jk}(t,\cdot))_{jk}\nabla u'(t)\Big) - \mathrm{div}\Big((a^2_{jk}(t,\cdot))_{jk}\nabla u(t)\Big) = f \quad t\text{-a.e.} \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases} \tag{49}$$

It is clear that

$$|\mathfrak{a}^l(t,u,v) - \mathfrak{a}^l(s,u,v)| \le C^l\, |t-s|^{\frac{1}{2}+\varepsilon} \|u\|_{\mathcal{V}} \|v\|_{\mathcal{V}},$$

where

$$C^l = \sup_{j,k} \|a^l_{jk}(\cdot,\cdot)\|_{C^{\frac{1}{2}+\varepsilon}(I;L^\infty(\mathbb{R}^d))}.$$

As we already mentioned before, the uniform Kato square root property required in Theorem 3.10 is satisfied in this setting, see [8, Theorem 6.1]. Then Proposition 5.2 follows from Theorem 3.10.

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