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Nonlocal evolution equations with $p[u(x, t)]$-Laplacian and lower-order terms

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Abstract

We study the homogeneous Dirichlet problem for a class of nonlocal singular parabolic equations

$$u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2}\nabla u\right) = f((x,t),u,l(u)) \quad \text{in } Q_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a smooth domain, $p[u] = p(l(u))$ is a given function with values in the interval $[p^-, p^+] \subset (\frac{2d}{d+2}, 2)$, and $l(u) = \int_\Omega |u(x,t)|^\alpha \, dx$, $\alpha \in [1,2]$, is a functional of the unknown solution. We prove the existence of a strong solution such that

$$u_t \in L^2(Q_T), \quad u \in L^\infty(0,T; W_0^{1,2}(\Omega)), \quad |D_{ij}^2 u|^{p[u]} \in L^1(Q_T).$$

Conditions of uniqueness of strong solutions are obtained.

Keywords Nonlocal equation · Singular parabolic equation · Variable nonlinearity · Strong solutions

Mathematics Subject Classification 35K67 · 35M99 · 35K55 · 35D35

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1 Introduction

We study the homogeneous Dirichlet problem

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p[u]-2}\nabla u\right) = f(z,u,l(u)) \quad \text{in } Q_T, \\ u = 0 \text{ on } \partial\Omega \times (0,T), \\ u(x,0) = u_0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $z = (x,t) \in Q_T = \Omega \times (0,T]$, $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a domain with the boundary $\partial\Omega$. The exponent of nonlinearity p is a functional of the unknown solution:

$$\begin{aligned} & p[u] = p(l(u)) : \mathbb{R} \mapsto [p^-, p^+] \subset (1,2), \\ & l(u) = \int_\Omega |u(x,t)|^\alpha \, dx : L^\infty(0,T;L^\alpha(\Omega)) \mapsto \mathbb{R}, \end{aligned} \tag{2}$$

with some constants $p^\pm$, $1 < p^- \leq p^+ < 2$, $\alpha \in [1,2]$. It is assumed that $f(z, s, r)$ is a Carathéodory function (measurable in $z \in Q_T$ for every $s, r \in \mathbb{R}$ and continuous in s, r for a.e. $z \in Q_T$) subject to the following growth conditions:

$$\begin{aligned} & \text{there exists a function } f_0(z) \in L^2(Q_T),\ f_0 \geq 0 \text{ a.e. in } Q_T, \text{ such that} \\ & |f(z,s,r)| \leq f_0(z) + N|s|^{\sigma(r)-1} \quad \text{for all } s \in \mathbb{R},\ r \in \mathbb{R}_+, \\ & \text{where } N = const \geq 0 \text{ and } \sigma(s) : \mathbb{R}_+ \mapsto [\sigma^-, \sigma^+] \subset (1,\infty),\ \sigma^\pm = const. \end{aligned} \tag{3}$$

We prove that problem (1) has a global in time strong solution. The results remain true for a wider class of functionals $l(u)$. For example, we may take

$$l(u) = \int_\Omega g(x)u(x,t)\,dx, \quad \text{with some} \quad g \in L^{\alpha'}(\Omega),\ \alpha \in [1,2].$$

The nonlocal evolution equations are widely used in modelling of various processes in physics and biology and are intensively studied, see, e.g., [1–4] and references therein. As for Eq. (1), we may regard it as the diffusion equation where the diffusion rate p and the nonlinear source f depend on the total mass of the substance given by $l(u)$ with $\alpha = 1$.

Nonlinear equations and system of equations whose structure may depend on the sought solution appear in the mathematical modelling of various real-life processes. In [5], a system of nonlinear equations, describing the stationary thermo-convective flow of a non-Newtonian fluid, was considered. The models of a thermistor was studied in [6, 7]. The models of electro-rheological fluids in which the character of nonlinearity in the governing Navier–Stokes equations varies according to the applied electromagnetic field were considered in [8]. The functionals with the growth condition depending on the solution or its gradient are successfully used for denoising of digital images—see, e.g., [9–11] for the models based on minimization of functionals with $p(|\nabla u|)$-growth and [12] for a discussion of the model of image denoising based on the minimization of a functional with the nonlinearity depending on u.

By now, the equations that involve the $p[u]$-Laplace operators where studied only in papers [13, 14]. These work address elliptic equations similar to $p(u)$-Laplacian, with local or nonlocal dependence on u, but their approach to the problem is different. Since the $p[u]$-Laplace equation can not be interpreted as a duality relation in a fixed Banach space, the authors of [13] reduce the study to the L^1 setting and obtain a solution using the Young measures. The authors of [14, 15] proceed in another way and develop the idea of [16] on the passing to the limit in a sequence of the form $\left\{|\nabla v_k(x)|^{q_k(x)}\right\}$. Both works offer a discussion of the uniqueness issue.

2 Assumption and results

2.1 The function spaces

For convenience, we collect here the needed information on the Lebesgue and Sobolev spaces with variable exponents. For a detailed presentation of the theory of these spaces we refer to the monograph [17], see also [18, Ch.1].

Let $\Omega \subset \mathbb{R}^d$ be a domain with the Lipschitz-continuous boundary $\partial\Omega$. Given a measurable function $p(x) : \Omega \mapsto [p^-, p^+] \subset (1,\infty)$, $p^\pm = const$, the set

$$L^{p(\cdot)}(\Omega) = \left\{ f : \Omega \mapsto \mathbb{R} : f \text{ is measurable on } \Omega,\ \int_\Omega |f|^{p(x)}\,dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|f\|_{p(\cdot),\Omega} := \inf\left\{ \alpha > 0 : \int_\Omega \left|\frac{f}{\alpha}\right|^{p(x)} dx \le 1 \right\}$$

becomes a Banach space. The relation between the modular $\int_\Omega |f|^{p(x)}\,dx$ and the norm follows from the definition:

$$\min\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right) \le \int_\Omega |f|^{p(x)}\,dx \le \max\left(\|f\|_{p(\cdot)}^{p^-}, \|f\|_{p(\cdot)}^{p^+}\right). \tag{4}$$

In case of $p(\cdot) = const > 1$ these inequalities transform into equalities. For all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1,\infty), \quad p'(x) = \frac{p(x)}{p(x)-1}$$

the generalized Hölder inequality holds:

$$\int_\Omega |f\,g|\,dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} \le 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)}. \tag{5}$$

If $p(x)$ is measurable and $1 < p^- \le p(x) \le p^+ < \infty$ in Ω, then $L^{p(\cdot)}(\Omega)$ is a reflexive and separable Banach space, and $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Let $p_1(x), p_2(x)$ be measurable on Ω functions such that $p_i(x) \in [p_i^-, p_i^+] \subset (1,\infty)$ a.e. in Ω. If $p_1(x) \geq p_2(x)$ a.e. in Ω, then the inclusion $L^{p_1(\cdot)}(\Omega) \subset L^{p_2(\cdot)}(\Omega)$ is continuous and

$$\|u\|_{p_2(\cdot),\Omega} \leq C\|u\|_{p_1(\cdot),\Omega} \qquad \forall\, u \in L^{p_1(\cdot)}(\Omega) \tag{6}$$

with a constant $C = C(|\Omega|, p_1^\pm, p_2^\pm)$.

The variable Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the collection of functions

$$W_0^{1,p(\cdot)}(\Omega) = \left\{u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega) \,:\, |\nabla u|^{p(x)} \in L^1(\Omega)\right\}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega} + \|u\|_{p(\cdot),\Omega}. \tag{7}$$

By $C_{\log}(\overline{\Omega})$ we denote the set of functions continuous on $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$|p(x_2) - p(x_1)| \leq \omega(|x_2 - x_1|) \tag{8}$$

where $\omega \geq 0$ satisfies the condition

$$\overline{\lim_{\tau\to 0^+}}\, \omega(\tau) \ln \frac{1}{\tau} = C < \infty, \qquad C = const.$$

It is known that for $p(x) \in C_{\log}(\overline{\Omega})$ the set $C_0^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the space $W_0^{1,p(\cdot)}(\Omega)$ coincides with the closure of $C_0^\infty(\Omega)$ with respect to the norm (7).

We will use the notation $p(z) \in C_{\log}(\overline{Q}_T)$ for the functions p of the arguments $z = (x,t)$ satisfying condition (8) in the cylinder $\underline{Q_T} = \Omega \times (0,T)$.

For the elements of $W_0^{1,p(\cdot)}(\Omega)$ with $p(x) \in C^0(\overline{\Omega})$ the Poincaré inequality holds:

$$\|u\|_{p(\cdot),\Omega} \leq C(d,\Omega)\|\nabla u\|_{p(\cdot),\Omega}. \tag{9}$$

An immediate consequence of the Poincaré inequality is that an equivalent norm of $W_0^{1,p()}(\Omega)$ can be defined by

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot),\Omega}.$$

Let $p(x), q(x) \in C^0(\overline{\Omega})$, $1 < p^- \leq p(x) \leq p^+ < \infty$, $d \geq 2$. If $q(x) < \frac{dp(x)}{d-p(x)}$ in Ω, then the embedding $W_0^{1,p(\cdot)}(\Omega) \subset L^{q(\cdot)}(\Omega)$ is continuous, compact, and

$$\|v\|_{q(\cdot),\Omega} \leq C\|\nabla v\|_{p(\cdot),\Omega} \qquad \forall v \in W_0^{1,p(\cdot)}(\Omega).$$

According to (6) $W_0^{1,p(\cdot)}(\Omega) \subset W_0^{1,p^-}(\Omega)$. If $p^- > \frac{2d}{d+2}$, then the embedding $W_0^{1,p^-}(\Omega) \subset L^2(\Omega)$ is compact.

Let us introduce the spaces of functions defined on the cylinder Q_T

$$\begin{aligned}
&\mathbf{V}_t(\Omega) = \{u \,:\, \Omega \mapsto \mathbb{R} | u \in L^2(\Omega) \cap W_0^{1,1}(\Omega),\ |\nabla u|^{p(x,t)} \in L^1(\Omega)\}, \quad t \in (0,T),\\
&\mathbf{W}(Q_T) = \{u \,:\, (0,T) \mapsto \mathbf{V}_t(\Omega) |\ u \in L^2(Q_T),\ |\nabla u|^{p(x,t)} \in L^1(Q_T)\}.
\end{aligned}$$

Given a measurable in Q_T function u and a functional p, we define the set

$$\mathbf{W}_u(Q_T) = \left\{ v \left| \begin{array}{l} v \in L^2(Q_T),\ |\nabla v|^{p[u]} \in L^1(Q_T), \\ v = 0 \text{ on } \partial\Omega \times (0,T) \text{ in the sense of traces} \end{array} \right. \right\}.$$

If we denote $\widetilde{p}(x,t) = p[u(x,t)]$, then $\mathbf{W}_u(Q_T)$ coincides with the space $\mathbf{W}(Q_T)$ with the given variable exponent $\widetilde{p}(x,t)$. The inclusion $u \in \mathbf{W}_u(Q_T)$ means that $u \in L^2(Q_T), |\nabla u|^{\widetilde{p}(x,t)} \in L^1(Q_T)$ and $u = 0$ on $\partial\Omega \times (0,T)$.

Notation. Throughout the text we use the notation

$$|v_{xx}|^q = \sum_{i,j=1}^{d} |D^2_{x_i x_j} v|^q$$

where the exponent q may depend on t. By C we denote the constants which can be computed or estimated through the data of the problem, but whose precise values are unimportant. The value of C may differ from line to line even in the same formula.

2.2 The main result and organization of the paper

Definition 1 A function u is called **strong solution** of problem (1) if

1. $u \in C^0([0,T];L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,2}(\Omega)),\ u_t \in L^2(Q_T)$;
2. $\|u(\cdot,t) - u_0\|_{2,\Omega} \to 0$ as $t \to 0+$;
3. for every test-function $\phi \in L^2(Q_T) \cap L^2(0,T;W_0^{1,2}(\Omega))$

$$\int_{Q_T} \left(u_t\phi + |\nabla u|^{p[u]-2}\nabla u \cdot \nabla\phi\right) dz = \int_{Q_T} f(z,u,l(u))\phi\, dz. \tag{10}$$

The main result of this work is given in the following theorem.

Theorem 1 *Assume that*

(a) Ω is a bounded domain with the boundary $\partial\Omega \in C^2$,

(b) $u_0 \in W_0^{1,2}(\Omega)$,

(c) $\sup_{\mathbb{R}} |p'(l)| \le C_*, \qquad C_* = const.$,

$$\text{(d)} \quad \begin{cases} |f(z,s,r)| \le f_0(z) + f_1(z)|s|^{\sigma(r)-1} \\ \text{with } f_0 \in L^{\frac{p^-}{p^- -1}}(Q_T),\ f_1 \in L^\infty(Q_T),\ f_0, f_1 \ge 0 \text{ a.e. in } Q_T, \\ \|f_1\|_{\infty,Q_T} \le N = const \ge 0, \\ \sigma(r) \in C^0[0,\infty), \quad \sigma : \mathbb{R}_+ \mapsto [\sigma^-,\sigma^+],\ \sigma^\pm = const > 1 \end{cases} \tag{11}$$

$$\text{(e)} \quad \begin{cases} \dfrac{2d}{d+2} < p^- \le p^+ < 2, \\ 1 < \sigma^- \le \sigma^+ \le \min\left\{2, 1 + \dfrac{2(d+1)}{d}\left(1 - \dfrac{1}{p^-}\right)\right\} \end{cases}$$

Then problem (1) *has a strong solution in the sense of Definition* 1 *and the following estimate holds:*

$$\sup_{(0,T)} \|\nabla u(\cdot,t)\|^2_{2,\Omega} + \|u_t\|^2_{2,Q_T} + \int_{Q_T} |u_{xx}|^{p[u]}\,dz \le C\Big(1 + \|\nabla u_0\|^2_{2,\Omega} + \|f_0\|_{\frac{p^-}{p^--1},Q_T}\Big). \tag{12}$$

Remark 1 Condition (11) on σ^+ can be omitted if the sign of the nonlinear source $f(z, u, l(u))$ coincides with the sign of $u(z)$. The assertion of Theorem remains true if, for example,

$$f(z,s,r) = f_0(z) - f_1(z)|s|^{\sigma(r)-2}s, \quad f_0 \in L^{\frac{p^-}{p^--1}}(Q_T), f_1 \ge 0 \text{ a.e. in } Q_T.$$

Organization of the paper. In Sect. 3 we consider the regularized non-singular problem (13). The solution of problem (13) is obtained as the limit of the sequence of Galerkin's approximations in the basis composed of the eigenfunctions of the Laplace operator. The bulk of Sect. 3 is devoted to deriving a priori estimates on the second-order space derivatives of the solutions of regularized problems, where we follow the technique developed in [19] for the given exponent $p(x,t)$. In Sect. 4 we justify first the passage to the limit in the sequence of Galerkin's approximations and obtain a solution of the regularized problem. We make use of monotonicity of the function $\gamma_\epsilon(q,\xi)\xi = (\epsilon^2 + |\xi|^2)^{\frac{q-2}{2}}\xi$ in ξ with a fixed q, continuity of $\gamma_\epsilon(q,\xi)\xi$ with respect to q with a fixed ξ, and the fact that in the singular case, $p^+ < 2$, the solutions u_ϵ of the regularized problems and their approximations possess extra regularity: $\|\nabla u_\epsilon(t)\|_{2,\Omega}$ are uniformly bounded for all $t \in (0,T)$.

To pass to the limit as $\epsilon \to 0$ in the sequence $\{u_\epsilon\}$ of solutions of the regularized problems we use the a priori estimates of Sect. 3. The procedure of passing to the limit in ϵ requires an additional step because now the exponent $p_\epsilon = p[u_\epsilon]$ also depends on ϵ.

The uniqueness is proven in Theorem 2 in Sect. 5. The study of uniqueness is practically independent of the issue of existence and requires some additional assumptions on the structure of the equation.

3 Regularized problem

We will obtain a solution of the singular problem (1) as the limit when $\epsilon \to 0$ of the family of solutions of the regularized problems

$$\begin{cases} u_{\epsilon t} = \operatorname{div}\Big((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p[u_\epsilon]-2}{2}}\nabla u_\epsilon\Big) + f(z, u_\epsilon, l(u_\epsilon)) & \text{in } Q_T, \\ u_\epsilon = 0 \text{ on } \partial\Omega \times (0,T), & \\ u_\epsilon(x,0) = u_0(x) \text{ in } \Omega, & \epsilon > 0. \end{cases} \tag{13}$$

3.1 Galerkin's approximations

The solution of problem (13) is understood in the sense of Definition 1. It is constructed as the limit of the sequence of finite-dimensional approximations

$$u_\epsilon \equiv u(x,t) = \lim_{m\to\infty} u^{(m)}, \quad u^{(m)} = \sum_{i=1}^{\infty} u_{i,m}(t)\psi_i(x),$$

where $\{\psi_i\}$ is the orthonormal basis of $L^2(\Omega)$ composed of the eigenfunctions of the Dirichlet problem for the Laplace operator

$$(\nabla\psi_i, \nabla\phi)_{2,\Omega} = \lambda_i(\psi_i, \phi)_{2,\Omega} \quad \forall\phi \in W_0^{1,2}(\Omega), \quad i = 1, 2, \dots . \tag{14}$$

The system $\left\{\frac{1}{\sqrt{\lambda_i}}\psi_i\right\}$ forms an orthogonal basis of $W_0^{1,2}(\Omega)$. Let us accept the notation

$$\begin{aligned} &\gamma_\epsilon(q,\mathbf{s}) = \left(\epsilon^2 + |\mathbf{s}|^2\right)^{\frac{q-2}{2}}, \quad \mathbf{s}\in\mathbb{R}^d,\ t\in(0,T),\ \epsilon>0,\ q\in(1,2],\\ &p_m(t) = p[u^{(m)}], \quad u^{(m)} = \sum_{i=1}^{m} u_{i,m}(t)\psi_i(x). \end{aligned} \tag{15}$$

The coefficients $u_{i,m}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary nonlinear differential equations

$$\begin{aligned} u'_{i,m}(t) &= -\int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})\nabla u^{(m)}\cdot\nabla\psi_i\,dx\\ &\quad + \int_\Omega f(z, u^{(m)}, l(u^{(m)}))\psi_i\,dx,\\ u_{i,m}(0) &= u_{0i}^{(m)}, \quad i = 1, 2, \dots, m, \end{aligned} \tag{16}$$

where the constants $v_i^{(m)}$ are the Fourier coefficients of u_0 in the basis $\{\psi_i\}$:

$$u_0^{(m)} = \sum_{i=1}^{m} u_{0i}^{(m)}\psi_i(x) \to u_0(x) \quad \text{in } L^2(\Omega).$$

By the Carathéodory theorem for every finite m system (16) has a continuous solution on an interval $(0, T_m)$. In the next subsection we derive the uniform estimates on $u^{(m)}$ and its derivatives, which show that the solutions of system (16) can be continued to the interval $(0, T)$.

3.2 Uniform a priori estimates

Throughout this section we denote by $u^{(m)}$ the finite-dimensional Galerkin approximation of the solution u_ϵ of problem (13) with $\epsilon > 0$. We assume that the data of problems (13) satisfy conditions (11) of Theorem 1.

Lemma 1 *The solutions of problem* (13) *satisfy the following estimates:*

$$\begin{aligned} &\sup_{(0,T)} \|u^{(m)}(t)\|^2_{2,\Omega} + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dz \\ &\quad \leq C\Big(1 + \|u_0^{(m)}\|^2_{2,\Omega} + \|f_0\|^2_{2,Q_T}\Big) = M_0, \end{aligned} \tag{17}$$

$$\begin{aligned} &\int_{Q_T} |\nabla u^{(m)}|^{p_m(t)}\, dz \\ &\quad \leq C\left(1 + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dz\right) = M_0' \end{aligned} \tag{18}$$

with absolute constants M_0, M_0'.

Proof Multiplying the ith equation of (16) by $u_i^{(m)}$ and summing the results leads to the energy relation

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(\|u^{(m)}(t)\|^2_{2,\Omega}\Big) + \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dx \\ &\quad = \int_\Omega u^{(m)} f\, dx =: I, \end{aligned} \tag{19}$$

where

$$\begin{aligned} |I| &\leq \int_\Omega \Big(f_0|u^{(m)}| + f_1|u^{(m)}|^{\sigma_m(t)}\Big)dx \\ &\leq \frac{1}{2}\|f_0\|^2_{2,\Omega} + \frac{1}{2}\|u^{(m)}\|^2_{2,\Omega} + N\int_\Omega |u^{(m)}|^{\sigma_m}\, dx \\ &\leq C + \|f_0\|^2_{2,\Omega} + \|u^{(m)}\|^2_{2,\Omega}, \quad \sigma_m(t) = \sigma(l(u^{(m)})). \end{aligned} \tag{20}$$

Let us prove (18) first. By Young's inequality, for every $\delta > 0$

$$\begin{aligned} &\int_{Q_T} |\nabla u^{(m)}|^{p_m}\, dz \\ &= \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})^{-\frac{p_m}{2}} \left(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\right)^{\frac{p_m}{2}}\, dz \\ &\leq \delta \int_{Q_T} (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m}{2}}\, dz + C_\delta \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dz \\ &\leq \delta \int_{Q_T} |\nabla u^{(m)}|^{p_m}\, dz + C_\delta\left(1 + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dz\right). \end{aligned}$$

Estimate (18) follows if we take $\delta = 1/2$. Substituting (20) into (19) and dropping the second nonnegative term on the left-hand side we obtain the differential inequality for $y(t) = \|u(t)\|_{2,\Omega}$:

$$\frac{1}{2}y'(t) \le C\Big(1 + y(t) + \|f_0\|_{2,\Omega}^2\Big).$$

Multiplying by e^{-2Ct} and integrating we arrive at the estimate

$$\|u(t)\|_{2,\Omega}^2 \le \|u_0\|_{2,\Omega}^2 \mathrm{e}^{2Ct} + 2C\mathrm{e}^{2Ct} \int_0^t \int_\Omega \left(1 + f_0^2\right) dz.$$

Substituting it into (20), returning to (19) and integrating, we obtain (17). □

Corollary 1

$$\int_{Q_T} (\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|)^{(p_m(t))'}\, dz \le C$$

uniformly with respect to m and ϵ.

Proof The estimate follows from (18) because for $1 < p^- \le p^+ \le 2$

$$\begin{aligned}\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}| &\le (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)-1}{2}} \\ &\le C(p^\pm)\big(1 + |\nabla u^{(m)}|^{p_m(t)-1}\big).\end{aligned}$$

□

Lemma 2 *The functions $u^{(m)}$ satisfy the estimate*

$$\begin{aligned}&\sup_{(0,T)} \|\nabla u^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \Big((\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)-2}{2}} |u_{xx}^{(m)}|^2 + |u_{xx}^{(m)}|^{p_m(t)}\Big)\, dz \\ &\quad \le C\bigg(\|\nabla u_0\|_{2,\Omega}^2 + \int_{Q_T} f_0^{(p^-)'}\, dz + 1\bigg)\, dz = M_1\end{aligned} \tag{21}$$

with an independent of m and ϵ constant M_1.

Proof Multiplying ith equation in (1) by $\lambda_i u_i^{(m)}$, summing up for $i = 1, \dots, m$ and then following the proof of [20, Lemma 2.2] we arrive at the equality

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla u^{(m)}\|_{2,\Omega}^2\Big) + \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u_{xx}^{(m)}|^2\, dx = -I - I_{\partial\Omega} + I_f, \tag{22}$$

where

$$I = \int_\Omega (p-2)\big(\epsilon^2 + |\nabla u^{(m)}|^2\big)^{\frac{p_m(t)-2}{2}-1} \left(\sum_{k=1}^d \left(\nabla u^{(m)} \cdot \nabla(D_k u^{(m)})\right)^2\right) dx, \tag{23}$$

$$I_f = \int_\Omega f(z, u^{(m)}, l(u^{(m)}))\Delta u^{(m)}\, dx, \tag{24}$$

$$\begin{aligned} I_{\partial\Omega} = \int_{\partial\Omega} & \gamma_\epsilon(p_m(t), \nabla u^{(m)}) \\ & \left(\Delta u^{(m)}(\nabla u^{(m)} \cdot \mathbf{n}) - \nabla u^{(m)} \cdot \nabla(\nabla u^{(m)} \cdot \mathbf{n})\right) dS. \end{aligned} \tag{25}$$

It is straightforward to check that

$$\begin{aligned} |I| \le & (2 - p^-) \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u_{xx}^{(m)}|^2\, dx, \\ |I_f| \le & \int_\Omega \left(f_0 + f_1|u^{(m)}|^{\sigma_m - 1}\right)|\Delta u^{(m)}| dx \equiv I_{f_0} + I_{f_1}. \end{aligned}$$

By the Young inequality

$$\begin{aligned} I_{f_0} \le & \int_\Omega f_0|u_{xx}^{(m)}|\, dx \le \delta \int_\Omega |u_{xx}^{(m)}|^{p_m(t)}\, dx + C(\delta) \int_\Omega f_0^{p'_m(t)}\, dx \\ \le & \delta \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})^{-\frac{p_m(t)}{2}} \left(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|u_{xx}^{(m)}|^2\right)^{\frac{p_m(t)}{2}} dx \\ & + C(\delta, p^\pm)\left(1 + \int_\Omega f_0^{(p^-)'}\, dx\right) \\ \le & \delta \int_\Omega \gamma_\epsilon(p_m(t), \nabla u^{(m)})|u_{xx}^{(m)}|^2\, dx + C'(\delta) \int_\Omega (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)}{2}}\, dx \\ & + C''(\delta)\left(1 + \int_\Omega f_0^{(p^-)'}\, dx\right), \\ I_{f_1} \le & \delta \int_\Omega \gamma_\epsilon |u_{xx}^{(m)}|^2\, dx + C(\delta) \int_\Omega f_1^2 \left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{2 - p_m(t)}{2}} |u^{(m)}|^{2(\sigma_m - 1)}\, dx \\ \le & \delta \int_\Omega \gamma_\epsilon |u_{xx}^{(m)}|^2\, dx + C \int_\Omega \left(\left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}} + N^{\frac{p_m}{p_m - 1}} |u^{(m)}|^{\frac{(\sigma_m - 1)p_m}{p_m - 1}}\right) dx \\ \le & \delta \int_\Omega \gamma_\epsilon |u_{xx}^{(m)}|^2\, dx + C(\delta, N) \int_\Omega \left(1 + |\nabla u^{(m)}|^{p_m(t)} + |u^{(m)}|^{\frac{(\sigma_m - 1)p_m}{p_m - 1}}\right) dx. \end{aligned}$$

According to condition (11) (e)

$$1 < \sigma^- \le \sigma^+ \le 1 + \frac{2(d+1)}{d}\left(1 - \frac{1}{p^-}\right),$$

which yields the inequalities

$$\theta = d\left(\frac{1}{2} - \frac{p_m - 1}{(\sigma_m - 1)p_m}\right) \in [0, 1], \qquad \frac{\theta}{2}\frac{(\sigma_m - 1)p_m}{p_m - 1} \le 1.$$

Using the Gagliardo-Nirenberg interpolation inequality we estimate

$$
\begin{aligned}
\int_\Omega |u^{(m)}|^{\frac{(\sigma_m-1)p_m}{p_m-1}}\,dx &\leq C\|\nabla u^{(m)}\|_{2,\Omega}^{\theta\frac{(\sigma_m-1)p_m}{p_m-1}}\|u^{(m)}\|_{2,\Omega}^{\frac{(\sigma_m-1)p_m}{p_m-1}(1-\theta)} \\
&\leq C\left(1+\|\nabla u^{(m)}\|_{2,\Omega}^2\right)
\end{aligned}
\tag{26}
$$

with a constant $C = C(p^\pm, \sigma^\pm, M_0)$ independent of $u^{(m)}$. Gathering the above inequalities we obtain

$$
\begin{aligned}
J = \int_\Omega |u_{xx}^{(m)}|^{p_m(t)}\,dx &\leq \delta\int_\Omega \gamma_\epsilon |u_{xx}^{(m)}|^2\,dx + C(\delta)\int_\Omega \left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\,dx \\
&\leq \delta\int_\Omega \gamma_\epsilon |u_{xx}^{(m)}|^2\,dx + C'\int_\Omega \left(|\nabla u^{(m)}|^2+1\right)dx.
\end{aligned}
$$

with an arbitrary $\delta > 0$. Adding J to the both sides of (19), choosing δ appropriately small and using (18), we arrive at the inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla u^{(m)}\|_{2,\Omega}^2\right) + \int_\Omega \left(\gamma_\epsilon(p_m(t),\nabla u^{(m)})|u_{xx}^{(m)}|^2 + |u_{xx}^{(m)}|^{p_m(t)}\right)dx \\
&\quad\leq C\left(1+|I_\Omega| + \int_\Omega f_0^{(p^-)'}\,dx + \int_\Omega |\nabla u^{(m)}|^2\,dx\right)
\end{aligned}
\tag{27}
$$

with a constant C which does not depend on m and ϵ. It is known (see [21, Ch.1, Sec.1.5] for the case $d = 2$ and [20, Lemma A.1] for the general case $d \geq 3$) that if $\partial\Omega \in C^2$, then there exist constants K, K', depending on $\partial\Omega$, such that

$$
|I_{\partial\Omega}| \leq K\int_{\partial\Omega}\gamma_\epsilon(z,\nabla u^{(m)})\left(\nabla u^{(m)}\cdot\mathbf{n}\right)^2 dS \leq K'\left(\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}\,dS + 1\right).
$$

Inequality (27) can be written in the form

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla u^{(m)}\|_{2,\Omega}^2\right) + \int_\Omega \left(\gamma_\epsilon(z,\nabla u^{(m)})|u_{xx}^{(m)}|^2 + |u_{xx}^{(m)}|^{p_m}\right)dx \\
&\quad\leq C\left(\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}\,dS + \int_\Omega f_0^{(p^-)'}\,dx + \int_\Omega |\nabla u^{(m)}|^2\,dx + 1\right).
\end{aligned}
\tag{28}
$$

To estimate the integral over $\partial\Omega$ we use the inequality that follows from [22, Theorem 1.5.1.10]: there exists a constant $L = L(d,\Omega)$ such that for every $\delta \in (0,1)$

$$
\int_{\partial\Omega}|\nabla u^{(m)}|^{p_m}dS \leq L\left(\delta^{1-\frac{1}{p^+}}\int_\Omega |u_{xx}^{(m)}|^{p_m}dx + \delta^{-\frac{1}{p^-}}\int_\Omega |\nabla u^{(m)}|^{p_m}dx\right). \tag{29}
$$

Combining (28) and (29) with $2\delta^{1-\frac{1}{p^+}}CK' \leq 1$, we obtain the inequality

$$
\begin{aligned}
&\frac{d}{dt}\|\nabla u^{(m)}\|^2_{2,\Omega} + \int_\Omega \left(\gamma_\epsilon(z,\nabla u^{(m)})|u^{(m)}_{xx}|^2 + |u^{(m)}_{xx}|^{p_m}\right) dx \\
&\quad \le C\Bigg(\int_\Omega \left(|\nabla u^{(m)}|^{p_m} + |\nabla u^{(m)}|^2\right) dx + \int_\Omega f_0^{(p^-)'}\, dx + 1\Bigg) \\
&\quad \le C\Bigg(\|\nabla u^{(m)}\|^2_{2,\Omega}\, dx + \int_\Omega f_0^{(p^-)'}\, dx + 1\Bigg).
\end{aligned}
$$

To complete the proof, we multiply this inequality by e^{-Ct}, integrate in t the resulting differential inequality for $\|\nabla u(t)\|^2_{2,\Omega}e^{-Ct}$, and plug in estimates (17), (18). □

Lemma 3 *The functions $u^{(m)}$ satisfy the estimates*

$$
\|u^{(m)}_t\|^2_{2,Q_T} + \sup_{(0,T)} \int_\Omega (\epsilon^2 + |\nabla u^{(m)}|^2)^{\frac{p_m(t)}{2}}\, dx \le M_2 \tag{30}
$$

with an independent of m and ϵ constant M_2.

Proof Estimates (30) follow upon multiplication the ith equation of (16) by $u'_{i,m}(t)$ and summation of the results. Following the proof of [20, Lemma 2.4] we arrive at the relations: for every $t \in [0,T]$

$$
\begin{aligned}
&\|u^{(m)}_t(t)\|^2_{2,\Omega} + \frac{d}{dt}\Bigg(\int_\Omega \left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}} dx\Bigg) \\
&= -\int_\Omega \frac{dp_m(t)}{dt}\frac{\left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}}{p_m^2(t)}\Bigg(1 - \frac{p_m(t)}{2}\ln\left(\epsilon^2 + |\nabla u^{(m)}|^2\right)\Bigg) dx \\
&\quad + \int_\Omega f u^{(m)}_t\, dx \le C\left|\frac{dp_m(t)}{dt}\right|\Bigg(1 + \int_\Omega |\nabla u^{(m)}|^{p_m(t)}\, dx \\
&\quad + \int_\Omega \left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}} \ln^2\left(\epsilon^2 + |\nabla u^{(m)}|^2\right) dx\Bigg) \\
&\quad + \delta\|u^{(m)}_t(t)\|^2_{2,\Omega} + C(\delta)\Bigg(\|f_0\|^2_{2,\Omega} + \int_\Omega |u^{(m)}|^{2(\sigma_m-1)}\, dx\Bigg)
\end{aligned} \tag{31}
$$

with $C = C(C^*, p^-, \sigma^+, N)$. Since $p_m < 2$ by assumption, it follows that $\frac{p_m}{p_m-1} \ge 2$, which allows one to estimate the last term by virtue of (26) and (21). Using the formula

$$
(|u|^\alpha)_t = \left((u^2)^{\frac{\alpha}{2}}\right)_t = \frac{\alpha}{2}(u^2)^{\frac{\alpha}{2}-1}2uu_t = \alpha u_t|u|^{\alpha-1}\operatorname{sign} u
$$

and (17) we have:

$$\left|\frac{dp_m(t)}{dt}\right| = \alpha\left|p'\left(\|u(\cdot,t)\|^{\alpha}_{\alpha,\Omega}\right)\right|\left|\int_{\Omega}|u^{(m)}|^{\alpha-1}u^{(m)}_t \operatorname{sign} u\,dx\right|$$
$$\le \alpha C_*\|u^{(m)}_t\|_{2,\Omega}\left(\int_{\Omega}|u^{(m)}|^{2(\alpha-1)}\,dx\right)^{\frac{1}{2}} \tag{32}$$
$$\le \alpha C_*|\Omega|^{1-\frac{\alpha}{2}}\|u^{(m)}_t\|_{2,\Omega}\|u^{(m)}\|^{\frac{\alpha-1}{2}}_{2,\Omega} \le C\|u^{(m)}_t\|_{2,\Omega}.$$

Then

$$\left|\frac{dp_m(t)}{dt}\right|\left|\int_{\Omega}\frac{\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}}{p^2_m(t)}\,dx\right|$$
$$\le \frac{C}{(p^-)^2}\|u^{(m)}_t\|_{2,\Omega}\left(1+\int_{\Omega}|\nabla u^{(m)}|^{p_m(t)}\,dx\right)$$
$$\le C'\left(1+\sup_{(0,T)}\int_{\Omega}|\nabla u^{(m)}|^2\,dx\right)\|u^{(m)}_t\|_{2,\Omega} \le C\|u^{(m)}_t\|_{2,\Omega},$$
$$\frac{1}{2}\left|\frac{dp_m(t)}{dt}\right|\left|\int_{\Omega}\frac{1}{p_m(t)}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx\right|$$
$$\le C''\|u^{(m)}_t\|_{2,\Omega}\int_{\Omega}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx = I.$$

For every $0<\mu<\min\{p^-/2,(2-p^+)/2\}$,

$$s^{\frac{p_m(t)}{2}}\ln^2 s \le \begin{cases} s^{\frac{p_m(t)-\mu}{2}}\left(s^{\mu/2}\ln^2 s\right) & \text{if } s\in(0,1),\\ s^{\frac{p_m(t)+\mu}{2}}\left(s^{-\mu/2}\ln^2 s\right) & \text{if } s>1 \end{cases} \tag{33}$$
$$\le C(\mu,p^{\pm})(1+s^{\frac{p^++\mu}{2}}) \le C(1+s).$$

Gathering (33) with (21) we obtain the estimate

$$\int_{\Omega}\left(\epsilon^2+|\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}}\ln^2\left(\epsilon^2+|\nabla u^{(m)}|^2\right)dx$$
$$\le C\int_{\Omega}(1+|\nabla u^{(m)}|^2)\,dx \tag{34}$$
$$\le C\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}f_0^{(p^-)'}\,dz+1\right)$$

for all $t\in(0,T)$. By Young's inequality

$$I \le C\|u_t(t)\|_{2,\Omega}\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}|f|^{(p^-)'}\,dz+1\right)$$
$$\le \delta\|u_t(t)\|^2_{2,\Omega}+C'\left(\|\nabla u_0\|^2_{2,\Omega}+\int_{Q_T}f_0^{(p^-)'}\,dz+1\right)^2. \tag{35}$$

Plugging (35), (17) and (18) into (31) with a sufficiently small δ, we rewrite (31) in the form

$$\frac{1}{4}\|u_t^{(m)}(t)\|_{2,\Omega}^2 + \frac{d}{dt}\left(\int_\Omega \left(\epsilon^2 + |\nabla u^{(m)}|^2\right)^{\frac{p_m(t)}{2}} dx\right) \leq C\left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|_{(p^-)',Q_T}^{(p^-)'}\right)$$

for every $t \in [0,T]$ with a constant depending on α, $p^\pm$, C^*, $|\Omega|$. Inequality (30) follows after integration in time. □

Corollary 2 *Under the conditions of Lemma 3*

(i) $p_m(t) \in C^{1/2}([0,T])$ *and*

$$\|p_m(t)\|_{C^{1/2}([0,T])} \leq C$$

with an independent of m and ϵ constant C,

(ii)

$$u^{(m)} \in W^{1,2}(Q_T) \cap L^\infty(0,T;W_0^{1,2}(\Omega)) \cap L^{\frac{2(d+1)}{d-1}}(Q_T),$$

and the sequence $\{u^{(m)}\}$ is compact in $L^q(Q_T)$, $2 < q < \frac{2(d+1)}{d-1}$.

Proof By virtue of (30) and (32), for every $0 \leq \tau \leq t \leq T$

$$|p_m(t) - p_m(\tau)| = \left|\int_\tau^t \frac{dp_m(s)}{dt}\,ds\right| \leq C\int_\tau^t \|u_t^{(m)}\|_{2,\Omega}\,ds \leq C'\|u_t^{(m)}\|_{2,Q_T}|t-\tau|^{\frac{1}{2}}$$

with an independent of m and ϵ constant C. The second assertion follows from the embedding theorem. □

4 Passing to the limit

4.1 Strong solution of the regularized problem

Lemma 4 *If the data satisfy conditions (11), then the regularized problem (13) has a strong solution $u_\epsilon = \lim u^{(m)}$ as $m \to \infty$. The solution satisfies the estimate*

$$\|u_{\epsilon t}\|_{2,Q_T}^2 + \operatorname*{ess\,sup}_{(0,T)} \|\nabla u_\epsilon(\cdot,t)\|_{2,\Omega}^2 + \int_{Q_T} |u_{\epsilon xx}|^{p[u_\epsilon]}\,dz \leq M_3. \tag{36}$$

For the sake of simplicity of notation, throughout this subsection we omit the subindex ϵ and denote by $u(z)$ the limit of the sequence $\{u^{(m)}\}$, which approximates the solution of the regularized problem (13). The uniform estimates (17), (18), (21), (30) allow one to extract from $\{u^{(m)}\}$ a subsequence (which we assume coinciding with the whole sequence) such that for some $u \in W^{1,2}(Q_T) \cap L^\infty(0,T;W_0^{1,2}(\Omega)) \cap L^{\frac{2(d+1)}{d-1}}(Q_T)$ and $\chi \in (L^{(p[u])'}(Q_T))^d$

$$\begin{aligned} & u^{(m)} \to u \in L^q(Q_T) \text{ with } 2 < q < \frac{2(d+1)}{d-1}, \\ & u_t^{(m)} \rightharpoonup u_t \text{ in } L^2(Q_T), \\ & \nabla u^{(m)} \rightharpoonup \nabla u \text{ in } (L^2(Q_T))^d, \\ & \gamma_\epsilon(p_m(t), \nabla u^{(m)})\nabla u^{(m)} \rightharpoonup \chi \text{ in } (L^{(p[u])'}(Q_T))^d. \end{aligned} \tag{37}$$

The first relation follows from Corollary 2. The second and third follow directly from (21) and (30). Let us check the last relation. According to [23, Th.5], the sequence $\{u^{(m)}\}$ is relatively compact in $C([0,T];L^2(\Omega))$:

$$u^{(m)} \to u \text{ in } C([0,T];L^2(\Omega)) \quad \text{and a.e. in} \quad Q_T. \tag{38}$$

Due to (38), for every $t \in [0,T]$ there exists

$$\lim_{m\to\infty} \|u^{(m)}(\cdot,t)\|^\alpha_{\alpha,\Omega} = \|u(\cdot,t)\|^\alpha_{\alpha,\Omega} \tag{39}$$

whence, by continuity of $p(\cdot)$,

$$p_m(t) = p\Big(\|u^{(m)}(\cdot,t)\|^\alpha_{\alpha,\Omega}\Big) \to p\Big(\|u(\cdot,t)\|^\alpha_{\alpha,\Omega}\Big) = p[u] \quad \forall t \in [0,T].$$

Fix some $\beta \in (0,1/2)$. By Corollary 2 the sequence $\{p_m(t)\}$ is equicontinous in $C^{0,1/2}[0,T]$. It follows then that $\{p_m(t)\}$ is precompact in $C^{0,\beta}[0,T]$:

$$p_m(t) \to p(t) \equiv p[u] \text{ in } C^{0,\beta}[0,T] \subset C_{\log}[0,T]. \tag{40}$$

Let us notice that

$$\begin{aligned} \big(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|\big)^{(p[u])'} & \le C\big(1 + |\nabla u^{(m)}|^{p_m(t)-1}\big)^{\frac{p[u]}{p[u]-1}} \\ & \le C\big(1 + |\nabla u^{(m)}|^{\lambda_m(t)}\big) \end{aligned}$$

with

$$\lambda_m(t) = (p_m(t) - 1)\frac{p[u]}{p[u]-1}.$$

It is easy to see that

$$\begin{aligned} & \lambda_m(t) < 2 \quad \Leftrightarrow \quad p_m(t) + \frac{2}{p[u]} < 3 \quad \Leftrightarrow \\ & (p[u]-1)(p[u]-2) < p[u](p[u] - p_m(t)), \end{aligned}$$

which is true for all sufficiently big m because for $1 < p^- \le p^+ < 2$ and $p_m(t) \to p[u]$ uniformly in $[0, T]$

$$(p[u]-1)(p[u]-2) \le (p^- - 1)(p^+ - 2) < 0 \quad \text{while} \quad p[u](p[u]-p_m(t)) \to 0$$

as $m \to \infty$. Hence,

$$\left(\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|\right)^{(p[u])'} \le C\left(1 + |\nabla u^{(m)}|^2\right)$$

and

$$\int_{Q_T} (\gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|)^{(p[u])'}\, dz \le C$$

by virtue of (21). These arguments prove the following assertion.

Lemma 5 *If conditions* (11) *are fulfilled, then there exist* $u \in L^2(Q_T) \cap L^\infty(0, T; W_0^{1,2}(\Omega))$ *and* $\chi \in (L^{(p[u])'}(Q_T))^d$ *such that relations* (37) *are fulfilled and*

$$\int_{Q_T} (\gamma_\epsilon(p[u], \nabla u)|\nabla u|)^{(p[u])'}\, dz \le C, \qquad (p[u])' = \frac{p[u]}{p[u]-1} \tag{41}$$

with a constant C *depending only on the data.*

By (11) (d), (38), (39) $f(z, u^{(m)}, l(u^{(m)})) \to f(z, u, l(u))$ for a.e. $z \in Q_T$. Since $\sigma_m(t) \le \sigma^+ \le 2$, the functions $F_m = f(z, u^{(m)}, l(u^{(m)}))$ are uniformly bounded in $L^2(Q_T)$, whence $F_{m_k} \rightharpoonup F$ in $L^2(Q_T)$ for a subsequence $\{u^{(m_k)}\}$. It is necessary then that $F = f(z, u, l(u))$ a.e. in Q_T.

By the method of construction of $u^{(m)}$, for every finite m and $\phi \in \mathcal{P}_k \equiv \operatorname{span}\{\psi_1, \dots, \psi_k\}$, $k \le m$,

$$\begin{aligned}\int_{Q_T} \left(u_t^{(m)}\phi + \gamma_\epsilon(p_m(t), \nabla u^{(m)})\nabla u^{(m)} \cdot \nabla\phi - f(z, u^{(m)}, l(u^{(m)}))\phi\right) dz \\ = 0.\end{aligned} \tag{42}$$

Relations (37) and (41) allow one to pass in (42) to the limit as $m \to \infty$, which leads to the equality

$$\int_{Q_T} \left(u_t\phi + \chi \cdot \nabla\phi - f(z, u, l(u))\phi\right) dz = 0 \qquad \forall \phi \in \mathcal{P}_k. \tag{43}$$

Lemma 6 *For every* $u \in L^2(0, T; W_0^{1,2}(\Omega))$ *there exists a sequence* $\{\phi_N\}$, $\phi_N \in \mathcal{P}_N$ *such that* $\phi_N \to u$ *in* $\mathbf{W}_u(Q_T)$.

The assertion follows from the inclusion $L^2(0, T; W_0^{1,2}(\Omega)) \subset \mathbf{W}_u(Q_T)$ and the fact that the system $\{\lambda_i^{-\frac{1}{2}}\psi_i\}$ is an orthonormal basis of $W_0^{1,2}(\Omega)$. Taking ϕ_N for the test-function in (43) and letting $N \to \infty$ we obtain the equality

$$\int_{Q_T} u_t u\, dz + \int_{Q_T} \chi \cdot \nabla u\, dz = \int_{Q_T} f(z, u, l(u))u\, dz. \tag{44}$$

Let us return to (42) and take for the test-function $\phi = u^{(m)}$: for every $\psi \in \mathcal{P}_k$ with $k \le m$

$$\begin{aligned} 0 &= \int_{Q_T} u_t^{(m)} u^{(m)}\, dz + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})|\nabla u^{(m)}|^2\, dz \\ &\quad - \int_{Q_T} f(z, u^{(m)}, l(u^{(m)}))u^{(m)}\, dz \\ &= \int_{Q_T} u_t^{(m)} u^{(m)}\, dz + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})\nabla u^{(m)} \cdot \nabla(u^{(m)} - \psi)\, dz \\ &\quad - \int_{Q_T} f(z, u^{(m)}, l(u^{(m)}))u^{(m)}\, dz \\ &\quad + \int_{Q_T} \gamma_\epsilon(p_m(t), \nabla u^{(m)})\nabla u^{(m)} \cdot \nabla\psi\, dz. \end{aligned} \tag{45}$$

We will use the following well-known inequality: if $q \in (1, 2]$, then for all $\xi, \zeta \in \mathbb{R}^d$, $\xi \ne \zeta$ and $\epsilon > 0$

$$(\gamma_\epsilon(q, \xi)\xi - \gamma_\epsilon(q, \zeta)\zeta) \cdot (\xi - \zeta) \ge (q - 1)(1 + |\xi|^2 + |\zeta|^2)^{\frac{q-2}{2}} |\xi - \zeta|^2. \tag{46}$$

By virtue of (46) for every $\psi \in \mathcal{P}_k$ with $k \le m$

$$\begin{aligned} &\int_{Q_T} \gamma_\epsilon(p_m, \nabla u^{(m)})\nabla u^{(m)} \cdot \nabla(u^{(m)} - \psi)\, dz \\ &\quad = \int_{Q_T} (\gamma_\epsilon(p_m, \nabla u^{(m)})\nabla u^{(m)} - \gamma_\epsilon(p_m, \nabla\psi)\nabla\psi) \cdot \nabla(u^{(m)} - \psi)\, dz \\ &\qquad + \int_{Q_T} \gamma_\epsilon(p_m, \nabla\psi)\nabla\psi \cdot \nabla(u^{(m)} - \psi)\, dz \\ &\quad \ge \int_{Q_T} \gamma_\epsilon(p_m, \nabla\psi)\nabla\psi \cdot \nabla(u^{(m)} - \psi)\, dz. \end{aligned} \tag{47}$$

Because of (40)

$$\Psi_m(\nabla\psi) \equiv \gamma_\epsilon(p_m(t), \nabla\psi)\nabla\psi - \gamma_\epsilon(p[u], \nabla\psi)\nabla\psi \to 0 \tag{48}$$

as $m \to \infty$ uniformly in Q_T. It follows from (21), (48) and (37) that

$$\begin{aligned}&\int_{Q_T} \gamma_\epsilon(p_m(t), \nabla\psi)\nabla\psi \cdot \nabla(u_m - \psi)\, dz\\&\quad = \int_{Q_T} \Psi_m(\nabla\psi) \cdot \nabla(u_m - \psi)\, dz + \int_{Q_T} \gamma_\epsilon(p[u], \nabla\psi)\nabla\psi \cdot \nabla(u_m - \psi)\, dz\\&\quad \equiv J_1 + J_2 \to \int_{Q_T} \gamma_\epsilon(p[u], \nabla\psi)\nabla\psi \cdot \nabla(u - \psi)\, dz \quad \text{as } m \to \infty\end{aligned}$$

because

$$\begin{aligned}&J_1 \le \|\sigma_m(\nabla\psi)\|_{\infty,Q_T} \|\nabla(u_m - \psi)\|_{1,Q_T} \le C\|\Psi_m(\nabla\psi)\|_{\infty,Q_T} \to 0,\\&J_2 \to \int_{Q_T} \gamma_\epsilon(p[u], \nabla\psi)\nabla\psi \cdot \nabla(u - \psi)\, dz.\end{aligned}$$

Let us accept the notation $\widetilde{\mathbf{W}} = L^2(0, T; W_0^{1,2}(\Omega))$. Using (47) in (45) and then letting $m \to \infty$ we find that for $\psi \in \mathcal{P}_k$ with any $k \in \mathbb{N}$

$$\begin{aligned}0 \ge \int_{Q_T} u_t u\, dz + \int_{Q_T} \gamma_\epsilon(p[u], \nabla\psi)\nabla\psi \cdot \nabla(u - \psi)\, dz\\ - \int_{Q_T} f(z, u, l(u))u\, dz + \int_{Q_T} \chi \cdot \nabla\psi\, dz.\end{aligned}$$

By Lemma 6 we may take $\psi = \psi^{(k)} \in \mathcal{P}_k \cap \widetilde{\mathbf{W}}$ and then let $k \to \infty$. Plugging (44) we arrive at the inequality

$$0 \ge \int_{Q_T} (\gamma_\epsilon(p[u], \nabla\psi)\nabla\psi - \chi) \cdot \nabla(u - \psi)\, dz \qquad \forall \psi \in \widetilde{\mathbf{W}}.$$

Take $\psi = u + \lambda\zeta$ with an arbitrary $\zeta \in \widetilde{\mathbf{W}}$ and $\lambda > 0$. Simplifying and then letting $\lambda \downarrow 0$ we obtain the inequality

$$I(u, \chi, \zeta) \equiv \int_{Q_T} (\gamma_\epsilon(p[u], \nabla u)\nabla u - \chi) \cdot \nabla\zeta\, dz \le 0 \qquad \forall \zeta \in \widetilde{\mathbf{W}}.$$

Since ζ is arbitrary, it is necessary that $I(u, \chi, \zeta) = 0$ for all $\zeta \in \widetilde{\mathbf{W}}$, whence

$$\int_{Q_T} \left(u_t\zeta + \gamma_\epsilon(p[u], \nabla u)\nabla u \cdot \nabla\zeta - f(z, u, l(u))\zeta\right) dz = 0. \tag{49}$$

Estimate (36) follows from the uniform in m and ϵ estimates (17), (18), (21), (30).

4.2 Strong solution of the singular problem

Let u_ϵ be the strong solution of problem (13) with $\epsilon > 0$ obtained as the limit of the sequence of Galerkin's approximations (see Lemma 4). The functions u_ϵ satisfy the independent of ϵ estimates (36). Therefore, there exist functions u and χ such that, up to a subsequence,

$$
\begin{aligned}
& u_\epsilon \to u \in L^q(Q_T), \quad 2 < q < \frac{2(d+1)}{d-1}, \\
& u_{\epsilon t} \rightharpoonup u_t \text{ in } L^2(Q_T), \\
& \nabla u_\epsilon \rightharpoonup \nabla u \text{ in } (L^2(Q_T))^d, \\
& \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \rightharpoonup \chi \text{ in } (L^1(Q_T))^d.
\end{aligned}
\tag{50}
$$

Moreover, $u \in C^0([0,T];L^2(\Omega))$. For every $\epsilon > 0$ the function u_ϵ satisfies equality (49): $\forall \phi \in \widetilde{\mathbf{W}}$

$$
\int_{Q_T} \left(u_{\epsilon t}\phi + \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla\phi - f(z,u,l(u))\phi \right) dz = 0. \tag{51}
$$

Since $u_\epsilon \to u$ in $C^0([0,T];L^2(\Omega))$, then $\|u_\epsilon(\cdot,t)\|^\alpha_{\alpha,\Omega} \to \|u(\cdot,t)\|^\alpha_{\alpha,\Omega}$ for every $t \in [0,T]$ and

$$
p\Big((\|u_\epsilon(\cdot,t)\|^\alpha_{\alpha,\Omega} \Big) \to p\Big(\|u(\cdot,t)\|^\alpha_{\alpha,\Omega} \Big) \quad \text{as } \epsilon \to 0
$$

by continuity. As in Corollary 2, one may check that the functions $p_\epsilon(t) := p[u_\epsilon]$ are equicontinuous in $C^{0,1/2}[0,T]$: by Lemma 4

$$
\begin{aligned}
|p_\epsilon(t) - p_\epsilon(\tau)| = \left| \int_\tau^t \frac{dp_\epsilon(s)}{ds}\, ds \right| & \le \alpha \sup_{\mathbb{R}} |p'| \int_\tau^t \int_\Omega |u_{\epsilon t}||u|^{\alpha-1}\, dxds \\
& \le C(\alpha, C_*)\|u_{\epsilon t}\|_{2,Q_T} \left(\int_\tau^t \int_\Omega |u_\epsilon|^{2(\alpha-1)}\, dz \right)^{\frac{1}{2}} \\
& \le C \sup_{(0,T)} \|u_\epsilon(t)\|^{\alpha-1}_{2,\Omega} |t-\tau|^{1/2} \le C'|t-\tau|^{1/2}
\end{aligned}
\tag{52}
$$

with an independent of ϵ constant C'. Hence,

$$
p[u_\epsilon] \to p[u] \text{ in} C^{0,\beta}[0,T] \text{ with some } \beta \in (0,1/2). \tag{53}
$$

It follows that $C^\infty(\overline{Q}_T)$ is dense in $\widetilde{\mathbf{W}}$, $\mathbf{W}_u(Q_T)$ and $\mathbf{W}_{u_\epsilon}(Q_T)$ with every ϵ. Let $\phi_\delta \in C^\infty(\overline{Q}_T)$ and $\phi_\delta \to u$ in $\widetilde{\mathbf{W}}$ as $\delta \to 0$. Repeating the proof of Lemma 5 we find that $\chi \in (L^{(p[u])'}(Q_T))^d$, and by (50)

$$
\int_{Q_T} \gamma_\epsilon(p[u_\epsilon], \nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla\phi_\delta\, dz \to \int_{Q_T} \chi \cdot \nabla\phi_\delta\, dz \quad \text{as } \epsilon \to 0.
$$

Taking ϕ_δ for the test-function in (51) and letting $\epsilon \to 0$ we obtain

$$
\int_{Q_T} \left(u_t\phi_\delta + \chi \cdot \nabla\phi_\delta - f(z,u,l(u))\phi_\delta \right) dz = 0.
$$

Letting now $\delta \to 0$ we arrive at the equality

$$\int_{Q_T} (u_t u + \chi \cdot \nabla u - f(z,u,l(u))u)\,dz = 0. \tag{54}$$

Choosing $u_\epsilon \in \widetilde{\mathbf{W}} \subset \mathbf{W}_{u_\epsilon}(Q_T)$ for the test-function in (51) we obtain

$$\int_{Q_T} u_{\epsilon t} u_\epsilon\,dz + \int_{Q_T} (\gamma_\epsilon(\gamma_\epsilon(p[u_\epsilon],,\nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon - f(z,u_\epsilon,p[u_\epsilon])u_\epsilon)\,dz = 0. \tag{55}$$

Let us take $\psi \in C^\infty([0,T];C_0^\infty(\Omega)) \subset \widetilde{\mathbf{W}}$ with any $\epsilon > 0$. By (46)

$$\begin{aligned}
&\int_{Q_T} \gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon \cdot \nabla u_\epsilon\,dz \\
&\quad\geq \int_{Q_T} \left(|\nabla\psi|^{p[u_\epsilon]-2}\nabla\psi - \gamma_\epsilon(p[u_\epsilon],\nabla\psi)\nabla\psi\right)\cdot\nabla(u_\epsilon-\psi)\,dz \\
&\qquad + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon],\nabla\psi)\nabla\psi\cdot\nabla(u_\epsilon-\psi)\,dz \\
&\qquad + \int_{Q_T} \gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\cdot\nabla\psi\,dz \equiv J.
\end{aligned}$$

We omit the proof of the convergence

$$J \to \int_{Q_T} \chi\cdot\nabla\psi\,dz + \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz \quad \text{as } \epsilon \to 0$$

which follows, save some minor details, the arguments of [20]. It follows from (54) and (55) as $\epsilon \to 0$ that for every $\psi \in C^\infty(0,T;C_0^\infty(\Omega))$

$$\begin{aligned}
0 &= \int_{Q_T} u_t u\,dz - \int_{Q_T} fu)\,dz + \lim_{\epsilon\to0}\int_{Q_T} \gamma_\epsilon(p[u_\epsilon],\nabla u_\epsilon)\nabla u_\epsilon\cdot\nabla u_\epsilon\,dz \\
&\geq -\int_{Q_T} \chi\cdot\nabla u\,dz + \int_{Q_T} \chi\cdot\nabla\psi\,dz \\
&\quad + \int_{Q_T} |\nabla\psi|^{p[u]-2}\nabla\psi\cdot\nabla(u-\psi)\,dz \\
&= \int_{Q_T} \left(|\nabla\psi|^{p[u]-2}\nabla\psi - \chi\right)\cdot\nabla(u-\psi)\,dz.
\end{aligned} \tag{56}$$

Let us take $\psi \equiv \psi_\delta + \lambda\zeta$ where $\lambda = const > 0$,

$$\zeta, \psi_\delta \in C^\infty([0,T];C_0^\infty(\Omega)) \text{ and } \psi_\delta \to u \text{ in } \widetilde{\mathbf{W}} \text{ as } \delta \to 0.$$

Inequality (56) takes the form

$$J_1 + J_2 \equiv \int_{Q_T} \left(|\nabla(\psi_\delta + \lambda\zeta|^{p[u]-2}\nabla(\psi_\delta + \lambda\zeta) - \chi \right) \cdot \nabla(u - \psi_\delta)\, dz$$
$$- \lambda \int_{Q_T} \left(|\nabla(\psi_\delta + \lambda\zeta|^{p[u]-2}\nabla(\psi_\delta + \lambda\zeta) - \chi \right) \cdot \nabla\zeta\, dz \le 0.$$

By the generalized Hölder inequality (5)

$$\begin{aligned}
|J_1| &\le 2\|u - \psi_\delta\|_{\mathbf{W}_u(Q_T)} \left\| |\nabla(\psi_\delta + \lambda\zeta|^{p[u]-2}\nabla(\psi_\delta + \lambda\zeta) - \chi \right\|_{p'[u],Q_T} \\
&\le 2\|u - \psi_\delta\|_{\mathbf{W}_u(Q_T)} \\
&\quad \left(\left\| |\nabla(\psi_\delta + \lambda\zeta|^{p[u]-1} \right\|_{p'[u],Q_T} + \|\chi\|_{p'[u],Q_T} \right) \\
&\le C\|u - \psi_\delta\|_{\mathbf{W}_u(Q_T)} \left(1 + \|\chi\|_{p'[u],Q_T} + \int_{Q_T} |\nabla\psi_\delta|^{p[u]}\, dz + \int_{Q_T} |\lambda\nabla\zeta|^{p[u]}\, dz \right) \\
&\le C\|u - \psi_\delta\|_{\widetilde{\mathbf{W}}} \to 0 \quad \text{as } \delta \to 0,
\end{aligned}$$

while

$$J_2 \to -\lambda \int_{Q_T} \left(|\nabla(u + \lambda\zeta|^{p[u]-2}\nabla(u + \lambda\zeta) - \chi \right) \cdot \nabla\zeta\, dz.$$

Hence,

$$\lambda \int_{Q_T} \left(|\nabla(u + \lambda\zeta|^{p[u]-2}\nabla(u + \lambda\zeta) - \chi \right) \cdot \nabla\zeta\, dz \ge 0.$$

Simplifying and letting $\lambda \to 0^+$ we obtain the inequality

$$\int_{Q_T} \left(|\nabla u|^{p[u]-2}\nabla u - \chi \right) \cdot \nabla\zeta\, dz \ge 0 \qquad \forall \zeta \in C^\infty([0,T]; C_0^\infty(\Omega)).$$

Because of the density of smooth functions in $\widetilde{\mathbf{W}}$, this inequality is possible only if

$$\int_{Q_T} \left(|\nabla u|^{p[u]-2}\nabla u - \chi \right) \cdot \nabla\phi\, dz = 0 \qquad \forall \phi \in \widetilde{\mathbf{W}}.$$

Returning to (51) and passing to the limit as $\epsilon \to 0$ we find that for every test-function $\phi \in \mathbf{W}_u(Q_T)$

$$\int_{Q_T} \left(u_t\phi + |\nabla u|^{p[u]-2}\nabla u \cdot \nabla\phi - f(z, u, l(u))\phi \right) dz = 0.$$

5 Uniqueness of strong solutions

Theorem 2 *Assume that* $p(\cdot)$, $l(u)$ *satisfy conditions* (2) *and* $\sup_{\mathbb{R}_+} p'(s) < \infty$. *If* $f(z,s,r)$

$$\sup_{Q_T\times\mathbb{R}\times\mathbb{R}_+}\left(\left|\frac{\partial f}{\partial u}\right|+\left|\frac{\partial f}{\partial l}\right|\right)\right| \leq K < \infty, \tag{57}$$

then problem (1) *has at most one strong solution in the class of functions*

$$\mathcal{S} = \left\{v\ :\ v \in C([0,T];L^2(\Omega)) \cap L^\infty(0,T;W_0^{1,2}(\Omega)),\ v_t \in L^2(Q_T)\right\}.$$

Proof Let $u_i \in \mathcal{S}$ be two different strong solution of problem (1). Denote

$$p_i = p[u_i], \qquad f^i = f(z,u_i,l(u_i)), \quad i = 1,2.$$

The function $u = u_1 - u_2 \in \mathcal{S} \subset \widetilde{\mathbf{W}}$ can be taken for for the test-function in the integral identities (44) for u_i. Combining these identities we arrive at the equality

$$\frac{1}{2}\|u(t)\|^2_{2,\Omega} + \int_{Q_t}\left(|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_2|^{p_2-2}\nabla u_2\right)\cdot\nabla u\,dz = D(t) \tag{58}$$

with

$$D(t) = \int_{Q_t}\left(f^1 - f^2\right)u\ dz.$$

We will prove first that the strong solution is unique on a time interval $[0,T^*]$ with some T^* depending only on the data. Writing

$$\begin{aligned}(|\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_2|^{p_2-2}\nabla u_2)\cdot\nabla u\\ = (\left|\nabla u_1\right|^{p_2-2}\nabla u_1 - \left|\nabla u_2\right|^{p_2-2}\nabla u_2)\cdot\nabla u\\ + (\left|\nabla u_1\right|^{p_1-2}\nabla u_1 - \left|\nabla u_1\right|^{p_2-2}\nabla u_1)\cdot\nabla u\end{aligned}$$

and using inequality (46) we transform (58) into the form

$$\frac{1}{2}\|u(t)\|^2_{2,\Omega} + (p_2^- - 1)\int_{Q_t}\Lambda|\nabla u|^2\,dz \leq I(t), \tag{59}$$

where

$$\begin{aligned}\Lambda &= (1 + \left|\nabla u_1\right|^{p_2} + \left|\nabla u_2\right|^{p_2})^{\frac{p_2-2}{p_2}},\\ I(t) &= \int_{Q_t}(\left|\nabla u_1\right|^{p_2-2}\nabla u_1 - \left|\nabla u_1\right|^{p_1-2}\nabla u_1)\cdot\nabla u\,dz.\end{aligned}$$

By Young's inequality

$$|I(t)| \le \delta \int_{Q_t} \Lambda |\nabla u|^2 \, dz + C(\delta) J(t) \tag{60}$$

with

$$J(t) = \int_{Q_t} \left| |\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla u_1|^{p_2-2} \nabla u_1 \right|^2 \Lambda^{-1} \, dz$$

and any $\delta > 0$. Plugging (60) into (59) and choosing δ appropriately small, we rewrite (60) in the form

$$\frac{1}{2} \|u(t)\|_{2,\Omega}^2 + (p^- - 1 - \delta) \int_{Q_t} |\nabla u|^2 \Lambda \, dz \le C(\delta) J(t) + D(t). \tag{61}$$

For every $q, r > 1$ and $\xi \in \mathbb{R}^d$, $|\xi| \neq 0$,

$$\begin{aligned} \left| |\xi|^{q-2}\xi - |\xi|^{r-2}\xi \right| &= \left| \left(|\xi|^{q-1} - |\xi|^{r-1} \right) \frac{\xi}{|\xi|} \right| \\ &\le \left| |\xi|^{q-1} - |\xi|^{r-1} \right| \left| \frac{\xi}{|\xi|} \right| = \left| |\xi|^{q-1} - |\xi|^{r-1} \right|. \end{aligned}$$

By the Lagrange theorem there exists $\theta \in (0,1)$ such that

$$\left| |\xi|^{q-1} - |\xi|^{r-1} \right| = |\xi|^{\theta q + (1-\theta) r - 1} |\ln |\xi|| |q - r|.$$

It follows that at every point $z \in Q_T$ either $|\nabla u_1| = 0$ and

$$\left| |\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla u_1|^{p_2-2} \nabla u_1 \right| = 0,$$

or $|\nabla u_1| \neq 0$ and

$$\left| |\nabla u_1|^{p_1-2} \nabla u_1 - |\nabla u_1|^{p_2-2} \nabla u_1 \right| \le |\nabla u_1|^{p-1} \left| \ln |\nabla u_1| \right| |p_1 - p_2| \tag{62}$$

with $p = \theta p_1 + (1-\theta) p_2$, $\theta \in (0,1)$. Recall that the exponents p_1, p_2 are independent of x. By Young's inequality, for a.e. $t \in (0,T)$

$$\begin{aligned} \|\Lambda^{-1}\|_{\frac{2}{2-p_2},\Omega}^{\frac{2}{2-p_2}} &= \int_\Omega \left(1 + |\nabla u_1|^{p_2} + |\nabla u_2|^{p_2} \right)^{\frac{2}{p_2}} dx \\ &\le C \int_\Omega (1 + |\nabla u_1|^2 + |\nabla u_2|^2) \, dx \le C' \end{aligned}$$

with a constant C' depending on d, $p^\pm$ and the constant in (17). Using the classical Hölder's inequality and then (62) we obtain

$$
\begin{aligned}
J(t) &\le \int_0^t \left\| \left| |\nabla u_1|^{p_1-2}\nabla u_1 - |\nabla u_1|^{p_2-2}\nabla u_1 \right|^2 \right\|_{\frac{2}{p_2(t)},\Omega} \|\Lambda^{-1}\|_{\frac{2}{2-p_2(t)},\Omega}\, dt \\
&\le C\int_0^t |p_1-p_2|^2 \left(\int_\Omega \left(|\nabla u_1|^{p-1} \left| \ln |\nabla u_1| \right| \right)^{\frac{4}{p_2}} dx \right)^{\frac{p_2}{2}} dt
\end{aligned}
\tag{63}
$$

with a constant $C = C(C', p^{\pm})$ and the exponent $p = \theta p_1 + (1-\theta)p_2$ where $\theta = \theta(t) \in (0,1)$. Set

$$
\kappa = \frac{4(p-1)}{p_2} = (\theta p_1 + (1-\theta)p_2 - 1)\frac{4}{p_2}.
$$

The assumption $p_i \le p^+ < 2$ yields the inequality

$$
\kappa \le \frac{4(p_2-1)}{p_2} < 2 \quad \text{if}\, p_2 \ge p_1.
$$

Let us also claim that

$$
\kappa < 2 \,\text{if}\, p_1 \ge p_2,
$$

that is,

$$
p_1 - p_2 < 2 - p^+ \le 2 - p_1 \quad \text{if}\, p_1 \ge p_2. \tag{64}
$$

Condition (64) is surely fulfilled on a sufficiently small time interval $(0, T^*)$ with T^* defined through the data. Indeed: repeating the derivation of (52) we obtain the inequalities

$$
|p_i(t) - p_i(\tau)| \le C'|t-\tau|^{\frac{1}{2}} \quad \forall\, t, \tau \in [0, T]
$$

with a constant C' depending only on u_0, f and d. It follows that

$$
|p_1(t) - p_2(t)| \le |p_1(t) - p[u_0]| + |(p_2 - p[u_0]| \le 2C' t^{\frac{1}{2}} < 2 - p^+
$$

for $t < T^* = \left(\frac{2-p^+}{2C'}\right)^2$. We will use inequality (33) in the following form: if $\mu \in (0,1)$ is so small that $\kappa(1+\mu) \le 2$, then for every $\xi > 0$

$$
(\xi |\ln \xi|)^\kappa = \begin{cases} \xi^{\kappa(1+\mu)}(\xi^{-\mu}\ln \xi)^\kappa & \text{if}\, \xi > 1, \\ \xi^{\kappa(1-\mu)}(\xi^{\mu}|\ln \xi|)^\kappa & \text{if}\, \xi \in (0,1] \end{cases} \le C(1+\xi^2)
$$

with a constant $C = C(\mu)$. This inequality together with (21) imply that for a.e. $t \in (0, T^*)$

$$
\left(\int_\Omega \left(|\nabla u_1| \left| \ln |\nabla u_1|^{\frac{1}{p-1}} \right| \right)^{\frac{4(p-1)}{p_2}} dx \right)^{\frac{p_2}{2}} \le C \left(1 + \int_\Omega |\nabla u_1|^2 dx \right)^{\frac{p^+}{2}} \le C,
$$

whence

$$J(t) \le C \int_0^t |p_1 - p_2|^2 \, dt, \quad t < T^*. \tag{65}$$

By Hölder's inequality and due to the assumption $\alpha \in [1, 2]$

$$\begin{aligned} |p_2 - p_1| &\le C\|u_2 - u_1\|_{2,\Omega} \left(\int_\Omega \left(|u_2|^{2(\alpha-1)} + |u_1|^{2(\alpha-1)} \right) dx \right)^{\frac{1}{2}} \\ &\le C\|u_2 - u_1\|_{2,\Omega} \left(1 + \|u_2\|_{2,\Omega}^{2(\alpha-1)} + \|u_1\|_{2,\Omega}^{2(\alpha-1)} \right)^{\frac{1}{2}} \\ &\le C\|u_2 - u_1\|_{2,\Omega}. \end{aligned}$$

To estimate the term $D(t)$ in (58) we use the inequalities

$$|f(z, u_1, l(u_1)) - f(z, u_2, l(u_2))| \le K\big(|u| + |l(u_1) - l(u_2)|\big)$$

with the constant K from condition (57) and

$$\begin{aligned} \big|l(u_1) - l(u_2)\big| &\le C(\alpha)\|u(\cdot, t)\|_{2,\Omega} \left(\|u_1(\cdot, t)\|_{2,\Omega}^{\alpha-1} + \|u_2(\cdot, t)\|_{2,\Omega}^{\alpha-1} \right) \\ &\le C'\|u(\cdot, t)\|_{2,\Omega}, \end{aligned}$$

whence

$$|D| \le K\left(\|u(\cdot, t)\|_{2,\Omega}^2 + C' \int_\Omega |u(s, t)| \, \|u(\cdot, t)\|_{2,\Omega} ds \right) \le C\|u(\cdot, t)\|_{2,\Omega}^2. \tag{66}$$

It follows now from (58), (61) and (65), (66) that $u = u_2 - u_1$ satisfies the inequality

$$\|u(t)\|_{2,\Omega}^2 \le C \int_0^t \|u(\tau)\|_{2,\Omega}^2 d\tau, \quad t \in (0, T^*). \tag{67}$$

By the Gronwall lemma $\|u(t)\|_{2,\Omega}^2 = 0$ for $t \in [0, T^*)$, which means that $u_2(x, T^*/2) = u_1(x, T^*/2)$ in Ω. Let us take $T^*/2$ for the initial instant and consider problem (1) in the cylinder $\Omega \times (T^*/2, T)$. As is already shown, the condition $u_2(x, T^*/2) - u_1(x, T^*/2) = 0$ in Ω yields the equality $u_2 = u_1$ in $\Omega \times (T^*/2, 3T^*/2)$. Repeating these arguments, in a finite number of steps of the length T^* we will exhaust the interval $(0, T)$. The proof of Theorem 2 is completed. □

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Compliance with ethical standards

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