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Linear parabolic equations with strong boundary degeneration

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Abstract
As an application of the theory of linear parabolic differential equations on noncompact Riemannian manifolds, developed in earlier papers, we prove a maximal regularity theorem for nonuniformly parabolic boundary value problems in Euclidean spaces. The new feature of our result is the fact that—besides of obtaining an optimal solution theory—we consider the 'natural' case where the degeneration occurs only in the normal direction.

Keywords Degenerate parabolic boundary value problems · Riemannian manifolds with bounded geometry

Mathematics Subject Classification 35K65 · 35K45 · 53C44

1 Introduction

Of concern in this paper are linear second order parabolic differential equations which are not uniformly parabolic but degenerate near (some part of) the boundary. In the main body of this work such equations are studied in the framework of Riemannian manifolds. Here we restrict ourselves to a simpler Euclidean setting.

We assume that Ω is a bounded domain in $\mathbb{R}^m$, $m \geq 1$, with a smooth boundary $\partial\Omega$ which lies locally on one side of Ω. We write

$$\partial\Omega = \Gamma \cup \Gamma_0 \cup \Gamma_1, \tag{1.1}$$

where Γ, Γ_0, and Γ_1 are pairwise disjoint and open and closed in $\partial\Omega$ with $\Gamma \neq \emptyset$. Either Γ_0 or Γ_1, or both, may be empty in which case obvious adaptions apply (as is the case if $m = 1$). We denote by ν the inner (unit) normal on $\partial\Omega$ and by γ the trace

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operator $u \mapsto u|\partial\Omega$. By $\cdot$ or $(\cdot|\cdot)$ we denominate the Euclidean inner product in $\mathbb{R}^m$ and $:$ stands for the Hilbert–Schmidt inner product in $\mathbb{R}^{m\times m}$. Moreover, ∇u is the m-vector of first order derivatives, and $\nabla^2 u$ is the $(m\times m)$-matrix of second order derivatives. As usual, C^k is used for spaces of C^k functions, B stands for 'bounded', and BUC for 'bounded and uniformly continuous'.

We set

$$M := \overline{\Omega}\backslash\Gamma$$

and consider on M a second order linear boundary value (BVP), denoted by $(\mathcal{A},\mathcal{B})$, where

$$\mathcal{A}u := -a : \nabla^2 u + a_1\cdot\nabla u + a_0 u \quad \text{on } M$$

and

$$\mathcal{B}u := \begin{cases} \gamma u & \text{on } \Gamma_0, \\ b\cdot\gamma\nabla u + b_0\gamma u & \text{on } \Gamma_1 \end{cases}$$

on $\partial M := \Gamma_0 \cup \Gamma_1$. It is assumed that

$$a = a^* \in C(M, \mathbb{R}^{m\times m}), \quad a_1 \in C(M, \mathbb{R}^m), \quad a_0 \in C(M), \tag{1.2}$$

and

$$b \in BC^1(\Gamma_1, \mathbb{R}^m), \quad b_0 \in BC^1(\Gamma_1).$$

We also suppose that $\mathcal{A}$ is strongly elliptic, that is, there exists $\underline{\alpha} : M \to (0,1]$ such that

$$\big(a(x)\xi\big|\xi\big) \ge \underline{\alpha}(x)\,|\xi|^2, \qquad x \in M,$$

and that $\mathcal{B}$ is normal, which means

$$\Big|\big(b(x)\big|\nu(x)\big)\Big| > 0, \qquad x \in \Gamma_1. \tag{1.3}$$

Note that $\mathcal{B}$ is the Dirichlet boundary operator on Γ_0 and a first order boundary operator on Γ_1.

We fix $T \in (0,\infty)$ and set $J := [0,T]$. In this paper we develop an L_p Sobolev space theory for the parabolic BVP on $M \times J$:

$$\begin{aligned} \partial_t u + \mathcal{A}u &= f && \text{on } M\times J, \\ \mathcal{B}u &= 0 && \text{on } \partial M \times J, \\ \gamma_0 u &= u_0 && \text{on } M\times\{0\}, \end{aligned} \tag{1.4}$$

where γ_0 is the trace operator at $t = 0$. Observe that (1.4) is *not* a BVP on $\overline{\Omega}$, since there is no boundary condition on Γ. Also note that $\mathcal{A}$ is *not* assumed to be uniformly elliptic.

In general, (1.4) will not be well-posed. We now introduce conditions for the behavior of a and a_1 near Γ which guarantee an optimal solvability theory. This is done by prescribing—by means of a singularity function—the way by which a and a_1 vanish as we approach Γ.

We call a function

$$R \in C^\infty((0,1],(0,\infty)) \quad \text{with} \quad \int_0^1 \frac{dy}{R(y)} = \infty$$

(*strong*) *singularity function*.

Example 1.1 Suppose $s \in \mathbb{R}$. Then the power function $R_s := (y \mapsto y^s)$ is a strong singularity function iff $s \geq 1$. Also $y \mapsto e^{-\beta y^{-\gamma}}$ is a strong singularity function if $\beta, \gamma > 0$. □

To specify the behavior of the coefficients of $\mathcal{A}$ near Γ we choose a normal collar for it. This means that we fix $0 < \varepsilon \leq 1$ such that, setting

$$S := \{\, q + y\nu(q)\ ;\ 0 < y \leq \varepsilon,\ q \in \Gamma \,\},$$

the map

$$\varphi : \overline{S} \to [0,\varepsilon] \times \Gamma, \quad q + y\nu(q) \mapsto (y,q) \tag{1.5}$$

is a smooth diffeomorphism. Hence

$$y = \operatorname{dist}(x,\Gamma) \quad \text{for} \quad x = q + y\nu(q) \in S.$$

We select $\rho \in C^\infty(M,(0,1])$ satisfying $\rho(x) = \operatorname{dist}(x,\Gamma)$ for $x \in S$ and set

$$r(x) := R(\rho(x)), \qquad x \in M.$$

We also define $\nu \in C^\infty(S,\mathbb{R}^n)$ by extending the normal vector field from Γ to S by setting

$$\nu(x) := \nu(q), \qquad x = q + y\nu(q) \in S.$$

The operator $\mathcal{A}$ is said to be *R-degenerate uniformly strongly elliptic on M* if

(i) $\mathcal{A}$ is strongly elliptic on M;

(ii) there exists $\underline{\alpha} \in (0,1)$ such that

$$\big(a(x)\xi \big| \xi\big) \geq \underline{\alpha}\big(r^2(x)\eta^2 + |\zeta|^2\big) \tag{1.6}$$

for all $x \in S$ and $\xi = \eta\nu(x) + \zeta \in \mathbb{R}^m$ with $\big(\zeta \big| \nu(x)\big) = 0$.

The boundary value problem $(\mathcal{A},\mathcal{B})$ is called *R-degenerate uniformly strongly elliptic on M* if $\mathcal{A}$ has this property and $\mathcal{B}$ is normal. It is *strongly degenerate near Γ* if (1.6) holds for some singularity function R.

Let $\lambda : V_\lambda \to \mathbb{R}^{m-1}$, $q \mapsto z = (z^2,\dots,z^m)$ be a local coordinate system for Γ. Set $U_\kappa := \varphi^{-1}([0,\varepsilon) \times V_\lambda) \subset \overline{\Omega}$. Then

$$\kappa := (\mathrm{id}_{[0,\varepsilon)} \times \lambda)\circ\varphi : U_\kappa \to \mathbb{H}^m := \mathbb{R}_+ \times \mathbb{R}^{m-1} \tag{1.7}$$

is a local boundary flattening chart for $\overline{\Omega}$. It follows from Sect. 7 that $\mathcal{A}_\kappa$, the local representation of $\mathcal{A}|U_\kappa$ in the coordinate system $\kappa = (y, z)$, is given by

$$\begin{aligned}\mathcal{A}_\kappa = &-\Big(\overline{a}_\kappa^{11}(R\partial_y)^2 + 2\overline{a}_\kappa^{1\alpha}(R\partial_y)\partial_{z^\alpha} + \overline{a}_\kappa^{\alpha\beta}\partial_{z^\alpha}\partial_{z^\beta}\Big)\\ &+ \overline{a}_\kappa^{1}(R\partial_y) + \overline{a}_\kappa^{\alpha}\partial_{z^\alpha} + \overline{a}_\kappa^{0},\end{aligned} \tag{1.8}$$

where we use the summation convention with α and β running from 2 to m. The operator $\mathcal{A}_\kappa$ on $\kappa(U_\kappa) \subset \mathbb{H}^m$ is *bc-regular* if

$$\overline{a}_\kappa^{ij} \in BUC\big(\kappa(U_\kappa)\big), \quad \overline{a}_\kappa^{k}, \overline{a}_\kappa^{0} \in BC\big(\kappa(U_\kappa)\big), \qquad 1 \le i, j, k \le m. \tag{1.9}$$

We call $\mathcal{A}$ *R-degenerate bc-regular* if

(i) (1.2) applies;

(ii) $\mathcal{A}_\kappa$ is *bc*-regular for each boundary flattening chart of the form (1.7).

Remark 1.2 The ellipticity condition (1.6)(ii) is equivalent to the statement: for each κ of the from (1.7), the matrix $[\overline{a}_\kappa^{ij}]$ is symmetric and uniformly positive definite on $\kappa(U_\kappa)$. □

Next we introduce weighted Sobolev spaces which are adapted to strongly degenerate differential operators. We assume throughout

- $1 < p < \infty$.

The representation of $u : S \to \mathbb{R}$ in the variables (y, q) is denoted by $\varphi_* u$, that is, $\varphi_* u = u\circ\varphi^{-1}$. Given $k = 0, 1, 2$ and $u \in C^2(S)$,

$$\|u\|_{W_p^k(S;R)} := \sum_{i=0}^{k} \Big(\int_0^\varepsilon \|(R(y)\partial_y)^i \varphi_* u(y, \cdot)\|_{W_p^{k-i}(\Gamma)}^p \frac{dy}{R(y)}\Big)^{1/p}.$$

The Sobolev space $W_p^k(S;R)$ is defined to be the completion in $L_{1,\mathrm{loc}}(S)$ of the set of smooth compactly supported functions with respect to this norm.

We choose a relatively compact open subset U of M such that $S \cup U = M$. Then the *Sobolev space* $W_p^k(M;R)$ consists of all $u \in L_{1,\mathrm{loc}}(M)$ for which

$$u|S \in W_p^k(S;R), \quad u|U \in W_p^k(U).$$

It is a Banach space with the norm

$$u \mapsto \|u|S\|_{W_p^k(S;R)} + \|u|U\|_{W_p^k(U)},$$

whose topology is independent of the particular choice of S and U.

For a concise formulation of our solvability result for problem (1.4) we recall some notation. Given Banach spaces E_0 and E_1, $\mathcal{L}(E_1, E_0)$ is the Banach space of bounded linear operators from E_1 into E_0, and $\mathcal{L}\mathrm{is}(E_1, E_0)$ is the set of isomorphisms therein. As usual, $E_1 \hookrightarrow E_0$ means that E_1 is continuously injected in E_0, and $E_1 \overset{d}{\hookrightarrow} E_0$ indicates that E_1 is also dense in E_0. We write $E_{1-1/p}$ for the real interpolation space $(E_0, E_1)_{1-1/p,p}$.

Suppose $E_1 \overset{d}{\hookrightarrow} E_0$ and $A \in \mathcal{L}(E_1, E_0)$. Then A is said to have *maximal L_p regularity* if, for each $(f, u_0) \in L_p(J, E_0) \times E_{1-1/p}$, the linear evolution equation in E_0,

$$\partial u + Au = f \text{ on } J, \quad \gamma_0 u = u_0,$$

has a unique solution $u \in L_p(J, E_1) \cap W_p^1(J, E_0)$ depending continuously on (f, u_0). By Banach's homomorphism theorem this is equivalent to

$$(\partial + A,\ \gamma_0) \in \mathcal{L}\mathrm{is}(L_p(J, E_1) \cap W_p^1(J, E_0),\ L_p(J, E_0) \times E_{1-1/p}).$$

This concept is independent of T.

Henceforth, we express maximal L_p regularity more precisely by saying

$$(L_p(J, E_1) \cap W_p^1(J, E_0),\ L_p(J, E_0))$$

is a *pair of maximal regularity for* A. It is known that this condition implies that $-A$, considered as a linear operator in E_0 with domain E_1, generates a strongly continuous analytic semigroup on E_0, that is, in $\mathcal{L}(E_0) = \mathcal{L}(E_0, E_0)$. For all this we refer to Chapter III in [2].

We suppose:

$$\begin{array}{rl} \text{(i)} & R \text{ is a strong singularity function.} \\ \text{(ii)} & (\mathcal{A}, \mathcal{B}) \text{ is an } R\text{-degenerate} \\ & \text{uniformly strongly elliptic BVP on } M. \\ \text{(iii)} & \mathcal{A} \text{ is } R\text{-degenerate } bc\text{-regular.} \end{array} \tag{1.10}$$

Then

$$W_{p,\mathcal{B}}^2(M;R) := \{\, u \in W_p^2(M;R)\ ;\ \mathcal{B}u = 0 \}$$

is a closed linear subspace of $W_p^2(M;R)$,

$$W_{p,\mathcal{B}}^2(M;R) \overset{d}{\hookrightarrow} L_p(M;R) := W_p^0(M;R),$$

and

$$A := \mathcal{A} | W_{p,\mathcal{B}}^2(M;R) \in \mathcal{L}(W_{p,\mathcal{B}}^2(M;R), L_p(M;R)).$$

Hence the parabolic BVP (1.4) can be interpreted, using standard identifications, as the linear evolution equation in $L_p(M;R)$:

$$\partial u + Au = f \text{ on } J, \quad \gamma_0 u = u_0.$$

Now we can formulate our well-posedness result for (1.4).

Theorem 1.3 *Let* (1.10) *be satisfied. Then*

$$(L_p(J, W_p^2(M;R)) \cap W_p^1(J, L_p(M;R)),\ L_p(J, L_p(M;R)))$$

is a pair of maximal regularity for A.

Proof See Sect. 7. □

Remark 1.4

(a) This theorem has an obvious generalization to situations in which R varies from connected component to connected component of Γ. It also applies verbatim to strongly elliptic systems.

(b) The weighted Sobolev space $W_p^2(M;R)$ satisfies embedding theorems analogous to the familiar ones for the unweighted spaces $W_p^2(M)$. This implies, in particular, that the solution u and its first derivatives are Hölder continuous if $p > m$. We refrain from giving details, since we would need to introduce appropriately weighted Hölder spaces.

It is also possible to establish a Hölder space analog of Theorem 1.3, as well as optimal solvability results for nonautonomous problems in parabolic space-time settings of the type $W_p^{2,1}(M \times J;R)$. All this will be found in the forthcoming book [9].

(c) For simplicity, we have restricted ourselves to bounded domains. However, Theorem 1.3 remains valid if it is only assumed that $\partial\Omega$ is uniformly regular in the sense of Browder [13] (also see [25, IV.§4] and Sect. 2 below).

□

It is worthwhile to have a closer look at a simple model problem, taking the last remark into account.

Example 1.5 Let $\overline{\Omega} := [0,1] \times \mathbb{R}^{m-1}$. Then $\partial\Omega$ is the union of $\partial_0\Omega \cup \partial_1\Omega$ with $\partial_i\Omega = \{i\} \times \mathbb{R}^{m-1}$. Set $\Gamma := \partial_0\Omega$ (identified with $\mathbb{R}^{m-1}$) and fix $s \geq 1$. On $M := (0,1] \times \mathbb{R}^{m-1}$ consider the Dirichlet BVP $(\mathcal{A}_s, \gamma)$ with

$$\begin{aligned}\mathcal{A}_s &:= -(y^s\partial_y(y^s\partial_y) + \Delta_{m-1}) \\ &= -(y^{2s}\partial_y^2 + \Delta_{m-1}) - (sy^{s-1})y^s\partial_y,\end{aligned} \tag{1.11}$$

where Δ_{m-1} is the Laplace operator on $\mathbb{R}^{m-1}$. Since $|sy^{s-1}| \leq s$ on M, it is obvious that $(\mathcal{A}_s, \gamma)$ is R_s-degenerate strongly uniformly elliptic on M. Here we can take $S = M$. Note that

$$L_p(M;R_s) = L_p(M, y^{-s}dy\,dz)$$

with $z \in \mathbb{R}^{m-1}$.

The operator $\mathcal{A}_s$ can be rewritten as

$$\mathcal{A}_s = -(y^{2s}\Delta_{g_s} + \Delta_{m-1}), \tag{1.12}$$

Δ_{g_s} being the Laplace–Beltrami operator on (0, 1] for the metric $g_s = y^{-2s}\,dy^2$ (see (6.1). □

The interpretation (1.12) is the first pivotal step on the way to an efficient and successful handling of strongly degenerate parabolic BVPs. The second step, which takes the theory off the ground, is the proof (in Sect. 5) that $((0, 1], g_s)$ is a uniformly regular Riemannian manifold (in the sense of Sect. 2).

Although there has been done much work on degenerate parabolic differential equations, there are only very few papers known to us dealing with strong boundary degenerations. We mention, in particular, Fursikov [16], Vespri [29], Krylov [22], Krylov and Lototsky [23], Lototsky [26], Kim [18], and Fornaro, Metafune, and Pallara [15]. In all but [16, 22], and [23], uniform boundary degenerations of type R_s, $s \geq 1$, are being considered. This means that the ellipticity condition

$$(a(x)\xi|\xi) \geq \underline{\alpha}\rho^{2s}(x)\,|\xi|^2, \qquad x \in M, \tag{1.13}$$

is imposed. Vespri, Fornaro et al., and also Kim, consider the operator

$$\widetilde{\mathcal{A}}_s u := -\rho^{2s}a : \nabla^2 u + \rho^s a_1 \cdot \nabla u + a_0 u \tag{1.14}$$

in $M = \Omega$, which means that $\Gamma = \partial\Omega$, with smooth coefficients, and a uniformly positive definite diffusion matrix a. They study $\widetilde{A}_s$ on the weighted Sobolev space

$$\widetilde{W}_p^2(\Omega;\rho^s) := \{\, u \in L_p(\Omega)\,;\, \rho^s\partial_i u, \rho^{2s}\partial_j\partial_k u \in L_p(\Omega),\ 1 \leq i,j,k \leq m\}.$$

Fornaro et al. give a new, functional analytically based proof for Vespri's result which says that $-\widetilde{A}_s$ generates a strongly continuous analytic semigroup on $L_p(\Omega)$. In a preparatory step they consider, in the setting of Example 1.5, the operator

$$-y^{2s}\Delta_m + y^s a_1 \cdot \nabla \tag{1.15}$$

with a constant vector a_1 and show that it has maximal $L_p(M)$ regularity. That proof uses the fact that second order equations are considered. It is not applicable to systems or higher order problems. There is no maximal regularity result for the general case. It should be mentioned that Vespri studies Hölder space settings also.

Kim [18] proves a maximal regularity theorem by employing weighted Bessel potential spaces, introduced originally by Krylov [21, 22] in connection with stochastic evolution equations. Krylov considers the half-space $\mathbb{H}^m$ and $s = 1$, and uses basically the fact that a logarithmic change of variables reduces the weighted spaces to the standard Bessel potential spaces on $\mathbb{R}^m$. Kim's proof is in the spirit of the classical theory of partial differential equations. He employs a priori estimates due to Krylov [22] and versions of the Krylov spaces for bounded domains, established by Lototsky [26]. A similar approach is used by the latter author for a related degenerate operator. However, Lototsky builds on techniques from stochastic differential equations.

Parabolic equations with strong boundary degeneration occur, in particular, in connection with Ito stochastic parabolic equations (e.g., Lototsky [27], Krylov and Lototsky [23, 24], Kim and Krylov [19, 20], and the references therein).

The obvious difference between (1.13) (resp. (1.14)) and (1.6) is the fact that, in the former case, the diffusion and drift coefficients decay uniformly in all variables, whereas in (1.6) only a degeneracy in the normal direction is taken into account. This sticks out particularly clearly by comparing (1.11) with (1.15). Our approach seems to be more natural since, a priori, there is no reason to expect that tangential derivatives blow up near Γ. (See [23] for a similar remark.)

The only results for parabolic equations with degeneracies in normal directions only are in [16, 22, 23]. Fursikov establishes an L_2 theory for general parabolic systems of arbitrary order which are of the type of Euler's differential equation. This means that, in the model half-space case, ∂_y carries the weight y. He uses a logarithmic change of variables and builds on the work of Agranovich and Vishik [1]. Krylov [22], resp. Krylov and Lototsky [23], establish a maximal regularity theory in the case of the one-dimensional half-line, resp. $\mathbb{H}^m$, in the weighted Bessel potential spaces introduced in [21], resp. [22]. Our paper is the first one in which the case of a general domain, in fact, a general Riemannian manifold, is being handled.

Section 2 contains a brief review of the relevant facts on uniformly regular Riemannian manifolds. In Sect. 3 we present the corresponding function space settings. In the subsequent section we recall the maximal regularity theorem for second order uniformly parabolic BVPs on uniformly regular Riemannian manifolds.

In Sect. 5 we introduce uniformly regular Riemannian manifolds with strong boundary singularities. Then, in Sect. 6, we prove a renorming theorem for Sobolev spaces on manifolds with strong boundary singularities. In the final section we investigate the concepts of uniform ellipticity and *bc*-regularity in the framework of strong boundary degeneracy and prove Theorem 1.3.

2 Uniformly regular Riemannian manifolds

In this section we recall the definition of uniformly regular Riemannian manifolds and collect those properties of which we will make use. Details can be found in [3–5], and in the comprehensive presentation [9]. Thus we shall be rather brief.

We use standard notation from differential geometry and function space theory. In particular, an upper, resp. lower, asterisk on a symbol for a diffeomorphism denominates the corresponding pull-back, resp. push-forward (of tensors).

By c, resp. $c(\alpha)$ etc., we denote constants ≥ 1 which can vary from occurrence to occurrence.

Assume S is a nonempty set. On the cone of nonnegative functions on S we define an equivalence relation $\sim$ by $f \sim g$ iff $f(s)/c \leq g(s) \leq cf(s)$, $s \in S$.

An m-dimensional manifold is a separable metrizable space equipped with an m-dimensional smooth structure. We always work in the smooth category.

Let M be an m-dimensional manifold with or without boundary. If κ is a local chart, then we use U_κ for its domain, the coordinate patch associated with κ. The chart is normalized if $\kappa(U_\kappa) = Q^m_\kappa$, where $Q^m_\kappa = (-1,1)^m$ if $U \subset \mathring{M}$, the interior of M, and $Q^m_\kappa = [0,1) \times (-1,1)^{m-1}$ otherwise. An atlas $\mathfrak{K}$ is normalized if it consists of normalized charts. It is shrinkable if it normalized and there exists $r \in (0,1)$ such that $\{\, \kappa^{-1}(rQ^m_\kappa) \,;\, \kappa \in \mathfrak{K}\}$ is a covering of M. It has finite multiplicity if there exists $k \in \mathbb{N}$ such that any intersection of more than k coordinate patches is empty.

The atlas $\mathfrak{K}$ is *uniformly regular* (ur) if

$$\begin{aligned} &\text{(i)} \quad \text{it is shrinkable and has finite multiplicity;} \\ &\text{(ii)} \quad \widetilde{\kappa}\circ\kappa^{-1} \in BUC^\infty(\kappa(U_{\kappa\widetilde{\kappa}}), \mathbb{R}^m) \text{ and} \\ &\qquad \|\widetilde{\kappa}\circ\kappa^{-1}\|_{k,\infty} \le c(k), \ \kappa, \widetilde{\kappa} \in \mathfrak{K}, \ k \in \mathbb{N}, \ \text{where } U_{\kappa\widetilde{\kappa}} := U_\kappa \cap U_{\widetilde{\kappa}}. \end{aligned} \tag{2.1}$$

Two ur atlases $\mathfrak{K}$ and $\widetilde{\mathfrak{K}}$ are equivalent if

(i) there exists $k \in \mathbb{N}$ such that each coordinate patch of $\mathfrak{K}$ meets at most k coordinate patches of $\widetilde{\mathfrak{K}}$, and vice versa;

(ii) condition (2.1) (ii) holds for all $(\kappa, \widetilde{\kappa})$ and $(\widetilde{\kappa}, \kappa)$ belonging to $\mathfrak{K} \times \widetilde{\mathfrak{K}}$.

This defines an equivalence relation in the class of all ur atlases. An equivalence class thereof is a ur structure. By a *ur manifold* we mean a manifold equipped with a ur structure. Each ur atlas $\mathfrak{K}$ defines a unique ur structure, namely the equivalence class to which it belongs. Thus, if we need to specify the ur structure, we write $(M, \mathfrak{K})$ for the ur manifold and say its ur structure is induced by $\mathfrak{K}$.

Let $(M, \mathfrak{K})$ be a ur manifold. A Riemannian metric g on M is *ur* if

$$\begin{aligned} &\text{(i)} \quad \kappa_* g \sim g_m, \ \kappa \in \mathfrak{K}; \\ &\text{(ii)} \quad \|\kappa_* g\|_{k,\infty} \le c(k), \ \kappa \in \mathfrak{K}, \ k \in \mathbb{N}, \end{aligned}$$

where $g_m := (\cdot|\cdot) = dx^2$ is the Euclidean metric[1] on $\mathbb{R}^m$ and (i) is understood in the sense of quadratic forms. This concept is well-defined, independently of the specific $\mathfrak{K}$. A *uniformly regular Riemannian* (urR) *manifold*, $(M, g) = (M, \mathfrak{K}, g)$, is a ur manifold, $M = (M, \mathfrak{K})$, endowed with a urR metric.

In the following examples we use the natural ur structure (e.g., the product ur structure in Example 2.1(c)) if nothing is mentioned.

Example 2.1

(a) Each compact Riemannian manifold is a urR manifold and its ur structure is unique.

[1] As usual, we use the same symbol for a Riemannian metric and its restrictions to submanifolds.

(b) Let Ω be a bounded domain in $\mathbb{R}^m$ with a smooth boundary such that Ω lies locally on one side of it. Then $(\overline{\Omega}, g_m)$ is a urR manifold.
(c) If (M_i, g_i), $i = 1, 2$, are urR manifolds and at most one of them has a nonempty boundary, then $(M_1 \times M_2,\ g_1 \times g_2)$ is a urR manifold.
(d) Assume (M, g) is a urR manifold with a nonempty boundary. Denote by $g_{\partial M}$ the Riemannian metric on ∂M induced by g. Then $(\partial M, g_{\partial M})$ is a urR manifold.
(e) Set $J_k := (k-1,\ k+1)$ and $\lambda_k(s) := s - k$ for $s \in J_k$ and $k \in \mathbb{Z}$. Then $\{\ \lambda_k\ ;\ k \in \mathbb{Z}\}$ is a ur atlas for $\mathbb{R}$ which induces the *canonical* ur structure. Its restriction $\{\ \lambda_k | \mathbb{R}_+\ ;\ k \in \mathbb{N}\ \}$ is a ur atlas for $\mathbb{R}_+$ inducing the *canonical* ur structure on $\mathbb{R}_+$. Unless explicitly said otherwise, $\mathbb{R}$ and $\mathbb{R}_+$ are always given the canonical ur structure. Then $(\mathbb{R}, dx^2)$ and $(\mathbb{R}_+, dx^2)$ are urR manifolds. Thus it follows from Example 2.1(c) that $(\mathbb{R}^m, g_m)$ and $(\mathbb{H}^m, g_m)$ are urR manifolds.
(f) Let M be a manifold, N a topological space, and $f : N \to M$ a homeomorphism. Let $\mathfrak{K}$ be an atlas for M. Then $f^*\mathfrak{K} := \{\ f^*\kappa\ ;\ \kappa \in \mathfrak{K}\ \}$ is an atlas for N which induces the smooth 'pull-back' structure on N. If $\mathfrak{K}$ is ur, then $f^*\mathfrak{K}$ also is ur.
Suppose $(M, g) = (M, \mathfrak{K}, g)$ is a urR manifold. Then

$$f^*(M, g) = f^*(M, \mathfrak{K}, g) := (N, f^*\mathfrak{K}, f^*g)$$

is a urR manifold and the map $f : (N, f^*g) \to (M, g)$ is an isometric diffeomorphism. □

It follows from these examples, for instance, that the cylinders $\mathbb{R} \times M_1$ or $\mathbb{R}_+ \times M_2$, where M_i are compact Riemannian manifolds with $\partial M_2 = \emptyset$, are urR manifolds. More generally, Riemannian manifolds with cylindrical ends are urR manifolds (see [5], where many more examples are discussed).

Without going into detail, we mention that a Riemannian manifold without boundary is a urR manifold iff it has bounded geometry (see [4] for one half of this assertion and [14] for the other half). Thus, for example, $(\mathring{\mathbb{H}}^m, g_m)$ is *not* a urR manifold.

A Riemannian manifold with boundary is a urR manifold iff it has bounded geometry in the sense of Schick [28] (also see [10–12, 17] for related definitions). Detailed proofs of these equivalences will be found in [9].

3 Function spaces

Let (M, g) be a Riemannian manifold. We consider the tensor bundles

$$T_0^1 M := TM, \quad T_1^0 M := T^*M, \quad T_0^0 := \mathbb{R},$$

and

$$T_\tau^\sigma M := (TM)^{\otimes\sigma} \otimes (T^*M)^{\otimes\tau}, \qquad \sigma, \tau \geq 1,$$

endow $T_\tau^\sigma M$ with the tensor bundle metric $g_\sigma^\tau := g^{\otimes\sigma} \otimes g^{*\otimes\tau}$, $\sigma, \tau \in \mathbb{N}$, and set[2]

$$|a|_{g_\sigma^\tau} = \sqrt{(a|a)_{g_\sigma^\tau}} := \sqrt{g_\sigma^\tau(a,a)}, \qquad a \in C(T_\tau^\sigma M). \tag{3.1}$$

By $\nabla = \nabla_g$ we denote the Levi–Civita connection and interpret it as covariant derivative. Then, given a smooth function u on M, $\nabla^k u \in C^\infty(T_k^0 M)$ is defined by $\nabla^0 u := u$, $\nabla^1 u = \nabla u := du$, and $\nabla^{k+1} u := \nabla(\nabla^k u)$ for $k \in \mathbb{N}$.

Let $\kappa = (x^1, \dots, x^m)$ be a local coordinate system and set $\partial_i := \partial/\partial x^i$. Then

$$\nabla^1 u = \partial_i u\, dx^i, \quad \nabla^2 u = \nabla_{ij} u\, dx^i \otimes dx^j = (\partial_i\partial_j u - \Gamma_{ij}^k \partial_k u) dx^i \otimes dx^j,$$

where

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}), \qquad 1 \le i,j,k \le m,$$

are the Christoffel symbols. It follows that

$$|\nabla u|_{g_0^1}^2 = |\nabla u|_{g^*}^2 = g^{ij}\partial_i u \partial_j u \tag{3.2}$$

and

$$|\nabla^2 u|_{g_0^2}^2 = g^{i_1 j_1} g^{i_2 j_2} \nabla_{i_1 i_2} u \nabla_{j_1 j_2} u. \tag{3.3}$$

As usual, $d\mathrm{vol}_g = \sqrt{g}\, dx$ is the Riemann–Lebesgue volume element on U_κ.

Let $\sigma, \tau \in \mathbb{N}$, put $V := T_\tau^\sigma M$, and write $|\cdot|_V := |\cdot|_{g_\sigma^\tau}$. Then $\mathcal{D}(V)$ is the linear subspace of $C^\infty(V)$ of compactly supported sections.

For $1 \le q \le \infty$ we set

$$\|u\|_{L_q(V,g)} := \begin{cases} \left(\int_M |u|_V^q\, d\mathrm{vol}_g\right)^{1/q}, & 1 \le q < \infty, \\ \sup_M |u|_V, & q = \infty. \end{cases}$$

Then

$$L_q(V,g) := \Big(\Big\{ u \in L_{1,\mathrm{loc}}(M) \,;\, \|\cdot\|_{L_q(M,g)} < \infty \Big\},\ \|\cdot\|_{L_q(M,g)} \Big)$$

is the usual Lebesgue space of L_q sections of V, and $L_q(M,g) = L_q(V,g)$ for $V = T_0^0 M = \mathbb{R}$. If $k \in \mathbb{N}$, then

$$\|u\|_{W_q^k(V,g)} := \sum_{j=0}^{k} \Big\| \, |\nabla^j v|_{g_\sigma^{\tau+j}} \Big\|_{L_q(M,g)}, \qquad 1 \le q < \infty,$$

and

[2] If V is a vector bundle over M, then $C^k(V)$ denotes the vector space of C^k sections of V.

$$\|u\|_{BC^k(V,g)} := \sum_{j=0}^{k} \Big\| \, |\nabla^j v|_{g_\sigma^{\tau+j}} \Big\|_\infty .$$

Suppose $1 \le q < \infty$. Then the Sobolev space $W_q^k(V,g)$ is the completion of $\mathcal{D}(V)$ in $L_q(V,g)$ with respect to the norm $\|\cdot\|_{W_q^k(V,g)}$.

We denote by $BC^k(V,g)$ the Banach space of all $u \in C^k(V)$ for which $\|u\|_{BC^k(V,g)}$ is finite. Then $bc^k(V,g)$ is the closure of $BC^{k+1}(V,g)$ in $BC^k(V,g)$.

In the classical Euclidean case, that is, if (M,g) is one of the Riemannian manifolds of Examples 2.1(b) or 2.1(e), it is well-known that the above definitions lead to the standard Sobolev spaces, resp. spaces of bounded and continuous, resp. bounded and uniformly continuous, F-valued functions, where $F := \mathbb{R}^{m^\sigma \times m^\tau}$ (cf. [3] or [9]).

Theorem 3.1 *Suppose (M,g) is a urR manifold. Then the Sobolev spaces of sections of V possess the same embedding, interpolation, and trace properties as their classical counterparts.*

Proof [3, 4, 9] (also cf. [17] for some of these results). □

It is possible and important to characterize these spaces locally.

Theorem 3.2 *Let (M,g) be a urR manifold, $\mathfrak{K}$ a ur atlas, $1 \le q < \infty$, and $k \in \mathbb{N}$. Then*

(i) $u \mapsto \sum_{\kappa\in\mathfrak{K}} \|\kappa_* u\|_{W_q^k(Q_\kappa^m,F)}$ is a norm for $W_q^k(V,g)$.

(ii) $u \mapsto \max_{\kappa\in\mathfrak{K}} \|\kappa_* u\|_{BC^k(Q_\kappa^m,F)}$ is a norm for $BC^k(V,g)$.

(iii) $u \in bc^k(V,g)$ iff $\kappa_* u \in BUC^k(Q_\kappa^m, F)$ uniformly with respect to $\kappa \in \mathfrak{K}$.

Proof [9]. Also see [3] and [4] for similar assertions which, however, additionally involve partitions of unity. □

4 Parabolic problems on uniformly regular Riemannian manifolds

Let (M,g) be an m-dimensional Riemannian manifold. In this section we do not mention g in the notation for function spaces. Thus $W_p^k(M) = W_p^k(M,g)$, etc.

We consider a second order differential operator $\mathcal{A}$, defined for $u \in C^2(M)$ by

$$\mathcal{A}u := -a_2 \cdot \nabla^2 u + a_1 \cdot \nabla u + a_0 \cdot u,$$

where

$$a_i \in C(T_0^i M), \qquad i = 0,1,2,$$

and $\cdot$ denotes complete contraction. Then $\mathcal{A}$ is *uniformly strongly elliptic* if there exists $\underline{\alpha} > 0$ such that

$$a_2(p) \cdot (\xi \otimes \xi) \geq \underline{\alpha}\, |\xi|^2_{g^*(p)}, \qquad \xi \in T_p^* M, \quad p \in M. \tag{4.1}$$

Remark 4.1 The following assumptions are equivalent:

(i) a_2 is uniformly bounded and satisfies (4.1).

(ii) $a_2(p) \cdot (\xi \otimes \xi) \sim |\xi|^2_{g^*(p)}$, $\xi \in T_p^* M$, $p \in M$.

Proof Let $(H, (\cdot|\cdot))$ be a Hilbert space and A a positive semidefinite symmetric linear operator on H. Then $\|A\| = \sup\{\, (Ax|x) \,;\, \|x\| = 1\}$. From this the assertion is obvious. □

Suppose $\partial M \neq \emptyset$. A first order boundary operator $\mathcal{B}_1$ is defined by

$$\mathcal{B}_1 := b_1 \cdot \gamma \nabla + b_0 \gamma,$$

where

$$b_0 \in C(\partial M), \quad b_1 \in C(T_{\partial M} M),$$

with $T_{\partial M}$ being the restriction of TM to ∂M.

We fix $\delta \in C(\partial M, \{0, 1\})$ and set

$$\mathcal{B} := \delta \mathcal{B}_1 + (1 - \delta)\gamma.$$

Thus $\mathcal{B}$ is the Dirichlet boundary operator on $\partial_0 M := \delta^{-1}\{0\}$ and the first order boundary operator $\mathcal{B}_1$ on $\partial_1 M := \delta^{-1}(1)$. Note that $\partial_0 M$ and $\partial_1 M$ are disjoint, open and closed in ∂M, and $\partial_0 M \cup \partial_1 M = \partial M$. Either $\partial_0 M$ or $\partial_1 M$ may be empty. Also note that δ is constant on the connected components of ∂M. Then $\mathcal{B}$ is a *uniformly normal* boundary operator if either $\delta = 0$ or

$$\inf_{q \in \partial_1 M} |(b_1(q) \big| \nu(q))_{g(q)}| > 0.$$

Finally, $(\mathcal{A}, \mathcal{B})$ is a *uniformly normally elliptic BVP* on (M, g) if

(i) $\mathcal{A}$ is uniformly strongly elliptic;

(ii) $\mathcal{B}$ is uniformly normal.

The BVP $(\mathcal{A}, \mathcal{B})$ is *bc-regular* if

$$a_2 \in bc(T_0^2 M), \quad a_1 \in BC(TM), \quad a_0 \in BC(M), \tag{4.2}$$

and

$$b_1 \in BC^1(T_{\partial M} M), \quad b_0 \in BC^1(\partial M).$$

Our interest in this section concerns the Sobolev space solvability of the BVP (1.4) in the present setting. Assuming $(\mathcal{A}, \mathcal{B})$ to be bc-regular, we set, as in the introduction,

$$W^2_{p,\mathcal{B}}(M) := \{ u \in W^2_p(M) ; \mathcal{B}u = 0 \}$$

and

$$A := \mathcal{A} | W^2_{p,\mathcal{B}}(M),$$

considered as a linear operator in $L_p(M)$.

Theorem 4.2 *Suppose* (M, g) *is a urR manifold,* $1 < p < \infty$, *and* $(\mathcal{A}, \mathcal{B})$ *is bc-regular and uniformly normally elliptic. Then*

(i) $(L_p(M), W^2_{p,\mathcal{B}}(M))$ is a densely injected Banach couple.

(ii) $A \in \mathcal{L}(W^2_{p,\mathcal{B}}(M), L_p(M))$.

(iii) $(L_p(J, W^2_{p,\mathcal{B}}(M)) \cap W^1_p(J, L_p(M)),\ L_p(J, L_p(M)))$ is a pair of maximal regularity for A.

Proof This is a special case of the much more general Theorem 1.2.3(i) of [7] (also see [9]). □

In order to reduce the technical requirements to a minimum, we restrict ourselves to autonomous second order problems with homogeneous boundary conditions.

There are similar results applying to more general situations: $(\mathcal{A}, \mathcal{B})$ can be nonautonomous, involve operators acting on vector bundles, and be of higher order, provided Shapiro–Lopatinskii conditions apply. Nonhomogeneous boundary conditions can also be admitted. Besides of the Sobolev space results, there is also a Hölder space solution theory of the same general nature. All this will be exposed in detail in [9]. The reader may also consult our earlier papers [6] and [7].

5 Uniformly regular manifolds with boundary singularities

Let R be a strong singularity function, $I := (0, \varepsilon]$, and set

$$\sigma(y) := \int_y^\varepsilon \frac{d\tau}{R(\tau)}, \qquad y \in I.$$

We denote the general point of $\mathbb{R}_+$ by s.

Lemma 5.1 *σ is a diffeomorphism from I onto $\mathbb{R}_+$ and $\sigma^*(ds^2) = dy^2/R^2$.*

Proof The first assertion follows since $\dot{\sigma}(y) = -1/R(y) < 0$ for $y \in I$. Hence $\sigma^* ds = d\sigma = \dot{\sigma} dy = -dy/R$. This implies the second claim. □

Corollary 5.2 $(I,\ dy^2/R^2) = \sigma^*(\mathbb{R}_+, ds^2)$ *is a urR manifold.*

Proof Examples 2.1(e) and 2.1(f). □

Now we assume

$$\begin{aligned} &\bullet\quad (M, g) \text{ is a urR manifold.} \\ &\bullet\quad \Gamma \text{ is a nonempty open and closed subset of } \partial M. \end{aligned} \tag{5.1}$$

By Example 2.1(d), (Γ, g_Γ) is a urR manifold. Thus, see [28] or [9], there exist $\varepsilon \in (0, 1]$ and a closed geodesic normal collar

$$\varphi : \overline{S} \to [0, \varepsilon] \times \Gamma.$$

This means that $\overline{S}$ is a closed neighborhood of Γ in M and φ is a diffeomorphism with

$$\varphi^{-1}(y, q) = \exp_q(y\nu(q)), \qquad (y, q) \in [0, \varepsilon] \times \Gamma.$$

Hence

$$v_q := (t \mapsto \varphi^{-1}(t, q)), \qquad 0 \le t \le \varepsilon, \tag{5.2}$$

is the unique geodesic starting at $q \in \Gamma$ in the direction of the inward normal vector $\nu(q)$. Moreover,

$$\varphi_* g = g_N := dy^2 \oplus g_\Gamma$$

is a product metric on $T([0, \varepsilon] \times \Gamma) = T[0, \varepsilon] \oplus T\Gamma$.

For $0 < r \le 1$ we set

$$I(r) := (0, r\varepsilon], \quad N(r) := I(r) \times \Gamma, \quad N := N(1),$$

and

$$S(r) := \varphi^{-1}(N(r)), \quad S := S(1) = \overline{S} \backslash \Gamma.$$

We equip S with a new metric, g_R, as follows: we choose $\chi \in C^\infty(I, [0, 1])$ with $\chi(y) = 1$ for $y \le \varepsilon/3$ and $\chi(y) = 0$ for $y \ge 2\varepsilon/3$. Then we put

$$1/\delta^2 := 1 - \chi + \chi/R^2, \quad \gamma_R := dy^2/\delta^2, \tag{5.3}$$

and

$$g_R := \varphi^*(\gamma_R \oplus g_\Gamma). \tag{5.4}$$

Lemma 5.3 *(S, g_R) is a urR manifold and $g_R(p) = g(p)$ for $p \in S \setminus S(2/3)$.*

Proof Corollary 5.2 and Examples 2.1(c) and 2.1(d) imply that $(N, \gamma_R \oplus g_\Gamma)$ is a urR manifold. Now the first claim follows by applying Example 2.1(f). The second one is obvious. □

Theorem 5.4 *Let* (5.1) *be satisfied. Put* $\widehat{M} := M\backslash\Gamma$. *Define*

$$\widehat{g}_R := \begin{cases} g & \text{on } \widehat{M}\backslash S, \\ g_R & \text{on } S. \end{cases} \tag{5.5}$$

Then $(\widehat{M}, \widehat{g}_R)$ *is a urR manifold.*

Proof This is clear by the preceding lemma. □

For easy reference we say that $(\widehat{M}, \widehat{g}_R)$ is an *R-singular model for* (M, g) (*near* Γ). Moreover, (M, g) is *R-singular* (*near* Γ), if it is equipped with an R-singular model.

6 A renorming theorem

In this section we derive a semi-local representation for the Sobolev norms on $(\widehat{M}, \widehat{g}_R)$.

First we observe that, in the local coordinate system $\mathrm{id}_{\mathring{I}}$ for $\mathring{I}$, the Christoffel symbol of ∇_{γ_R} equals $-\dot{\delta}/\delta$. Hence

$$\nabla^2_{\gamma_R} = \Big(\frac{\partial}{\partial y}\Big)^2 + \frac{\dot{\delta}}{\delta}\frac{\partial}{\partial y} = \frac{1}{\delta^2}\Big(\delta\frac{\partial}{\partial y}\Big)^2. \tag{6.1}$$

To simplify the writing, we set

$$h := g_\Gamma, \quad \widetilde{g} := \gamma_R \oplus h = dy^2/\delta^2 \oplus h. \tag{6.2}$$

It follows from (3.2) and $\nabla_{\widetilde{\gamma}} = \nabla_{\gamma_R} \oplus \nabla_h$ that

$$|\nabla_{\widetilde{\gamma}} v|^2_{\widetilde{g}^*} = \Big|\delta\frac{\partial v}{\partial y}\Big|^2 + |\nabla_h v|^2_{h^*}. \tag{6.3}$$

Similarly, using (6.1) and $\nabla^2_{\widetilde{g}} = \nabla^2_{\gamma_R} \oplus \nabla^2_h$,

$$|\nabla^2_{\widetilde{\gamma}} v|^2_{\widetilde{g}^2_0} = \Big|\Big(\delta\frac{\partial}{\partial y}\Big)^2 v\Big|^2 + |\nabla^2_h v|^2_{h^2_0}. \tag{6.4}$$

Also note that

$$\sqrt{\widetilde{g}} = \sqrt{h}/\delta. \tag{6.5}$$

Each urR manifold possesses a ur atlas whose coordinate patches are smaller than any prescribed positive number (cf. [5, Section 3] or [9]). Thus we can choose a ur atlas $\mathfrak{K}$ for M such that $U_\kappa \subset S\backslash S(1/3)$ for each $\kappa \in \mathfrak{K}$ for which U_κ meets the boundary of $S(2/3)$. Then we set

$$\mathfrak{K}(W) := \big\{\kappa \in \mathfrak{K}\ ;\ U_\kappa \cap \big(M\backslash\mathring{S}(2/3)\big) \neq \emptyset\big\}$$

and

$$W := \bigcup_{\kappa \in \mathfrak{K}(W)} U_\kappa.$$

For $k \in \mathbb{N}$ and $u \in C^k(\widehat{M})$ we define

$$\|u\|_{k,p}(W) := \sum_{\kappa \in \mathfrak{K}(W)} \|\kappa_* u\|_{W_p^k(Q_\kappa^m)}$$

and

$$\|u\|_{k,p}(S,R)$$
$$:= \sum_{j=0}^{k} \left(\int_0^1 \left(\left| \left(R(y) \frac{\partial}{\partial y} \right)^j \varphi_* u(y,\cdot) \right|^p + |\nabla_h^j \varphi_* u(y,\cdot)|_{h_0^j}^p \right) d\mathrm{vol}_h \frac{dy}{R(y)} \right)^{1/p}.$$

Theorem 6.1 *$u \mapsto \|u\|_{k,p}(S,R) + \|u\|_{k,p}(W)$ is a norm for $W_p^k(\widehat{M}, \widehat{g}_R)$.*

Proof This is a consequence of Theorem 3.2, Lemma 5.3, Theorem 5.4, (6.3), (6.4), and (6.5). We leave it to the reader to fill in the details. □

Since, according to Sect. 3, the norm of $W_p^k(\widehat{M}, \widehat{g}_R)$ is defined in a coordinate free manner, it follows from this theorem that the topology of $W_p^k(\widehat{M}, \widehat{g}_R)$ is independent of the particular choice of the collar neighborhood (that is, of ε) and the cut-off function χ.

7 Elliptic operators on singular manifolds

Let $(\widehat{M}, \widehat{g}_R)$ be an R-singular model for (M, g) near Γ and set

$$\widehat{g} := \widehat{g}_R, \quad \widehat{\nabla} := \widehat{\nabla}_{\widehat{g}}.$$

Assume that

$$\widehat{\mathcal{A}} = \mathcal{A}(\widehat{\nabla}) := -a_2 \cdot \widehat{\nabla}^2 + a_1 \cdot \widehat{\nabla} + a_0$$

is a linear differential operator on $(\widehat{M}, \widehat{g})$ with continuous coefficients. Due to Theorem 5.4, we can apply Theorem 4.2, provided $\widehat{\mathcal{A}}$ is uniformly strongly elliptic and bc-regular on $(\widehat{M}, \widehat{g})$ and $\mathcal{B}$ is uniformly normal on $\partial\widehat{M} = \partial M \setminus \Gamma$. It follows from the definition of $\widehat{g}$ that $\widehat{\mathcal{A}}$, considered as a differential operator on $(\widehat{M}, \widehat{g})$, has singular coefficients. It is the purpose of the following considerations to describe the assumptions on $\widehat{\mathcal{A}}$ in this singular setting.

Recalling (5.2), we extend the normal vector field over S by setting

$$\nu(p) := \dot{\nu}_q(y) \in T_p S \quad \text{if} \quad \varphi(p) = (y, q).$$

Now we define $\nu^*(p) \in T_p^* S$ by

$$\langle \nu^*(p), X\rangle_p := \big(\nu(p)\big|X\big)_{g(p)}, \qquad X \in T_pS,$$

where $\langle\cdot,\cdot\rangle_p : T_p^*M \times T_pM \to \mathbb{R}$ is the canonical duality pairing. Thus $\nu(p)$, resp. $\nu^*(p)$, is at $p \in \varphi^{-1}(y,q)$ obtained from the normal vector $\nu(q)$, resp. conormal vector $\nu^*(q)$, by parallel transport along the geodesic curve $v_q(t)$, $0 \le t \le y$. Hence

$$|\nu(p)|_{g(p)} = |\nu^*(p)|_{g^*(p)} = 1, \qquad p \in S.$$

In abuse of language we call ν^* conormal vector (field) *on* S.

We denote by $\rho(p) := \mathrm{dist}_g(p, \Gamma)$ the distance in (S, g) from p to Γ. Thus $\rho(p) = y$ if $\varphi(p) = (y,q)$. Then

$$r(p) := R(\rho(p)), \qquad p \in S.$$

For shorter writing we also set

$$w[\xi]^2 := w\cdot(\xi\otimes\xi), \qquad w \in C(T_0^2M), \quad \xi \in C(T^*M).$$

Theorem 7.1 *$\widehat{\mathcal{A}}$ is uniformly strongly elliptic on $(\widehat{M}, \widehat{g})$ iff*

$$a_2(p)[\xi]^2 \sim |\xi|^2_{g^*(p)}, \qquad p \in \widehat{M}\backslash S, \quad \xi \in T_p^*M,$$

and

$$a_2(p)[\xi]^2 \sim \Big(r^2(p)\eta^2 + |\zeta|^2_{g^*(p)}\Big), \qquad p \in S, \tag{7.1}$$

for $\xi = \eta\nu^*(p) + \zeta \in T_p^*M$ with $\zeta \perp \nu^*(p)$.

Proof Set $\Gamma_y := \varphi^{-1}(y \oplus \Gamma)$ for $0 < y \le \varepsilon$. Then

$$T_p^*\Gamma_{r(p)} = \nu^*(p)^\perp. \tag{7.2}$$

It follows from (5.4) and (5.5) that it does not matter whether we take the orthogonal complement with respect to $g^*(p)$ or to $\widehat{g}^*(p)$. Thus, given

$$\xi = \eta\nu^*(p) + \zeta \in T_p^*S \quad \text{with} \quad \zeta \in \nu^*(p)^\perp,$$

we find

$$\varphi_*\xi = \eta \oplus \widetilde{\zeta} \in T^*_{(y,q)}N = \mathbb{R} \oplus T_q^*\Gamma,$$

where $\varphi(p) = (y,q)$. We deduce from (5.3) and (6.2) that

$$\widetilde{g}^* = \delta^2 dy^2 \oplus h^*. \tag{7.3}$$

Hence

$$|\varphi_*\xi|^2_{\widetilde{g}^*(y,q)} = \delta^2(y)\eta^2 + |\widetilde{\zeta}|^2_{h^*(q)}.$$

Note that $\delta(y) \sim R(y)$ for $1/3 \leq y \leq 1$. Thus, since $\delta(y) = R(y)$ if $0 < y \leq 1/3$, we get

$$|\varphi_*\xi|^2_{\widetilde{g}^*(y,q)} \sim \Big(R^2(y)\eta^2 + |\widetilde{\zeta}|^2_{h^*(q)}\Big), \tag{7.4}$$

uniformly with respect to ξ. Observe that

$$|\xi|^2_{\widehat{g}^*(p)} = \varphi^*\Big(\varphi_*(|\xi|^2_{\widehat{g}^*(p)})\Big) = \varphi^*(|\varphi_*\xi|^2_{\widetilde{g}^*(y,q)}).$$

From this and (7.4) we obtain

$$|\xi|^2_{\widehat{g}^*(p)} \sim \Big(\rho^2(p)\eta^2 + |\zeta|^2_{g^*(p)}\Big), \qquad \xi \in T^*S.$$

Now the assertion is an obvious consequence of (5.5) and Remark 4.1. □

We introduce tensor fields $\widetilde{a}_i \in C(T^i_0 N)$, $i = 0, 1, 2$, by setting

$$\widetilde{a}_2(y,q)\cdot(\xi_1\otimes\xi_2) := (\varphi_* a_2)(y,q)\cdot\left(\left(\frac{\eta_1}{R(y)}\oplus\zeta_1\right)\otimes\left(\frac{\eta_2}{R(y)}\oplus\zeta_2\right)\right)$$
$$\text{for } \xi_i = \eta_i\oplus\zeta_i \in \mathbb{R}\oplus T^*_q\Gamma,\ i=1,2,\ (y,q)\in N,$$
$$\widetilde{a}_1(y,q)\cdot\xi := (\varphi_* a_1)(y,q)\cdot\left(\frac{\eta}{R(y)}\oplus\zeta\right)$$
$$\text{for } \xi = \eta\oplus\zeta \in \mathbb{R}\oplus T^*_q\Gamma,\ (y,q)\in N,$$

and $\widetilde{a}_0 := \varphi_* a_0$.

Theorem 7.2 *We set $S^c := \widehat{M}\backslash S$. Then $\widehat{\mathcal{A}}$ is bc-regular on $(\widehat{M}, \widehat{g})$ iff*

(i) $a_2 \in bc(T^2_0 S^c, g)$, $\qquad a_i \in BC(T^i_0 S^c, g),\ i = 0, 1;$

(ii) $\widetilde{a}_2 \in bc(T^2_0 N, g_N)$, $\qquad \widetilde{a}_i \in BC(T^i_0 N, g_N),\ i = 0, 1.$

Proof

(1) Since, by (5.5), $(T^k_0 S^c, \widehat{g}) = (T^k_0 S^c, g)$ for $k \in \mathbb{N}$, we can restrict our considerations to S.

(2) We denote by $\varphi_*\widehat{\mathcal{A}}$ the push-forward of $\widehat{\mathcal{A}}$ by φ. Thus $\varphi_*\widehat{\mathcal{A}}$ is a linear operator on N, defined by

$$(\varphi_*\widehat{\mathcal{A}})v := \varphi_*\Big(\widehat{\mathcal{A}}(\varphi^* v)\Big), \qquad v \in C^2(N).$$

It follows that (see (5.4), (5.5), and (6.2))

$$\varphi_*\widehat{\nabla} = \nabla_{\varphi_*\widehat{g}} = \nabla_{\widetilde{g}} = \nabla_{g_R} \oplus \nabla_h.$$

Hence

$$\varphi_*\widehat{\mathcal{A}} = -(\varphi_* a_2)\cdot\nabla^2_{\widetilde{g}} + (\varphi_* a_1)\cdot\nabla_{\widetilde{g}} + \varphi_* a_0.$$

Using (6.1), we find

$$\varphi_*\widehat{\mathcal{A}} = -(\varphi_* a_2)\cdot\left(\frac{1}{\delta^2}\left(\delta\frac{\partial}{\partial y}\right)^2 \oplus \nabla_h^2\right) + (\varphi_* a_1)\cdot\left(\frac{1}{\delta}\left(\delta\frac{\partial}{\partial y}\right)\oplus\nabla_h\right) + \varphi_* a_0.$$

Note that, by (5.3),

$$R^2/\delta^2 = \chi + R^2(1-\chi)$$

and $1/c \le \partial^j R(y) \le c$ for $\varepsilon/3 \le y \le \varepsilon$ and $j = 0, 1$. Thus we can rewrite $\varphi_*\widehat{\mathcal{A}}$ as

$$\varphi_*\widehat{\mathcal{A}} = -\widehat{a}_2\cdot\left(R\frac{\partial}{\partial y}\oplus\nabla_h\right)^2 + \widehat{a}_1\cdot\left(R\frac{\partial}{\partial y}\oplus\nabla_h\right) + \widehat{a}_0, \tag{7.5}$$

where

$$\widehat{a}_2 \in bc(T^2_0 N, \widetilde{g}), \qquad \widehat{a}_i \in BC(T^i_0 N, \widetilde{g}), \quad i = 0, 1,$$

iff

$$\widetilde{a}_2 \in bc(T^2_0 N, g_N), \qquad \widetilde{a}_i \in BC(T^i_0 N, g_N), \quad i = 0, 1.$$

It is a consequence of the definition of $\widetilde{a}_i$ that

$$\|\widetilde{a}_i\|_{BC(T^i_0 N, g_N)} = \|\varphi_* a_i\|_{BC(T^i_0 N,\widetilde{g})}, \qquad i = 0, 1, 2.$$

Consequently, we derive from (7.5) that $\widehat{\mathcal{A}}$ is bc-regular on $(S, \widehat{g})$ iff assumption (ii) is satisfied. From this and step (1) we get the assertion.

□

Finally, we prove Theorem 1.3 by specializing our general results to the specific setting of the introduction.

Proof of Theorem 1.3 Example 2.1(b) guarantees that $(M, g) := (\overline{\Omega}, g_m)$ is a urR manifold. It follows from Theorem 6.1 that

$$W^k_p(\overline{\Omega}\setminus\Gamma; R) = W^k_p(\widehat{M}, \widehat{g}).$$

Theorem 7.1 shows that the R-degenerate uniform strong ellipticity (1.6) implies that $\mathcal{A}$ is uniformly strongly elliptic on $(\widehat{M}, \widehat{g})$. By taking the compactness of Γ into account, we deduce from (1.8), (1.9), and Theorem 7.2 that $\mathcal{A}$ is bc-regular on $(\widehat{M}, \widehat{g})$. Due to (1.3) and the compactness of Γ_1, we see that $\mathcal{B}$ is uniformly normal on $\partial\widehat{M}$. Now the assertion is implied by Theorem 4.2. □

Remark 1.2 is an easy consequence of the proof of Theorem 7.2, using once more the compactness of Γ.

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by the author.

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