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A review on reaction–diffusion approximation

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Abstract

Recently reaction–diffusion approximation has been intensively studied. This paper reviews the studies in reaction–diffusion approximation and is based on Iida and Ninomiya (Sugaku 66:225–248, 2014) added recent studies to. This paper explains reaction–diffusion approximation of the Stefan problem, nonlinear diffusion, nonlocal dispersal and wave equations. Moreover, some instabilities of equations mentioned above will be explained by the Turing instability through reaction–diffusion approximations.

Keywords Singular limit · Fast reaction limit · Reaction–diffusion system

Mathematics Subject Classification 35K57 · 35B25 · 35A35

1 Introduction

To express the distributions of the coexisting chemical substances, the following partial differential equation is often used:

$$\boldsymbol{u}_t = D\Delta \boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}) \tag{1}$$

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where $\boldsymbol{u} = \boldsymbol{u}(x, t)$ is a vector function with m-components, $\boldsymbol{D}$ is a diagonal matrix with non-negative components, N is a spatial dimension, Δ is the Laplace operator for $x \in \mathbb{R}^N$, $\boldsymbol{F}$ is a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^m$. This system belongs to the parabolic system. In this paper, we assume that the variable x for $\boldsymbol{u}(x, t)$ belongs to a bounded domain Ω in $\mathbb{R}^N$ with smooth boundary and the homogeneous Neumann boundary condition

$$\frac{\partial \boldsymbol{u}}{\partial \nu} = 0, \quad x \in \partial\Omega,\ t > 0$$

is always equipped with (1), if the boundary condition is not specified, where ν is an outer normal vector of $\partial\Omega$. It is well-known that the local existence of the classical solutions of (1) with $\boldsymbol{u}(x, 0) = \boldsymbol{u}_0$ under the suitable boundary conditions follows from the standard theory. This system (1) consists of reaction terms and diffusion terms only. Therefore, it is often called a *reaction–diffusion system*. The reaction-diffusion systems are used in chemistry, physics and ecology as mathematical models describing non-equilibrium chemical reactions etc. On the other hand, this notion does not fit to mathematical notion. By the change of variables $\boldsymbol{v} = P^{-1}\boldsymbol{u}$ with a $m \times m$ regular matrix P, (1) is transformed into

$$\boldsymbol{v}_t = P^{-1}DP\Delta\boldsymbol{v} + P^{-1}\boldsymbol{F}(P\boldsymbol{v}).$$

If D is a scalar matrix, the above equation also belongs to a class of reaction–diffusion system. However, if D is not a scalar matrix, it does not belong to the class. Namely, this notion is not invariant under the linear transformation. This inconvenience produces interesting phenomena. This is one of the reasons of the Turing instability [47]. For example, the ordinary differential equations

$$\frac{d}{dt}\boldsymbol{u} = \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\boldsymbol{u}$$

has a stable equilibrium (0, 0), while $\boldsymbol{u} = (0, 0)$ is unstable for the corresponding reaction–diffusion system

$$\frac{\partial}{\partial t}\boldsymbol{u} = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix}\Delta\boldsymbol{u} + \begin{pmatrix} 2 & -1 \\ 7 & -3 \end{pmatrix}\boldsymbol{u} \tag{2}$$

equipped with Neumann boundary condition where $\Omega = (0, \pi)$. One may think that the diffusion homogenizes the solution, but this implies that the diffusion sometimes induces spatial periodic pattern.

Next, consider a simple example with a small parameter ε:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}\Delta\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} u_2 - u_1 \\ u_1 - u_2 \end{pmatrix}. \tag{3}$$

Note that the first component u_1 does not diffuse by itself. How does the solution of this reaction–diffusion system with small parameter ε behave as $\varepsilon \to +0$? We denote the solution of (3) by $(u_1^\varepsilon, u_2^\varepsilon)$. By multiplying ε by both sides of (3) and taking a limit formally, we can expect that $u_1^\varepsilon - u_2^\varepsilon \to 0$. Namely, if the limits exist, we get

$$u_1^* = u_2^*$$

where

$$u_j^* := \lim_{\varepsilon\to 0} u_j^\varepsilon \qquad (j = 1, 2).$$

Moreover, set $w^\varepsilon = u_1^\varepsilon + u_2^\varepsilon$, $w^* = u_1^* + u_2^*$. Adding two components of (3) yields

$$w_t^\varepsilon = 2\Delta u_2^\varepsilon.$$

Using $w^* = u_1^* + u_2^* = 2u_2^*$, we expect that

$$w_t^* = \Delta w^*.$$

This means that u_1 is transformed into u_2, u_2 moves randomly and then u_2 becomes u_1 again, though u_1 does not move by itself randomly. In other words, this can be regarded as one of facilitated diffusion of u_1 by a transporter u_2.

Consider the general reaction–diffusion system with a small parameter ε:

$$\boldsymbol{u}_t = \boldsymbol{D}\Delta\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}) + \frac{1}{\varepsilon}\boldsymbol{G}(\boldsymbol{u}) \tag{4}$$

in Ω with Neumann boundary condition where $\boldsymbol{u} \in \mathbb{R}^m$ and $\boldsymbol{G}$ is a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^m$. How does the solution behave when ε tends to 0 ? In this paper, we review this problem including the recent studies [19, 40–42].

As seen in the above example, some solutions of (4) often exhibit extraordinary behaviors which are out of our expectation when ε is very small. These behaviors will be explained in Sect. 3.4 (see Figs. 7, 8, 9). We denote a solution of (4) by $\boldsymbol{u}^\varepsilon$. Let us take a formal limit of (4). Multiplying ε to both sides of (4), we get

$$\varepsilon\boldsymbol{u}_t^\varepsilon = \varepsilon\boldsymbol{D}\Delta\boldsymbol{u}^\varepsilon + \varepsilon\boldsymbol{F}(\boldsymbol{u}^\varepsilon) + \boldsymbol{G}(\boldsymbol{u}^\varepsilon).$$

Thus we formally obtain

$$\boldsymbol{0} = \boldsymbol{G}(\lim_{\varepsilon\to 0}\boldsymbol{u}^\varepsilon). \tag{5}$$

Namely, we expect that any solution $\boldsymbol{u}^\varepsilon$ of (4) converges to the nullset of $\boldsymbol{G}$:

$$\mathcal{A} := \{\boldsymbol{u} \in \mathbb{R}^m \mid \boldsymbol{G}(\boldsymbol{u}) = 0\}.$$

This set must be important to study the limit problem. We call it a *reaction limit set* [13, 37]. To explain the importance of this set, let us consider the Lotka-Volterra competition-diffusion system:

$$\begin{cases} u_{1,t} &= d_1\Delta u_1 + (r_1 - a_1u_1 - b_1u_2)u_1, \qquad x \in \Omega,\ t > 0, \\ u_{2,t} &= d_2\Delta u_2 + (r_2 - a_2u_2 - b_2u_1)u_2, \qquad x \in \Omega,\ t > 0, \end{cases} \tag{6}$$

where r_1, r_2, a_1, a_2 are non-negative constants and d_1, d_2, b_1, b_2 are positive constants. Setting $b_j = s_j/\varepsilon$ and $f(u_1) = (r_1 - a_1u_1)u_1$, $g(u_2) = (r_2 - a_2u_2)u_2$ with some positive constants s_j $(j = 1, 2)$, we can rewrite the system into

$$\begin{cases} u_{1,t} &= d_1\Delta u_1 + f(u_1) - \dfrac{s_1}{\varepsilon}u_1u_2, \qquad x \in \Omega,\ t > 0, \\ u_{2,t} &= d_2\Delta u_2 + g(u_2) - \dfrac{s_2}{\varepsilon}u_1u_2, \qquad x \in \Omega,\ t > 0. \end{cases} \tag{7}$$

For simplicity, we may take $f = g = 0$. If ε tends to 0, then the profiles of solutions with suitable initial values are shown in Fig. 1. Namely, the supports of u_1 and u_2 are segregated from each other as ε tends to 0 (see Sect. 2.2 for more details). Note that if the initial distributions are non-negative, then all solutions of the reaction–diffusion system (6) are also non-negative. Then, the reaction limit set of (7) consists of two segments:

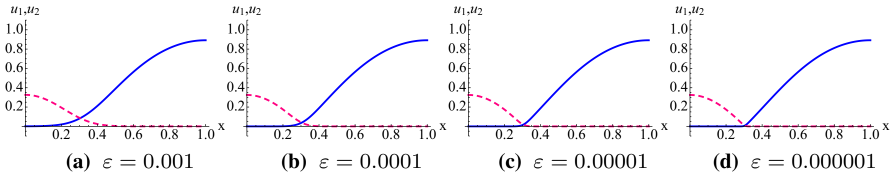
$$\mathcal{A}_1 := \{(u_1, u_2) \mid u_1 \geq 0,\ u_2 = 0\} \cup \{(u_1, u_2) \mid u_1 = 0,\ u_2 \geq 0\}.$$

It is easily seen that the reaction limit set $\mathcal{A}_1$ is not smooth at the origin. The fluxes of the limit functions u_1, u_2 possess discontinuity at $(u_1, u_2) = (0, 0)$. As a result, the free boundary appears as $\varepsilon \to +0$. We will explain that the limit problem is reduced to a two-phase Stefan problem in Sect. 2.2. Conversely, we can say that the Stefan problem can be approximated by a competition-diffusion system (7). Hence, some partial differential equations which do not belong to a class of reaction–diffusion systems can be approximated by reaction–diffusion systems. We call it *reaction–diffusion approximation*. The reaction–diffusion approximation is one of the singular limit problems and it corresponds to studying the boundary of a class of reaction–diffusion systems.

We note that (5) does not always hold. Indeed, if we consider the Allen–Cahn–Nagumo equation:

$$u_t = \Delta u + \frac{1}{\varepsilon}(u - u^3),$$

then $\mathcal{A}$ consists of three isolated points, namely, $\mathcal{A} = \{-1, 0, 1\}$. If ε is sufficiently small, $u^\varepsilon(\cdot, t)$ is continuous but becomes close to step functions connecting -1 and



(a) $\varepsilon = 0.001$ **(b)** $\varepsilon = 0.0001$ **(c)** $\varepsilon = 0.00001$ **(d)** $\varepsilon = 0.000001$

Fig. 1 Dependence of numerical solutions of (7) (solid curve:u_1, dashed curve:u_2) on ε when $N = 1, d_1 = d_2 = s_1 = s_2 = 1, f = g = 0$ and $t = 0.06$

1. Then u^ε does not stay in $\mathcal{A}$. This profile of $u^\varepsilon(\cdot, t)$ is called a *transition layer*. See [43] for the detailed dynamics of transition layers.

Hereafter let us consider the case where the reaction limit set of (4) consists only of neutral stable equilibria of

$$\boldsymbol{u}_t = \boldsymbol{G}(\boldsymbol{u}) \tag{8}$$

and consider the relationship between (4) and its limit problem. In the successive section, we explain several examples to facilitate the readers to understand the whole picture of reaction–diffusion approximation.

2 Reaction–diffusion approximation

2.1 One-phase Stefan problem

Hilhorst, Hout and Peletier [8] considered a two-component reaction–diffusion system:

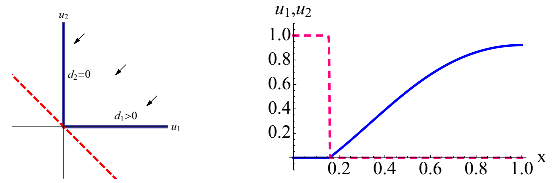
$$\begin{cases} u_{1,t} = \Delta u_1 - \dfrac{1}{\varepsilon} u_1 u_2, \\ u_{2,t} = -\dfrac{1}{\varepsilon} u_1 u_2 \end{cases} \tag{9}$$

for $x \in \Omega$ and $0 < t \le T$ with any positive constant T. Let $(u_1^\varepsilon, u_2^\varepsilon)$ be a solution of (9) with

$$u_j^\varepsilon(x, 0) = u_{0j}(x) \ge 0 \quad \text{for } x \in \Omega,\ \varepsilon > 0,\ j = 1, 2.$$

See Fig. 2b for the profile of a solution to (9). We shall explain that the limit of the solution of (9) as $\varepsilon \to 0$ is represented by the solution of the so-called one-phase Stefan problem. First we expect that

$$\lim_{\varepsilon \to 0} u_1^\varepsilon u_2^\varepsilon = 0.$$



(a) solid segments:$\mathcal{A}_2$, dashed line: $\boldsymbol{P}\mathcal{A}_2$ **(b)** solid curve:u_1, dashed curve u_2.

Fig. 2 Reaction limit set **a** and numerical solution **b** of (9) when $\varepsilon = 0.000001$, $t = 0.06$. The reaction limit set $\mathcal{A}_1$ for (7) is same as **a**

This result means that the supports of the limit functions u_j $(j = 1, 2)$ are separated by an interface. The difference between the two equations of (9) yields

$$(u_1^\varepsilon - u_2^\varepsilon)_t = \Delta u_1^\varepsilon.$$

We note that this equation does not include the parameter ε explicitly. Multiplying the above equation by a test function ζ and integrating by parts over $Q_T := \Omega \times (0, T)$, we get

$$\int\int_{Q_T} \left\{-\left(u_1^\varepsilon - u_2^\varepsilon\right)\zeta_t + \nabla u_1^\varepsilon \cdot \nabla \zeta\right\} dxdt = \int_\Omega \left(u_{01} - u_{02}\right)\zeta(x, 0)dx$$

where $\zeta \in C^\infty(\overline{Q_T})$ with $\zeta(x, T) = 0$. We may assume that $u_j^\varepsilon \to u_j$ $(j = 1, 2)$ strongly in $L^2(Q_T)$ and $u_1^\varepsilon \to u_1$ weakly in $L^2(0, T; H^1(\Omega))$ as $\varepsilon \to 0$. Then,

$$\int\int_{Q_T} \left\{-\left(u_1 - u_2\right)\zeta_t + \nabla u_1 \cdot \nabla \zeta\right\} dxdt = \int_\Omega \left(u_{01} - u_{02}\right)\zeta(x, 0)dx.$$

We denote $u_1 - u_2$ by w. Since $u_1 \geq 0$, $u_2 \geq 0$ and $u_1 u_2 \equiv 0$, the above equation is rewritten as

$$\int\int_{Q_T} \left\{-w\zeta_t + \nabla d(w) \cdot \nabla \zeta\right\} dxdt = \int_\Omega w_0 \zeta(x, 0)dx, \tag{10}$$

where $d(r) := \max(r, 0)$ and $w_0 = u_{01} - u_{02}$. This is the weak form of one-phase Stefan problem

$$w_t = \Delta d(w) \qquad \text{in } Q_T. \tag{11}$$

Next we derive the strong form from (11) under the assumption of the smoothness of the weak solution and the interface. Let w be a weak solution of (11). We denote the region where $w(\cdot, t) > 0$ (resp. $w(\cdot, t) < 0$) for $t \in [0, T]$ by $\Omega_1(t)$ (resp. $\Omega_2(t)$). Set

$$u_1 := w^+, \quad u_2 := w^-, \qquad \text{where } r^+ := \max(r, 0), \quad r^- := -\min(r, 0).$$

We also assume that the interface

$$\Gamma(t) := \Omega \setminus (\Omega_1(t) \cup \Omega_2(t)) = \{x \in \Omega \mid w(x, t) = 0\}$$

is a hypersurface. The first term on the left hand-side of (10) can be rewritten as

$$\int\int_{Q_T} w\zeta_t dxdt = \int_0^T \left\{\int_{\Omega_1(t)} u_1 \zeta_t dx - \int_{\Omega_2(t)} u_2 \zeta_t dx\right\} dt.$$

Recalling $u_1 \in L^2(0,T;H^1(\Omega))$ and $u_1 u_2 \equiv 0$, we see that $u_1 = 0$ on $\Gamma(t)$, which leads us to

$$\left[\int_{\Omega_1(t)} u_1 \zeta dx\right]_{t=0}^{t=T} = \int_0^T \frac{d}{dt}\left(\int_{\Omega_1(t)} u_1 \zeta dx\right) dt = \int_0^T \int_{\Omega_1(t)} (u_1 \zeta_t + u_{1,t}\zeta) dxdt.$$

By using the normal velocity V_n of $\Gamma(t)$ from $\Omega_1(t)$ to $\Omega_2(t)$, we obtain

$$\left[\int_{\Omega_2(t)} u_2 \zeta dx\right]_{t=0}^{t=T} = \int_0^T \int_{\Omega_2(t)} (u_2 \zeta_t + u_{2,t}\zeta) dxdt - \int_0^T \int_{\Gamma(t)} V_n u_2 \zeta d\sigma dt.$$

Taking the test function ζ which vanishes at $t = 0, T$ yields

$$-\iint_{Q_T} w\zeta_t dxdt = \int_0^T\int_{\Omega_1(t)} u_{1,t}\zeta dxdt - \int_0^T\int_{\Omega_2(t)} u_{2,t}\zeta dxdt + \int_0^T\int_{\Gamma(t)} V_n u_2 \zeta d\sigma dt. \tag{12}$$

Next let us consider the second term of (10). The unit normal vector on $\Gamma(t)$ oriented from $\Omega_1(t)$ to $\Omega_2(t)$ is denoted by n. We have

$$\begin{aligned} &\int\int_{Q_T} \nabla d(w) \cdot \nabla \zeta dxdt \\ &= \int_0^T \int_{\Omega_1(t)} \nabla u_1 \cdot \nabla \zeta dxdt \\ &= \int_0^T \int_{\Gamma(t)} \frac{\partial u_1}{\partial n} \zeta d\sigma dt + \int_0^T \int_{\partial\Omega} \frac{\partial u_1}{\partial \nu} \zeta d\sigma dt - \int_0^T \int_{\Omega_1(t)} \Delta u_1 \, \zeta dxdt. \end{aligned} \tag{13}$$

Summarizing (10), (12) and (13), we get

$$\begin{aligned} &\int_0^T \int_{\Gamma(t)} \left\{ V_n u_2 + \frac{\partial u_1}{\partial n} \right\} \zeta d\sigma dt + \int_0^T \int_{\Omega_1(t)} \{u_{1,t} - \Delta u_1\} \zeta dxdt \\ &\quad - \int_0^T \int_{\Omega_2(t)} u_{2,t} \zeta dxdt + \int_0^T \int_{\partial\Omega} \frac{\partial u_1}{\partial \nu} \zeta d\sigma dt = 0 \end{aligned}$$

for all $\zeta \in C^\infty(\overline{Q_T})$ such that $\zeta(x,0) = \zeta(x,T) = 0$. Since the test functions are arbitrary, we deduce

$$
\begin{cases}
u_{1,t} = \Delta u_1 & \text{in } Q_1 := \bigcup_{t\in[0,T]} \Omega_1(t) \times \{t\}, \\
u_1 = 0 & \text{on } \Gamma := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}, \\
u_2 V_n = -\dfrac{\partial u_1}{\partial n} & \text{on } \Gamma, \\
\dfrac{\partial u_1}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,T]
\end{cases}
\tag{14}
$$

and $u_2(x,t) = u_2(x,0)$ in $Q_2 := \bigcup_{t\in[0,T]} \Omega_2(t) \times \{t\}$. Namely, the strong form of (11) is the classical one-phase Stefan problem (14).

We remark that Hilhorst, Hout and Peletier [9, 10] and Eymard, Hilhorst, Hout and Peletier [7] extended this study to the following system:

$$
\begin{cases}
u_{1,t} = \Delta\phi(u_1) - \dfrac{1}{\varepsilon}F(u_1,u_2) \\
u_{2,t} = -\dfrac{1}{\varepsilon}F(u_1,u_2)
\end{cases}
\tag{15}
$$

where ϕ is a nondecreasing smooth function and F is smooth and nondecreasing in u_j (see [7, 9, 10] for the details). The limit problem of (15) is also described by the one-phase Stefan problem, provided that $\phi(u) = u$ and $F(u,v) = u^{p_1}v^{p_2}$ where $p_1, p_2 \geq 1$.

2.2 Competition-diffusion systems and Stefan problems

The previous example can be regarded as a special case of the Lotka-Volterra competition-diffusion system (7) introduced in Sect. 1. In (7), u_1, u_2 correspond to the population densities of competing two species, a_1, a_2 are intraspecific competition rates; s_1/ε, s_2/ε are interspecific competition rates. The situation where $\varepsilon \to +0$ means that the interspecific competition rates are so large. By strong competition, two species can hardly coexist and the exclusion principle works. Namely, the habitats of two species are separated from each other (see Fig. 1). Then we encounter the following question

How do the separated habitats develop in time ?

To answer this question, we first specify the initial condition of (7):

$$
u_1(x,0) = u_{01}(x), \quad u_2(x,0) = u_{02}(x), \qquad x \in \Omega,
$$

where

$$
\begin{aligned}
& u_{01}, u_{02} \in C(\overline{\Omega}), \quad u_{01} \geq 0, \quad u_{02} \geq 0, \\
& \{x \in \Omega \mid u_{01} > 0\} \cap \{x \in \Omega \mid u_{02} > 0\} = \emptyset
\end{aligned}
$$

and the set where $u_{01} = u_{02} = 0$ is a $N - 1$ dimensional hypersurface. We can relax these assumptions, but we assume the above conditions for simplicity.

Theorem 1 (Dancer, Hilhorst, Mimura and Peletier [6, Lemmas 3.1, 3.4]) *For any* $T > 0$, *set* $Q_T := \Omega \times (0, T)$. *Let* $(u_1^\varepsilon, u_2^\varepsilon)$ *be a solution of* (7). *Then*

$$u_1^\varepsilon \longrightarrow u_1 := s_1 w^+, \quad u_2^\varepsilon \longrightarrow u_2 := s_2 w^-$$

strongly in $L^2(Q_T)$ *as* $\varepsilon \to +0$. *Moreover, w is a weak solution of*

$$\begin{cases} w_t = \Delta d(w) + h(w) & \text{in } \ \Omega \times \mathbb{R}_+, \\ \dfrac{\partial d(w)}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times \mathbb{R}_+, \\ w(\cdot, 0) = w_0 := \dfrac{u_{01}}{s_1} - \dfrac{u_{02}}{s_2} & \text{in } \ \Omega, \end{cases} \tag{16}$$

where

$$d(r) := d_1 r^+ - d_2 r^-, \quad h(r) = \frac{f(s_1 r^+)}{s_1} - \frac{g(s_2 r^-)}{s_2}. \tag{17}$$

See [5] for the corresponding results under the Dirichlet boundary condition:

$$u_1 = m_1^\varepsilon, \quad u_2 = m_2^\varepsilon, \qquad x \in \partial\Omega, \ t > 0$$

instead of the Neumann boundary condition. Note that the limit problem (16) consists of one component $w = u_1/s_1 - u_2/s_2$, though the original reaction–diffusion system (7) consists of two components $u_1^\varepsilon, u_2^\varepsilon$. In the limit problem, a set $\{x \in \Omega \mid w(x, t) = 0\}$ forms an interface which divides Ω into the regions $\{x \in \Omega \mid u_j(x, t) > 0\}$ $(j = 1, 2)$. We notice that the interface develops in time and that (16) becomes a free boundary problem. If the weak solution and the interface are sufficiently smooth, we can integrate the equation by parts in the argument similar to the previous subsection and we obtain the free boundary problem

$$\begin{cases} u_{1,t} = d_1 \Delta u_1 + f(u_1) & \text{in } \ Q_1 := \bigcup_{t \in [0,T]} \Omega_1(t) \times \{t\}, \\ u_{2,t} = d_2 \Delta u_2 + g(u_2) & \text{in } \ Q_2 := \bigcup_{t \in [0,T]} \Omega_2(t) \times \{t\}, \\ u_1 = u_2 = 0 & \text{on } \ \Gamma := \bigcup_{t \in [0,T]} \Gamma(t) \times \{t\}, \\ \dfrac{d_1}{s_1} \dfrac{\partial u_1}{\partial n} = -\dfrac{d_2}{s_2} \dfrac{\partial u_2}{\partial n} & \text{on } \ \Gamma, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times [0, T], \end{cases} \tag{18}$$

and the initial conditions

$$\frac{u_1(x,0)}{s_1} = \left(\frac{u_{01}(x)}{s_1} - \frac{u_{02}(x)}{s_2}\right)^+, \quad \frac{u_2(x,0)}{s_2} = \left(\frac{u_{01}(x)}{s_1} - \frac{u_{02}(x)}{s_2}\right)^-, \qquad x \in \Omega.$$

(see [6] for details). Note that both u_1 and u_2 vanish at the interface $\Gamma(t)$ and are continuous at $\Gamma(t)$, while their derivatives $\partial u_1/\partial n$, $\partial u_2/\partial n$ become discontinuous at the interface. The layer of this type is called a *corner layer*. This limit problem is a sort of two-phase Stefan problems, though the latent heat vanishes. Therefore we encounter the following natural question: *are there any reaction–diffusion approximation to a two-phase Stefan problem with positive latent heat ?*

To answer this question, we introduce a weak form of two-phase Stefan problem with positive latent heat first. The two-phase Stefan problem with positive latent heat becomes

$$\begin{cases} w_t = \Delta d(\varphi_\lambda(w)) + h(\varphi_\lambda(w)) & \text{in } \Omega \times \mathbb{R}_+, \\ \dfrac{\partial d(\varphi_\lambda(w))}{\partial \nu} = 0 & \text{on } \partial\Omega \times \mathbb{R}_+, \\ w(\cdot,0) = \dfrac{u_{01}}{s_1} - \dfrac{u_{02}}{s_2} + \lambda H\Big(\dfrac{u_{01}}{s_1} - \dfrac{u_{02}}{s_2}\Big) & \text{in } \Omega, \end{cases} \tag{19}$$

where $d(r)$, $h(r)$ are given by (17), H is the Heaviside function and

$$\varphi_\lambda(r) = (r - \lambda)^+ - r^-.$$

Let us return to the previous question. Introducing the new variable u_3^ε which characterizes the habitat of u_1^ε, we propose the following three-component reaction–diffusion system:

$$\begin{cases} u_{1,t} = d_1 \Delta u_1 + f(u_1) - \dfrac{s_1}{\varepsilon} u_1 u_2 - \dfrac{\lambda s_1}{\varepsilon}(1 - u_3)u_1, & (x,t) \in \Omega \times \mathbb{R}_+, \\ u_{2,t} = d_2 \Delta u_2 + g(u_2) - \dfrac{s_2}{\varepsilon} u_1 u_2 - \dfrac{\lambda s_2}{\varepsilon} u_3 u_2, & (x,t) \in \Omega \times \mathbb{R}_+, \\ u_{3,t} = \dfrac{1}{\varepsilon}(1 - u_3)u_1 - \dfrac{1}{\varepsilon} u_3 u_2, & (x,t) \in \Omega \times \mathbb{R}_+, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u_1(x,0) = u_{01}(x),\ u_2(x,0) = u_{02}(x),\ u_3(x,0) = H\Big(\dfrac{u_{01}}{s_1} - \dfrac{u_{02}}{s_2}\Big), & x \in \Omega, \end{cases} \tag{20}$$

where λ is a non-negative constant which corresponds to a coefficient of the latent. Under the same assumption as in Theorem 1 for the initial data besides $0 \le u_{03} \le 1$, the set $\{(u_1, u_2, u_3) \mid u_1 \ge 0,\ u_2 \ge 0,\ 0 \le u_3 \le 1\}$ becomes an invariant region by the flow of (20).

Theorem 2 (Hilhorst, Iida, Mimura and Ninomiya [11, Theorem 3.6][12, Theorem 1.1]) *Let* $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ *be a solution of* (20). *Then as* $\varepsilon \to +0$,

$$u_1^\varepsilon \longrightarrow u_1, \quad u_2^\varepsilon \longrightarrow u_2, \quad u_3^\varepsilon \longrightarrow u_3$$

strongly in $L^2(Q_T)$. *Moreover,*

$$u_1 = s_1(w-\lambda)^+, \quad u_2 = s_2 w^-, \quad u_3 = \frac{w - \varphi_\lambda(w)}{\lambda}$$

where w *is a weak solution of* (19).

Theorem 1 is the special case of Theorem 2 when $\lambda = 0$. Theorem 2 is shown by introducing $w^\varepsilon = u_1^\varepsilon/s_1 - u_2^\varepsilon/s_2 + \lambda u_3^\varepsilon$. By the similar argument for (11) and (16), (19) is a free boundary problem. By rewriting (19) into the strong form, the equation on the free boundary is explicitly derived. We denote the region where $\varphi_\lambda(w(\cdot,t)) > 0$ and the region where $\varphi_\lambda(w(\cdot,t)) < 0$ by $\Omega_1(t)$, and $\Omega_2(t)$ respectively. Set

$$\Gamma(t) = \Omega \backslash (\Omega_1(t) \cup \Omega_2(t)) = \{x \in \Omega \mid 0 \le w(x,t) \le \lambda\}.$$

We also assume that the weak solution of (19) and the interface $\Gamma(t)$ are sufficiently smooth. Then the limit functions u_1 and u_2 satisfy the following two-phase Stefan problem:

$$\begin{cases} u_{1,t} = d_1 \Delta u_1 + f(u_1) & \text{in } Q_1 := \bigcup_{t\in[0,T]} \Omega_1(t) \times \{t\}, \\ u_{2,t} = d_2 \Delta u_2 + g(u_2) & \text{in } Q_2 := \bigcup_{t\in[0,T]} \Omega_2(t) \times \{t\}, \\ u_1 = u_2 = 0 & \text{on } \Gamma := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}, \\ \lambda V_n = -\dfrac{d_1}{s_1}\dfrac{\partial u_1}{\partial n} - \dfrac{d_2}{s_2}\dfrac{\partial u_2}{\partial n} & \text{on } \Gamma, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = 0 & \text{on } \partial\Omega \times [0,T], \end{cases} \tag{21}$$

where V_n is a normal velocity of $\Gamma(t)$ from $\Omega_1(t)$ to $\Omega_2(t)$ (see also [31, 32]). Moreover,

$$\Gamma(0) = \left\{ x \in \Omega \,\middle|\, \frac{u_{01}(x)}{s_1} = \frac{u_{02}(x)}{s_2} \right\}.$$

While u_3^ε has a transition layer at the interface, u_1^ε and u_2^ε have a corner layer at the interface and the corresponding limit functions u_1, u_2 are continuous. The function u_3^ε represents a characteristic function of a territory of the species u_1^ε. When u_2^ε (resp. u_1^ε) invades the territory of the other species u_1^ε (resp. u_2^ε), the cost required between u_3^ε (resp. $1 - u_3^\varepsilon$) and u_2^ε (resp. u_1^ε) induces a latent heat λ.

Murakawa [35, 36] obtained the rate of the convergence in Theorems 1 and 2 as follows.

Theorem 3 (Murakawa [35, 36]) *Let* $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ *be a solution of* (20) *with the initial datum* (u_{01}, u_{02}, u_{03}) *and set* $w^\varepsilon := u_1^\varepsilon/s_1 - u_2^\varepsilon/s_2 + \lambda u_3^\varepsilon$. *When* $\lambda = 0$, *let* $(u_1^\varepsilon, u_2^\varepsilon)$ *be a solution of* (7) *with the initial datum* (u_{01}, u_{02}) *and set* $w^\varepsilon := u_1^\varepsilon/s_1 - u_2^\varepsilon/s_2$. *Then there is a positive constant* C *independent of* λ *such that*

$$\begin{aligned}&\|u_1^\varepsilon - u_1\|_{L^2(Q_T)} + \|u_2^\varepsilon - u_2\|_{L^2(Q_T)} + \|w^\varepsilon - w\|_{L^\infty(0,T;(H^1(\Omega))^*)}\\ &\quad\le C\Big(\varepsilon^{1/4} + \|w^\varepsilon(\cdot,0) - w(\cdot,0)\|_{L^2(\Omega)}\Big).\end{aligned}$$

We refer to Iida, Nakashima and Yanagida [17] for the further study of asymptotic expansions of solutions to (7).

2.3 Cross-diffusion system

In this subsection, let us consider the relation between microscopic models and macroscopic models from the viewpoints of reaction–diffusion approximation. If the individuals move randomly, the probability density is ruled by a diffusion equation. If the competition among two species is also considered, the competition-diffusion system (6) is often used. However, it is not necessary that the individual moves randomly and the movement may be influenced by the environment or the distributions of the other species. For example, the transition probabilities in the movements of bio-individuals often depend on the light intensity, the density of the nutrient and the heat [23]. The system (6) is not suitable any more under this situation. To propose an appropriate model, we classify the transition probabilities of individuals which move to a next point on a lattice into three types:

(i) neutral transition: the transition probabilities between adjacent points on a lattice are assumed to be equal, where the probability density $p = p(x,t)$ that an individual exists at a position x at a time t satisfies

$$p_t = \nabla(D(x)\nabla p);$$

(ii) repulsive transition: the transition probabilities depend only on conditions at the point of departure, where p satisfies

$$p_t = \Delta(D(x)p);$$

(iii) attractive transition: the transition probabilities depend on conditions at the point of arrival, where p satisfies

$$p_t = \nabla\left[D(x)^2\nabla\left(\frac{p}{D(x)}\right)\right].$$

These equations for p are derived from the continuum limit of transition probabilities under a suitable spatial and temporal scaling. Since the sum of probability densities p of respective individuals is a population density and the equation is linear, the population density n satisfies the same equation as the probability density p. See Okubo and Levine [44, §5.4] for the details. It is natural that transition probabilities in the movements of bio-individuals depend on the point of departure. Then, following (ii), we get the system (6) with $d_1\Delta u_1$ and $d_2\Delta u_2$ replaced by $\Delta[D_1(x)u_1]$ and $\Delta[D_2(x)u_2]$ respectively where $D_i(x)$ is a motility of u_i at x (see Turchin [46, §4.2.3]). Namely, where $D_i(x)$ is large, u_i has a tendency to move faster [44]. Shigesada, Kawasaki and Teramoto [45] explained the spatial segregation of competing two species by using the environmental pressure and by replacing $D_i(x)$ with the function which depends not on the position but on the density of the species. As a special case, they introduced the following system:

$$\begin{cases} w_{1,t} &= \Delta[(d_1+\alpha w_2)w_1] + (r_1 - a_1 w_1 - b_1 w_2)w_1, \qquad x\in\Omega, t>0, \\ w_{2,t} &= \Delta[(d_2+\beta w_1)w_2] + (r_2 - b_2 w_1 - a_2 w_2)w_2, \qquad x\in\Omega, t>0 \end{cases} \tag{22}$$

where w_1 and w_2 are the population densities of species. This system is called a *cross-diffusion system* for competing two species where α, β are non-negative constants which represent the pressures by cross-diffusivity. Therefore α and β are called *cross-diffusion coefficients*. Because (22) is a quasilinear parabolic system, the local existence of a solution follows from Amann [1]. The global existence of solutions are also studied under several assumptions. We cite Lou, Ni and Wu [26] and Choi, Lui and Yamada [4] for instance. From the biological viewpoint, it is important whether there exist inhomogeneous stable stationary solutions of (22) or not, because they correspond to spatial segregation. There are three mathematical methods to treat them: bifurcation theory [29], singular limit method [21, 27, 30], the theory of elliptic equations [25]. From these studies, we see that the structure of stationary solutions of (22) deeply depends on the parameters and is complicated due to the cross-diffusivity.

Let us consider the cross-diffusivity of (22) from the viewpoint of reaction–diffusion approximations (see also [2, 3, 14]). To do so, first, we consider a simple example: the distribution of species whose transition probability depends only on the departure place in inhomogeneous media. Let $V(x)$ $(0 \le V(x) \le 1)$ indicate a spatial inhomogeneity. From (ii) it follows that the population density n satisfies

$$\begin{cases} n_t = \Delta\Big[\Big(d+\alpha V(x)\Big)n\Big], & x\in\Omega,\ t>0, \\ \dfrac{\partial n}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0, \\ n(0,x) = n_0(x), & x\in\Omega, \end{cases} \tag{23}$$

where $d + \alpha V(x)$ represents the motility at x (see [44, 45]). Because

$$(d+\alpha V(x))n = d\cdot\{1-V(x)\}n + (d+\alpha)\cdot V(x)n,$$

we can split n into two parts: $n_1 := \{1 - V(x)\}n$ and $n_2 := V(x)n$. Then, (23) is rewritten into

$$(n_1+n_2)_t = \Delta\big[dn_1 + (d+\alpha)n_2\big].$$

This means that n ruled by (23) is represented by the sum of n_1 with motility d and n_2 with motility $d+\alpha$. Namely, n_1 (resp. n_2) corresponds to the density of the inactive (resp. active) individuals. Further we assume that each individual switches its activity depending on the environment: from the active state to the inactive one and vice versa. Here we can regard $V(x)$ as the probability that each individual has become in the active state. To derive the respective equations for n_1 and n_2, we introduce the switching rate between an inactive state and an active state. More precisely, the switching rate from the active state to the inactive one is denoted by $h_{2\to1}(x)$ and that from the inactive state to the active one is denoted by $h_{1\to2}(x)$. Assuming that each individual switches its state quickly, we derive the system for n_1 and n_2 as follows:

$$\begin{cases} n_{1,t} &= d\Delta n_1 + \dfrac{1}{\varepsilon}[h_{2\to1}(x)n_2 - h_{1\to2}(x)n_1], \\ n_{2,t} &= (d+\alpha)\Delta n_2 + \dfrac{1}{\varepsilon}[h_{1\to2}(x)n_1 - h_{2\to1}(x)n_2] \end{cases} \tag{24}$$

with a small parameter $\varepsilon > 0$. To determine $h_{1\to2}(x)$ and $h_{2\to1}(x)$ from $V(x)$, let us consider the limit problem. Adding two equations of (24) and using $n = n_1 + n_2$ yield

$$\begin{aligned} n_t =& \Delta\left[dn_1 + (d+\alpha)n_2\right], \\ n_{2,t} =& (d+\alpha)\Delta n_2 + \frac{1}{\varepsilon}\left[h_{1\to2}(x)n - \Big(h_{1\to2}(x) + h_{2\to1}(x)\Big)n_2\right]. \end{aligned}$$

By taking the limit $\varepsilon \to +0$ in the second equation, we get

$$n_1 = \frac{h_{2\to1}(x)}{h_{1\to2}(x) + h_{2\to1}(x)}n, \qquad n_2 = \frac{h_{1\to2}(x)}{h_{1\to2}(x) + h_{2\to1}(x)}n$$

formally. Therefore, we obtain the limiting equation:

$$n_t = \Delta\left[\left(d + \alpha\frac{h_{1\to2}(x)}{h_{1\to2}(x) + h_{2\to1}(x)}\right)n\right].$$

Hence, if $V(x) = h_{1\to2}(x)/\{h_{1\to2}(x) + h_{2\to1}(x)\}$, then the limit equation is equal to (23). This formal derivation implies that the transition probability depending only on the departure points can be represented by the random walk and the switching between the incative state and the active state.

Let us apply this method to (22). For simplicity, we assume $\beta = 0$. We split one species represented by w_1 into inactive individuals and active ones, and the densities of the inactive and active individuals are denoted by $u_1(x,t)$ and $u_2(x,t)$ respectively. We also denote w_2 by $u_3(x,t)$. Let us construct the reaction–diffusion system for (u_1, u_2, u_3) which will approximate (w_1, w_2) by using the above idea. Assume the boundedness of solutions:

$$0 \le w_1(x,t) \le M_1, \quad 0 \le w_2(x,t) \le M_2, \qquad x \in \Omega,\ 0 \le t \le T, \tag{25}$$

which immediately implies

$$d_1 \leq d_1 + \alpha w_2(x,t) \leq d_1 + \alpha M_2.$$

Replacing n, n_1, n_2, d, $d+\alpha$, V by w_1, u_1, u_2, d_1, $d_1 + \alpha M_2$, w_2/M_2 respectively and assuming that the switching possibilities $h_{1\to 2}$, $h_{2\to 1}$ is a function of $u_3(x,t)$, we obtain the following reaction–diffusion system:

$$\begin{cases} u_{1,t} &= d_1 \Delta u_1 + f_1(u_1,u_2,u_3) + \dfrac{1}{\varepsilon}[h_{2\to 1}(u_3)u_2 - h_{1\to 2}(u_3)u_1], \\ u_{2,t} &= (d_1 + \alpha M_2)\Delta u_2 + f_2(u_1,u_2,u_3) + \dfrac{1}{\varepsilon}[h_{1\to 2}(u_3)u_1 - h_{2\to 1}(u_3)u_2], \\ u_{3,t} &= d_2 \Delta u_3 + f_3(u_1,u_2,u_3), \qquad x \in \Omega,\ t > 0, \end{cases} \tag{26}$$

where

$$\begin{cases} f_1(u_1,u_2,u_3) &= [r_1 - a_1(u_1+u_2) - b_1 u_3]u_1, \\ f_2(u_1,u_2,u_3) &= [r_1 - a_1(u_1+u_2) - b_1 u_3]u_2, \\ f_3(u_1,u_2,u_3) &= [r_2 - b_2(u_1+u_2) - a_2 u_3]u_3. \end{cases}$$

Especially, if we take

$$h_{2\to 1}(s) = 1 - \frac{s}{M_2}, \quad h_{1\to 2}(s) = \frac{s}{M_2},$$

it is expected that (26) is an approximation of a cross-diffusion system

$$\begin{cases} w_{1,t} &= \Delta[(d_1 + \alpha w_2)w_1] + (r_1 - a_1 w_1 - b_1 w_2)w_1, & x \in \Omega,\ t > 0, \\ w_{2,t} &= d_2 \Delta w_2 + (r_2 - b_2 w_1 - a_2 w_2)w_2, & x \in \Omega,\ t > 0, \\ \dfrac{\partial w_1}{\partial \nu} &= \dfrac{\partial w_2}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w_1(x,0) &= w_{01}(x), \qquad w_2(x,0) = w_{02}(x), & x \in \Omega. \end{cases} \tag{27}$$

Next we investigate the conditions for $h_{1\to 2}$ and $h_{2\to 1}$. The approximation of the cross-diffusion system requires the condition

$$h_{1\to 2}(s) \equiv \Big(h_{1\to 2}(s) + h_{2\to 1}(s)\Big)\frac{s}{M_2}, \qquad s \in [0, M_2] \tag{28}$$

only. Namely, the ratio $h_{2\to 1}(s)/h_{1\to 2}(s)$ is uniquely determined by (28), but there are many degrees of freedom. For example,

(i) $\quad h_{1\to 2}(u_3) := \dfrac{u_3}{M_2},\ \ h_{2\to 1}(u_3) := 1 - \dfrac{u_3}{M_2},$

$$\text{(ii)} \qquad h_{1\to 2}(u_3) := \frac{u_3}{M_2+u_3}, \quad h_{2\to 1}(u_3) := \frac{M_2-u_3}{M_2+u_3}.$$

Moreover, if we take

$$\text{(iii)} \qquad h_{1\to 2}(u_3) := \frac{\varphi(u_3)}{1+\varphi(u_3)}, \quad h_{2\to 1}(u_3) := \frac{1}{1+\varphi(u_3)},$$

$$\text{(iv)} \qquad h_{1\to 2}(u_3) := \varphi(u_3), \quad h_{2\to 1}(u_3) := 1,$$

then both of their limit systems become

$$\begin{cases} w_{1,t} &= \Delta\left[\left(d_1 + \dfrac{\alpha M_2 \varphi(w_2)}{1+\varphi(w_2)}\right) w_1\right] + (r_1 - a_1 w_1 - b_1 w_2) w_1, \qquad x \in \Omega,\ t > 0, \\ w_{2,t} &= d_2 \Delta w_2 + (r_2 - b_2 w_1 - a_2 w_2) w_2, \qquad x \in \Omega,\ t > 0. \end{cases}$$

If we take $\varphi(s) = s/M_2$, then this cross-diffusion system is a good approximation of (27) with small w_2. The above system might be regarded as the natural introduction of the cross-diffusion mechanism.

Though the above calculations are formal, it can be confirmed rigorously that the solution of (26) is an approximation of the solution of (27).

Theorem 4 (Iida and Ninomiya [18, Theorem 1.1], Iida, Mimura and Ninomiya [15, Theorem 1]) *Assume that the solution* (w_1, w_2) *of* (27) *starting from* (w_{01}, w_{02}) *is smooth on* $\overline{\Omega} \times [0, T]$, *there are positive constants* $M_1(\geq r_1/a_1)$, $M_2(\geq r_2/a_2)$ *satisfying* (25) *and that* $h_{1\to 2}$,$h_{2\to 1}$ *satisfy* (28) *and*

$$h_{1\to 2}(s) \geq 0, \quad h_{2\to 1}(s) \geq 0, \quad h_{1\to 2}(s) + h_{2\to 1}(s) > 0 \tag{29}$$

for any $s \in [0, M_2]$ and the initial data (u_{01}, u_{02}, u_{03}) of (26) satisfies

$$u_{01}(x) \equiv \left\{1 - \frac{w_{02}(x)}{M_2}\right\} w_{01}(x), \quad u_{02}(x) \equiv \frac{w_{02}(x)}{M_2} w_{01}(x), \quad u_{03}(x) \equiv w_{02}(x)$$

in Ω. If there are positive constants ε_0 and M_0 such that the solution $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ of (26) satisfies

$$\sup_{\overline{\Omega}\times[0,T]} \left(|u_1^\varepsilon(x,t)| + |u_2^\varepsilon(x,t)| + |u_3^\varepsilon(x,t)|\right) \leq M_0 \tag{30}$$

for $0 < \varepsilon \leq \varepsilon_0$, then there is a positive constant $C = C(w_1, w_2, \varepsilon_0, M_0, T)$ independent of ε satisfying

$$\begin{cases} \sup_{t\in[0,T]} \|u_1^\varepsilon(\cdot,t)+u_2^\varepsilon(\cdot,t)-w_1(\cdot,t)\|_{L^2(\Omega)} \le C\varepsilon, \\ \sup_{t\in[0,T]} \|u_3^\varepsilon(\cdot,t)-w_2(\cdot,t)\|_{L^2(\Omega)} \le C\varepsilon. \end{cases}$$

This theorem implies that the complicated biological diffusion such as cross-diffusion can be composed of the regular random walk and the reaction. The proof is essentially based on [15, 18] extends the result of [18] to more general system. Though neither systems (27) nor (26) seem to possess Lyapunov functions, the theorem is shown by the construction of the energy functional. Moreover, instead of the boundedness (30), it is possible to assume the uniform Lipschitz continuity. Murakawa [35] guaranteed an approximation of cross-diffusion system with both positive cross-diffusivities by assuming the uniform Lipschitz continuity of the growth terms in (27).

Next we introduce the result on the stationary solutions.

Theorem 5 (Izuhara and Mimura [20, Theorem 2]) *Let* (w_1, w_2) *be a smooth stationary solution of* (27). *Assume that the linearized operator for the right-hand sides of* (27) *near* (w_1, w_2) *is bijective from* $(W^{2,p}(\Omega))^2$ *to* $(L^p(\Omega))^2$. *Then, there are positive constants* ε_0 *and* C *such that there is a unique stationary solution* $(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon)$ *of* (26) *satisfying*

$$\|w_1-(u_1^\varepsilon+u_2^\varepsilon)\|_{W^{2,p}(\Omega)}+\|w_2-u_3^\varepsilon\|_{W^{2,p}(\Omega)} \le C\varepsilon$$

for $0<\varepsilon<\varepsilon_0$.

2.4 Reaction limit sets

First we recall the reaction–diffusion system (4) and denote the solution by $\boldsymbol{u}^\varepsilon$. Assume that there is a $m\times m$ matrix $\boldsymbol{P}$ with a rank $m_0(<m)$ such that

$$\boldsymbol{PG}(\boldsymbol{u})=0 \qquad \text{for } \boldsymbol{u}\in\mathbb{R}^m$$

and that the reaction limit set $\mathcal{A}$ for (4) is represented by a graph of a function $\boldsymbol{h}$ from a subset U of $\boldsymbol{P}\mathbb{R}^m$ to $\mathbb{R}^m$. Usualy we take $U=\boldsymbol{P}\overline{\mathbb{R}_+^m}$ or $\boldsymbol{P}\mathbb{R}^m$. Then

$$\mathcal{A}=\{\boldsymbol{u}=\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w})\in\mathbb{R}^m \mid \boldsymbol{w}\in U\subset\boldsymbol{P}\mathbb{R}^m\cong\mathbb{R}^{m_0}\}.$$

As seen in (5), we can expect that $\lim_{\varepsilon\to+0}\boldsymbol{u}^\varepsilon$ stays in $\mathcal{A}$, that is,

$$\lim_{\varepsilon\to+0}\boldsymbol{u}^\varepsilon=\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w}), \qquad \boldsymbol{w}=\boldsymbol{P}(\lim_{\varepsilon\to+0}\boldsymbol{u}^\varepsilon).$$

Therefore, operating $\boldsymbol{P}$ to (4) and taking the limit $\varepsilon\to+0$, we formally get

$$\boldsymbol{w}_t=\Delta\boldsymbol{PD}(\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w}))+\boldsymbol{PF}(\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w})), \qquad x\in\Omega,\ t>0. \tag{31}$$

This equation is the formal limit problem of (4) as $\varepsilon \to +0$. The Eq. (31) is a partial differential equation for $\boldsymbol{w} \in U \subset \boldsymbol{P}\mathbb{R}^m \cong \mathbb{R}^{m_0}$.

We revert to (9). The corresponding reaction limit set is

$$\mathcal{A}_2 = \{(u_1, 0) \mid u_1 \geq 0\} \cup \{(0, u_2) \mid u_2 \geq 0\}.$$

See Fig. 2. It can obviously be regarded as a one-dimensional graph. For $u_1 := \lim_{\varepsilon\to+0} u_1^\varepsilon$, $u_2 := \lim_{\varepsilon\to+0} u_2^\varepsilon$, we set $w = u_1 - u_2$. In this case,

$$\boldsymbol{D} = \begin{pmatrix} d_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{F}(\boldsymbol{u}) \equiv \boldsymbol{0}.$$

Set

$$\boldsymbol{P} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \; \boldsymbol{w} = \frac{w}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; \boldsymbol{h}(\boldsymbol{w}) = \frac{|w|}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then the rank m_0 of the matrix $\boldsymbol{P}$ is equal to 1 and

$$\boldsymbol{PD}(\boldsymbol{w} + \boldsymbol{h}(\boldsymbol{w})) = \frac{d(w)}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; \boldsymbol{w} + \boldsymbol{h}(\boldsymbol{w}) = \begin{pmatrix} w^+ \\ w^- \end{pmatrix}.$$

The limit problem (31) as $\varepsilon \to +0$ can be rewritten as

$$\frac{w_t}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\Delta d(w)}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

where $d(r) := d_1 r^+$ Hence we can derive the limit problem

$$w_t = \Delta d(w) \qquad \text{in } \; \Omega \times \mathbb{R}_+. \tag{32}$$

The reaction limit set $\mathcal{A}_1$ for (7) is the same as $\mathcal{A}_2$. We omit the explanation for (7) because we can treat it similarly to the case of (20).

Next we consider (20). The reaction limit set for (20) is given by

$$\mathcal{A}_3 = \{(u_1, 0, 1) \mid u_1 \geq 0\} \cup \{(0, 0, u_3) \mid 0 \leq u_3 \leq 1\} \cup \{(0, u_2, 0) \mid u_2 \geq 0\},$$

which can be also regarded as a one-dimensional graph (see Fig. 3). We see that

$$\boldsymbol{D} = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} f(u_1) \\ g(u_2) \\ 0 \end{pmatrix}, \; m_0 = 1.$$

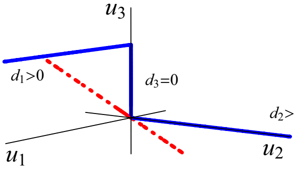


Fig. 3 Reaction limit sets $\mathcal{A}_3$ for (20) (solid segments: reaction limit sets $\mathcal{A}_3$, dashed line: the projection $\boldsymbol{P}\mathcal{A}_3$)

Set $w = u_1/s_1 - u_2/s_2 + \lambda u_3$ and

$$\boldsymbol{P} = \frac{1}{2}\begin{pmatrix} 1 & -\dfrac{s_1}{s_2} & s_1\lambda \\ -\dfrac{s_2}{s_1} & 1 & -s_2\lambda \\ \dfrac{1}{s_1\lambda} & -\dfrac{1}{s_2\lambda} & 1 \end{pmatrix}, \quad \boldsymbol{w} = \frac{w}{2}\begin{pmatrix} s_1 \\ -s_2 \\ \dfrac{1}{\lambda} \end{pmatrix},$$

$$\boldsymbol{h}(\boldsymbol{w}) = \begin{pmatrix} \dfrac{s_1(|w-\lambda|-\lambda)}{2} \\ \dfrac{s_2|w|}{2} \\ \dfrac{|w|-w+\lambda-|w-\lambda|}{2\lambda} \end{pmatrix}.$$

Then we can obtain

$$\boldsymbol{PD}(\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w})) = \frac{d(\varphi_\lambda(w))}{2}\begin{pmatrix} s_1 \\ -s_2 \\ \dfrac{1}{\lambda} \end{pmatrix}, \quad \boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w}) = \begin{pmatrix} s_1(w-\lambda)^+ \\ s_2 w^- \\ \dfrac{w-(w-\lambda)^+ + w^-}{\lambda} \end{pmatrix}.$$

Therefore the limit problem (31) becomes (19) and is regarded as the two-phase Stefan problem with the latent heat λ.

In the limit equation (31) for $\boldsymbol{w}$ on the reaction limit set $\mathcal{A}$, the diffusion term includes the function $\boldsymbol{h}(\boldsymbol{w})$ which represents the reaction limit set $\mathcal{A}$. As in the above three examples (7), (20), (9), the singularity appears when the derivative of $\boldsymbol{h}(\boldsymbol{w})$ is not continuous. We may say that the "corner" of $\mathcal{A}$ corresponds to the interface. For example, for (7) or (9), the set where $u_1 = u_2 = 0$ performs the interface. For (20), there are two corners at $(u_1, u_2, u_3) = (0, 0, 1)$ and $(u_1, u_2, u_3) = (0, 0, 0)$, which perform one interface. Here the derivative of $\boldsymbol{PD}(\boldsymbol{w}+\boldsymbol{h}(\boldsymbol{w}))$ with respect to $\boldsymbol{w}$ is called the diffusion coefficient $\boldsymbol{d}(\boldsymbol{w})$ of (31) on the reaction limit set. The diffusion coefficients $d_1, d_2(, d_3)$ are also shown in Figs. 2, 3. Note that the diffusion coefficient of (7) is positive in $\mathcal{A}_1$, while the diffusion coefficient of (9) (resp. (20)) vanishes for some region of $\mathcal{A}_2$ (resp. $\mathcal{A}_3$). This difference determines whether the interface is a corner layer or a transition one.

For the above three examples, the diffusion coefficients are piecewise constant, while it is not for the case of (26) which approximates the cross-diffusion system. For simplicity, let us consider the porous medium equation

$$w_t = d_1 \Delta w^k \qquad \text{in} \quad \Omega \times \mathbb{R}_+ \tag{33}$$

with $k > 1$ instead of cross-diffusion system. The reaction–diffusion approximation for (33) is studied in [35]. The reaction–diffusion approximation is

$$\begin{cases} u_{1,t} &= d_1 \Delta u_1 - \dfrac{1}{\varepsilon}\left\{u_1 - (u_1+u_2)^k\right\} \quad \text{in } \Omega \times \mathbb{R}_+, \\ u_{2,t} &= \dfrac{1}{\varepsilon}\left\{u_1 - (u_1+u_2)^k\right\} \qquad\qquad \text{in } \Omega \times \mathbb{R}_+. \end{cases} \tag{34}$$

The reaction limit set for (34) is

$$\mathcal{A}_4 = \{(u_1,u_2) \mid u_1 \geq 0,\ u_2 \geq 0,\ u_1 = (u_1+u_2)^k\}.$$

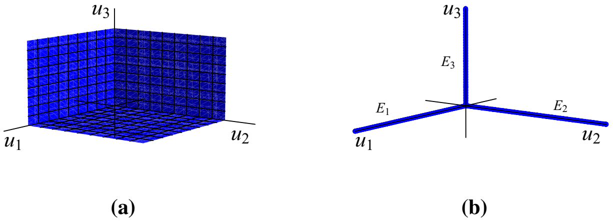
Especially, the diffusion coefficient on $\mathcal{A}_4$ is $d_1 k(u_1+u_2)^{k-1}$ and it is not piecewise constant. Though the derivative of $\boldsymbol{h}(\boldsymbol{w})$ is continuous on $\mathcal{A}_4$, it vanishes at $(u_1,u_2)=(0,0)$. This causes the appearance of the interface of (33). Note that the diffusion coefficient is continuous and non-degenerate on the reaction limit set for the reaction–diffusion approximation (26) of (27), which implies that the interface does not appear in (27).

The reaction limit set is one-dimensional for (7), (20), (9) and (34). Next we give an example where the reaction limit set is two-dimensional. Consider the reaction–diffusion system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1 + f_1(u_1,u_2,u_3) - \dfrac{1}{\varepsilon} u_1 u_2 u_3, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2 + f_2(u_1,u_2,u_3) - \dfrac{1}{\varepsilon} u_1 u_2 u_3, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_3}{\partial t} = d_3 \Delta u_3 + f_3(u_1,u_2,u_3) - \dfrac{1}{\varepsilon} u_1 u_2 u_3, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u_1}{\partial \nu} = \dfrac{\partial u_2}{\partial \nu} = \dfrac{\partial u_3}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{35}$$

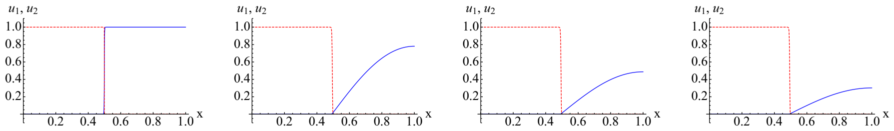
It is easily seen that the corresponding reaction limit set is

$$\begin{aligned} \mathcal{A}_5 &= \{(u_1,u_2,0) \mid u_1 \geq 0,\ u_2 \geq 0\} \cup \{(0,u_2,u_3) \mid u_2 \geq 0,\ u_3 \geq 0\} \\ &\quad \cup \{(u_1,0,u_3) \mid u_1 \geq 0,\ u_3 \geq 0\} \end{aligned}$$



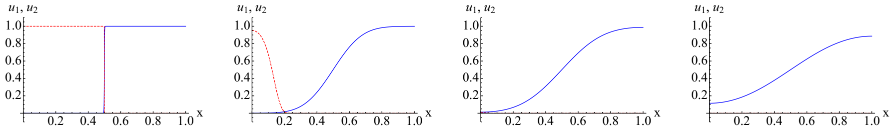
(a) **(b)**

Fig. 4 Reaction limit sets $\mathcal{A}_5$ (**a**) and $\mathcal{A}_6$ (**b**) for (35) and (36) respectively



(a) $t = 0$ **(b)** $t = 0.05$ **(c)** $t = 0.1$ **(d)** $t = 0.15$

Fig. 5 Snapshots of numerical solutions of (37) when $m_1 = m_2 = m_4 = 1$, $m_3 = 2$, $d_1 = 1$, $d_2 = 0$ and $\varepsilon = 0.00001$ (solid: u_1, dashed: u_2)



(a) $t = 0$ **(b)** $t = 0.01$ **(c)** $t = 0.02$ **(d)** $t = 0.05$

Fig. 6 Snapshots of numerical solutions of (37) when $m_1 = 4$, $m_2 = m_3 = m_4 = 1$, $d_1 = 1$, $d_2 = 0$ and $\varepsilon = 0.00001$ (solid: u_1, dashed: u_2)

(see Fig. 4a). Since this set includes points where the derivative of $\boldsymbol{h}$ is discontinuous, the interface forms. The weak form of the limit problem is derived in [37], but it is still open to determine its strong form.

There is an example where the reaction limit set is one-dimensional but cannot be represented by a graph. Indeed, consider the m-component reaction–diffusion system:

$$u_{it} = d_i \Delta u_i + f_i(u_i) - \frac{1}{\varepsilon} \sum_{j \neq i} a_{ij} u_j u_i (x \in \Omega,\ t > 0,\ i = 1, \cdots, m). \tag{36}$$

The corresponding reaction limit set is

$$\mathcal{A}_6 = \bigcup_{i=1}^{m} \{(0, \cdots, 0, u_i, 0, \cdots, 0) \mid u_i \geq 0\}.$$

For example, when $m = 3$, this set is seen as in Fig. 4b and it cannot be represented by any graph. Therefore it is difficult to obtain the limit problem. The convergence of the solution of the reaction–diffusion system including (36) has been shown in [12]. However, it is still open to derive the limit problem for (36). For the related studies, we also refer to Keller, Sternberg and Rubinstein [24] when the reaction limit set is a unit sphere.

In most of the above examples, we can find a projection onto a base space U which is linear and represented by a matrix $\boldsymbol{P}$ satisfying $\boldsymbol{PG}(\boldsymbol{u}) = \boldsymbol{0}$ for $\boldsymbol{u} \in \mathbb{R}^m$. Seeing these examples, you might imagine that the limit problem of (4) as $\varepsilon \to 0$ is uniquely determined by the dynamics on the reaction limit set $\mathcal{A}$. However, it is not true. Indeed, let us consider the following reaction–diffusion system:

$$\begin{cases} u_{1,t} &= d_1 \Delta u_1 - \dfrac{s_1}{\varepsilon} u_1^{m_1} u_2^{m_2}, \qquad x \in \Omega,\ t > 0, \\ u_{2,t} &= d_2 \Delta u_2 - \dfrac{s_2}{\varepsilon} u_1^{m_3} u_2^{m_4}, \qquad x \in \Omega,\ t > 0. \end{cases} \tag{37}$$

It is easily seen that (37) is equal to (7) with $f = g = 0$ when $m_j = 1$ $(j = 1, \dots, 4)$. When $m_j > 0$, the reaction limit set $\mathcal{A}$ of (37) is equal to $\mathcal{A}_1$ independent of m_j. When $m_1 = m_3$ and $m_2 = m_4$, (37) possesses a linear projection which maps $\boldsymbol{G}(\boldsymbol{u})$ onto $\boldsymbol{0}$ for all $\boldsymbol{u} \in \mathbb{R}_+^2$. Note that such a projection erasing the effect of $\boldsymbol{G}(\boldsymbol{u})$ becomes nonlinear for some other combinations of m_j. This situation leads us to various limit problems of (37) for fixed d_1, d_2, s_1, s_2, although its dynamics on the reaction limit set $\mathcal{A}_1$ is unique. For example, Iida, Monobe, Murakawa and Ninomiya [16] showed that $V_n = 0$ and that the interface does not move when $d_1 > 0, d_2 = 0, m_1 = m_2 = m_4 = 1, m_3 > 1$ (see Fig. 5), while the initial interface vanishes instantaneously when $d_1 > 0, d_2 = 0, m_1 > 3, m_2 = m_3 = m_4 = 1$ (see Fig. 6). Thus the limit problem can not be determined only by the dynamics on $\mathcal{A}$, but also by the dynamics of (8) in a neighborhood of $\mathcal{A}$. See [16] for the details.

2.5 Reaction–diffusion approximation of nonlocal effect

If we consider the reaction–diffusion system (4) where $\boldsymbol{D}$ may depend on $\varepsilon > 0$, then the limit problem may include a wide class of partial differential equations. Here we introduce the reaction–diffusion approximation of a nonlocal evolution equation.

In Sect. 2.3, we have considered the continuum limit of the transition probability in the movements on a lattice. Here we consider the transition probability in the movements between any two points on $\mathbb{R}$. If the transition probability from $x \in \mathbb{R}$ to $y \in \mathbb{R}$ is given by $J(x - y)$ at each moment, then the density $n(x, t)$ decreases by departing to any point y and increases by arriving from any point y. Namely, we obtain

$$n_t(x, t) = \int_{\mathbb{R}} J(y - x) n(y, t) dy - n(x, t) \int_{\mathbb{R}} J(x - y) dy, \quad x \in \mathbb{R},\ t > 0.$$

This is a typical nonlocal dispersal model often used in mathematical biology. We consider the following generalized nonlocal evolution equation:

$$\begin{cases} u_t = d_u u_{xx} + g(u, J * u) & x \in I,\ t > 0, \\ u(x, 0) = u_0(x) & x \in I, \end{cases} \tag{38}$$

with the periodic boundary condition where $I = (-L, L)$, d_u is a positive constant, $J \in L^1(I)$ with $J(x + 2L) = J(x)$ for $x \in \mathbb{R}$, g is a C^1-function from $\mathbb{R}^2$ to $\mathbb{R}$. Here

$$J * h(x) := \int_{-L}^{L} J(x - y) h(y)\, dy$$

for any $2L$-periodic function $h \in L^1(I)$. Typical examples of the nonlinear term $g(u,v)$ are

$$g(u,v) = uv + au(1-u^2) \quad \text{and} \quad g(u,v) = v + au(1-|u|).$$

The explicit nonlinearity of g is written in [40, 41]. Let us introduce a result on reaction–diffusion approximation of (38):

Theorem 6 (Ninomiya, Tanaka and Yamamoto [40, Theorem 4]) *Assume that the initial datum $u_0 \in H^1(I)$ satisfies $u_0(x+2L) = u_0(x)$ for any $x \in \mathbb{R}$. For any even $2L$-periodic continuous function $J \in L^1(I)$, any sufficiently small constant $\varepsilon > 0$ and any $T > 0$, there exists an $(M+1)$-component reaction–diffusion system* (39) *such that*

$$\sup_{x\in[-L,L],\, t\in[0,T]} |u(x,t) - u^\varepsilon(x,t)| \leq \varepsilon,$$

where u is a solution of (38) *and u^ε is the first component of the solution to*

$$\begin{cases} u_t &= d_u u_{xx} + g\left(u, \displaystyle\sum_{j=1}^{M} \alpha_j v_j\right), \\ v_{jt} &= \dfrac{1}{\varepsilon}\left(d_j v_{jxx} + u - v_j\right), \end{cases} \qquad x \in I,\ t > 0 \tag{39}$$

with the periodic boundary condition and the initial datum $(u, v_1, \dots, v_M)(x,0) = (u_0, k^{d_1} * u_0, \dots, k^{d_M} * u_0)(x)$, $d_j = j^{-2}$, α_j is a constant depending on J and for any $d > 0$,

$$k^d(x) := \frac{1}{2\sqrt{d}\sinh\frac{L}{\sqrt{d}}} \cosh\frac{L-|x|}{\sqrt{d}}. \tag{40}$$

We give a heuristic explanation of this theorem. First we note that $k^d(x)$ is the Green function of a Poisson equation $-d\Delta v + v = f$ under the periodic boundary condition, i.e., $v = k^d * f$. In order to obtain a reaction–diffusion approximation of (38), it is desirable to replace the convolution $J * u$ with new variables which becomes a solution to an appropriate reaction–diffusion system including u. Due to the Weirestrass theorem, J can be approximated by the linear sum of k^{d_j} with some positive constants d_j. Moreover, since $\varepsilon v_t = dv_{xx} + u - v$ is close to $dv_{xx} + u - v = 0$ for small ε, $J * u$ can be approximated by the linear sum of the solutions v_j of $\varepsilon v_{j,t} = d_j v_{j,xx} + u - v_j$. Thus a solution of (38) can be approximated by a solution of (39).

It is natural to encounter the following question: *how the coefficients* α_j *are determined ?* To answer this question, we consider the case $I = \mathbb{R}$. In the following theorem, the coefficients α_j in the approximation equation (39) are explicitly calculated.

Theorem 7 (Ninomiya, Tanaka and Yamamoto [41, Theorems 2 and 3]) *Assume that* J *is even and continuous in* $\mathbb{R}$, *the limit of* $J(x)e^{|x|}$ *as* $|x| \to \infty$ *exists finitely and an initial datum* $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. *For any sufficiently small* $\varepsilon > 0$ *and any* $T > 0$, *there exist constants* $M \in \mathbb{N}$, $\alpha_j \in \mathbb{R}$ *and* $d_j > 0\,(j = 1, \dots, M)$ *such that*

$$\begin{aligned}\sup_{t\in[0,T]} \|u(\cdot,t) - u^\varepsilon(\cdot,t)\|_{H^1(\mathbb{R})} &\le C\varepsilon,\\ \sup_{t\in[0,T]} \|v_j^\varepsilon(\cdot,t) - K^{d_j} * u(\cdot,t)\|_{H^1(\mathbb{R})} &\le C\varepsilon,\end{aligned}$$

were C *is a constant depending on* $T > 0$, u *is a solution of* (38) *and* $(u^\varepsilon, v_j^\varepsilon)_{1\le j\le M}$ *is a solution of* (39) *with the initial datum* $(u, v_1, \dots, v_M)(x,0) = (u_0, K^{d_1} * u_0, \dots, K^{d_M} * u_0)(x)$, $d_j = j^{-2}$,

$$K^d(x) := \frac{1}{2\sqrt{d}} e^{-|x|/\sqrt{d}} \quad \text{for any } d > 0$$

and for all $j \in \mathbb{N}$

$$\begin{aligned}\alpha_j &= \frac{1}{j} \sum_{\ell=j}^{M} p_{j-1,\ell-1} \sum_{k=1}^{\ell} p_{k-1,\ell-1} \int_{\mathbb{R}} J(x)e^{-k|x|}\,dx,\\ p_{k-1,\ell-1} &= \frac{(-1)^{k+\ell}\sqrt{2\ell}\,(k+\ell-1)!}{(\ell-k)!k!(k-1)!} \qquad \text{for } \ell \in \mathbb{N},\ 1 \le k \le \ell.\end{aligned}$$

2.6 Reaction–diffusion approximation of finite propagation

Next, we consider wave equations. It is well-known that a solution of a wave equation have a property of finite propagation. Indeed, the solution for the initial problem of the wave equation

$$w_{tt} = c^2 w_{xx} \tag{41}$$

in $\mathbb{R}$ is represented by

$$w(x,t) = \frac{1}{2}\{w_0(x+ct) + w_0(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} w_1(r)\,dr$$

where $w_0(x) = w(x, 0)$ and $w_1(x) = w_t(x, 0)$ are C^2-functions. Thus, the value of $w(x,\ t)$ only in light cone $\{(x,t) \mid |x - x_0| \le c|t - t_0|\}$ is affected by $w(x_0, t_0)$ and $w_t(x_0, t_0)$, where (x_0, t_0) is any fixed point. Conversely, the parabolic equation including reaction–diffusion system has a property of infinite propagation. For example, any solution of a heat equation immediately becomes positive everywhere even if the initial function has compact support and is non-negative. Thus we encounter the following question: *can a semilinear wave equation be approximated by reaction–diffusion systems ?* Ninomiya and Yamamoto [42] gives a reaction–diffusion approximation of a semilinear wave equation.

First we give a heuristic explanation of the reaction–diffusion approximation of a wave equation (41). Comparing the wave equation with the diffusion equation, we see that the wave equation has the second-order derivative of the solution with respect to time t. Hence, we have to produce the derivative with respect to t from reaction term of the system. Set $v_1(x,t) := w(x,t)$ and $v_2(x,t) := w(x, t + \varepsilon)$. Then, we see that

$$v_{1,t}(x,t) \approx \frac{v_2(x,t) - v_1(x,t)}{\varepsilon}. \tag{42}$$

By using (41), we can expect that

$$v_{1,tt} = c^2 \Delta v_1 \approx \frac{v_{2,t} - v_{1,t}}{\varepsilon},$$

which implies that

$$v_{2,t} \approx \varepsilon c^2 \Delta v_1 + v_{1,t} \approx \varepsilon c^2 \Delta v_1 + \frac{v_2(t) - v_1(t)}{\varepsilon}. \tag{43}$$

Combing two Eqs. (42) and (43), we can obtain the candidate system for (v_1, v_2), but it is not a reaction–diffusion system. Therefore we modify the candidate system to

$$\begin{cases} u_{1,t} &= \varepsilon \Delta u_1 + \dfrac{u_2 - u_1}{\varepsilon}, \\ u_{2,t} &= (\varepsilon + c^2\varepsilon)\Delta u_2 + \dfrac{u_2 - u_1}{\varepsilon}. \end{cases} \tag{44}$$

Indeed, the difference between the two equations yields

$$\frac{u_{2,t} - u_{1,t}}{\varepsilon} = (1 + c^2)\Delta u_2 - \Delta u_1.$$

Since

$$u_2 = u_1 + \varepsilon u_{1,t} - \varepsilon^2 \Delta u_1,$$

we can rewrite it as

$$\begin{aligned}\frac{u_{2,t} - u_{1,t}}{\varepsilon} =&(1 + c^2)\Delta(u_1 + \varepsilon u_{1,t} - \varepsilon^2 \Delta u_1) - \Delta u_1\\ =&c^2 \Delta u_1 + \varepsilon(1 + c^2)\Delta(u_{1,t} - \varepsilon \Delta u_1).\end{aligned}$$

Differentiating the first equation of (44) with respect to t, we get

$$\begin{aligned}u_{1,tt} =&\frac{u_{2,t} - u_{1,t}}{\varepsilon} + \varepsilon \Delta u_{1,t}\\ =&c^2 \Delta u_1 + \varepsilon(1 + c^2)\Delta(u_{1,t} - \varepsilon \Delta u_1) + \varepsilon \Delta u_{1,t}.\end{aligned}$$

If the last three terms of the right-hand side are small as ε tends to 0, we can expect that u_1 converges to the solution of the wave equation (41).

We extend this idea to the semilinear wave equation. Consider the following initial problem:

$$\begin{cases} w_{tt} = d\Delta w + f(w) & \text{in } \mathbb{R}^N \times (0,T),\\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{45}$$

where $d > 0$ and $T > 0$ are constants, w_0 and w_1 are the initial data. Moreover, we assume that the nonlinear term f is a C^1-function in $\mathbb{R}$ and satisfies $f(u)u \leq f_1|u|^2$ and $-f_2(|u|^2 + |u|^{p+1}) \leq F(u) \leq f_3|u|^2$, where $p > 1$, $f_j \geq 0$ are constants for $j = 1, 2, 3$ and

$$F(u) := \int_0^u f(x)\, ds.$$

As an approximation of (45) we propose the following initial problem of the reaction–diffusion system:

$$\begin{cases} u_{1,t} &= d_1 \varepsilon \Delta u_1 + \dfrac{u_2 - u_1}{\varepsilon}, \\ u_{2,t} &= d_2 \varepsilon \Delta u_2 + \dfrac{u_2 - u_1}{\varepsilon} + \varepsilon f(u_1) \end{cases} \quad \text{in } \mathbb{R}^N \times (0,T), \\ u_1(x,0) = w_0,\ u_2(x,0) = w_0 + \varepsilon w_1 \qquad \text{in } \mathbb{R}^N, \tag{46}$$

where $\varepsilon > 0$ is a small constant, w_0 and w_1 is the initial data of (45), $d_1 > 0$ and $d_2 = d + d_1 > 0$.

Theorem 8 (Ninomiya and Yamamoto [42]) *Let $N = 1, 2$ and f be a linear function if $N \geq 3$. Assume $d_2 > d_1 > 0$ and $w_0 \in H^4(\mathbb{R}^N)$, $w_1 \in H^2(\mathbb{R}^N)$. Then, for any $T > 0$, there exists a unique solution $(u_1^\varepsilon, u_2^\varepsilon)$ of* (46) *in $\mathbb{R}^N \times (0,T)$ such that for sufficiently small ε,*

$$\sup_{t\in[0,T]} \left\{ \|w - u_1^\varepsilon\|_{H^1(\mathbb{R}^N)} + \|w_t - u_{1,t}^\varepsilon\|_{L^2(\mathbb{R}^N)} \right\} \leq C\sqrt{\varepsilon}.$$

3 Applications

3.1 Relation between cross-diffusion induced instability and the Turing instability

Here we apply reaction–diffusion approximation to the pattern formation. Turing [47] proposed the theoretical mechanism which explains how spatial patterns form autonomously. It is called the *diffusion-induced instability* or *the Turing instability*. As seen in Sect. 1, the stable steady state of the ordinary differential equations may become unstable by the addition of a certain diffusion effect and the stable periodic patterns emerge. One of the sufficient conditions for the diffusion-induced instability is "local activation with long-range inhibition". Namely, we consider two chemical substances: an activator and an inhibitor. The activator locally enhances itself and the inhibitor diffuses faster than the activator and inhibits the activator at the place where the activator is not distributed yet. Thus the spatially periodic pattern will be generated.

Let us consider the competition-diffusion system (6) under the weak competition condition, namely,

$$\frac{b_1}{a_2} < \frac{r_1}{r_2} < \frac{a_1}{b_2}.$$

In this case, the diffusion-free system has a stable steady state

$$(e_1, e_2) := \left(\frac{a_2 r_1 - b_1 r_2}{a_1 a_2 - b_1 b_2}, \frac{-b_2 r_1 + a_1 r_2}{a_1 a_2 - b_1 b_2} \right)$$

and three unstable steady states: (0, 0), $(r_1/a_1, 0)$ and $(0, r_2/a_2)$. Under this condition, (e_1, e_2) is stable also in the competition-diffusion system (6). In fact, we can easily show that all solutions of (6) with positive components converge to (e_1, e_2) by using the comparison principle. Namely, the diffusion-induced instability never occurs.

Mimura and Kawasaki [29] indicated that if we consider the cross-diffusion system (27) instead of (6), an inhomogeneous stable stationary solution bifurcates from (e_1, e_2) under the same weak competition setting. Of course, when α is small, (e_1, e_2) is still stable also in (27), while it becomes unstable if α increases (see also [21, 22, 28]). This means that the effect of the cross-diffusion generates the spatial segregation of the species. Since this instability is never observed for (6) as stated above, it is often called the *cross-diffusion induced instability*. Here we explain the relation between the cross-diffusion induced instability and the Turing instability by using the reaction–diffusion approximation.

Recall the system

$$\begin{cases} w_{1,t} &= \Delta[(d_1 + \alpha w_2)w_1] + f(w_1, w_2), \qquad x \in \Omega,\ t > 0, \\ w_{2,t} &= d_2 \Delta w_2 + g(w_1, w_2), \qquad x \in \Omega,\ t > 0 \end{cases} \tag{47}$$

where $\alpha > 0$ and

$$\begin{cases} f(w_1, w_2) &= (r_1 - a_1 w_1 - b_1 w_2)w_1, \\ g(w_1, w_2) &= (r_2 - b_2 w_1 - a_2 w_2)w_2. \end{cases}$$

Note that the steady state (e_1, e_2) is a homogeneous stationary solution of (47). The linearized operator for the right-hand sides of (47) around (e_1, e_2) is

$$\begin{pmatrix} d_1 \Delta + \alpha e_2 \Delta + f_{w_1}(e_1, e_2) & \alpha e_1 \Delta + f_{w_2}(e_1, e_2) \\ g_{w_1}(e_1, e_2) & d_2 \Delta + g_{w_2}(e_1, e_2) \end{pmatrix}.$$

Each eigenvalue μ of the linearized operator is a root of the characteristic polynomial

$$\Xi^*(\mu) := \begin{vmatrix} -d_1\sigma - \alpha e_2 \sigma + f_{w_1}(e_1, e_2) - \mu & -\alpha e_1 \sigma + f_{w_2}(e_1, e_2) \\ g_{w_1}(e_1, e_2) & -d_2 \sigma + g_{w_2}(e_1, e_2) - \mu \end{vmatrix},$$

where σ is an eigenvalue of $-\Delta$ under the homogeneous Neumann boundary condition.

Next let us consider the reaction–diffusion approximation (26) of the cross-diffusion system (47). By (28), the corresponding homogeneous stationary solution of (26) is (e_1^*, e_2^*, e_3^*) where

$$e_3^* = e_2, \quad e_2^* = \frac{h_{1\to 2}(e_3^*)e_1}{h_{1\to 2}(e_3^*) + h_{2\to 1}(e_3^*)} = \frac{e_3^*}{M_2} e_1 = \frac{e_2}{M_2} e_1, \quad e_1^* = \left(1 - \frac{e_2}{M_2}\right) e_1.$$

Denote the characteristic polynomial which determines each eigenvalue μ of the linearized operator for the right-hand sides of (26) around (e_1^*, e_2^*, e_3^*) by $\Xi^\varepsilon(\mu)$ like $\Xi^*(\mu)$. Then we obtain the following relation between them:

$$\Xi^\varepsilon(\mu) = -\frac{h_{1\to 2}(e_2) + h_{2\to 1}(e_2)}{\varepsilon} \Xi^*(\mu) + \Xi(\mu). \tag{48}$$

We can also calculate the principal parts of them:

$$\Xi^*(\mu) = \mu^2 + O(\mu), \qquad \Xi(\mu) = -\mu^3 + O(\mu^2) \qquad (|\mu| \to \infty)$$

(see [15, §3] for the details). If ε is small, then it follows from (29) and (48) that the three roots of $\Xi^\varepsilon(\mu) = 0$ consist of the two roots of $\Xi^*(\mu) = 0$ and one negative one. Therefore, when ε is small, the following holds: When $\alpha = 0$, $(e_1^* + e_2^*, e_3^*)$ is a stable homogeneous solution of (47), which implies that (e_1^*, e_2^*, e_3^*) is also a stable homogeneous stationary solution of (26); on the other hand the linearized operator corresponding to (26) has a positive eigenvalue when α is large.

In (26) with a small parameter ε, u_3 plays as an activator and u_2 does as an inhibitor if $h_{1\to 2}(s)$ is increasing in s and if $h_{2\to 1}(s)$ is decreasing in s. Moreover, because the diffusion coefficient for u_2 is $d_1 + \alpha M_2$, the Turing instability takes place when α is large. As we have seen above, this instability still holds in the limit problem as $\varepsilon \to 0$. Thus, the cross-diffusion induced instability can be regarded as the diffusion-induced instability for the reaction–diffusion approximation.

3.2 Instability induced by nonlocal effect

Here we consider (38) and we assume that

$$J(x) := \mu(k^{d_1}(x) - k^{d_2}(x)) \tag{49}$$

where $0 < d_1 < d_2$ and k^d is given in (40). Note that J is a Mexican-hat kernel. Then J satisfies

$$\int_{-L}^{L} J(x)dx = 0 \tag{50}$$

It is easily seen that the solution of the ordinary differential equation

$$U_t = g(U, 0) \tag{51}$$

can be regarded as a spatial homogeneous solution of (38). In addition we assume that

$$g(1,0) = 0, \quad g_u(1,0) < 0, \quad g_v(1,0) > 0. \tag{52}$$

Then $U^* \equiv 1$ is a stable equilibrium of (51) and is a constant steady-state solution of (38). To study the stability, we linearize (38) at $U^* \equiv 1$. Noting $J * 1 = 0$ by (50), we have the associated eigenvalue problem:

$$\lambda\varphi = d_u\varphi_{xx} + g_u(1,0)\varphi + g_v(1,0)J * \varphi =: L\varphi. \tag{53}$$

By using Fourier series, we can easily check that the spectrum set of the linear operator L defined by (53) consists of eigenvalues. It follows from (49) that the eigenvalue λ is given by

$$\begin{aligned} \lambda_k^0 &= -d_u\sigma_k + \left(\frac{\mu}{d_1\sigma_k + 1} - \frac{\mu}{d_2\sigma_k + 1}\right)g_v(1,0) + g_u(1,0) \\ &= \frac{(d_2 - d_1)\sigma_k g_v(1,0)}{(d_1\sigma_k + 1)(d_2\sigma_k + 1)}\mu - d_u\sigma_k + g_u(1,0), \end{aligned} \tag{54}$$

where $\sigma_k := (k\pi/L)^2$ for $k \in \mathbb{Z}$. Thus we can show the following nonlocal interaction induced instability. Namely, for any positive constants d_u and d_1, if $d_2 > d_1$, there exists a positive constant $\mu^* = \mu^*(d_u, d_1, d_2)$ such that $U^* \equiv 1$ is stable in (38) if $0 \le \mu < \mu^*$, but is unstable in (38) if $\mu > \mu^*$. Hereafter we fix the constants d_1, d_2

and d_u as above, namely, we fix the profile of J. Then, when the magnitude μ of J is large, we can destabilize $U^* \equiv 1$ in (38). See Murray [38, Chap 12] and [40, Proposition 1] for the details.

Next let us consider the reaction–diffusion approximation of (38) with the Mexican hat kernel (49). The corresponding reaction–diffusion system (39) in Theorem 6 becomes

$$\begin{cases} u_t &= d_u u_{xx} + g(u, v_1 - v_2), \\ v_{1,t} &= \dfrac{1}{\varepsilon}\left(d_1 v_{1,xx} + \mu u - v_1\right), \\ v_{2,t} &= \dfrac{1}{\varepsilon}\left(d_2 v_{2,xx} + \mu u - v_2\right), \end{cases} \tag{55}$$

by taking $\alpha_1 = 1,\ \alpha_2 = -1$ where $0 < d_1 < d_2$. Note that this reaction–diffusion system consists of two activators and one inhibitor. We investigate the instability of the homogeneous stationary solution $(u, v_1, v_2) = (1, \mu, \mu)$ to (55) when ε is close to 0. Linearizing (55) near $(u, v_1, v_2) = (1, \mu, \mu)$, we obtain the following eigenvalue problem:

$$\lambda \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -d_u\sigma_k + g_u(1,0) & g_v(1,0) & -g_v(1,0) \\ \frac{\mu}{\varepsilon} & \frac{-1-d_1\sigma_k}{\varepsilon} & 0 \\ \frac{\mu}{\varepsilon} & 0 & \frac{-1-d_2\sigma_k}{\varepsilon} \end{pmatrix} \begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix},$$

where $\sigma_k := (k\pi/L)^2,\ k \in \mathbb{Z}$ and λ is an eigenvalue of the linearized matrix. The characteristic polynomial $\Xi_k(\varepsilon, \lambda)$ is calculated as

$$\begin{aligned} \Xi_k(\varepsilon, \lambda) =& \frac{1}{\varepsilon^2}\left(\mu g_v(1,0)(d_2 - d_1)\sigma_k - (d_u\sigma_k - g_u(1,0) + \lambda)(1 + d_1\sigma_k)(1 + d_2\sigma_k)\right) \\ &- \frac{\lambda}{\varepsilon}(d_u\sigma_k - g_u(1,0) + \lambda)(2 + d_1\sigma_k + d_2\sigma_k) \\ &- \lambda^2(d_u\sigma_k - g_u(1,0) + \lambda). \end{aligned} \tag{56}$$

Solving $\lim_{\varepsilon\to 0} \varepsilon^2 \Xi_k(\varepsilon, \lambda) = 0$, we have

$$\begin{aligned} \lambda =& \frac{\mu g_v(1,0)(d_2 - d_1)\sigma_k - (d_u\sigma_k - g_u(1,0))(1 + d_1\sigma_k)(1 + d_2\sigma_k)}{(1 + d_1\sigma_k)(1 + d_2\sigma_k)} \\ =& - d_u\sigma_k + \left(\frac{\mu g_v(1,0)}{1 + d_1\sigma_k} - \frac{\mu g_v(1,0)}{1 + d_2\sigma_k}\right) + g_u(1,0). \end{aligned}$$

We remark that this eigenvalue is equal to λ_k^0 given in (54). The implicit function theorem guarantees that, for small ε, $\Xi_k(\varepsilon, \lambda) = 0$ possesses the unique root $\lambda^\varepsilon(k)$ satisfying $\lim_{\varepsilon\to 0} \lambda^\varepsilon(k) = \lambda_k^0$. Moreover, we can confirm that there exists no eigenvalues with positive real part except for $\lambda^\varepsilon(k)$ when $\varepsilon \to +0$. It turns out that $\lambda^\varepsilon(k) > 0$ (resp. $\lambda^\varepsilon(k) < 0$) for sufficiently small $\varepsilon > 0$ if $\lambda_k^0 > 0$ (resp. $\lambda_k^0 < 0$). Note that $\lambda_0^0 = g_u(1,0) < 0$. This implies that all eigenvalues of the linearized matrix corresponding to (55) without diffusion around $(1, \mu, \mu)$ are negative. Namely, $(1, \mu, \mu)$ is stable for (55) without diffusion whenever $\mu > 0$. Next let us consider the system (55) with diffusion. For positive constants d_1, d_2 and d_u as above, we take a constant μ satisfying $\mu > \mu^*(d_u, d_1, d_2)$. Since λ_k^0 becomes positive for some k, $\lambda^\varepsilon(k) > 0$ for sufficiently small $\varepsilon > 0$. Hence, the homogeneous solution $(1, \mu, \mu)$ of (55) becomes unstable for any sufficiently small ε, while $(1, \mu, \mu)$ is stable for (55) without diffusion. Therefore the nonlocal interaction induced instability of (38) can be regarded as the diffusion-induced instability of (55). See [40, Section 5.2] for the details.

3.3 Application to the numerical analysis

It is sometimes difficult to determine the position of the interface when we solve the free boundary problem numerically. Recall the porous medium equation (33). The support of the solution of this equation starting from the compactly supported initial distribution is compact and the boundary of the support develops in time. However, it is difficult to determine the boundary of the support numerically because it may be influenced by the errors. Nakaki and Murakawa [39] proposed to use a reaction–diffusion approximation to determine the boundary of the support of the solution to (33) with $d_1 = 1$ and $k = 2$ by introducing the ideal variable u_2 satisfying

$$\begin{cases} u_{1,t} &= \Delta u_1^2 - \dfrac{2}{\varepsilon} u_1^{1.5} u_2^2, \\ u_{2,t} &= -\dfrac{1}{\varepsilon} u_1 u_2. \end{cases} \tag{57}$$

Then, u_1 forms a corner layer, while u_2 provides a transition layer as a one-phase Stefan problem. We can expect that the position of the transition layer of u_2 is a good approximation of the position of the boundary of the support of u_1. Indeed, in [39], they studied numerically the motion of the transition layer of u_2 by using the explicit Barenblatt solution and they have chosen the exponents of the nonlinear terms to minimize errors. Since the interaction terms in (57) are not cancelled by subtraction, it is difficult to handle these terms by the rigorous calculation (see Sect. 2.4 and [16]). Let us consider the limit problem heuristically. Because of the corner layer of u_1, u_1 is small near the boundary. Thus $u_1^{1.5} u_2^2$ is smaller than $u_1 u_2$ near the interface. This suggests that the influence of $u_1^{1.5} u_2^2$ to $u_{1,t}$ is much smaller than that of $u_1 u_2$ to $u_{2,t}$ and that the interaction term in the first equation of (57) does not provide any influence to the motion of the boundary of the support of u_1. Thus, it is expected that the limit equation is $u_{1,t} = \Delta u_1^2$. However, this explanation does not provide any reasons why the indices are the specific numbers 1.5 and 2.

Next we discuss the relation between the penalty method and the reaction–diffusion approximation by using one-phase Stefan problem in the one-dimensional interval $I = [0, 1]$. Denote the position of the interface by $x = \ell(t)$. Let u_0 be the initial function which satisfies

$$u_0(x) > 0 \qquad (0 \le x < \ell_0), \qquad u_0(x) = 0 \qquad (\ell_0 \le x < 1).$$

Recall the one-phase Stefan problem:

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell(t),\ t > 0, \\ u(x,0) = u_0(x), & 0 < x < \ell_0, \\ u_x(0,t) = 0, \quad u(\ell(t),t) = 0, & t > 0, \\ \ell'(t) = -u_x(\ell(t),t), & t > 0, \\ \ell(0) = \ell_0. \end{cases} \tag{58}$$

The classical solution of this problem always satisfies $\ell'(t) > 0$. Integrating (58) over $\cup_{0\le t\le T}[0,\ell(t)] \times \{t\}$ yields

$$\ell(T) = \ell_0 - \int_0^{\ell(T)} u(x,T)dx + \int_0^{\ell_0} u_0(x)dx. \tag{59}$$

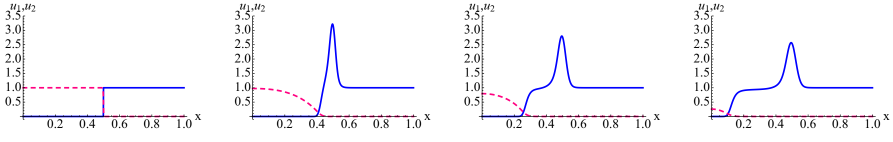
The penalty method of (58) is to consider the following problem in $I = [0,1]$ by adding the penalty term to (58):

$$\begin{cases} u_t^\varepsilon = u_{xx}^\varepsilon - \dfrac{1}{\varepsilon}\chi(x,t)u^\varepsilon, & x \in I,\ t > 0, \\ u^\varepsilon(x,0) = u_0(x), & x \in I, \\ u_x^\varepsilon(0,t) = 0, \quad u_x^\varepsilon(1,t) = 0, & t > 0, \end{cases} \tag{60}$$

where χ is a characteristic function of the set $[\ell^\varepsilon(t),1]$ and the position of the approximating interface $\ell^\varepsilon(t)$ is determined by

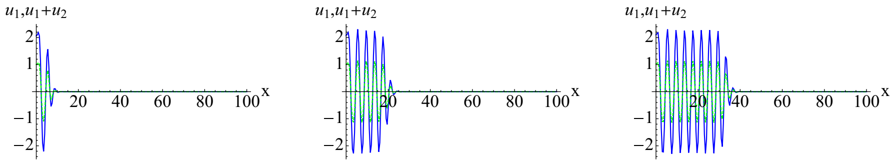
$$\begin{cases} \dfrac{d}{dt}\ell^\varepsilon(t) = \dfrac{1}{\varepsilon}\displaystyle\int_0^1 \chi(x,t)u^\varepsilon(x,t)dx, \\ \ell^\varepsilon(0) = \ell_0. \end{cases} \tag{61}$$

Indeed, integrating (60) over $I \times [0,T]$ and using (61) yield



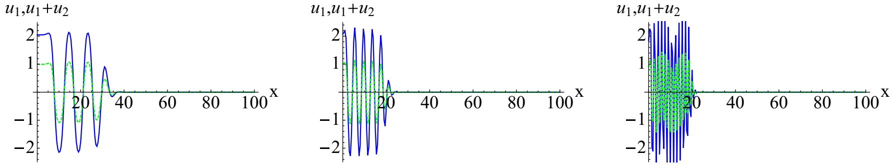
(a) $t = 0$ **(b)** $t = 0.01$ **(c)** $t = 0.02$ **(d)** $t = 0.03$

Fig. 7 Snapshots of numerical solutions of (63) when $\varepsilon = 0.000001$ (solid: u_1, dashed: u_2)



(a) $t = 5$ **(b)** $t = 15$ **(c)** $t = 25$

Fig. 8 Snapshots of numerical solutions of (64) with appropriate saturation terms when $\varepsilon = 0.01$ (solid: u_1, dashed: $u_1 + u_2$). The periodic structure propagate in time



(a) $\varepsilon = 0.1$ **(b)** $\varepsilon = 0.01$ **(c)** $\varepsilon = 0.001$

Fig. 9 Dependence of numerical solutions of (64) with appropriate saturation terms on ε when $t = 15$ (solid: u_1, dashed: $u_1 + u_2$)

$$\ell^\varepsilon(t) = \ell_0 - \int_0^1 u^\varepsilon(x,t)dx + \int_0^{\ell_0} u_0(x)dx. \tag{62}$$

Because u^ε is expected to be small in the interval $[\ell^\varepsilon(t), 1]$, (62) is a good approximation of (59).

Now, recalling Fig. 2b, we realize that (9) is quite similar to (60). Actually, combining (60) with the equality

$$\frac{d}{dt}\int_0^1 \chi dx = \frac{d}{dt}(1 - \ell^\varepsilon) = -\frac{1}{\varepsilon}\int_0^1 \chi(x,t)u^\varepsilon(x,t)dx,$$

we see that the equation for u_2 is similar to that of χ. This suggests the deep relation between the penalty method and the reaction–diffusion approximation.

3.4 Related topics

First we explain that the situation is completely different if we change the sign of nonlinear terms of (9). Consider the following reaction–diffusion system:

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0.01 & 0 \\ 0 & 2 \end{pmatrix}\Delta\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} u_1u_2 \\ -u_1u_2 \end{pmatrix} \tag{63}$$

Since the corresponding ordinary differential equation without diffusion does not change $u_1 + u_2$, its solution converges to $(u_1(0) + u_2(0), 0)$ as t tends to ∞. The dynamics of the corresponding ordinary differential equation is quite simple. However, the profiles of solutions of (63) are seen in Fig. 7. The smaller ε is, the larger the maximum of solutions is. It is still difficult to consider the limit problem and we need some modification of the nonlinear terms see Monneau and Weiss [34] for the details).

Next we propose the example where the limit problem becomes ill-posed even if $\boldsymbol{G}$ is linear. Consider

$$\frac{\partial}{\partial t}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \Delta \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} + \frac{1}{\varepsilon}\begin{pmatrix} u_2 + \alpha(u_3 + u_4) - u_1 \\ u_1 - \alpha(u_3 + u_4) - u_2 \\ u_4 + \beta(u_1 + u_2) - u_3 \\ u_3 - \beta(u_1 + u_2) - u_4 \end{pmatrix}. \tag{64}$$

Similar to (3), we set $w_1 = u_1 + u_2$, $w_2 = u_3 + u_4$. Then the reaction limit set is

$$\mathcal{A} = \left\{ \left(\frac{w_1 + \alpha w_2}{2}, \frac{w_1 - \alpha w_2}{2}, \frac{\beta w_1 + w_2}{2}, \frac{-\beta w_1 + w_2}{2} \right) \,\middle|\, w_1, w_2 \in \mathbb{R} \right\},$$

and we obtain

$$\begin{cases} w_{1,t} &= 3\Delta w_1 - \alpha\Delta w_2\,, \\ w_{2,t} &= -\beta\Delta w_1 + 3\Delta w_2 \end{cases}$$

as the limit system. However, this system is ill-posed when $\alpha\beta > 9$. By adding some appropriate saturation terms to (64), we can obtain the numerical solutions of (64) added suitable saturation terms to as seen in Figs. 8, 9. This means that the reaction–diffusion system (4) is always well-posed for any $\varepsilon > 0$, while the limit problem is not necessary. See [33] for the details.

We are still far from the complete understanding of the general theory of reaction–diffusion approximation.

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