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Galerkin method for time fractional diffusion equations

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Abstract
We propose a Galerkin method for solving time fractional diffusion problems under quite general assumptions. Our approach relies on the theory of vector valued distributions. As an application, the "ℓ goes to plus infinity" issue for these problems is investigated.

Keywords Galerkin method · Time fractional diffusion equations

Mathematics Subject Classification 35K05

1 Introduction

The Galerkin approximation method in an efficient and robust tool for solving linear and nonlinear partial differential equations (see for instance [12, 16]). In this paper, we implement this method for solving time fractional diffusion problems. Our implementation allows non trivial initial conditions and the functional framework is quite simple.

There are two drawbacks for solving time fractional PDE's with the Galerkin method. First, an estimate from below is needed for integrals of the form

$$\int_0^T \int_\Omega \mathbf{D}^\alpha u(t,x)u(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

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Here $u = u(t,x)$ is the solution of some time fractional PDE set on $[0,T] \times \Omega \subset [0,\infty) \times \mathbb{R}^d$, and $\alpha \in (0,1)$. The positivity of that integral can be achieved by assuming, roughly speaking, that $u(0,x) = 0$ (see [17]). However, this hypothesis is clearly too restrictive. Also, in the integer setting (i.e. when $\alpha = 1$), the above integral equals

$$\frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)}.$$

Hence, it has no sign. Thus, in the fractional case, we may expect to control this integral in a similar way. This is indeed the case: in the proof of Theorem 4.1, we decompose that integral into the sum of a bad term, which turns out to be positive (by an estimate due to Nohel and Shea; see Theorem 2.1), and a quantity with no sign but controllable.

The second difficulty concerns functional spaces. In many papers, fractional Gagliardo-Sobolev spaces are used. These spaces are quite complicated to handle, and the necessity of their use in the Galerkin method, seems not obvious to the authors. Moreover, in order to have a continuation property from the time interval $[0, T]$ into $\mathbb{R}$, a trivial initial condition is needed (see [11, 13]).

In this paper, we use simple functional spaces which are natural generalization of the spaces involved in the integer setting (see Definition 4.1). We consider Riemann-Liouville derivatives.

In the two forcoming sections, we give the background on weak fractional derivatives. The Galerkin method is implemented in Sect. 4 for solving a time fractional model problem. Finally, in Sect. 5, we apply our result to the "ℓ goes to plus infinity" issue. It is about to study the asymptotic behavior of the solution $u = u(t,x)$ when the domain $\Omega = \Omega_\ell$ becomes unbounded in one or several directions as $\ell \to \infty$.

2 Preliminaries

For $(X, \|\cdot\|)$ a real Banach space, let us introduce the convolution of functions and the (formal) adjoint of the convolution.

Definition 2.1 Let $g \in L^1_{loc}([0,\infty))$, $T > 0$ and $f \in L^1(0,T;X)$. Then the *convolution* of g and f is the function of $L^1(0,T;X)$ defined by

$$g * f(t) := \int_0^t g(t-y)f(y)\mathrm{d}y, \quad \text{a.e. } t \in [0,T].$$

Also, we define

$$g *' f(t) := \int_t^T g(y-t)f(y)\mathrm{d}y, \quad \text{a.e. } t \in [0,T].$$

Remark 2.1 Roughly speaking, $f \mapsto g *' f$ is the adjoint of $f \mapsto g * f$. Indeed, for f, g as above and $\psi \in C([0,T])$, one has, by Fubini's Theorem

$$\int_0^T g * f(t)\psi(t)\,\mathrm{d}t = \int_0^T f(t) g *' \psi(t)\,\mathrm{d}t.$$

The following kernel is of fundamental importance in the theory of fractional derivatives.

Definition 2.2 For $\beta \in (0,\infty)$, let us denote by g_β the function of $L^1_{loc}([0,\infty))$ defined for a.e. $t > 0$ by

$$g_\beta(t) = \frac{1}{\Gamma(\beta)} t^{\beta-1}.$$

For each $\alpha, \beta \in (0,\infty)$, the following identity holds.

$$g_\alpha * g_\beta = g_{\alpha+\beta}, \quad \text{in } L^1_{\mathrm{loc}}\big([0,\infty)\big). \tag{2.1}$$

Let us recall the following well-known result: if $f \in L^2(0,T;X)$ and $g \in L^1(0,T)$ then

$$g * f \in L^2(0,T;X) \quad \text{and} \quad \|g * f\|_{L^2(0,T;X)} \le \|g\|_{L^1(0,T)} \|f\|_{L^2(0,T;X)}. \tag{2.2}$$

The following deep result due to Nohel and Shea ([14], Theorem 2 and Corollary 2.2]) is a crucial tool for estimating. That result was originally stated for scalar valued functions but can easily be extended into an Hilbertian setting.

Theorem 2.1 *Let $(H,(\cdot,\cdot))$ be a real Hilbert space, $f \in L^2(0,T;H)$ and $\alpha \in (0,1)$. Then*

$$\int_0^T \big(f(t), g_\alpha * f(t)\big)\,\mathrm{d}t \ge 0.$$

3 Riemann fractional derivatives

We will introduce fractional derivatives and weak fractional derivatives, that is, fractional derivatives in the sense of distributions. Let us start with the well-known *fractional forward* and *backward derivatives* of a function in the sense of Riemann and Liouville. We refer to [15] for more details on fractional derivatives.

Definition 3.1 Let $\alpha \in (0,1)$, $T > 0$ and $f \in L^2(0,T;X)$. We say that f *admits a (forward) derivative of order* α in $L^2(0,T;X)$ if

$$g_{1-\alpha} * f \in H^1(0,T;X).$$

In this case, its *(forward) derivative of order* α is the function of $L^2(0,T;X)$ defined by

$$ {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f := \frac{\mathrm{d}}{\mathrm{d}t}\{g_{1-\alpha} * f\}.$$

Definition 3.2 Let $\alpha \in (0,1)$, $T > 0$ and $f \in W^{1,1}(0,T;X)$. Then we say that f *admits a backward derivative of order* α in $L^2(0,T;X)$ if

$$g_{1-\alpha} *' \frac{\mathrm{d}}{\mathrm{d}t} f \in L^2(0,T;X).$$

In this case, its *backward derivative of order* α is the function of $L^2(0,T;X)$ defined by

$$ {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T} f := g_{1-\alpha} *' \frac{\mathrm{d}}{\mathrm{d}t} f.$$

Remark 3.1 If $f \in H^1(0,T;X)$ then $g_{1-\alpha} *' \frac{\mathrm{d}}{\mathrm{d}t} f$ lies in $L^2(0,T;X)$, according to (2.2). Hence f admits a fractional backward derivative of order α in $L^2(0,T;X)$.

Proposition 3.1 *Let* $\alpha \in (0,1)$, $f \in L^2(0,T;X)$ *and* $\psi \in H^1(0,T)$. *Assume that* f *admits a derivative of order* α in $L^2(0,T;X)$. *Then*

$$\int_0^T \mathbf{D}^{\alpha}_{0,t} f(t)\psi(t)\,\mathrm{d}t = -\int_0^T f(t)\,{}^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\psi(t)\,\mathrm{d}t + \left[g_{1-\alpha} * f\,\psi\right]_0^T. \tag{3.1}$$

Moreover, if, in addition, $\psi \in \mathcal{D}(0,T)$ *then*

$$\left\| \int_0^T f(t)\mathbf{D}^{\alpha}_{t,T}\psi(t)\,\mathrm{d}t \right\| \le \sqrt{T} g_{2-\alpha}(T)\|f\|_{L^2(0,T;X)}\|\psi'\|_{\infty,[0,T]}. \tag{3.2}$$

Proof Starting to integrate by part, we obtain

$$\begin{aligned}\int_0^T \mathbf{D}^{\alpha}_{0,t} f(t)\psi(t)\,\mathrm{d}t &= -\int_0^T g_{1-\alpha} * f(t)\frac{\mathrm{d}}{\mathrm{d}t}\psi(t)\,\mathrm{d}t + \left[g_{1-\alpha} * f\,\psi\right]_0^T \\ &= -\int_0^T f(t) g_{1-\alpha} *' \frac{\mathrm{d}}{\mathrm{d}t}\psi(t)\,\mathrm{d}t + \left[g_{1-\alpha} * f\,\psi\right]_0^T \quad \text{(by Rem. 2.1)} \\ &= -\int_0^T f(t)^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\psi(t)\,\mathrm{d}t + \left[g_{1-\alpha} * f\,\psi\right]_0^T \quad \text{(by Def. 3.2).}\end{aligned}$$

In order to prove (3.2), we use Cauchy-Schwarz inequality and the estimate

$$\|\mathbf{D}^{\alpha}_{t,T}\psi\|_{L^\infty(0,T)} \le g_{2-\alpha}(T)\|\psi'\|_{L^\infty(0,T)}.$$

□

That property allows us to define fractional derivative in the sense of distributions. Indeed, (3.2) shows that the linear map

$$\mathcal{D}(0,T) \to X, \quad \varphi \mapsto -\int_0^T f(t)^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\varphi(t)\,\mathrm{d}t$$

is a distribution, whose order is (at most) 1. The set of distributions with values in X is denoted by $\mathcal{D}'(0,T;X)$. That allows us to set the following definition.

Definition 3.3 Let $\alpha \in (0,1)$ and $f \in L^2(0,T;X)$. Then the *weak derivative of order* α of f is the vector valued distribution, denoted by ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$, and defined, for all $\varphi \in \mathcal{D}(0,T)$, by

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, \varphi \right\rangle = -\int_0^T f(t)^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\varphi(t)\,\mathrm{d}t.$$

If we want to highlight the duality taking place in the above bracket, we will write

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, \varphi \right\rangle_{\mathcal{D}'(0,T;X),\mathcal{D}(0,T)}$$

instead of $\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, \varphi \right\rangle$. The following result states that weak derivative extend fractional derivatives in $L^2(0,T;X)$. That justifies the use of the same notation in Definitions 3.1 and 3.3.

Proposition 3.2 *Let $\alpha \in (0,1)$ and $f \in L^2(0,T;X)$.*

(i) *If f admits a derivative of order α in $L^2(0,T;X)$ (in the sense of* Definition 3.1) *then that derivative is equal to the weak derivative of f.*
(ii) If the weak derivative of *f* belongs to $L^2(0,T;X)$ *then f admits a derivative in $L^2(0,T;X)$ and these two derivatives are equal.*

Proof (i) Let ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$ be the derivative of *f* in $L^2(0,T;X)$. Then, for each $\varphi \in \mathcal{D}(0,T)$, Proposition 3.1 leads to

$$\int_0^T {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f(t)\varphi(t)\,\mathrm{d}t = -\int_0^T f(t){}^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\varphi(t)\,\mathrm{d}t.$$

Then Definition 3.3 tells us that ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$ is the weak derivative of *f*.

(ii) Let ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$ denote the weak derivative of *f* (in the sense of Definition 3.3). Then

$$\begin{aligned}\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, \varphi \right\rangle_{\mathcal{D}'(0,T;X),\mathcal{D}(0,T)} &= -\int_0^T f(t) g_{1-\alpha} *' \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\,\mathrm{d}t \\ &= -\int_0^T g_{1-\alpha} * f(t)\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\,\mathrm{d}t,\end{aligned}$$

by Remark 2.1. Since, by assumption, ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$ lies in $L^2(0,T;X)$ we deduce that $g_{1-\alpha} * f$ is in $H^1(0,T;X)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\{g_{1-\alpha} * f\} = {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, \quad \text{in } L^2(0,T;X).$$

□

Proposition 3.3 *Let $\alpha \in (0,1)$, V be a real Banach space and $f \in L^2(0,T;V')$. We assume that f admits a derivative of order α in $L^2(0,T;V')$. Then, for each v* in *V*, *$\langle f, v\rangle_{V',V}$ admits a derivative of order α in $L^2(0,T)$ and*

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f(\cdot), v \right\rangle_{V',V} = {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t}\left\{\langle f, v\rangle_{V',V}\right\}, \quad \text{in } L^2(0,T). \tag{3.3}$$

Above V' denotes the dual space of V and $\langle\cdot,\cdot\rangle_{V',V}$, the duality between V' and V.

Proof Let $\varphi \in \mathcal{D}(0,T)$. Since, for each $v \in V$, the linear map $\langle\cdot, v\rangle_{V',V}$ is bounded on V', we have (see for instance [1], Proposition 1.1.6])

$$I := \int_0^T \left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f(t), v \right\rangle_{V',V} \varphi(t)\,\mathrm{d}t = \left\langle \int_0^T {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f(t)\varphi(t)\,\mathrm{d}t, v \right\rangle_{V',V}.$$

Then, with Proposition 3.1,

$$\begin{aligned} I &= \left\langle -\int_0^T f(t)\,{}^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\varphi(t)\,\mathrm{d}t, v \right\rangle_{V',V} \\ &= -\int_0^T \langle f(t), v\rangle_{V',V}\,{}^{\mathrm{R}}\mathbf{D}^{\alpha}_{t,T}\varphi(t)\,\mathrm{d}t. \end{aligned}$$

Then, we infer from Definition 3.3 that

$$I = \left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t}\langle f(\cdot), v\rangle_{V',V}, \varphi \right\rangle_{\mathcal{D}'(0,T),\mathcal{D}(0,T)}.$$

Hence

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f, v \right\rangle_{V',V} = {}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t}\langle f(\cdot), v\rangle_{V',V}, \quad \text{in } \mathcal{D}'(0,T).$$

By assumption, ${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t} f$ belongs to $L^2(0,T;V')$, thus that identity holds in $L^2(0,T)$. By Proposition 3.2 (ii), we deduce that $\langle f(t), v\rangle_{V',V}$ admits a derivative of order α in $L^2(0,T)$. That completes the proof. □

Proposition 3.4 *Let* $\alpha \in (0,1)$ *and* $u \in L^2(0,T;X)$. *If* u *admits a derivative of order* α *in* $L^2(0,T;X)$, *then*

$$u = (g_{1-\alpha} * u)(0)g_\alpha + g_\alpha * {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u \quad \text{in } L^1(0,T;X). \tag{3.4}$$

Proof The proof is rather standard, we just emphasize the functional spaces involved. By integration, we have

$$g_{1-\alpha} * u = (g_{1-\alpha} * u)(0) + g_1 * {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u \quad \text{in } H^1(0,T;X).$$

By [9], Proposition 2.6] or [1], Proposition 1.3.6], we know that

$$U \in H^1(0,T;X) \Longrightarrow g_\alpha * U \in W^{1,1}(0,T;X).$$

Thus, with (2.1)

$$g_1 * u = (g_{1-\alpha} * u)(0)g_{1+\alpha} + g_{1+\alpha} * {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u \quad \text{in } W^{1,1}(0,T;X).$$

By differentiation and using a slight variant of [9], Proposition 2.6], we get (3.4). □

Proposition 3.5 *Let* $\alpha \in (0,1)$ *and* $u \in C([0,T];X)$ *be such that* ${}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u$ *lies in* $C([0,T];X)$. *Then* $u(0) = 0$.

Proof Since u is continuous on $[0,T]$, there holds $(g_{1-\alpha} * u)(0) = 0$. Thus, with (3.4),

$$u = g_\alpha * {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u.$$

By continuity of ${}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u$, we get $u(0) = 0$. □

4 Galerkin method for a time fractional PDE

Let $d \geq 1$ and Ω be an open bounded subset of $\mathbb{R}^d$. We refer to [2] for the definition of the standard Sobolev spaces $H^1_0(\Omega)$ and $H^{-1}(\Omega)$.

Definition 4.1 Let $\alpha \in (0,1)$ and $T > 0$. Then we denote by

$$\mathcal{H}^\alpha\big(0,T;H^1_0(\Omega),H^{-1}(\Omega)\big),$$

the set of all functions in $L^2(0,T;H^1_0(\Omega))$ whose weak fractional derivative of order α belongs to $L^2(0,T;H^{-1}(\Omega))$.

Let $f \in L^2(0,T;H^{-1}(\Omega))$ and $v \in L^2(\Omega)$. We will focus on the following model problem.

$$\begin{cases} \text{Find } \ u \in \mathcal{H}^{\alpha}\big(0,T;H_0^1(\Omega),H^{-1}(\Omega)\big) & \text{such that} \\ {}^{\mathrm{R}}\mathbf{D}_{0,t}^{\alpha} u - \Delta u = f & \text{in } \ L^2(0,T;H^{-1}(\Omega)) \\ (g_{1-\alpha} * u)(0) = v & \text{in } \ L^2(\Omega). \end{cases} \tag{4.1}$$

In (4.1), the initial condition means that

$$(g_{1-\alpha} * u)(t) \xrightarrow[t\to 0^+]{} v \quad \text{in } \ L^2(\Omega).$$

4.1 Well posedness

Theorem 4.1 *Let $f \in L^2(0,T;H^{-1}(\Omega))$ and $v \in H_0^1(\Omega)$.*

(i) *If $\alpha \in (\frac{1}{2},1)$ then (4.1) has a unique solution.*
(ii) *If $\alpha \in (0,\frac{1}{2}]$ then*

  (a) *if $v \neq 0$ then (4.1) has no solution.*
  (b) *if $v = 0$ then (4.1) has a unique solution.*

Proof Combining (3.4) and (2.2), we derive that (4.1) has no solution if $\alpha \leq 1/2$ and $v \neq 0$. On the other hand, if $v = 0$ then the solvability of (4.1) can be achieved as in the case where $\alpha \in (\frac{1}{2},1)$. Thus we will assume in the sequel that $\alpha > 1/2$.

Existence of a solution. We will implement the Galerkin approximation method. For, let us introduce some notation. Let $V := H_0^1(\Omega)$ and

$$A : H_0^1(\Omega) \to H^{-1}(\Omega), \quad u \mapsto -\Delta u.$$

For $k = 1,2,\ldots$, let $(w_k,\lambda_k) \in H_0^1(\Omega) \times (0,\infty)$ be a k^{th} mode of A such that $(w_k)_{k\geq 1}$ forms an hilbertian basis of $L^2(\Omega)$.

For $n = 1,2,\ldots$, we denote by F_n the vector space generated by $w_1,\ldots,w_n$. Finally, we decompose the initial condition v, by writing

$$v = \sum_{k\geq 1} b_k w_k \quad \text{in } \ H_0^1(\Omega),$$

and we set

$$v_n := \sum_{k=1}^{n} b_k w_k. \tag{4.2}$$

Whence $v_n \in F_n$ and $v_n \to v$ in $H_0^1(\Omega)$.

For each integer $n \geq 1$, our approximated problem takes the form

(i) *Solvability of the approximated problem* . The decomposition and notation

$$\begin{cases} \text{Find } u_n \in L^2(0,T;F_n) \text{ such that } {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n \in L^2\big(0,T;H^{-1}(\Omega)\big) \\ \big\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n, w\rangle_{V',V} + \big\langle Au_n, w\rangle_{V',V} = \langle f, w\rangle_{V',V} \quad \text{in } L^2(0,T), \quad \forall w \in F_n \\ (g_{1-\alpha} * u_n)(0) = v_n. \end{cases} \tag{4.3}$$

$$u_n(t) = \sum_{k=1}^{n} x_k(t) w_k, \quad f_k(t) := \langle f(t), w_k\rangle_{V',V},$$

lead to the equivalent system:

$$\begin{cases} {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} x_k + \lambda_k x_k = f_k \text{ in } L^2(0,T) \\ (g_{1-\alpha} * x_k)(0) = b_k \end{cases}, \quad \forall k = 1, \dots, n. \tag{4.4}$$

Surprisingly, we have not found a well-posedness result for (4.4) in the literature. However, the local well-posedness in $L^2(0,\tau)$, for small positive τ can be obtained by standard fix point method (see [8], Chap 5] where another functional setting is used).

Regarding global well-posedness i.e. well-posedness on [0, T] for all $T > 0$, adapting to our framework, Lemma 4.2 in [18] and Theorem 10 of [7], we may obtain a *blow-up alternative*. Namely, if the maximal existence time T_m is finite then the corresponding maximal solution u to (4.4), fulfills

$$\|u\|_{L^2(0,\tau)} \to \infty, \quad \text{as } \tau \to T_m^-. \tag{4.5}$$

Let us notice that

$$\sup_{\tau\in(0,T_m)} \|u\|_{L^2(0,\tau)} < \infty$$

implies by the monotone convergence theorem, that u lies in $L^2(0,T_m)$.

So, in order to get global well-posedness, we assume that T_m is finite. Then, for each $\tau \in (0,T_m)$, we have by (4.4), Proposition 3.4 and (2.2),

$$x_k + \lambda_k g_\alpha * x_k = b_k g_\alpha + g_\alpha * f_k, \quad \text{in } L^2(0,\tau).$$

We multiply that equation by x_k and integrate on $[0,\tau]$. By Theorem 2.1,

$$\int_0^\tau x_k(t) g_\alpha * x_k(t)\,\mathrm{d}t \geq 0.$$

Thus since $\lambda_k \geq 0$ and $\alpha > 1/2$, we get

$$\|x_k\|^2_{L^2(0,\tau)} \leq |b_k| \|g_\alpha\|_{L^2(0,T_m)} \|x_k\|_{L^2(0,\tau)} + \|g_\alpha * f_k\|_{L^2(0,T_m)} \|x_k\|_{L^2(0,\tau)}.$$

Then $\|x_k\|_{L^2(0,\tau)}$ remains bounded as τ approaches T_m. That contradicts (4.5), so that $T_m = \infty$. Thus (4.3) admits an unique solution for all positive time T.

(ii) *Estimates*. Using $g_\alpha \in L^2(0, T)$ and taking $w = v_n$ in (4.3), we derive

$$\int_0^T \left\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n, u_n - g_\alpha v_n \right\rangle_{V',V} \mathrm{d}t + \int_0^T \langle A u_n, u_n - g_\alpha v_n \rangle_{V',V} \, \mathrm{d}t = \int_0^T \langle f, u_n - g_\alpha v_n \rangle_{V',V} \, \mathrm{d}t. \tag{4.6}$$

Let us show that the first integral above is non negative; this is the key point of our proof. For, in view of Proposition 3.4, there holds

$$u_n - g_\alpha v_n = g_\alpha * {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n \quad \text{in } L^2(0, T; H^1_0(\Omega)). \tag{4.7}$$

Thus, setting for simplicity $\mathbf{D}^\alpha$ instead of ${}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}$,

$$\begin{aligned}
&\int_0^T \langle \mathbf{D}^\alpha u_n, u_n - g_\alpha v_n \rangle_{V',V} \, \mathrm{d}t \\
&= \int_0^T \langle \mathbf{D}^\alpha u_n, g_\alpha * \mathbf{D}^\alpha u_n \rangle_{V',V} \, \mathrm{d}t \\
&= \int_0^T \mathrm{d}t \int_0^t g_\alpha(t-y) \langle \mathbf{D}^\alpha u_n(t), (\mathbf{D}^\alpha u_n)(t-y) \rangle_{V',V} \mathrm{d}y \\
&= \sum_{k=1}^n \int_0^T \mathrm{d}t \int_0^t g_\alpha(t-y) \mathbf{D}^\alpha x_k(t) (\mathbf{D}^\alpha x_k)(t-y) \mathrm{d}y,
\end{aligned}$$

since $\langle w_k, w_j \rangle_{V',V} = \delta_{k,j}$. By Theorem 2.1, the latter right hand side is the sum of non-negative numbers. Hence

$$\int_0^T \left\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n, u_n - g_\alpha v_n \right\rangle_{V',V} \mathrm{d}t \geq 0.$$

Going back to (4.6), we derive

$$\int_0^T \langle Au_n, u_n\rangle_{V',V}\,\mathrm{d}t \le \int_0^T |\langle Au_n, v_n\rangle_{V',V}| g_\alpha(t)\,\mathrm{d}t$$
$$+\int_0^T |\langle f, u_n\rangle_{V',V}|\,\mathrm{d}t + \int_0^T |\langle f, v_n\rangle_{V',V}| g_\alpha(t)\,\mathrm{d}t.$$

Since

$$\int_0^T \langle Au_n, u_n\rangle_{V',V}\,\mathrm{d}t = \|u_n\|^2_{L^2(0,T;H_0^1(\Omega))}$$

and $v_n \to v$ in $H_0^1(\Omega)$, we derive in a standard way that

$$\|u_n\|_{L^2(0,T;H_0^1(\Omega))} \le C,$$

where the constant C is independent of n. Then there exists some $u \in L^2(0,T;H_0^1(\Omega))$ such that, up to a subsequence,

$$u_n \rightharpoonup u \quad \text{in} \;\; L^2(0,T;H_0^1(\Omega))\text{–}weak. \tag{4.8}$$

(iii) Equation of (4.1). Let $k \ge 1$ be fixed and $n \ge k$. For each $\varphi \in \mathcal{D}(0,T)$, we derive from (4.3) and Proposition 3.1 that

$$\left\langle \int_0^T -u_n(t)^{\mathrm{R}}\mathbf{D}^\alpha_{t,T}\varphi(t) + \big(Au_n - f(t)\big)\varphi(t)\,\mathrm{d}t, w_k \right\rangle_{V',V} = 0.$$

Passing to the limit in n and using Definition 3.3, we get

$$\mathbf{D}^\alpha u + Au - f = 0 \quad \text{in} \;\; \mathcal{D}'(0,T;H^{-1}(\Omega)).$$

Since Au and f belong to $L^2(0,T;H^{-1}(\Omega))$, we derive from Proposition 3.2 (ii), that u lies in $\mathcal{H}^\alpha\big(0,T;H_0^1(\Omega),H^{-1}(\Omega)\big)$ and

$$\mathbf{D}^\alpha u + Au = f \quad \text{in} \;\; L^2(0,T;H^{-1}(\Omega)).$$

(iv) *Initial condition*. Let $k, n \ge 1$ and $\varphi \in \mathcal{D}(0,T)$ with $\varphi(T) = 0$. Then, due to Propositions 3.3 and 3.1,

$$\begin{aligned}
&\int_0^T \langle \mathbf{D}^\alpha u_n(t), w_k\rangle_{V',V}\varphi(t)\,\mathrm{d}t \\
&= -\int_0^T \langle u_n(t), w_k\rangle_{V',V}\, {}^{\mathrm{R}}\mathbf{D}^\alpha_{t,T}\varphi(t)\,\mathrm{d}t - \langle g_{1-\alpha} * u_n(0), w_k\rangle_{V',V}\varphi(0) \\
&\xrightarrow[n\to\infty]{} -\int_0^T \langle u(t), w_k\rangle_{V',V}\, {}^{\mathrm{R}}\mathbf{D}^\alpha_{t,T}\varphi(t)\,\mathrm{d}t - \langle v, w_k\rangle_{V',V}\varphi(0),
\end{aligned}$$

by (4.8) and (4.2). Moreover, using Propositions 3.1 and 3.3 once again, the latter limit is equal to

$$\int_0^T \langle \mathbf{D}^\alpha u(t), w_k\rangle_{V',V}\,\mathrm{d}t + \langle g_{1-\alpha} * u(0), w_k\rangle_{V',V}\varphi(0) - \langle v, w_k\rangle_{V',V}\varphi(0).$$

Then, we get in a usual way (see for instance [2], Chap 11]) that $g_{1-\alpha} * u(0) = v$. That completes the proof of the existence part.

Uniqueness of the solution. By linearity, it is enough to prove that any function u in $\mathcal{H}^\alpha\big(0,T;H^1_0(\Omega),H^{-1}(\Omega)\big)$, solution to

$$\begin{aligned}
&{}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t}u - \Delta u = 0 \quad \text{in } L^2(0,T;H^{-1}(\Omega)) \\
&(g_{1-\alpha} * u)(0) = 0 \quad \text{in } L^2(\Omega),
\end{aligned}$$

is trivial. For, testing the above equation with

$$u_\alpha := g_{1-\alpha} * u \in L^2\big(0,T;H^1_0(\Omega)\big),$$

we get

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t}u_\alpha, u_\alpha\right\rangle_{V',V}\mathrm{d}t + \int_0^T\int_\Omega \nabla u \nabla u_\alpha\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Moreover, since $u_\alpha(0) = 0$,

$$\int_0^T \left\langle \frac{\mathrm{d}}{\mathrm{d}t} u_\alpha, u_\alpha \right\rangle_{V',V} \mathrm{d}t = \frac{1}{2} \|u_\alpha(T)\|^2_{L^2(\Omega)}$$

and, by Theorem 2.1,

$$\int_0^T \int_\Omega \nabla u \nabla u_\alpha \,\mathrm{d}x\,\mathrm{d}t = \sum_{k=1}^n \int_\Omega \int_0^T \partial_{x_k} u(t,x) g_{1-\alpha} * \partial_{x_k} u(\cdot,x)(t)\,\mathrm{d}t\,\mathrm{d}x \geq 0.$$

Then, for all $t \in [0,T]$, we deduce $g_{1-\alpha} * u(t) = 0$. Thus, with (2.1)

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\{g_\alpha * g_{1-\alpha} * u\}(t) = 0.$$

That completes the proof of the þ. □

4.2 Regularity

Similarly to the case $\alpha = 1$, regularity of the solution to (4.1) is obtained assuming some smoothness conditions on the data. However, no additional assumption is made on the domain Ω. Let us recall that the operator A is defined by

$$A : H_0^1(\Omega) \to H^{-1}(\Omega), \quad u \mapsto -\Delta u.$$

Theorem 4.2 *Let $\alpha \in (0,1)$, Ω be an open bounded subset of $\mathbb{R}^d$, f be in $L^2(0,T;L^2(\Omega))$, and v belong to $H_0^1(\Omega)$.*

(i) *If $\alpha \in (\frac{1}{2}, 1)$ then assume that Av lies in $L^2(\Omega)$;*
(ii) *If $\alpha \in (0, \frac{1}{2}]$ then assume that $v = 0$.*

Then the solution u to (4.1) satisfies

$$Au,\ {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u \in L^2(0,T;L^2(\Omega))$$
$${}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u + Au = f \quad \text{in } L^2(0,T;L^2(\Omega)).$$

Remark 4.1 Theorem 4.2 is not a regularity result in $H^2(\Omega)$. Indeed, we do not claim that u belongs to $L^2(0,T;H^2(\Omega))$. Moreover, since Ω is only assumed to be a bounded open set, the eigenfuntion w_k does not belong to $H^2(\Omega)$, in general.

$H^2(\Omega)$-regularity results may be obtained by assuming for instance, that Ω is convex (see [10], Theorem 3.2.1.2]).

Proof Arguing as in the proof of Theorem 4.1, we will focus on the case $\alpha > \frac{1}{2}$. Let us recall that $(w_k, \lambda_k) \in H_0^1(\Omega) \times (0, \infty)$ denotes a k^{th} mode of A and that v_n is defined by (4.2). Let u_n be the solution to (4.3). Since $Aw_k = \lambda_k w_k$, it is clear that $A(u_n(t) - g_\alpha(t)v_n)$ belongs to F_n for all $t \in (0,T)$. Thus (4.3) leads to

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n, A(u_n - g_\alpha v_n)\right\rangle_{V',V} + \langle Au_n, A(u_n - g_\alpha v_n)\rangle_{V',V} = \langle f, A(u_n - g_\alpha v_n)\rangle_{V',V}.$$

In view of (4.7), we derive

$$\left\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n(t), A(u_n(t) - g_\alpha(t)v_n)\right\rangle_{V',V} = \sum_{k=1}^n \lambda_k \mathbf{D}^\alpha x_k(t) g_\alpha * \mathbf{D}^\alpha x_k(t).$$

Then, Theorem 2.1 leads to

$$\int_0^T \left\langle {}^{\mathrm{R}}\mathbf{D}^\alpha_{0,t} u_n, A(u_n - g_\alpha v_n)\right\rangle_{V',V} \mathrm{d}t \geq 0.$$

In order to estimate $\langle f, g_\alpha A v_n\rangle_{V',V}$, we recall that

$$\langle f, h\rangle_{V',V} = \int_\Omega f(x)h(x)\,\mathrm{d}x, \quad \forall f \in L^2(\Omega), \quad \forall\, h \in H_0^1(\Omega).$$

Thus

$$\left|\langle f(t), g_\alpha(t) A v_n\rangle_{V',V}\right| \leq g_\alpha(t)\|f(t)\|_{L^2(\Omega)}\|Av_n\|_{L^2(\Omega)}.$$

Moreover, $Av_n = \sum_{k=1}^n \lambda_k b_k w_k$ and $Av \in L^2(\Omega)$, thus Lemma 4.3 below implies that $\|Av_n\|_{L^2(\Omega)}$ is bounded.

Thus, estimating in a standard way, we obtain that a subsequence of (Au_n) converges weakly in $L^2(0,T;L^2(\Omega))$. Hence, by the uniqueness of the limit, Au belongs to $L^2(0,T;L^2(\Omega))$. □

The following lemma is used is the proof of Theorem 4.2.

Lemma 4.3 *Let Ω be an open bounded subset of $\mathbb{R}^d$. For each $v \in H_0^1(\Omega)$ with $v = \sum b_k w_k$ in $H_0^1(\Omega)$, one has*

$$Av \in L^2(\Omega) \Leftrightarrow \sum (b_k \lambda_k)^2 < \infty.$$

Proof Let us assume that Av lies in $L^2(\Omega)$. Since (w_k) is an hilbertian basis of $L^2(\Omega)$, there exists a sequence $(c_k)_{k\geq 0} \subset \mathbb{R}$ such that

$$Av = \sum_{k\geq 1} c_k w_k, \quad \sum (c_k)^2 < \infty, \quad c_k = \int_\Omega Av(x) w_k(x)\,\mathrm{d}x.$$

Moreover,

$$\int_\Omega Av(x) w_k(x)\,\mathrm{d}x = \int_\Omega \nabla v(x) \nabla w_k(x)\,\mathrm{d}x = b_k \lambda_k.$$

Hence, $\sum (b_k \lambda_k)^2 < \infty$.

Conversely, let $v_n := \sum_{k\geq 1}^n b_k w_k$. Since $Aw_k = \lambda_k w_k$, we know that Aw_k is in $L^2(\Omega)$. Thus, for $1 \leq m < n$,

$$\|Av_n - Av_m\|^2_{L^2(\Omega)} = \sum_{k=m+1}^n (b_k \lambda_k)^2.$$

By assumption $\sum (b_k \lambda_k)^2$ converges; so that there exists some $f \in L^2(\Omega)$ such that

$$Av_n \to f \quad \text{in} \quad L^2(\Omega).$$

However, $v_n \to v$ in $H_0^1(\Omega)$ and the operator A is continuous from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$. Thus $Av_n \to Av$ in $H^{-1}(\Omega)$. Whence Av lies in $L^2(\Omega)$. □

5 The "ℓ goes to plus infinity" issue

In many physical situations, three dimensional problems are sometimes approximated by two dimensional problems. That procedure simplifies the mathematical analysis and decreases the computational cost of discretisation algorithms.

The issue is then to estimate the error made by replacing the solution of the 3D problem by the solution of some 2D problem. We refer to the books [3, 4] for more informations on that subject. Basically, if the 3D problem is set on a cylinder with large height, then the solution will be locally well approximated by the solution of the "same problem " set on the section of the cylinder.

Regarding fractional derivatives, the paper [6] is concerned with the fractional Laplacian. Here, we will look at linear time fractional diffusion problems. More precisely, let $p < d$ be positive integers, ω be an open bounded subset of $\mathbb{R}^{d-p}$ and

$$\Omega_\ell := (-\ell, \ell)^p \times \omega \subset \mathbb{R}^p \times \mathbb{R}^{d-p}.$$

We write any $x \in \Omega_\ell$ as $x = (X_1, X_2)$, where $X_1 \in \mathbb{R}^p$ and $X_2 \in \mathbb{R}^{d-p}$.

For f in $L^2(0,T;L^2(\omega))$ and $v \in H_0^1(\omega)$, we consider the problem

$$\begin{cases} \text{Find } u_\ell \in \mathcal{H}^\alpha\big(0,T;H_0^1(\Omega_\ell),H^{-1}(\Omega_\ell)\big) & \text{such that} \\ {}^{\mathrm{R}}\mathbf{D}_{0,t}^{\alpha} u_\ell - \Delta u_\ell = f & \text{in } L^2(0,T;H^{-1}(\Omega_\ell)) \\ (g_{1-\alpha} * u_\ell)(0) = v & \text{in } L^2(\Omega_\ell). \end{cases} \tag{5.1}$$

Then the problem set on the section ω is

$$\begin{cases} \text{Find } u_\infty \in \mathcal{H}^\alpha\big(0,T;H_0^1(\omega),H^{-1}(\omega)\big) & \text{such that} \\ {}^{\mathrm{R}}\mathbf{D}_{0,t}^{\alpha} u_\infty - \Delta u_\infty = f & \text{in } L^2(0,T;H^{-1}(\omega)) \\ (g_{1-\alpha} * u_\infty)(0) = v & \text{in } L^2(\omega). \end{cases} \tag{5.2}$$

Theorem 5.1 *Let ω and Ω_ℓ as above, $\alpha \in (0,1)$, f belong to $L^2(0,T;L^2(\omega))$ and $v \in H_0^1(\omega)$.*

(i) *If $\alpha \in (\frac{1}{2}, 1)$ then assume that Av lies in $L^2(\Omega)$;*
(ii) *If $\alpha \in (0, \frac{1}{2}]$ then assume that $v = 0$.*

Then there exists two positive constants ε and C such that, for all $\ell > 0$, the solutions u_ℓ and u_∞ to (5.1) and (5.2) satisfy

$$\int_0^T \int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\mathrm{e}^{-\varepsilon\ell}. \tag{5.3}$$

Of course, we deduce from the above result (using also Poincaré inequality) that, for any fixed $\ell_0 > 0$,

$$u_\ell \xrightarrow[\ell\to\infty]{} u_\infty \quad \text{in } L^2\big(0,T;H^1(\Omega_{\ell_0})\big),$$

with exponential convergence rate. Let us recall that the Poincaré constant is independent of ℓ: see for instance [5], Lemma 2.1.

Proof As in the previous section, we will only study the case where $\alpha > 1/2$. For $\ell \geq 1$ and $\ell_1 \leq \ell - 1$, consider a function $\rho = \rho_{\ell_1} : \mathbb{R}^p \to [0, \infty)$ such that

$$\rho_{\ell_1} = 1 \quad \text{on } \Omega_{\ell_1}, \quad \rho_{\ell_1} = 0 \quad \text{on } \mathbb{R}^p \backslash \Omega_{\ell_1+1} \tag{5.4}$$

$$\left|\nabla \rho_{\ell_1}(X_1)\right| \leq C, \quad \forall X_1 \in \mathbb{R}^p, \tag{5.5}$$

where C is independent of X_1 and ℓ_1. Also, by Theorem 4.2, one has

$${}^{\mathrm{R}}\mathbf{D}^{\alpha}_{0,t}(u_\ell - u_\infty) + A(u_\ell - u_\infty) = 0 \quad \text{in } L^2(0, T; L^2(\Omega_\ell)). \tag{5.6}$$

Moreover, $(u_\ell - u_\infty)\rho$ is in $L^2(0, T; H^1_0(\Omega_\ell))$ by (5.45.5). Thus testing (5.6) with this function, we get

$$\int_{\Omega_\ell} \rho(X_1)\,\mathrm{d}x \int_0^T \mathbf{D}^\alpha(u_\ell - u_\infty)\,(u_\ell - u_\infty)\,\mathrm{d}t + \int_0^T \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \rho(X_1)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^T \int_{\Omega_{\ell_1+1}\backslash\Omega_{\ell_1}} \nabla(u_\ell - u_\infty)\nabla\rho(X_1)\,(u_\ell - u_\infty)\,\mathrm{d}x\,\mathrm{d}t.$$

The first integral above is positive due to Theorem 2.1, since $u_\ell - u_\infty = g_\alpha * \mathbf{D}^\alpha(u_\ell - u_\infty)$, by Proposition 3.4. Next, using $|\nabla\rho| \leq C$, Young and Poincaré inequalities, we get in a standard way, the following bound of the latter integral:

$$C \int_0^T \int_{\Omega_{\ell_1+1}\backslash\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, since $\rho = 1$ on Ω_{ℓ_1}, we derive

$$\int_0^T \int_{\Omega_{\ell_1}} |\nabla(u_\ell - u_\infty)|^2\,\mathrm{d}x\,\mathrm{d}t \leq \frac{C}{1+C} \int_0^T \int_{\Omega_{\ell_1+1}} |\nabla(u_\ell - u_\infty)|^2\,\mathrm{d}x\,\mathrm{d}t.$$

There results (see [4] Section 1.7) that

$$\int_0^T \int_{\Omega_{\ell/2}} |\nabla(u_\ell - u_\infty)|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\mathrm{e}^{-2\varepsilon\ell} \int_0^T \int_{\Omega_\ell} |\nabla(u_\ell - u_\infty)|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

There remains to estimate the latter integral. For, using the regularity result of Theorem 4.2 and testing (5.1) with $u_\ell - g_\alpha(t)v$, we get

$$\int_0^T \int_{\Omega_\ell} \mathbf{D}^\alpha u_\ell (u_\ell - g_\alpha(t)v) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_{\Omega_\ell} f(t, X_2)(u_\ell - g_\alpha(t)v) \,\mathrm{d}x\,\mathrm{d}t + \int_0^T g_\alpha(t)\,\mathrm{d}t \int_{\Omega_\ell} \nabla u_\ell \nabla v \,\mathrm{d}x.$$

The first term is positive by Theorem 2.1 and Proposition 3.4. Moreover, Young and Poincaré inequalities yield

$$\left| \int_0^T \int_{\Omega_\ell} f(t, X_2)(u_\ell - g_\alpha(t)v) \,\mathrm{d}x\,\mathrm{d}t \right| \le C_{\varepsilon'} (2\ell)^p \|f\|^2_{L^2(0,T;\,L^2(\omega))}$$
$$+ \varepsilon' C \int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{(2\ell)^p}{2} \|g_\alpha\|^2_{L^2(0,T)} \|v\|^2_{L^2(\omega)}.$$

Choosing the positive constant ε' sufficiently small, there results that

$$\int_0^T \int_{\Omega_\ell} |\nabla u_\ell|^2 \,\mathrm{d}x\,\mathrm{d}t \le C\ell^p, \quad \forall \ell \ge 1.$$

Performing the same computation with u_∞, we obtain (5.3). That completes the proof of the Theorem. □

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