CrossMark

On the longtime behavior of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions

Pierluigi Colli[1] · Gianni Gilardi[1] · Jürgen Sprekels[2,3]

Abstract

In this paper, we study the longtime asymptotic behavior of a phase separation process occurring in a three-dimensional domain containing a fluid flow of given velocity. This process is modeled by a viscous convective Cahn–Hilliard system, which consists of two nonlinearly coupled second-order partial differential equations for the unknown quantities, the chemical potential and an order parameter representing the scaled density of one of the phases. In contrast to other contributions, in which zero Neumann boundary conditions were assumed for both the chemical potential and the order parameter, we consider the case of dynamic boundary conditions, which model the situation when another phase transition takes place on the boundary. The phase transition processes in the bulk and on the boundary are driven by free energy functionals that may be nondifferentiable and have derivatives only in the sense of (possibly set-valued) subdifferentials. For the resulting initial-boundary value system of Cahn–Hilliard type, general well-posedness results have been established in a recent contribution by the same authors. In the present paper, we investigate the asymptotic behavior of the solutions as times approaches infinity. More precisely, we study the ω-limit (in a suitable topology) of every solution trajectory. Under the assumptions that the viscosity coefficients are strictly positive and that at least one of the underlying free energies is differentiable, we prove that the ω-limit is meaningful and that all of its elements are solutions to the corresponding stationary system, where the component representing the chemical potential is a constant.

Dedicated to our friend Prof. Dr. Alexander Mielke on the occasion of his 60th birthday with best wishes.

✉ Pierluigi Colli
   pierluigi.colli@unipv.it

Extended author information available on the last page of the article

⌂ Springer

1 Introduction

The recent paper [16] addresses an initial-boundary value problem for the Cahn–Hilliard system with convection

$$\partial_t \rho + \nabla \rho \cdot u - \Delta \mu = 0 \quad \text{and} \quad \tau_\Omega \partial_t \rho - \Delta \rho + f'(\rho) = \mu \qquad \text{in } Q_T := \Omega \times (0, T), \tag{1}$$

in the unknowns ρ, the order parameter, and μ, the chemical potential. In the above equations, τ_Ω is a nonnegative constant, f' is the derivative of a double-well potential f, and u is a given velocity field.

Typical and significant examples of f are the so-called *classical regular potential*, the *logarithmic double-well potential*, and the *double obstacle potential*, which are given by

$$f_{reg}(r) := \frac{1}{4} (r^2 - 1)^2, \quad r \in \mathbb{R}, \tag{2}$$

$$f_{log}(r) := ((1 + r) \ln(1 + r) + (1 - r) \ln(1 - r)) - c_1 r^2, \quad r \in (-1, 1), \tag{3}$$

$$f_{2obs}(r) := c_2(1 - r^2) \quad \text{if } |r| \leq 1 \qquad \text{and} \qquad f_{2obs}(r) := +\infty \quad \text{if } |r| > 1, \tag{4}$$

where the constants in (3) and (4) satisfy $c_1 > 1$ and $c_2 > 0$, so that f_{log} and f_{2obs} are nonconvex. In cases like (4), one has to split f into a nondifferentiable convex part (the indicator function of $[-1, 1]$ in the present example) and a smooth perturbation. Accordingly, one has to replace the derivative of the convex part by the subdifferential and interpret the second identity in (1) as a differential inclusion.

As far as the conditions on the boundary $\Gamma := \partial\Omega$ are concerned, instead of the classical homogeneous Neumann boundary conditions, the dynamic boundary condition for both μ and ρ are considered, namely,

$$\partial_t \rho_\Gamma + \partial_n \mu - \Delta_\Gamma \mu_\Gamma = 0 \quad \text{and}$$
$$\tau_\Gamma \partial_t \rho_\Gamma + \partial_n \rho - \Delta_\Gamma \rho_\Gamma + f'_\Gamma(\rho_\Gamma) = \mu_\Gamma \quad \text{on } \Sigma_T := \Gamma \times (0, T), \tag{5}$$

where $\mu_\Gamma = \mu_{|\Sigma_T}$, $\rho_\Gamma = \rho_{|\Sigma_T}$, are the traces of μ and ρ, respectively, ∂_n and Δ_Γ denote the outward normal derivative and the Laplace–Beltrami operator on Γ, τ_Γ is a nonnegative constant, and f'_Γ is the derivative of another potential f_Γ.

The equations in (1) aim to describe a class of evolution phenomena with phase separation and fluid convection, in which the convection is represented by the term $\nabla \rho \cdot u$ for a known velocity vector u. Regarding this system of partial differential equations, some boundary conditions are usually prescribed and the standard approach leads to the no-flux conditions

$$\partial_n \mu = 0, \quad \partial_n \rho = 0 \qquad \text{on } \Sigma_T,$$

for both μ and ρ. On the contrary, as already announced, here we are interested to handle the dynamic boundary conditions (5), which set a Cahn–Hilliard type system also on the boundary. The two potentials $f = \hat{\beta} + \hat{\pi}$ in the bulk and $f_\Gamma = \hat{\beta}_\Gamma + \hat{\pi}_\Gamma$ on the boundary are both the sum of a convex and lower semicontinuous part and a (possibly concave) perturbation; they are not completely independent but related by a suitable growth condition. Within the framework given by (1) and (5), initial conditions should be prescribed for ρ both in the bulk and on the boundary.

All in all, the resulting initial and boundary value problem reads

$$\partial_t \rho + \nabla \rho \cdot u - \Delta \mu = 0 \quad \text{in } Q_T, \tag{6}$$

$$\tau_\Omega \partial_t \rho - \Delta \rho + \beta(\rho) + \pi(\rho) \ni \mu \quad \text{in } Q_T, \tag{7}$$

$$\rho_\Gamma = \rho_{|_{\Sigma_T}}, \quad \mu_\Gamma = \mu_{|_{\Sigma_T}} \quad \text{and} \quad \partial_t \rho_\Gamma + \partial_n \mu - \Delta_\Gamma \mu_\Gamma = 0 \qquad \text{on } \Sigma_T, \tag{8}$$

$$\tau_\Gamma \partial_t \rho_\Gamma + \partial_n \rho - \Delta_\Gamma \rho_\Gamma + \beta_\Gamma(\rho_\Gamma) + \pi_\Gamma(\rho_\Gamma) \ni \mu_\Gamma \quad \text{on } \Sigma_T, \tag{9}$$

$$\rho(0) = \rho_0 \quad \text{in } \Omega \qquad \text{and} \qquad \rho_\Gamma(0) = \rho_{0|\Gamma} \quad \text{on } \Gamma. \tag{10}$$

The paper [16] is devoted to the study of the initial-boundary value problem (6)–(10). Under suitable assumptions and compatibility conditions on the potentials, well-posedness and regularity results are proved.

The aim of the present paper is investigating the longtime behavior. More precisely, we study the ω-limit (in a suitable topology) of every trajectory (ρ, ρ_Γ). Under the additional assumptions that the viscosity coefficients τ_Ω and τ_Γ are strictly positive and that at least one of the potentials f and f_Γ is differentiable, we prove that the ω-limit is meaningful and that every element $(\rho^\omega, \rho_\Gamma^\omega)$ of it is a stationary solution (ρ^s, ρ_Γ^s) of the system for (ρ, ρ_Γ) with some constant value μ^s of the chemical potential.

Let us now review some related literature. About Cahn–Hilliard problems, we quote the pioneering contributions [1, 4, 20, 21, 35] and observe that for this class of evolution processes it turns out that the phases do not diffuse, but they separately concentrate and form the so-called spinodal decomposition. A discussion on the modeling aspects of phase separation, spinodal decomposition and mobility of atoms between cells can be found in [7, 12, 22, 31, 36]). Up to our knowledge, in the case of a pure Cahn–Hilliard system, that is, with $\tau_\Omega = \tau_\Gamma = 0$, and without convective term ($u = 0$), the problem (6)–(10) has been introduced by Gal [24] and formulated by Goldstein, Miranville and Schimperna [29]. It has been studied from various viewpoints in other contributions (see [6–8, 25, 29, 30]). In the case of general potentials, existence, uniqueness and regularity of the weak solution have been shown in [11] (see also [23] for an optimal control problem) by using an abstract approach. In the problem considered by Gal [24] the Laplace–Beltrami term was missing in the third condition in (8) (thus, the boundary condition was of Wentzell type); on the other hand, the presence of the term $-\Delta_\Gamma \mu_\Gamma$ actually enhances the dissipation mechanism in (6)–(10) and is helpful in order to recover a better regularity for the solution. However, it is worth to point

out that in [16] the presence of the convective term $\nabla \rho \cdot u$ gives rise to further complications in the analysis.

Some class of Cahn–Hilliard systems, possibly including dynamic boundary conditions, have collected an increasing interest in recent years: we can quote [9, 32, 34, 37, 38, 40] among other contributions. In case of no convective term in (6), and assuming the homogeneous boundary condition $\partial_n \mu = 0$ and the condition (9) with $\tau_\Gamma > 0$ and μ_Γ as a given datum, the problem has been first addressed in [26]: the well-posedness and the large time behavior of solutions have been studied for regular potentials f and f_Γ, as well as for various singular potentials like the ones in (3) and (4). One can see [26, 27]: in these two papers, the authors were able to overcome the difficulties due to singularities using a set of assumptions for f and f_Γ that gives the role of the dominating potential to f and entails some technical difficulties. The subsequent papers [13–15] follow a different approach, which was firstly considered in [5] and [19] to investigate the Allen–Cahn equation with dynamic boundary conditions. This approach consists in letting f_Γ be the leading potential with respect to f: by this, the analysis turns out to be simpler. In particular, [13] contains many results about existence, uniqueness and regularity of solutions for general potentials that include (2)–(3), and are valid for both the viscous and pure cases, i.e., by assuming just $\tau_\Omega \geq 0$. The paper [10] deals with the well-posedness of the same system, but in which also an additional mass constraint on the boundary is imposed. The recent contribution [33] deals with the physical derivation of some Cahn–Hilliard systems in the bulk and on the boundary, arriving at the study of a model in which the two chemical potentials are completely independent. Finally, let us point out that the optimal control problems for (6)–(10) with the velocity as the control is thoroughly discussed in [17, 18].

The present paper is organized as follows. In the next section, we list our assumptions and notations, recall the properties already known and state our result on the longtime behavior. The last section is devoted to the corresponding proof.

2 Statement of the problem and results

In this section, we state precise assumptions and notations and present our results. First of all, the set $\Omega \subset \mathbb{R}^3$ is assumed to be bounded, connected and smooth. As in the introduction, ν is the outward unit normal vector field on $\Gamma := \partial \Omega$, and ∂_n and Δ_Γ stand for the corresponding normal derivative and the Laplace–Beltrami operator, respectively. Furthermore, we denote by ∇_Γ the surface gradient and write $|\Omega|$ and $|\Gamma|$ for the volume of Ω and the area of Γ, respectively. Moreover, we widely use the notations

$$Q_t := \Omega \times (0, T) \quad \text{and} \quad \Sigma_t := \Gamma \times (0, T) \qquad \text{for } 0 < t \leq +\infty. \tag{11}$$

Next, if X is a Banach space, then $\| \cdot \|_X$ denotes both its norm and the norm of X^3, and the symbols X^* and $\langle \cdot, \cdot \rangle_X$ stand for the dual space of X and the duality pairing between X^* and X. The only exception from the convention for the norms is given by the Lebesgue spaces L^p, for $1 \leq p \leq \infty$, whose norms will be denoted by $\| \cdot \|_p$. Furthermore, we put

$$H := L^2(\Omega), \quad V := H^1(\Omega) \quad \text{and} \quad W := H^2(\Omega), \tag{12}$$

$$H_\Gamma := L^2(\Gamma), \quad V_\Gamma := H^1(\Gamma) \quad \text{and} \quad W_\Gamma := H^2(\Gamma), \tag{13}$$

$$\mathcal{H} := H \times H_\Gamma, \quad \mathcal{V} := \{(v, v_\Gamma) \in V \times V_\Gamma : \ v_\Gamma = v_{|\Gamma}\}$$
$$\text{and} \quad \mathcal{W} := (W \times W_\Gamma) \cap \mathcal{V}. \tag{14}$$

In the following, we work in the framework of the Hilbert triplet $(\mathcal{V}, \mathcal{H}, \mathcal{V}^*)$. Thus, we have

$$\langle (g, g_\Gamma), (v, v_\Gamma) \rangle_\mathcal{V} = \int_\Omega gv + \int_\Gamma g_\Gamma v_\Gamma \quad \text{for every } (g, g_\Gamma) \in \mathcal{H} \text{ and } (v, v_\Gamma) \in \mathcal{V}.$$

Now, we list our assumptions. For the structure of our system, we postulate the following properties, which are slightly stronger than those requested in [16]:

$$\tau_\Omega \text{ and } \tau_\Gamma \text{ are strictly positive real numbers.} \tag{15}$$

$$\hat{\beta}, \hat{\beta}_\Gamma : \mathbb{R} \to [0, +\infty] \quad \text{are convex, proper, and l.s.c., with } \hat{\beta}(0) = \hat{\beta}_\Gamma(0) = 0. \tag{16}$$

$$\hat{\pi}, \hat{\pi}_\Gamma : \mathbb{R} \to \mathbb{R} \quad \text{are of class } C^2 \text{ with Lipschitz continuous first derivatives.} \tag{17}$$

$$\text{The functions } f := \hat{\beta} + \hat{\pi} \text{ and } f_\Gamma := \hat{\beta}_\Gamma + \hat{\pi}_\Gamma \text{ are bounded from below.} \tag{18}$$

Thus, in contrast to [16], the constants τ_Ω and τ_Γ are strictly positive, here, and (18) holds in addition. However, we remark that this assumption is very reasonable and fulfilled by all of the potentials (2)–(4). We set, for convenience,

$$\beta := \partial\hat{\beta}, \quad \beta_\Gamma := \partial\hat{\beta}_\Gamma, \quad \pi := \hat{\pi}', \quad \text{and} \quad \pi_\Gamma := \hat{\pi}'_\Gamma, \tag{19}$$

and assume that, with some positive constants C and η,

$$D(\beta_\Gamma) \subseteq D(\beta) \quad \text{and} \quad |\beta^\circ(r)| \leq \eta|\beta_\Gamma^\circ(r)| + C \qquad \text{for every } r \in D(\beta_\Gamma). \tag{20}$$

In (20), the symbols $D(\beta)$ and $D(\beta_\Gamma)$ denote the domains of β and β_Γ, respectively. More generally, we use the notation $D(\mathcal{G})$ for every maximal monotone graph $\mathcal{G}$ in $\mathbb{R} \times \mathbb{R}$, as well as for the associated maximal monotone operators induced on L^2 spaces. Moreover, for $r \in D(\mathcal{G})$, $\mathcal{G}^\circ(r)$ stands for the element of $\mathcal{G}(r)$ having minimum modulus.

Remark 2.1 Notice that, physically speaking, the compatibility condition (20) means that the thermodynamic force driving the phase separation on the surface is stronger than the one in the bulk.

For the data, we make the following assumptions:

$$u \in (L^2(0, +\infty; L^3(\Omega)))^3 \quad \text{with } \partial_t u \in (L^2(0, +\infty; L^{3/2}(\Omega)))^3. \tag{21}$$

$$\text{div } u = 0 \quad \text{in } Q_\infty \qquad \text{and} \qquad u \cdot v = 0 \quad \text{on } \Sigma_\infty. \tag{22}$$

$$(\rho_0, \rho_{0|\Gamma}) \in \mathcal{W}, \quad \beta^\circ(\rho_0) \in H \quad \text{and} \quad \beta_\Gamma^\circ(\rho_{0|\Gamma}) \in H_\Gamma. \tag{23}$$

$$m_0 := \frac{\int_\Omega \rho_0 + \int_\Gamma \rho_{0|\Gamma}}{|\Omega| + |\Gamma|} \quad \text{belongs to the interior of } D(\beta_\Gamma). \tag{24}$$

Since we are assuming (15) and (21)–(24), the regularity level required for the notion of solution on a finite time interval is higher than the one in [16]. Namely, a solution on $(0, T)$ is a triple of pairs $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\zeta, \zeta_\Gamma))$ that satisfies

$$(\mu, \mu_\Gamma) \in L^\infty(0, T; \mathcal{W}), \tag{25}$$

$$(\rho, \rho_\Gamma) \in W^{1,\infty}(0, T; \mathcal{H}) \cap H^1(0, T; \mathcal{V}) \cap L^\infty(0, T; \mathcal{W}), \tag{26}$$

$$(\zeta, \zeta_\Gamma) \in L^\infty(0, T; \mathcal{H}). \tag{27}$$

However, we write $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ instead of $((\mu, \mu_\Gamma), (\rho, \rho_\Gamma), (\zeta, \zeta_\Gamma))$, in order to simplify the notation. As far as the problem under study is concerned, we still state it in a weak form as in [16], on account of the assumptions (22) on u. Namely, we require that

$$\int_\Omega \partial_t \rho\, v + \int_\Gamma \partial_t \rho_\Gamma\, v_\Gamma - \int_\Omega \rho u \cdot \nabla v + \int_\Omega \nabla \mu \cdot \nabla v + \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma v_\Gamma = 0 \tag{28}$$
a.e. in $(0, T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$,

$$\tau_\Omega \int_\Omega \partial_t \rho\, v + \tau_\Gamma \int_\Gamma \partial_t \rho_\Gamma\, v_\Gamma + \int_\Omega \nabla \rho \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_\Gamma \cdot \nabla_\Gamma v_\Gamma$$
$$+ \int_\Omega (\zeta + \pi(\rho))v + \int_\Gamma (\zeta_\Gamma + \pi_\Gamma(\rho_\Gamma))v_\Gamma = \int_\Omega \mu v + \int_\Gamma \mu_\Gamma v_\Gamma \tag{29}$$
a.e. in $(0, T)$ and for every $(v, v_\Gamma) \in \mathcal{V}$,

$$\zeta \in \beta(\rho) \quad \text{a.e. in } Q_T \qquad \text{and} \qquad \zeta_\Gamma \in \beta_\Gamma(\rho_\Gamma) \quad \text{a.e. on } \Sigma_T, \tag{30}$$

$$\rho(0) = \rho_0 \quad \text{a.e. in } \Omega. \tag{31}$$

However, every solution also satisfies the boundary value problem presented in the introduction. The basic well-posedness and regularity results are given by [16, Thms. 2.3 and 2.6]. We collect them in the following

Theorem 2.2 *Assume* (15)–(20) *for the structure and* (21)–(24) *for the data, and let $T \in (0, +\infty)$. Then problem* (28)–(31) *has at least one solution* $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ *satisfying* (25)–(27)*. Moreover, the solution is unique if at least one of the operators β and β_Γ is single-valued.*

We obviously deduce the following consequence:

Corollary 2.3 *In addition to* (15)–(20)*, assume that at least one of the operators β and β_Γ is single-valued. Moreover, assume* (21)–(24) *for the data. Then there exists a unique 6-tuple* $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ *defined on* $(0, +\infty)$ *that fulfils* (25)–(27) *and solves* (28)–(31) *for every $T \in (0, +\infty)$.*

At this point, given a solution $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$, our aim is investigating its longtime behavior, namely, the ω-limit (which we simply term ω for brevity) of the component (ρ, ρ_Γ). We notice that the property (26) holds for every $T \in (0, +\infty)$, which implies that (ρ, ρ_Γ) belongs to $C^0([0, +\infty); \mathcal{V})$. Hence, the next definition is meaningful. We set

$$\omega := \{(\rho^\omega, \rho_\Gamma^\omega) = \lim_{n \to \infty} (\rho, \rho_\Gamma)(t_n) \quad \text{in the weak topology of } \mathcal{V}$$
$$\text{for some sequence } \{t_n \nearrow +\infty\}\}, \tag{32}$$

and look for the relationship between ω and the set of stationary solutions to the system obtained from (28)–(30) by ignoring the convective term. Indeed, assumption (21) implies that

$$u \in (H^1(0, +\infty; L^{3/2}(\Omega)))^3, \tag{33}$$

whence $u(t)$ tends to zero strongly in $L^{3/2}(\Omega)$ as t tends to infinity. It is immediately seen from (28) that the components μ and μ_Γ of every stationary solution are constant functions and that the constant values they assume are the same. Therefore, by a stationary solution we mean a quadruplet $(\rho^s, \rho_\Gamma^s, \zeta^s, \zeta_\Gamma^s)$ satisfying for some $\mu^s \in \mathbb{R}$ the conditions

$$(\rho^s, \rho_\Gamma^s) \in \mathcal{V} \quad \text{and} \quad (\zeta, \zeta_\Gamma^s) \in \mathcal{H}, \tag{34}$$

$$\int_\Omega \nabla \rho^s \cdot \nabla v + \int_\Gamma \nabla_\Gamma \rho_\Gamma^s \cdot \nabla_\Gamma v_\Gamma + \int_\Omega (\zeta^s + \pi(\rho^s))v + \int_\Gamma (\zeta_\Gamma^s + \pi_\Gamma(\rho_\Gamma^s))v_\Gamma$$
$$= \int_\Omega \mu^s v + \int_\Gamma \mu^s v_\Gamma \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}, \tag{35}$$

$$\zeta^s \in \beta(\rho^s) \quad \text{a.e. in } \Omega \qquad \text{and} \qquad \zeta_\Gamma^s \in \beta_\Gamma(\rho_\Gamma^s) \quad \text{a.e. on } \Gamma. \tag{36}$$

It is not difficult to show that (34)–(35) imply that the pair (ρ^s, ρ_Γ^s) belongs to $\mathcal{W}$ and satisfies the boundary value problem

$$-\Delta \rho^s + \zeta^s + \pi(\rho^s) = \mu^s \quad \text{a.e. in } \Omega,$$
$$\partial_n \rho^s - \Delta_\Gamma \rho_\Gamma^s + \zeta_\Gamma^s + \pi_\Gamma(\rho_\Gamma^s) = \mu^s \quad \text{a.e. on } \Gamma.$$

Here is our result:

Theorem 2.4 *Let the assumptions of Corollary* 2.3 *be satisfied, and let* $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ *be the unique global solution on* $(0, +\infty)$. *Then the ω-limit* (32) *is nonempty. Moreover, for every* $(\rho^\omega, \rho_\Gamma^\omega) \in \omega$, *there exist* $\mu^s \in \mathbb{R}$ *and a solution* $(\rho^s, \rho_\Gamma^s, \zeta^s, \zeta_\Gamma^s)$ *to* (34)–(36) *such that* $(\rho^\omega, \rho_\Gamma^\omega) = (\rho^s, \rho_\Gamma^s)$.

3 Auxiliary material

The proof of Theorem 2.4 will be performed in the last section. Our argument needs some tools which we collect in the present section. In particular, we use the generalized mean value, the related spaces and the operator $\mathcal{N}$ which we introduce now. However, we proceed very shortly and refer to [16, Sect. 2] for further details. We set

$$\text{mean } g^* := \frac{\langle g^*, (1, 1) \rangle_\mathcal{V}}{|\Omega| + |\Gamma|} \quad \text{for } g^* \in \mathcal{V}^* \tag{37}$$

and observe that

$$\text{mean}(v, v_\Gamma) = \frac{\int_\Omega v + \int_\Gamma v_\Gamma}{|\Omega| + |\Gamma|} \quad \text{if } g^* = (v, v_\Gamma) \in \mathcal{H}. \tag{38}$$

Notice that the constant m_0 appearing in our assumption (24) is nothing but the mean value $\text{mean}(\rho_0, \rho_{0|\Gamma})$, and that taking $(v, v_\Gamma) = (|\Omega| + |\Gamma|)^{-1}(1, 1)$ in (28) yields the conservation property for the component (ρ, ρ_Γ) of the solution,

$$\partial_t \text{mean}(\rho, \rho_\Gamma) = 0, \quad \text{whence} \quad \text{mean}(\rho, \rho_\Gamma)(t) = m_0 \qquad \text{for every } t \in [0, T]. \tag{39}$$

We also stress that the function

$$\mathcal{V} \ni (v, v_\Gamma) \mapsto \left(\|\nabla v\|_H^2 + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}^2 + |\text{mean}(v, v_\Gamma)|^2 \right)^{1/2} \tag{40}$$

yields a Hilbert norm on $\mathcal{V}$ which is equivalent to the natural one. Now, we set

$$\mathcal{V}_{*0} := \{g^* \in \mathcal{V}^* : \text{ mean } g^* = 0\}, \quad \mathcal{H}_0 := \mathcal{H} \cap \mathcal{V}_{*0}, \quad \text{and} \quad \mathcal{V}_0 := \mathcal{V} \cap \mathcal{V}_{*0}, \tag{41}$$

and notice that the function

$$\mathcal{V}_0 \ni (v, v_\Gamma) \mapsto \|(v, v_\Gamma)\|_{\mathcal{V}_0} := (\|\nabla v\|_H^2 + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}^2)^{1/2} \tag{42}$$

is a Hilbert norm on $\mathcal{V}_0$ which is equivalent to the usual one. Next, we define the operator $\mathcal{N} : \mathcal{V}_{*0} \to \mathcal{V}_0$ (which will be applied to $\mathcal{V}_{*0}$-valued functions as well) as follows. For every element $g^* \in \mathcal{V}_{*0}$,

$$\mathcal{N} g^* = (\mathcal{N}_\Omega g^*, \mathcal{N}_\Gamma g^*) \text{ is the unique pair } (\xi, \xi_\Gamma) \in \mathcal{V}_0 \text{ such that}$$
$$\int_\Omega \nabla \xi \cdot \nabla v + \int_\Gamma \nabla_\Gamma \xi_\Gamma \cdot \nabla_\Gamma v_\Gamma = \langle g^*, (v, v_\Gamma) \rangle_\mathcal{V} \quad \text{for every } (v, v_\Gamma) \in \mathcal{V}. \tag{43}$$

It turns out that $\mathcal{N}$ is well defined, linear, symmetric, and bijective. Therefore, if we set

$$\|g\|_*^* := \|\mathcal{N} g^*\|_{\mathcal{V}_0}, \quad \text{for } g^* \in \mathcal{V}_{*0}, \tag{44}$$

then we obtain a Hilbert norm on $\mathcal{V}_{*0}$ (equivalent to the norm induced by the norm of $\mathcal{V}^*$). Furthermore, we notice that

$$\langle \partial_t g^*, \mathcal{N} g^* \rangle_\mathcal{V} = \frac{1}{2} \frac{d}{dt} \|g^*\|_*^2 \quad \text{a.e. in } (0, T), \qquad \text{for every } g^* \in H^1(0, T; \mathcal{V}_{*0}). \tag{45}$$

Finally, it is easy to see that $\mathcal{N} g^*$ belongs to $\mathcal{W}$ whenever $g^* \in \mathcal{H}_0$, and that

$$\|\mathcal{N} g^*\|_\mathcal{W} \leq C_\Omega \|g^*\|_\mathcal{H} \quad \text{for every } g^* \in \mathcal{H}_0, \tag{46}$$

where C_Ω depends only on Ω.

In performing our estimates, we will repeatedly use the Young inequality

$$a b \leq \delta a^2 + \frac{1}{4\delta} b^2 \quad \text{for all } a, b \in \mathbb{R} \text{ and } \delta > 0, \tag{47}$$

as well as Hölder's inequality and the Sobolev inequality

$$\|v\|_p \leq C_\Omega \|v\|_V \quad \text{for every } p \in [1, 6] \text{ and } v \in V, \tag{48}$$

which is related to the continuous embedding $V \subset L^p(\Omega)$ for $p \in [1, 6]$ (since Ω is three-dimensional, bounded and smooth). In particular, by also using the equivalent norm (40) on $\mathcal{V}$, we have that

$$\|v\|_6^2 \leq C_\Omega \Big(\|\nabla v\|_H^2 + \|\nabla_\Gamma v_\Gamma\|_{H_\Gamma}^2 + |\text{mean}(v, v_\Gamma)|^2 \Big) \tag{49}$$

for every $(v, v_\Gamma) \in \mathcal{V}$. In both (48) and (49), the constant C_Ω depends only on Ω. Furthermore, we will owe to a well-known fact from interpolation theory. By [3, Thm. 5.2.1, p. 109], we have that

$$(L^3(\Omega), L^{3/2}(\Omega))_{1/2,2} = L^2(\Omega),$$

where the interpolation can be understood also in the sense of the trace method (see [3, Sect. 3.12] for the equivalence between various interpolation methods). It follows that there hold the continuous embedding and the related inequality

$$H^1(0, +\infty; L^{3/2}(\Omega)) \cap L^2(0, +\infty; L^3(\Omega)) \subset L^\infty(0, +\infty; L^2(\Omega)),$$
$$\|u\|_{L^\infty(0,+\infty; L^2(\Omega))} \leq C_\Omega \left(\|u\|_{L^2(0,+\infty; L^3(\Omega))} + \|\partial_t u\|_{L^2(0,+\infty; L^{3/2}(\Omega))} \right) \quad (50)$$
$$\text{for every } v \in H^1(0, +\infty; L^{3/2}(\Omega)) \cap L^2(0, +\infty; L^3(\Omega)),$$

where C_Ω depends only on Ω.

Finally, as far as constants are concerned, we employ the following general rule: the small-case symbol c stands for different constants which depend only on Ω, the structure of our system and the norms of the data involved in the assumptions (21)–(24). A notation like c_δ (in particular, with $\delta = T$) allows the constant to depend on the positive parameter δ, in addition. Hence, the meaning of c and c_δ may change from line to line and even within the same chain of inequalities. On the contrary, we mark the constants that we want to refer to by using a different notation (e.g., a capital letter).

4 Longtime behavior

This section is devoted to the proof of Theorem 2.4. Thus, we fix any global solution $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ once and for all. Our arguments relies on some global a priori estimates and on the study of the behavior of such a solution on intervals of a fixed length T whose endpoints tend to infinity.

To keep the paper at a reasonable length, we often proceed formally. However, we notice that the estimates to be obtained in this way can be performed rigorously by acting on the solution to a proper regularized or discrete problem. Indeed, the solution found in [16] was constructed in this way: first, a regularized ε-problem was introduced by replacing the graphs β and β_Γ by their Yosida regularizations β_ε and $\beta_{\Gamma, \varepsilon}$. However, the same argument would work if β_ε and $\beta_{\Gamma, \varepsilon}$ were smooth approximation of β and β_Γ with analogous boundedness and convergence properties (like the C^∞ approximations introduced in [28, Sect. 3]). In order to solve such an approximating problem, a Faedo–Galerkin scheme depending on a parameter $n \in \mathbb{N}$ can be used. Its solution is smooth according to the smoothness of the nonlinearities that appear in the ε-problem. Then, the solution to the original problem is constructed by first letting n tend to infinity and then letting ε tend to zero. Thus, our procedure would be completely rigorous if it were performed on one of the above approximating solutions. Indeed, the estimates established in this way would be uniform with respect to the parameters involved. We now start proving such global estimates.

First global estimate We write the Eqs. (28) and (29) at the time s and test them by $(\mu, \mu_\Gamma)(s) \in \mathcal{V}$ and $\partial_t(\rho, \rho_\Gamma)(s) \in \mathcal{V}$, respectively. Then, we integrate with respect to s over $(0, t)$ with an arbitrary $t > 0$, sum up and rearrange. We obtain

$$
\int_{Q_t} \partial_t \rho \, \mu + \int_{\Sigma_t} \partial_t \rho_\Gamma \, \mu_\Gamma + \int_{Q_t} |\nabla \mu|^2 + \int_{\Sigma_t} |\nabla_\Gamma \mu_\Gamma|^2 + \tau_\Omega \int_{Q_t} |\partial_t \rho|^2 + \tau_\Gamma \int_{\Sigma_t} |\partial_t \rho_\Gamma|^2
$$
$$
+ \frac{1}{2} \int_\Omega |\nabla \rho(t)|^2 + \frac{1}{2} \int_\Gamma |\nabla_\Gamma \rho_\Gamma(t)|^2 + \int_\Omega f(\rho(t)) + \int_\Gamma f_\Gamma(\rho_\Gamma(t))
$$
$$
= \int_{Q_t} \rho u \cdot \nabla \mu + \int_{Q_t} \mu \partial_t \rho + \int_{\Sigma_t} \mu_\Gamma \partial_t \rho_\Gamma
$$
$$
+ \frac{1}{2} \int_\Omega |\nabla \rho_0|^2 + \frac{1}{2} \int_\Gamma |\nabla_\Gamma \rho_{0|\Gamma}|^2 + \int_\Omega f(\rho_0) + \int_\Gamma f_\Gamma(\rho_{0|\Gamma}).
$$

Four integrals obviously cancel out, the ones on the left-hand side containing f and f_Γ are bounded from below by (18), and the terms on the right-hand side involving the initial values are finite by (23). We deal with the convective term using the Young and Hölder inequalities, the Sobolev type inequality (49), and the conservation property (39). We have

$$
\int_{Q_t} \rho u \cdot \nabla \mu \leq \frac{1}{2} \int_{Q_t} |\nabla \mu|^2 + \frac{1}{2} \int_0^t \|u(s)\|_3^2 \|\rho(s)\|_6^2 \, ds
$$
$$
\leq \frac{1}{2} \int_{Q_t} |\nabla \mu|^2 + c \int_0^t \|u(s)\|_3^2 (\|\nabla \rho(s)\|_H^2 + \|\nabla_\Gamma \rho_\Gamma(s)\|_{H_\Gamma}^2 + m_0^2) \, ds.
$$

Since the function $s \mapsto \|u(s)\|_3^2$ belongs to $L^1(0, +\infty)$ by (21), we can apply the Gronwall lemma on $(0, +\infty)$ and obtain that

$$
\int_{Q_\infty} |\nabla \mu|^2 + \int_{\Sigma_\infty} |\nabla_\Gamma \mu_\Gamma|^2 + \int_{Q_\infty} |\partial_t \rho|^2 + \int_{\Sigma_\infty} |\partial_t \rho_\Gamma|^2 < +\infty, \tag{51}
$$

$$
f(\rho) \in L^\infty(0, +\infty; L^1(\Omega)) \quad \text{and} \quad f_\Gamma(\rho_\Gamma) \in L^\infty(0, +\infty; L^1(\Gamma)), \tag{52}
$$

as well as $(\nabla \rho, \nabla_\Gamma \rho_\Gamma) \in (L^\infty(0, +\infty; H))^3$. From this, by accounting for the conservation property (39) once more, we conclude that

$$
(\rho, \rho_\Gamma) \in L^\infty(0, +\infty; \mathcal{V}). \tag{53}
$$

Consequence By using the quadratic growth of $\hat{\pi}$ and $\hat{\pi}_\Gamma$ implied by the Lipschitz continuity of their derivatives, and combining with (52) with (53), we deduce that

$$
\hat{\beta}(\rho) \in L^\infty(0, +\infty; L^1(\Omega)) \quad \text{and} \quad \hat{\beta}_\Gamma(\rho_\Gamma) \in L^\infty(0, +\infty; L^1(\Gamma)). \tag{54}
$$

Second global estimate We formally differentiate the Eqs. (28) and (29) with respect to time, where we argue as if β and β_Γ were smooth functions and write $\beta(\rho)$ and $\beta_\Gamma(\rho_\Gamma)$ instead of ζ and ζ_Γ (see (30)). We obtain that

$$
\int_\Omega \partial_t^2 \rho \, v + \int_\Gamma \partial_t^2 \rho_\Gamma \, v_\Gamma + \int_\Omega \nabla \partial_t \mu \cdot \nabla v + \int_\Gamma \nabla_\Gamma \partial_t \mu_\Gamma \cdot \nabla_\Gamma v_\Gamma
$$
$$
= \int_\Omega (\partial_t \rho \, u + \rho \partial_t u) \cdot \nabla v \tag{55}
$$

$$
\tau_\Omega \int_\Omega \partial_t^2 \rho \, v + \tau_\Gamma \int_\Gamma \partial_t^2 \rho_\Gamma \, v_\Gamma + \int_\Omega \nabla \partial_t \rho \cdot \nabla v + \int_\Gamma \nabla_\Gamma \partial_t \rho_\Gamma \cdot \nabla_\Gamma v_\Gamma
$$
$$
+ \int_\Omega \beta'(\rho) \partial_t \rho \, v + \int_\Gamma \beta'_\Gamma(\rho_\Gamma) \partial_t \rho_\Gamma \, v_\Gamma \tag{56}
$$
$$
= \int_\Omega \partial_t \mu \, v + \int_\Gamma \partial_t \mu_\Gamma \, v_\Gamma - \int_\Omega \pi'(\rho) \partial_t \rho \, v - \int_\Gamma \pi'_\Gamma(\rho_\Gamma) \partial_t \rho_\Gamma \, v_\Gamma
$$

a.e. in $(0, +\infty)$ and for every $(v, v_\Gamma) \in \mathcal{V}$. By recalling that $\partial_t(\rho, \rho_\Gamma)$ is $\mathcal{V}_0$-valued by (39), so that $\mathcal{N} \partial_t(\rho, \rho_\Gamma)$ is well defined, we write the above equations at the time s and test them by $\mathcal{N} \partial_t(\rho, \rho_\Gamma)(s)$ and $\partial_t(\rho, \rho_\Gamma)(s)$, respectively. Then, we integrate with respect to s over $(0, t)$ with an arbitrary $t > 0$ and sum up. We have

$$
\int_0^t \langle \partial_t^2(\rho, \rho_\Gamma)(s), \mathcal{N} \partial_t(\rho, \rho_\Gamma)(s) \rangle_\mathcal{V} \, ds
$$
$$
+ \int_{Q_t} \nabla \partial_t \mu \cdot \nabla \mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) + \int_{\Sigma_t} \nabla_\Gamma \partial_t \mu_\Gamma \cdot \nabla_\Gamma \mathcal{N}_\Gamma(\partial_t(\rho, \rho_\Gamma))
$$
$$
+ \frac{\tau_\Omega}{2} \int_\Omega |\partial_t \rho(t)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t \rho_\Gamma(t)|^2 + \int_{Q_t} |\nabla \partial_t \rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma \partial_t \rho_\Gamma|^2
$$
$$
+ \int_{Q_t} \beta'(\rho) |\partial_t \rho|^2 + \int_{\Sigma_t} \beta'_\Gamma(\rho_\Gamma) |\partial_t \rho_\Gamma|^2
$$
$$
= \int_{Q_t} (\partial_t \rho \, u + \rho \partial_t u) \cdot \nabla \mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t \rho(0)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t \rho_\Gamma(0)|^2
$$
$$
+ \int_{Q_t} \partial_t \mu \, \partial_t \rho + \int_{\Sigma_t} \partial_t \mu_\Gamma \, \partial_t \rho_\Gamma - \int_{Q_t} \pi'(\rho) |\partial_t \rho|^2 - \int_{\Sigma_t} \pi'_\Gamma(\rho_\Gamma) |\partial_t \rho_\Gamma|^2 .
$$

The integrals containing $\partial_t \mu$ and $\partial_t \mu_\Gamma$ cancel out by the definition (43) of $\mathcal{N}$ (with the choices $g^* = \partial_t(\rho, \rho_\Gamma)(s)$ and $(v, v_\Gamma) = \partial_t(\mu, \mu_\Gamma)(s)$), and the terms involving β' and β'_Γ are nonnegative. Moreover, the last two integrals on the right-hand side are

bounded by (51) and the Lipschitz continuity of π and π_Γ. Therefore, owing to (45) for the first term on the left-hand side, we deduce that

$$\frac{1}{2} \|\partial_t(\rho, \rho_\Gamma)(t)\|_*^2 + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t\rho(t)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t\rho_\Gamma(t)|^2 + \int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2$$

$$\leq \frac{1}{2} \|\partial_t(\rho, \rho_\Gamma)(0)\|_*^2 + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t\rho(0)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t\rho_\Gamma(0)|^2$$

$$+ \int_{Q_t} \partial_t\rho \, u \cdot \nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) + \int_{Q_t} \rho\partial_t u \cdot \nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) + c.$$

$$\tag{57}$$

Thus, is suffices to obtain a bound for the $\mathcal{H}$-norm of $\partial_t(\rho, \rho_\Gamma)(0)$ and to estimate the last two integrals. For the first aim, we write the Eqs. (28) and (29) at the time $t = 0$, test them by $(\mu, \mu_\Gamma)(0)$ and $\partial_t(\rho, \rho_\Gamma)(0)$, respectively, and sum up. Then, we account for the regularity of ρ_0 ensured by (23) and integrate by parts the terms involving $\nabla\rho_0$ and $\nabla_\Gamma\rho_{0|\Gamma}$. We obtain

$$\int_\Omega \partial_t\rho(0) \, \mu(0) + \int_\Gamma \partial_t\rho_\Gamma(0) \, \mu_\Gamma(0) + \int_\Omega |\nabla\mu(0)|^2 + \int_\Gamma |\nabla_\Gamma\mu_\Gamma(0)|^2$$

$$+ \tau_\Omega \int_\Omega |\partial_t\rho(0)|^2 + \tau_\Gamma \int_\Gamma |\partial_t\rho_\Gamma(0)|^2 - \int_\Omega \Delta\rho_0 \, \partial_t\rho(0) - \int_\Gamma \Delta_\Gamma\rho_{0|\Gamma} \, \partial_t\rho_\Gamma(0)$$

$$+ \int_\Omega (\beta + \pi)(\rho_0)\partial_t\rho(0) + \int_\Gamma (\beta_\Gamma + \pi_\Gamma)(\rho_{0|\Gamma})\partial_t\rho_\Gamma(0)$$

$$= \int_\Omega \rho_0 u(0) \cdot \nabla\mu(0) + \int_\Omega \mu(0)\partial_t\rho(0) + \int_\Gamma \mu_\Gamma(0)\partial_t\rho_\Gamma(0).$$

Four integrals obviously cancel each other. Now, we rearrange and use the Young inequality and the Lipschitz continuity of π and π_Γ. In view of the full (23), we infer that

$$\int_\Omega |\nabla\mu(0)|^2 + \int_\Gamma |\nabla_\Gamma\mu_\Gamma(0)|^2 + \frac{\tau_\Omega}{2} \int_\Omega |\partial_t\rho(0)|^2 + \frac{\tau_\Gamma}{2} \int_\Gamma |\partial_t\rho_\Gamma(0)|^2$$

$$\leq c\left(\|(\rho_0, \rho_{0|\Gamma})\|_{\mathcal{W}}^2 + 1 + \|\beta(\rho_0)\|_H^2 + \|\beta_\Gamma(\rho_{0|\Gamma})\|_{H_\Gamma}^2 \right) + \int_\Omega \rho_0 u(0) \cdot \nabla\mu(0)$$

$$\leq c + \int_\Omega \rho_0 u(0) \cdot \nabla\mu(0).$$

In order to estimate the last integral, we use the Hölder and Young inequalities, the continuous embedding $W \subset L^\infty(\Omega)$, and the interpolation inequality (50), to conclude that

$$\int_\Omega \rho_0 u(0) \cdot \nabla\mu(0) \leq \|\rho_0\|_\infty \|u(0)\|_2 \|\nabla\mu(0)\|_2$$

$$\leq \frac{1}{2} \int_\Omega |\nabla\mu(0)|^2 + c \|\rho_0\|_W^2 \left(\|u\|_{L^2(0,+\infty; L^3(\Omega))}^2 + \|\partial_t u\|_{L^2(0,+\infty; L^{3/2}(\Omega))}^2 \right),$$

and we observe that all of the last norms are finite by virtue of the assumptions (23) and (21) on ρ_0 and u. By combining this and the above inequality, we obtain the desired bound for the $\mathcal{H}$-norm of $\partial_t(\rho, \rho_\Gamma)(0)$.

Finally, we estimate the last two integrals on the right-hand side of (57) by using the Hölder, Sobolev and Young inequalities (in particular (49)), the interpolation inequality (50), the conservation property (39), the regularity inequality (46) for $\mathcal{N}$, and the already established estimate (53). We have that

$$\int_{Q_t} \partial_t \rho\, u \cdot \nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) \leq \int_0^t \|\partial_t\rho(s)\|_4 \|u(s)\|_2 \|\nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)(s))\|_4\, ds$$

$$\leq c \|u\|_{L^\infty(0,+\infty; H)} \int_0^t \|\partial_t\rho(s)\|_V \|\mathcal{N}(\partial_t(\rho, \rho_\Gamma)(s))\|_W\, ds$$

$$\leq \frac{1}{2} \left(\int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2 \right) + c \int_0^t \|\partial_t(\rho, \rho_\Gamma)(s)\|_{\mathcal{H}}^2\, ds$$

$$\leq \frac{1}{2} \left(\int_{Q_t} |\nabla\partial_t\rho|^2 + \int_{\Sigma_t} |\nabla_\Gamma\partial_t\rho_\Gamma|^2 \right) + c,$$

as well as, with a similar argument,

$$\int_{Q_t} \rho\partial_t u \cdot \nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)) \leq \int_0^t \|\rho(s)\|_6 \|\partial_t u(s)\|_{3/2} \|\nabla\mathcal{N}_\Omega(\partial_t(\rho, \rho_\Gamma)(s))\|_6\, ds$$

$$\leq c \|\rho\|_{L^\infty(0,+\infty; V)} \|\partial_t u\|_{L^2(0,+\infty; L^{3/2}(\Omega))} \|\partial_t(\rho, \rho_\Gamma)\|_{L^2(0,+\infty; \mathcal{H})} \leq c.$$

Coming back to (57) and taking these estimates into account, we conclude that

$$\partial_t(\rho, \rho_\Gamma) \in L^\infty(0,+\infty; \mathcal{H}) \cap L^2(0,+\infty; \mathcal{V}). \tag{58}$$

Third global estimate Also in this step, we argue as if the graphs β and β_Γ were smooth functions (in particular, we write $\zeta = \beta(\rho)$ and $\zeta_\Gamma = \beta(\rho_\Gamma)$) and first notice that the inclusion $D(\beta_\Gamma) \subseteq D(\beta)$ (see (20)) and assumption (24) imply that

$$\beta(r)(r - m_0) \geq \delta_0 |\beta(r)| - C_0 \quad \text{and} \quad \beta_\Gamma(r)(r - m_0) \geq \delta_0 |\beta_\Gamma(r)| - C_0 \tag{59}$$

for every r belonging to the respective domains, where δ_0 and C_0 are some positive constants that depend only on β, β_Γ and on the position of m_0 in the interior of $D(\beta_\Gamma)$ and of $D(\beta)$ (see, e.g., [26, p. 908]). Now, we recall the conservation property (39)

and test (28) and (29) by $\mathcal{N}(\rho - m_0, \rho_\Gamma - m_0)$ and $(\rho - m_0, \rho_\Gamma - m_0)$, respectively. Then, we sum up without integrating with respect to time. We obtain that, almost everywhere in $(0, +\infty)$,

$$
\begin{aligned}
&\langle \partial_t(\rho, \rho_\Gamma), \mathcal{N}(\rho - m_0, \rho_\Gamma - m_0) \rangle_\mathcal{V} \\
&\quad + \int_\Omega \nabla \mu \cdot \nabla \mathcal{N}_\Omega(\rho - m_0, \rho_\Gamma - m_0) + \int_\Gamma \nabla_\Gamma \mu_\Gamma \cdot \nabla_\Gamma \mathcal{N}_\Gamma(\rho - m_0, \rho_\Gamma - m_0) \\
&\quad + \tau_\Omega \int_\Omega \partial_t \rho \, (\rho - m_0) + \tau_\Gamma \int_\Gamma \partial_t \rho_\Gamma \, (\rho_\Gamma - m_0) + \int_\Omega |\nabla \rho|^2 + \int_\Gamma |\nabla_\Gamma \rho_\Gamma|^2 \\
&\quad + \int_\Omega \beta(\rho)(\rho - m_0) + \int_\Gamma \beta_\Gamma(\rho_\Gamma)(\rho_\Gamma - m_0) \\
&= \int_\Omega \rho u \cdot \nabla(\mathcal{N}_\Omega(\rho - \rho_0, \rho_\Gamma - \rho_{0|\Gamma})) + \int_\Omega \mu(\rho - m_0) + \int_\Gamma \mu_\Gamma(\rho_\Gamma - m_0) \\
&\quad - \int_\Omega \pi(\rho)(\rho - m_0) - \int_\Gamma \pi_\Gamma(\rho_\Gamma)(\rho_\Gamma - m_0).
\end{aligned}
$$

All of the integrals involving μ and μ_Γ cancel out by (43). Now, we owe to (59), keep just the positive contributions on the left-hand side and move the other terms on the right-hand side. By also accounting for the Lipschitz continuity of π and π_Γ, the Sobolev inequality related to the continuous embedding $V \subset L^4(\Omega)$, the interpolation inequality (50) on u, the regularity inequality (46) for $\mathcal{N}$, (53) and (58), we deduce that

$$
\begin{aligned}
&\int_\Omega |\nabla \rho|^2 + \int_\Gamma |\nabla_\Gamma \rho_\Gamma|^2 + \delta_0 \int_\Omega |\beta(\rho)| + \delta_0 \int_\Gamma |\beta_\Gamma(\rho_\Gamma)| \\
&\leq \|\partial_t(\rho, \rho_\Gamma)\|_{L^\infty(0,+\infty; \mathcal{V}_0^*)} \|\mathcal{N}(\rho - m_0, \rho_\Gamma - m_0)\|_{L^\infty(0,+\infty; \mathcal{V}_0)} \\
&\quad + c \|\partial_t(\rho, \rho_\Gamma)\|_{L^\infty(0,+\infty; \mathcal{H})} \|(\rho - m_0, \rho_\Gamma - m_0)\|_{L^\infty(0,+\infty; \mathcal{H})} \\
&\quad + c \|(\pi(\rho), \pi_\Gamma(\rho_\Gamma))\|_{L^\infty(0,+\infty; \mathcal{H})} \|(\rho - m_0, \rho_\Gamma - m_0)\|_{L^\infty(0,+\infty; \mathcal{H})} \\
&\quad + \|\rho\|_{L^\infty(0,+\infty; L^4(\Omega))} \|u\|_{L^\infty(0,+\infty; L^2(\Omega))} \|\nabla(\mathcal{N}_\Omega(\rho - \rho_0, \rho_\Gamma - \rho_{0|\Gamma}))\|_{L^\infty(0,+\infty; L^4(\Omega))} \\
&\leq c + c \|\mathcal{N}(\rho - \rho_0, \rho_\Gamma - \rho_{0|\Gamma})\|_{L^\infty(0,+\infty; \mathcal{W})} \\
&\leq c + c \|(\rho - \rho_0, \rho_\Gamma - \rho_{0|\Gamma})\|_{L^\infty(0,+\infty; \mathcal{H})} \leq c.
\end{aligned}
$$

Since this holds a.e. in $(0, +\infty)$, we have (in particular) that

$$
\zeta \in L^\infty(0, +\infty; L^1(\Omega)) \quad \text{and} \quad \zeta_\Gamma \in L^\infty(0, +\infty; L^1(\Gamma)).
$$

Now, we test (29) by $(1, 1)$ and obtain a.e. in $(0, +\infty)$

$$(|\Omega| + |\Gamma|)\mathrm{mean}(\mu, \mu_\Gamma) = \tau_\Omega \int_\Omega \partial_t \rho + \tau_\Gamma \int_\Gamma \partial_t \rho_\Gamma + \int_\Omega \zeta + \int_\Gamma \zeta_\Gamma + \int_\Omega \pi(\rho) + \int_\Gamma \pi_\Gamma(\rho_\Gamma).$$

Thus, we infer that

$$\mathrm{mean}(\mu, \mu_\Gamma) \in L^\infty(0, +\infty). \tag{60}$$

It was already clear from (53) that the ω-limit ω is nonempty. Indeed, the continuous $\mathcal{V}$-valued function (ρ, ρ_Γ) is also bounded, so that there exists a sequence $t_n \nearrow +\infty$ such that the sequence $\{(\rho, \rho_\Gamma)(t_n)\}$ is weakly convergent in $\mathcal{V}$. More precisely, any sequence of times that tends to infinity contains a subsequence of this type. Thus, it remains to prove the second part of the statement. Therefore, we fix an element $(\rho^\omega, \rho_\Gamma^\omega) \in \omega$ and a corresponding sequence $\{t_n\}$ like in the definition (32). We also fix $T \in (0, +\infty)$ and set, for a.a. $t \in (0, T)$,

$$\mu^n(t) := \mu(t_n + t), \quad \rho^n(t) := \rho(t_n + t), \quad \zeta^n(t) := \zeta(t_n + t)$$
$$\mu_\Gamma^n(t) := \mu_\Gamma(t_n + t), \quad \rho_\Gamma^n(t) := \rho_\Gamma(t_n + t), \quad \zeta_\Gamma^n(t) := \zeta_\Gamma(t_n + t)$$
$$u^n(t) := u(t_n + t)$$

and notice that (21) and the interpolation inequality (50) applied to u^n imply that

$$u^n \to 0 \quad \text{strongly in } L^\infty(0, T; L^2(\Omega)). \tag{61}$$

Moreover, it is clear that the 6-tuple $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ satisfies the regularity conditions (25)–(27) and the equations (28)–(30) with u replaced by u^n, as well as the initial condition $\rho^n(0) = \rho(t_n)$. In particular, by construction, we have that

$$(\rho^n, \rho_\Gamma^n)(0) \to (\rho^\omega, \rho_\Gamma^\omega) \quad \text{weakly in } \mathcal{V}. \tag{62}$$

Furthermore, the global estimates already performed on $(\mu, \mu_\Gamma, \rho, \rho_\Gamma, \zeta, \zeta_\Gamma)$ immediately imply some estimates on $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ that are uniform with respect to n. Here is a list. From (53) and (58), we infer that

$$\|(\rho^n, \rho_\Gamma^n)\|_{H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V})} \leq c. \tag{63}$$

By (51), we also deduce that

$$(\nabla \mu^n, \nabla_\Gamma \mu_\Gamma^n) \to 0 \quad \text{strongly in } (L^2(0, T; \mathcal{H}))^3 \tag{64}$$

$$(\partial_t \rho^n, \partial_t \rho_\Gamma^n) \to 0 \quad \text{strongly in } L^2(0, T; \mathcal{H}). \tag{65}$$

On the other hand, (60) yields a uniform estimate on the mean value $\mathrm{mean}(\mu^n, \mu_\Gamma^n)$. By combining this with (64), we conclude that

$$\|(\mu^n, \mu_\Gamma^n)\|_{L^2(0,T;\mathcal{V})} \leq c_T. \tag{66}$$

In the next steps, we perform further estimates that ensure some additional convergence properties for $(\mu^n, \mu_\Gamma^n, \rho^n, \rho_\Gamma^n, \zeta^n, \zeta_\Gamma^n)$ on the interval $(0, T)$.

First auxiliary estimate Once again, we treat β and β_Γ as single-valued functions and write $\beta(\rho^n)$ and $\beta_\Gamma(\rho^n)$ in place of ζ^n and ζ_Γ^n, respectively. We test (29), written for

$(\mu^n, \mu^n_\Gamma, \rho^n, \rho^n_\Gamma, \zeta^n, \zeta^n_\Gamma)$ at the time s, by $(\beta(\rho^n), \beta(\rho^n_\Gamma))(s) \in \mathcal{V}$ and integrate over $(0, t)$ with any $t \in (0, T)$. We have

$$
\begin{aligned}
\frac{\tau_\Omega}{2} & \int_\Omega \hat{\beta}(\rho^n(t)) + \frac{\tau_\Gamma}{2} \int_\Gamma \hat{\beta}_\Gamma(\rho^n_\Gamma(t)) + \int_{Q_t} \beta'(\rho^n) |\nabla \rho^n|^2 + \int_{\Sigma_t} \beta'(\rho^n) |\nabla_\Gamma \rho^n_\Gamma|^2 \\
& + \int_{Q_t} |\beta(\rho^n)|^2 + \int_{\Sigma_t} \beta_\Gamma(\rho^n_\Gamma) \beta(\rho^n_\Gamma) \\
= & \int_{Q_t} \left(\mu^n - \pi(\rho^n)\right) \beta(\rho^n) + \int_{\Sigma_t} (\mu^n_\Gamma - \pi_\Gamma(\rho^n_\Gamma)) \beta(\rho^n_\Gamma) \\
& + \frac{\tau_\Omega}{2} \int_\Omega \hat{\beta}(\rho(t_n)) + \frac{\tau_\Gamma}{2} \int_\Gamma \hat{\beta}_\Gamma(\rho_\Gamma(t_n)).
\end{aligned}
\tag{67}
$$

All of the integrals on the left-hand side are nonnegative but the last one, which we treat in the following way: we notice that (20), and the fact that β and β_Γ have the same sign by (16), imply that $\beta(r)\beta_\Gamma(r) \geq (2\eta)^{-1}\beta^2(r) - c$ for all r. It follows that

$$
\int_{\Sigma_t} \beta_\Gamma(\rho^n_\Gamma) \beta(\rho^n_\Gamma) \geq \frac{1}{2\eta} \int_{\Sigma_t} |\beta(\rho^n_\Gamma)|^2 - c_T.
$$

Let us come to the right-hand side. The first integral can obviously be dealt with using the Young inequality. We treat the second one this way:

$$
\begin{aligned}
\int_{\Sigma_t} & (\mu^n_\Gamma - \pi_\Gamma(\rho^n_\Gamma)) \beta(\rho^n_\Gamma) \\
& \leq \frac{1}{4\eta} \int_{\Sigma_t} |\beta(\rho^n_\Gamma)|^2 + c \int_{\Sigma_t} |\mu^n_\Gamma|^2 + c \int_{\Sigma_t} (1 + |\rho^n_\Gamma|^2) \leq \frac{1}{4\eta} \int_{\Sigma_t} |\beta(\rho^n_\Gamma)|^2 + c,
\end{aligned}
$$

thanks to (66). Finally, the last two terms of (67) are bounded by (54). By combining all these inequalities, we derive that

$$
\|\zeta^n\|_{L^2(0,T;H)} \leq c_T,
\tag{68}
$$

as well as an estimate for $\|\beta(\rho^n_\Gamma)\|_{L^2(0,T;H_\Gamma)}$ as a by-product.

Second auxiliary estimate We apply [16, Lem. 3.1] a.e. in $(0, +\infty)$ to (29), written for $(\mu^n, \mu^n_\Gamma, \rho^n, \rho^n_\Gamma, \zeta^n, \zeta^n_\Gamma)$ in the form

$$
\begin{aligned}
\int_\Omega \nabla \rho^n \cdot \nabla v & + \int_\Gamma \nabla_\Gamma \rho^n_\Gamma \cdot \nabla_\Gamma v_\Gamma + \int_\Gamma \beta_\Gamma(\rho^n_\Gamma) v_\Gamma \\
& = \int_\Omega \mu^n v + \int_\Gamma \mu^n_\Gamma v_\Gamma - \tau_\Omega \int_\Omega \partial_t \rho^n v - \tau_\Gamma \int_\Gamma \partial_t \rho^n_\Gamma v_\Gamma \\
& \quad - \int_\Omega (\zeta^n + \pi(\rho^n)) v - \int_\Gamma \pi_\Gamma(\rho^n_\Gamma) v_\Gamma.
\end{aligned}
$$

We obtain, in particular, that

$$\|\beta_\Gamma(\rho_\Gamma^n(t))\|_{H_\Gamma} \le c \left(\|(\mu^n, \mu_\Gamma^n)(t)\|_{\mathcal{H}} + \|\partial_t(\rho^n, \rho_\Gamma^n)(t)\|_{\mathcal{H}} + \|(\zeta^n + \pi(\rho^n))(t)\|_H \right)$$

for a.a. $t \in (0, +\infty)$, where c depends only on Ω. By accounting for (63), (66) and (68), we conclude that

$$\|\zeta_\Gamma^n\|_{L^2(0,T;H_\Gamma)} \le c_T. \tag{69}$$

Limits We collect the estimates (63), (66), (68) and (69) and use standard weak and weak star compactness results. For a subsequence, still labeled by n, we have

$$(\rho^n, \rho_\Gamma^n) \to (\rho^\infty, \rho_\Gamma^\infty) \quad \text{weakly star in } H^1(0,T;\mathcal{H}) \cap L^\infty(0,T;\mathcal{V}), \tag{70}$$

$$(\mu^n, \mu_\Gamma^n) \to (\mu^\infty, \mu_\Gamma^\infty) \quad \text{weakly in } L^2(0,T;\mathcal{V}), \tag{71}$$

$$(\zeta^n, \zeta_\Gamma^n) \to (\zeta^\infty, \zeta_\Gamma^\infty) \quad \text{weakly in } L^2(0,T;\mathcal{H}). \tag{72}$$

We now prove that $(\mu^\infty, \mu_\Gamma^\infty, \rho^\infty, \rho_\Gamma^\infty, \zeta^\infty, \zeta_\Gamma^\infty)$ satisfies the integrated version of (28)–(29), where we read $u = 0$, with time-dependent test functions $(v, v_\Gamma) \in L^2(0,T;\mathcal{V})$, and that (30) holds true as well. First of all, we notice that ρ^n converges to ρ^∞ weakly star in $L^\infty(0,T;L^6(\Omega))$, by (70) and the continuous embedding $V \subset L^6(\Omega)$. Owing to (33), we see that u^n tends to zero strongly in $(L^\infty(0,T;L^{3/2}(\Omega)))^3$ and deduce that $\rho^n u^n$ converges to zero weakly star in $(L^\infty(0,T;L^{6/5}(\Omega)))^3$. Next, from (70) we derive the strong convergence

$$(\rho^n, \rho_\Gamma^n) \to (\rho^\infty, \rho_\Gamma^\infty) \quad \text{strongly in } C^0([0,T];\mathcal{H}), \tag{73}$$

which follows from the compact embedding $\mathcal{V} \subset \mathcal{H}$ and from applying, e.g., [39, Sect. 8, Cor. 4]. Then, $(\pi(\rho^n), \pi_\Gamma(\rho_\Gamma^n))$ converges to $(\pi(\rho^\infty), \pi_\Gamma(\rho_\Gamma^\infty))$ strongly in the same space, by Lipschitz continuity. This concludes the proofs that (28) and (29) hold for the limiting 6-tuple in an integrated form, which is equivalent to the pointwise formulation. In order to derive (30), i.e., $\zeta^\infty \in \beta(\rho^\infty)$ and $\zeta_\Gamma^\infty \in \beta_\Gamma(\rho_\Gamma^\infty)$, we combine the weak convergence (72) with the strong convergence (73) and apply, e.g., [2, Lemma 2.3, p. 38].

5 Conclusion

It remains to prove that the above limit leads to a stationary solution with the properties specified in the statement. To this end, we first derive that $(\rho^\infty, \rho_\Gamma^\infty)$ belongs to $L^2(0,T;\mathcal{W})$ and solves the boundary value problem

$$-\Delta \rho^\infty + \zeta^\infty + \pi(\rho^\infty) = \mu^\infty \quad \text{a.e. in } Q_T, \tag{74}$$

$$\partial_n \rho^\infty - \Delta_\Gamma \rho_\Gamma^\infty + \zeta_\Gamma^\infty + \pi_\Gamma(\rho_\Gamma^\infty) = \mu_\Gamma^\infty \quad \text{a.e. on } \Sigma_T. \tag{75}$$

From (65) and (70), we see that $\partial_t(\rho^\infty, \rho_\Gamma^\infty)$ vanishes identically. Thus, we are dealing with a time-dependent elliptic problem in a variational form and can use the

following well-known estimates from trace theory and from the theory of elliptic equations. For any v and v_Γ that make the right-hand sides meaningful, we have that

$$\|\partial_n v\|_{H^{-1/2}(\Gamma)} \le C_\Omega\big(\|v\|_{H^1(\Omega)} + \|\Delta v\|_{L^2(\Omega)}\big), \tag{76}$$

$$\|\partial_n v\|_{L^2(\Gamma)} \le C_\Omega\big(\|v\|_{H^{3/2}(\Omega)} + \|\Delta v\|_{L^2(\Omega)}\big), \tag{77}$$

$$\|v\|_{H^2(\Omega)} \le C_\Omega\big(\|v_{|\Gamma}\|_{H^{3/2}(\Gamma)} + \|\Delta v\|_{L^2(\Omega)}\big), \tag{78}$$

$$\|v_\Gamma\|_{H^2(\Gamma)} \le C_\Omega\big(\|v_\Gamma\|_{H^1(\Gamma)} + \|\Delta_\Gamma v_\Gamma\|_{L^2(\Gamma)}\big), \tag{79}$$

$$\|v_\Gamma\|_{H^{3/2}(\Gamma)} \le C_\Omega\big(\|v_\Gamma\|_{H^1(\Gamma)} + \|\Delta_\Gamma v_\Gamma\|_{H^{-1/2}(\Gamma)}\big), \tag{80}$$

where the positive constant C_Ω depends only on Ω. By taking test functions $(v, 0)$ with $v \in H_0^1(\Omega)$, we derive that (74) holds in the sense of distributions on Q_T. This implies that $\Delta\rho^\infty \in L^2(0, T; H)$, so that $\partial_n \rho^\infty$ is a well-defined element of $L^2(0, T; H^{-1/2}(\Gamma))$ (see (76)) satisfying the integration–by–parts formula in a generalized sense. Coming back to (29) written for our limiting solution and arbitrary test functions $(v, v_\Gamma) \in \mathcal{V}$, we deduce that (75) holds in a generalized sense. From this, we infer that $\Delta_\Gamma \rho_\Gamma^\infty \in L^2(0, T; H^{-1/2}(\Gamma))$, so that $\rho_\Gamma^\infty \in L^2(0, T; H^{3/2}(\Gamma))$ (see (80)). It follows that $\rho^\infty \in L^2(0, T; W)$ by (78). In particular, $\partial_n \rho^\infty \in L^2(0, T; H_\Gamma)$ by (77) so that $\Delta_\Gamma \rho_\Gamma^\infty \in L^2(0, T; H_\Gamma)$ and $\rho_\Gamma^\infty \in L^2(0, T; W_\Gamma)$ by (79). Finally, as all of the ingredients are L^2 functions, it is also clear that equations (74)–(75) hold almost everywhere.

At this point, we are ready to conclude. Since $\partial_t(\rho^\infty, \rho_\Gamma^\infty)$ vanishes and the same holds for $(\nabla\mu^\infty, \nabla_\Gamma \mu_\Gamma^\infty)$ by (64), there exist $(\rho^s, \rho_\Gamma^s) \in \mathcal{V}$ and $\mu_\infty \in L^2(0, T)$ such that

$$(\rho^\infty, \rho_\Gamma^\infty)(x, t) = (\rho^s, \rho_\Gamma^s)(x) \quad \text{and} \quad (\mu^\infty, \mu_\Gamma^\infty)(x, t) = (\mu_\infty(t), \mu_\infty(t))$$

$$\text{for a.a. } (x, t) \in Q_T.$$

We prove that $(\zeta^\infty, \zeta_\Gamma^\infty)$ is time independent as well and that μ_∞ is a constant, by accounting for our assumptions on the graphs β and β_Γ: at least one of them is single-valued. Assume first that β is single-valued. Then $\zeta^\infty = \beta(\rho^\infty)$ is time independent and attains the value $\zeta^s := \beta(\rho^s)$ at any time. From (74), it follows that μ^∞ is time independent as well, so that the function μ_∞ is a constant that we term μ^s. Thus, the right-hand side of (75) is the same constant μ^s. As this does not depend on time, the same holds for ζ_Γ^∞. Hence, it attains some value $\zeta_\Gamma^s \in H_\Gamma$ a.e. in $(0, T)$. Assume now that β_Γ is single-valued. Then, we first use (75) to derive that $\zeta_\Gamma^\infty = \beta_\Gamma(\rho_\Gamma^\infty)$ and μ_Γ^∞ are time independent. In particular, μ^∞ takes some constant value μ^s, so that ζ^∞ is time independent, by comparison in (74). As in both cases (36) holds as a consequence of (30) for the limiting solution, the quadruplet $(\rho^s, \rho_\Gamma^s, \zeta^s, \zeta_\Gamma^s)$ is a stationary solution corresponding to the value μ^s of the chemical potential. Finally, we prove that (ρ^s, ρ_Γ^s) coincides with the given $(\rho^\omega, \rho_\Gamma^\omega)$. Indeed, (70) implies weak convergence also in $C^0([0, T]; \mathcal{H})$, whence

$$(\rho^n, \rho_\Gamma^n)(0) \to (\rho^\infty, \rho_\Gamma^\infty)(0) = (\rho^s, \rho_\Gamma^s) \quad \text{weakly in } \mathcal{H}.$$

By comparison with (62), we conclude that $(\rho^s, \rho_\Gamma^s) = (\rho^\omega, \rho_\Gamma^\omega)$, and the proof is complete.

References

1. Bai, F., Elliott, C.M., Gardiner, A., Spence, A., Stuart, A.M.: The viscous Cahn–Hilliard equation. I. Computations. Nonlinearity **8**, 131–160 (1995)
2. Barbu, V.: Nonlinear Differential Equations of Monotone Type in Banach Spaces. Springer, London (2010)
3. Bergh, J., Löfström, J.: Interpolation Spaces. An Introduction. Grundlehren der mathematischen Wissenschaften, vol. 223. Springer, Berlin (1976)
4. Cahn, J.W., Hilliard, J.E.: Free energy of a nonuniform system I. Interfacial free energy. J. Chem. Phys. **2**, 258–267 (1958)
5. Calatroni, L., Colli, P.: Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions. Nonlinear Anal. **79**, 12–27 (2013)
6. Cherfils, L., Gatti, S., Miranville, A.: A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls. J. Math. Sci. (N.Y.) **189**, 604–636 (2013)
7. Cherfils, L., Miranville, A., Zelik, S.: The Cahn–Hilliard equation with logarithmic potentials. Milan J. Math. **79**, 561–596 (2011)
8. Cherfils, L., Petcu, M.: A numerical analysis of the Cahn–Hilliard equation with non-permeable walls. Numer. Math. **128**, 518–549 (2014)
9. Chill, R., Fašangová, E., Prüss, J.: Convergence to steady states of solutions of the Cahn–Hilliard equation with dynamic boundary conditions. Math. Nachr. **279**, 1448–1462 (2006)
10. Colli, P., Fukao, T.: Cahn–Hilliard equation with dynamic boundary conditions and mass constraint on the boundary. J. Math. Anal. Appl. **429**, 1190–1213 (2015)
11. Colli, P., Fukao, T.: Equation and dynamic boundary condition of Cahn–Hilliard type with singular potentials. Nonlinear Anal. **127**, 413–433 (2015)
12. Colli, P., Gilardi, G., Podio-Guidugli, P., Sprekels, J.: Well-posedness and long-time behaviour for a nonstandard viscous Cahn–Hilliard system. SIAM J. Appl. Math. **71**, 1849–1870 (2011)
13. Colli, P., Gilardi, G., Sprekels, J.: On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential. J. Math. Anal. Appl. **419**, 972–994 (2014)
14. Colli, P., Gilardi, G., Sprekels, J.: A boundary control problem for the pure Cahn–Hilliard equation with dynamic boundary conditions. Adv. Nonlinear Anal. **4**, 311–325 (2015)
15. Colli, P., Gilardi, G., Sprekels, J.: A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions. Appl. Math. Optim. **73**, 195–225 (2016)
16. Colli, P., Gilardi, G., Sprekels, J.: On a Cahn–Hilliard system with convection and dynamic boundary conditions. Ann. Mat. Pura Appl. (4), 1–31 (2018). https://doi.org/10.1007/s10231-018-0732-1 (see also preprint arXiv:1704.05337 [math.AP])
17. Colli, P., Gilardi, G., Sprekels, J.: Optimal velocity control of a viscous Cahn–Hilliard system with convection and dynamic boundary conditions. SIAM J. Control Optim. **56**, 1665–1691 (2018)
18. Colli, P., Gilardi, G., Sprekels, J.: Optimal velocity control of a convective Cahn–Hilliard system with double obstacles and dynamic boundary conditions: a 'deep quench' approach. J. Convex Anal. 1–30 (2018) (see also preprint arXiv:1709.03892 [math.AP])
19. Colli, P., Sprekels, J.: Optimal control of an Allen–Cahn equation with singular potentials and dynamic boundary condition. SIAM J. Control Optim. **53**, 213–234 (2015)
20. Elliott, C.M., Stuart, A.M.: Viscous Cahn–Hilliard equation. II. Analysis. J. Differ. Equ. **128**, 387–414 (1996)
21. Elliott, C.M., Zheng, S.: On the Cahn–Hilliard equation. Arch. Ration. Mech. Anal. **96**, 339–357 (1986)
22. Fried, E., Gurtin, M.E.: Continuum theory of thermally induced phase transitions based on an order parameter. Physica D **68**, 326–343 (1993)
23. Fukao, T., Yamazaki, N.: A boundary control problem for the equation and dynamic boundary condition of Cahn–Hilliard type. In: Colli, P., Favini, A., Rocca, E., Schimperna, G., Sprekels, J. (eds.)

Solvability, Regularity, Optimal Control of Boundary Value Problems for PDEs. Springer INdAM Series, vol. 22, pp. 255–280. Springer, Milan (2017)

24. Gal, C.G.: A Cahn–Hilliard model in bounded domains with permeable walls. Math. Methods Appl. Sci. **29**, 2009–2036 (2006)

25. Gal, C.G., Wu, H.: Asymptotic behavior of a Cahn–Hilliard equation with Wentzell boundary conditions and mass conservation. Discret. Contin. Dyn. Syst. **22**, 1041–1063 (2008)

26. Gilardi, G., Miranville, A., Schimperna, G.: On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions. Commun. Pure Appl. Anal. **8**, 881–912 (2009)

27. Gilardi, G., Miranville, A., Schimperna, G.: Long-time behavior of the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions. Chin. Ann. Math. Ser. B **31**, 679–712 (2010)

28. Gilardi, G., Rocca, E.: Well posedness and long time behaviour for a singular phase field system of conserved type. IMA J. Appl. Math. **72**, 498–530 (2007)

29. Goldstein, G.R., Miranville, A., Schimperna, G.: A Cahn–Hilliard model in a domain with non-permeable walls. Physica D **240**, 754–766 (2011)

30. Goldstein, G.R., Miranville, A.: A Cahn–Hilliard–Gurtin model with dynamic boundary conditions. Discret. Contin. Dyn. Syst. Ser. S **6**, 387–400 (2013)

31. Gurtin, M.: Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance. Physica D **92**, 178–192 (1996)

32. Kubo, M.: The Cahn–Hilliard equation with time-dependent constraint. Nonlinear Anal. **75**, 5672–5685 (2012)

33. Liu, C., Wu, H.: An energetic variational approach for the Cahn–Hilliard equation with dynamic boundary conditions: derivation and analysis, pp. 1–68 (2017). Preprint arXiv:1710.08318 [math.AP]

34. Miranville, A., Zelik, S.: Robust exponential attractors for Cahn–Hilliard type equations with singular potentials. Math. Methods Appl. Sci. **27**, 545–582 (2004)

35. Novick-Cohen, A.: On the viscous Cahn–Hilliard equation. In: Material Instabilities in Continuum Mechanics (Edinburgh, 1985–1986), pp. 329–342. Oxford Science Publications, Oxford University Press, New York (1988)

36. Podio-Guidugli, P.: Models of phase segregation and diffusion of atomic species on a lattice. Ric. Mat. **55**, 105–118 (2006)

37. Prüss, J., Racke, R., Zheng, S.: Maximal regularity and asymptotic behavior of solutions for the Cahn–Hilliard equation with dynamic boundary conditions. Ann. Mat. Pura Appl. **185**(4), 627–648 (2006)

38. Racke, R., Zheng, S.: The Cahn–Hilliard equation with dynamic boundary conditions. Adv. Differ. Equ. **8**, 83–110 (2003)

39. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl. **146**(4), 65–96 (1987)

40. Wu, H., Zheng, S.: Convergence to equilibrium for the Cahn–Hilliard equation with dynamic boundary conditions. J. Differ. Equ. **204**, 511–531 (2004)

Affiliations

Pierluigi Colli[1] ⬥ · **Gianni Gilardi[1]** · **Jürgen Sprekels[2,3]**

Gianni Gilardi
gianni.gilardi@unipv.it

Jürgen Sprekels
sprekels@wias-berlin.de

[1] Dipartimento di Matematica "F. Casorati", Università di Pavia and Research Associate at the IMATI-C.N.R. Pavia, via Ferrata 5, 27100 Pavia, Italy

[2] Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany

[3] Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany