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On time-periodic Navier–Stokes flows with fast spatial decay in the whole space

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Abstract We investigate the pointwise behavior of time-periodic Navier–Stokes flows in the whole space. We show that if the time-periodic external force is sufficiently small in an appropriate sense, then there exists a unique time-periodic solution $\{u, p\}$ of the Navier–Stokes equation such that $|u(t,x)| = O(|x|^{1-n})$, $|\nabla u(t,x)| = O(|x|^{-n})$ and $|p(t,x)| = O(|x|^{-n})$ uniformly in $t \in \mathbb{R}$ as $|x| \to \infty$. Our solution decays more rapidly than the time-periodic Stokes fundamental solution. The proof is based on the representation formula of a solution via the time-periodic Stokes fundamental solution and its properties.

Keywords Navier–Stokes equation · Time-periodic solution · Asymptotic property

Mathematics Subject Classification 35Q30 · 35B10 · 76D05 · 76D03

1 Introduction

We consider the time-periodic problem for the Navier–Stokes equation in $\mathbb{R} \times \mathbb{R}^n$ with $n \geq 3$:

$$\begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = \operatorname{div} F & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(\cdot, x) \to 0 & \text{as } |x| \to \infty, \\ u(t, \cdot) = u(t+T, \cdot) & \text{forall } t \in \mathbb{R}. \end{cases} \tag{1}$$

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Here $u = (u_1(t,x), \cdots, u_n(t,x))$ and $p = p(t,x)$ denote, respectively, the unknown velocity and pressure of a viscous incompressible fluid, while $F = (F_{ij}(t,x))_{i,j=1}^n$ is a given periodic tensor with div $F = \left(\sum_{i=1}^n \partial_{x_i} F_{ij}(t,x)\right)_{j=1}^n$ denoting the periodic external force. Furthermore, T denotes a fixed period.

Existence and uniqueness of solutions to (1) are studied in many literatures such as [2, 3, 5, 7, 10, 13–15]. The time-periodic problem (1) is studied in various functional settings, and, in particular, it is known that the L^2 theory enables us to construct a solution v with $\nabla v \in L^2((0,T) \times \mathbb{R}^n)$ of (1) without restricting the size of the external force, see [7, Theorem 6.3.1] for instance. The asymptotic property of the solution v is, however, still an open problem since the construction of v gives us little information on it. One method for describing the asymptotic behavior of v, under the smallness of the external force in a sense, is to establish the existence of solutions, say w, with desired decay properties to (1) and then apply an appropriate uniqueness theorem so that we conclude $v = w$. The purpose of this paper is to prove the existence of solutions with pointwise decay properties to (1). Such results were given by Galdi and Sohr [3] and Kang et al. [4]. They considered the Navier–Stokes equation in three-dimensional exterior domains and established the existence of time-periodic solutions satisfying

$$|\nabla^j u(t,x)| = O(|x|^{-j-1}), \quad |\nabla^j p(t,x)| = O(|x|^{-j-2}) \quad (j = 0, 1) \tag{2}$$

uniformly in time as $|x| \to \infty$. Recently, Kyed [9] introduced the notion of the time-periodic Stokes fundamental solution, and its leading term is given by the steady Stokes fundamental solution. The spatial decay rate (2) coincides with that of the Stokes fundamental solution and thus it seems to be natural. It is known that, in the three-dimensional exterior problem, (2) is the best spatial decay rate of periodic solutions expected in general [4], however, we do not know whether it is also optimal in other unbounded domains. Miyakawa [11] and the author [12] constructed a unique solution $\{u, p\}$ of the stationary Navier–Stokes equation in $\mathbb{R}^n$ such that

$$|\nabla^j u(x)| = O(|x|^{1-n-j}), \quad |\nabla^j p(x)| = O(|x|^{-n-j}) \quad (j = 0, 1, \ldots) \tag{3}$$

as $|x| \to \infty$. The stationary solution in [11, 12] decays more rapidly than the steady Stokes fundamental solution, and we can expect that the time-periodic problem (1) also admits a solution decaying like (3) since stationary solutions can be regarded as time-periodic ones with arbitrary period.

In this paper, we shall show that there exists a unique solution $\{u, p\}$ of (1) such that

$$|u(t,x)| = O(|x|^{1-n}), \quad |\nabla u(t,x)| = O(|x|^{-n}), \quad |p(t,x)| = O(|x|^{-n})$$

uniformly in $t \in \mathbb{R}$ as $|x| \to \infty$, provided that F and div F are small in a suitable sense. Our solution has the same spatial decay as (3) and we emphasize that it decays more rapidly than the time-periodic Stokes fundamental solution. Furthermore, the structure of the time-periodic Stokes fundamental solution enables us to write the solution u as the sum of the steady and time-periodic parts. The decay properties of each parts of solutions are also studied in this paper.

The proof relies upon the representation formula of solutions via the time-periodic Stokes fundamental solution. We transform (1) into the integral equation via the fundamental solution, and then we estimate both the steady and time-periodic parts of a solution in order to apply the contraction mapping principle. In the proof, the well known properties of steady Stokes fundamental solution and the results of [1, 9] play an important role.

2 Main results

Before stating our results, we introduce some function spaces. In what follows, we adopt the same symbols for vector and scalar function spaces. Let $1 \leq q \leq \infty$ and let X be a Banach space. We denote by $L^q_{per}(\mathbb{R}; X)$ the Banach space of all T-periodic functions $u : \mathbb{R} \to X$ such that the restriction $u|_{[0,T)} \in L^q(0, T; X)$. The norm in $L^q_{per}(\mathbb{R}; X)$ is given by $\|u\|_{q,X} := \|u\|_{L^q(0,T;X)}$. In the case $X = L^q(\mathbb{R}^n)$, we write simply $L^q_{per}(\mathbb{R} \times \mathbb{R}^n)$ with norm $\|\cdot\|_q$. The space $W^{1,2,q}_{per}(\mathbb{R} \times \mathbb{R}^n)$ is defined by $W^{1,2,q}_{per}(\mathbb{R} \times \mathbb{R}^n) := \{u \in L^q_{per}(\mathbb{R} \times \mathbb{R}^n); \|u\|_{1,2,q} < \infty\}$ where $\|u\|_{1,2,q} := (\|\partial_t u\|^q_q + \sum_{|\alpha| \leq 2} \|\partial^\alpha_x u\|^q_q)^{1/q}$. For $\mu > 0$, we define the Banach space X_μ by

$$X_\mu := \{u \in L^\infty(\mathbb{R}^n); \sup_{x \in \mathbb{R}^n} (|x| + 1)^\mu |u(x)| < \infty\}$$

with the norm

$$\|u\|_{X_\mu} := \sup_{x \in \mathbb{R}^n} (|x| + 1)^\mu |u(x)|.$$

It is easy to check that $X_{\mu_1} \subset X_{\mu_2}$ if $\mu_2 < \mu_1$.

Let $\{E, Q\}$ be the time-periodic Stokes fundamental solution introduced in [9]. The form and properties of the fundamental solution shall be reviewed in the next section. We consider (1) in the form

$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \nabla E(t-s, x-y)(F - u \otimes u)(s,y)\, dsdy, \tag{4}$$

where $u \otimes u := (u_i u_j)^n_{i,j=1}$ and ∇ denotes the gradient with respect to the spatial variable. If F and u decay rapidly at spatial infinity, then (4) can be written as

$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T E(t-s, x-y)(\operatorname{div} F - u \cdot \nabla u)(s,y)\, dsdy. \tag{5}$$

As we shall see later, we should interpret E in (4) and (5) as the composition $E \circ \pi$ where π is a map introduced in the next section. In this paper, we use the notation E instead of $E \circ \pi$ as long as we consider (4) and (5). Also, it shall be seen that the integral equations (4) and (5) are equivalent to (1) in an appropriate sense. The associated pressure p is given by

$$p(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T Q(t-s, x-y) \cdot (\operatorname{div} F - u \cdot \nabla u)(s,y)\, dsdy.$$

The main result of this paper is stated in the following theorem. The leading term of $\{E, Q\}$ is the steady Stokes fundamental solution as we shall see in the next section, and our solution in the following theorem decays more rapidly than the time-periodic Stokes fundamental solution.

Theorem 1 *Let $n \geq 3$ and $0 < \delta < 1$. Suppose*

$$F \in L^\infty_{per}(\mathbb{R}; X_{n+\delta}) \quad \textit{and} \quad \operatorname{div} F \in L^\infty_{per}(\mathbb{R}; X_{n+1}). \tag{6}$$

If F and $\operatorname{div} F$ are sufficiently small in $L^\infty_{per}(\mathbb{R}; X_{n+\delta})$ and $L^\infty_{per}(\mathbb{R}; X_{n+1})$ respectively, then (4) admits a unique solution u such that

$$u \in L^\infty_{per}(\mathbb{R}; X_{n-1}), \quad \nabla u \in L^\infty_{per}(\mathbb{R}; X_n)$$

and

$$\partial_t u, \nabla^2 u \in L^q_{per}(\mathbb{R} \times \mathbb{R}^n) \quad \textit{for all } 1 < q < \infty.$$

Furthermore, the associated pressure p satisfies

$$p \in L^\infty_{per}(\mathbb{R}; X_n).$$

Remark 1 The constant δ is introduced so that we can apply the inequality (10) below and $F(t, \cdot)$ is integrable, see also [12, Remark 2.1].

Remark 2 If F is independent of t, then we can verify that u in Theorem 1 is a stationary solution and thus coincides with the one constructed in [11, 12].

Remark 3 It is possible to show that the solution u in Theorem 1 satisfies $\partial_t u \in L^\infty_{per}(\mathbb{R}; X_{n+1})$ provided that $\partial_t F$ is sufficiently small in $L^\infty_{per}(\mathbb{R}; X_{n+1})$, see Remark 4. However, the pointwise behavior of $\partial_t E$ is still an open problem and we do not know whether the decay rate of $\partial_t u$ is faster than that of $\partial_t E$.

We shall see in the next section that the time-periodic Stokes fundamental solution E is defined as the sum of the steady Stokes fundamental solution E_s and the time-periodic remainder E_p. Hence every solution u of (4) is written as the sum of the steady part u_s:

$$u_s(x) := \int_{\mathbb{R}^n} \nabla E_s(x-y) \frac{1}{T} \int_0^T (F - u \otimes u)(s,y)\, dsdy$$

and time-periodic part u_p:

$$u_p(t,x) := \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T \nabla E_p(t-s,x-y)(F-u\otimes u)(s,y)\,dsdy.$$

The next theorem describes the asymptotic behavior of each parts of solutions.

Theorem 2 *Let $n\geq 3$ and $0<\delta<1$. Suppose that F satisfies (6). Every solution $u\in L^\infty_{per}(\mathbb{R};X_{n-1})$ with $\nabla u\in L^\infty_{per}(\mathbb{R};X_n)$ of (4) is written as the sum of the steady part u_s and time-periodic one u_p such that*

$$u_s\in X_{n-1},\quad \nabla u_s\in X_n,\quad \nabla^2 u_s\in L^q(\mathbb{R}^n)\quad \text{for all } 1<q<\infty,$$

and

$$\begin{aligned}&u_p\in L^\infty_{per}(\mathbb{R};X_{n+\delta}),\quad \nabla u_p\in L^\infty_{per}(\mathbb{R};X_{n+1}),\\ &\partial_t u_p,\nabla^2 u_p,\in L^q_{per}(\mathbb{R}\times\mathbb{R}^n)\quad \text{for all } 1<q<\infty.\end{aligned}$$

If

$$F\in L^\infty_{per}(\mathbb{R};X_{n+1})\quad \text{and}\quad \operatorname{div} F\in L^\infty_{per}(\mathbb{R};X_{n+2}), \tag{7}$$

then we have

$$u_p\in L^\infty_{per}(\mathbb{R};X_{n+1})\quad \text{and}\quad \nabla u_p\in L^\infty_{per}(\mathbb{R};X_{n+2}).$$

3 Proof of main theorems

3.1 Time-periodic Stokes fundamental solution

In this subsection, we review the theory for the time-periodic Stokes fundamental solution. For this purpose, we need the following notation. Set $\mathbb{T} := \mathbb{R}/T\mathbb{Z}$ and $G := \mathbb{T}\times\mathbb{R}^n$. We define the map $\pi:\mathbb{R}\times\mathbb{R}^n\to G$ by $\pi(t,x) := ([t],x)$ and let $\Pi := \pi|_{[0,T)\times\mathbb{R}^n}$. The restriction Π is a bijection from $[0,T)\times\mathbb{R}^n$ to G, and via Π we identify G with $[0,T)\times\mathbb{R}^n$. The Haar measure dg on the locally compact abelian group G, unique up to a normalization factor, is chosen as the product of the Lebesgue measures on $\mathbb{R}^n$ and $[0, T)$, and we have

$$\int_G u(g)\,dg := \frac{1}{T}\int_0^T\int_{\mathbb{R}^n}(u\circ\Pi)(s,y)\,dyds.$$

Derivatives on G are defined by

$$\partial_t\partial_{x_i}u = (\partial_t\partial_{x_i}(u\circ\pi))\circ\Pi^{-1}.$$

The differentiable structure on G is inherited from $\mathbb{R}\times\mathbb{R}^n$ and we can formulate $(1)_{1,2,4}$ as

$$\begin{cases} \partial_t \tilde{u} - \Delta \tilde{u} + \tilde{u} \cdot \nabla \tilde{u} + \nabla \tilde{p} = \operatorname{div} \tilde{F} & \text{in G}, \\ \operatorname{div} \tilde{u} = 0 & \text{in G}. \end{cases} \tag{8}$$

We can verify that if u is a solution of $(1)_{1,2,4}$, then $\tilde{u} = u \circ \Pi^{-1}$ is a solution of (8) with $\tilde{F} = F \circ \Pi^{-1}$. Conversely, if $\tilde{u}$ is a solution of (8), then $u = \tilde{u} \circ \pi$ is a solution of $(1)_{1,2,4}$ with $F = \tilde{F} \circ \pi$. For more details on the analysis on G, see [7, 8].

According to Kyed [9], the time-periodic Stokes fundamental solution $\{E, Q\}$ is given by

$$E = E_s \otimes 1_{\mathbb{T}} + E_p, \quad Q = Q_s \otimes \delta_{\mathbb{T}},$$

where $1_{\mathbb{T}}$ is the constant distribution 1 and $\delta_{\mathbb{T}}$ the Dirac delta distribution on $\mathbb{T}$. Here $\{E_s, Q_s\}$ is the steady Stokes fundamental solution, that is,

$$E_s = \left(E_{s,ij}\right)_{i,j=1}^n, \qquad E_{s,ij}(x) = \frac{1}{2\omega_n}\left(\frac{\delta_{ij}}{n-2}|x|^{2-n} + \frac{x_i x_j}{|x|^n}\right),$$
$$Q_s = \left(Q_{s,i}\right)_{i=1}^n, \qquad Q_{s,i}(x) = \frac{x_i}{\omega_n |x|^n},$$

with ω_n denoting the surface area of the unit sphere in $\mathbb{R}^n$. The time-periodic remainder E_p is defined as a tempered distribution on G via the Fourier transform, see Kyed [9] and Eiter and Kyed [1] for its precise form. Properties of E_s are well-known and those of E_p are studied in [1, 9]. The properties of E_p are stated within the functional framework on G in [1, 9], however, they are still valid even if we replace E_p by $E_p \circ \pi$, since $L^q(G)$ and $L^q_{per}(\mathbb{R} \times \mathbb{R}^n)$ are isometrically homeomorphic and so are $W^{1,2,q}(G)$ and $W^{1,2,q}_{per}(\mathbb{R} \times \mathbb{R}^n)$. For the sake of convenience, we shall simply write E_p instead of $E_p \circ \pi$ below. Also, we denote by $C = C(\cdot, \cdots, \cdot)$ various constants depending only on the quantities in parentheses.

Proposition 1 [1, 9] *The time-periodic remainder E_p satisfies*

$$E_p \in L^q_{per}(\mathbb{R} \times \mathbb{R}^n) \quad \text{for all } q \in \left(1, \frac{n+2}{n}\right),$$
$$\nabla E_p \in L^q_{per}(\mathbb{R} \times \mathbb{R}^n) \quad \text{for all } q \in \left[1, \frac{n+2}{n+1}\right),$$
$$\|\nabla^j E_p(\cdot, x)\|_{L^q(0,T)} \leq \frac{C}{|x|^{n+j}} \quad \text{for all } j = 0, 1, \dots \text{ and } q \in [1, \infty),$$

with $C = C(n, q, T)$. Furthermore, for $f \in L^q_{per}(\mathbb{R} \times \mathbb{R}^n)$ there exists a constant $C = C(n, q, T)$ such that

$$\left\| \int_0^T \int_{\mathbb{R}^n} E_p(\cdot - s, \cdot - y) f(s, y)\, dyds \right\|_{1,2,q} \leq C\|f\|_q \quad \text{for } 1 < q < \infty.$$

The time-periodic Stokes fundamental solution is a tempered distribution on G and thus we should consider the convolution integral on G. However, we can apply it in the classical setting as stated in [9, Remark 1.2]. To see this, we note that in (5) the periodicity of E implies that of u and we may assume $t \in [0, T)$. Let $f \in G$ with $\Pi^{-1}(f) = (t, x)$ and let $\tilde{u}$ be a solution of

$$\tilde{u}(f) = \int_G E(f-g)(\mathrm{div}\tilde{F} - \tilde{u}\cdot\nabla\tilde{u})(g)\,dg. \tag{9}$$

Then $\tilde{u}$ is a solution of (8) in an appropriate sense. Lifting (9) by π yields

$$(\tilde{u}\circ\pi)(t,x) = \int_{\mathbb{R}^n}\frac{1}{T}\int_0^T (E\circ\pi)(t-s, x-y)(\mathrm{div}\,\tilde{F} - \tilde{u}\cdot\nabla\tilde{u})(\pi(s,y))\,dsdy,$$

which is (5) with $E = E\circ\pi$, $u = \tilde{u}\circ\pi$ and $F = \tilde{F}\circ\pi$. Conversely, let u be a solution of (5) and we write

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^n}\frac{1}{T}\int_0^t (E\circ\pi)(t-s, x-y)(\mathrm{div}\,F - u\cdot\nabla u)(s,y)\,dsdy \\ &\quad + \int_{\mathbb{R}^n}\frac{1}{T}\int_t^T (E\circ\pi)(t-s+T, x-y)(\mathrm{div}\,F - u\cdot\nabla u)(s,y)\,dsdy. \end{aligned}$$

For $(t,x) \in [0,T)\times\mathbb{R}^n$, we lift this equation by Π^{-1} to get

$$(u\circ\Pi^{-1})(f) = \int_G E(f-g)(\mathrm{div}\,F - u\cdot\nabla u)(\Pi^{-1}(g))\,dg,$$

which is (9) with $\tilde{u} = u\circ\Pi^{-1}$ and $\tilde{F} = F\circ\Pi^{-1}$. Here we have used $\Pi^{-1}(f-g) = (t-s, x-y)$ for $0 \le s \le t$ and $\Pi^{-1}(f-g) = (t-s+T, x-y)$ for $t < s < T$. Therefore, there is a natural correspondence between the integral equations (5) and (9), and we readily see that a function u satisfying (5) is a solution of (1) in a suitable sense. If F and u decay rapidly at spatial infinity, we can obtain (4) by integrating (5) by parts and hence it suffices to consider (4) for our purpose.

In view of the form of E, we note again that every solution u of (4) can be written as the sum of the steady part u_s:

$$u_s(x) = \int_{\mathbb{R}^n}\nabla E_s(x-y)\frac{1}{T}\int_0^T (F - u\otimes u)(s,y)\,dsdy$$

and time-periodic part u_p:

$$u_p(t,x) = \int_{\mathbb{R}^n}\frac{1}{T}\int_0^T \nabla E_p(t-s, x-y)(F - u\otimes u)(s,y)\,dsdy.$$

Furthermore, the associated pressure p is written as

$$p(t,x) = \int_{\mathbb{R}^n} Q_s(x-y)\cdot(\mathrm{div}\,F - u\cdot\nabla u)(t,y)\,dy.$$

3.2 Proof of main results

With properties of the fundamental solution in hand, we estimate the steady and time-periodic parts. Let Y be a Banach space defined by

$$Y := \left\{ u \in L^\infty_{per}(\mathbb{R}; X_{n-1}); \nabla u \in L^\infty_{per}(\mathbb{R}; X_n) \right\}$$

with the norm

$$\|u\|_Y := \|u\|_{\infty, X_{n-1}} + \|\nabla u\|_{\infty, X_n}.$$

We define the operator K_s on Y by

$$(K_s u)(x) := \int_{\mathbb{R}^n} \nabla E_s(x-y) \frac{1}{T} \int_0^T (F - u \otimes u)(s, y)\, dsdy.$$

Estimates for $K_s u$ are studied in the next lemma.

Lemma 1 *Let $0 < \delta < 1$ and suppose F satisfies (6). For $u \in Y$ we have*

$$K_s u \in X_{n-1} \quad \textit{and} \quad \nabla(K_s u) \in X_n,$$

and there exists a constant $C = C(n, \delta)$ such that

$$\|K_s u\|_{X_{n-1}} + \|\nabla(K_s u)\|_{X_n} \le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|\operatorname{div} F\|_{\infty, X_{n+1}} + \|u\|_Y^2 \Big).$$

Proof We first recall the basic estimate

$$\int_{\mathbb{R}^n} \frac{dy}{|x-y|^{n-1}(|y|+1)^\mu} \le C(|x|+1)^{1-n} \quad \text{if } \mu > \text{n} \tag{10}$$

(see [12, Lemma 3.1]). Since $|\nabla^j E_s(x-y)| \le C|x-y|^{2-n-j}$ $(j = 0, 1, \ldots)$, it follows from (10) that

$$\begin{aligned} |(K_s u)(x)| &\le \int_{\mathbb{R}^n} \frac{C}{|x-y|^{n-1}} \left\{ \frac{\|F\|_{\infty, X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty, X_{n-1}}}{(|y|+1)^{2n-2}} \right\} dy \\ &\le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}} \Big)(|x|+1)^{1-n}. \end{aligned}$$

Hence we derive

$$K_s u \in X_{n-1}$$

with the estimate

$$\|K_s u\|_{X_{n-1}} \le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|_Y^2 \Big).$$

Next, we write

$$\begin{aligned}\nabla(K_s u)(x) &= \int_{\mathbb{R}^n} \nabla E_s(x-y) \frac{1}{T}\int_0^T (\operatorname{div}(F-u\otimes u))(s,y)\,dsdy \\ &= \int_{|x-y|<\frac{|x|+1}{2}} \nabla E_s(x-y)\frac{1}{T}\int_0^T (\operatorname{div}(F-u\otimes u))(s,y)\,dsdy \\ &\quad + \int_{|x-y|>\frac{|x|+1}{2}} \nabla E_s(x-y)\frac{1}{T}\int_0^T (\operatorname{div}(F-u\otimes u))(s,y)\,dsdy \\ &=: I_1 + I_2.\end{aligned}$$

Since $|x-y|<(|x|+1)/2$ implies $(|x|-1)/2<|y|<(3|x|+1)/2$, we see

$$\begin{aligned}|I_1| &\le \int_{|x-y|<\frac{|x|+1}{2}} \frac{C}{|x-y|^{n-1}}\left\{\frac{\|\operatorname{div} F\|_{\infty,X_{n+1}}}{(|y|+1)^{n+1}} + \frac{\|u\|_{\infty,X_{n-1}}\|\nabla u\|_{\infty,X_n}}{(|y|+1)^{2n-1}}\right\}dy \\ &\le \frac{C\Big(\|\operatorname{div} F\|_{\infty,X_{n+1}} + \|u\|_{\infty,X_{n-1}}\|\nabla u\|_{\infty,X_n}\Big)}{(|x|+1)^{n+1}} \int_{|x-y|<\frac{|x|+1}{2}} \frac{dy}{|x-y|^{n-1}} \\ &\le C\Big(\|\operatorname{div} F\|_{\infty,X_{n+1}} + \|u\|_{\infty,X_{n-1}}\|\nabla u\|_{\infty,X_n}\Big)(|x|+1)^{-n}.\end{aligned} \tag{11}$$

We integrate I_2 by parts to get

$$|I_2| \le I_{21} + I_{22},$$

where

$$\begin{aligned} I_{21} &:= \int_{|x-y|>\frac{|x|+1}{2}} \big|\nabla^2 E_s(x-y)\big| \frac{1}{T}\int_0^T |(F-u\otimes u)(s,y)|\,dsdy, \\ I_{22} &:= \int_{|x-y|=\frac{|x|+1}{2}} |\nabla E_s(x-y)| \frac{1}{T}\int_0^T |(F-u\otimes u)(s,y)|\,dsdS_y.\end{aligned}$$

The assumption $\delta>0$ leads us to

$$\begin{aligned} I_{21} &\le \int_{|x-y|>\frac{|x|+1}{2}} \frac{C}{|x-y|^n}\left\{\frac{\|F\|_{\infty,X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty,X_{n-1}}}{(|y|+1)^{2n-2}}\right\}dy \\ &\le \frac{C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)}{(|x|+1)^n}\int_{\mathbb{R}^n}\frac{dy}{(|y|+1)^{n+\delta}} \\ &\le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)(|x|+1)^{-n}.\end{aligned} \tag{12}$$

Furthermore, $|x-y|=(|x|+1)/2$ implies $(|x|-1)/2\le|y|\le(3|x|+1)/2$ and we thus obtain

$$
\begin{aligned}
I_{22} &\le \int_{|x-y|=\frac{|x|+1}{2}} \frac{C}{|x-y|^{n-1}} \left\{ \frac{\|F\|_{\infty,X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty,X_{n-1}}}{(|y|+1)^{2n-2}} \right\} dS_y \\
&\le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)(|x|+1)^{1-n}(|x|+1)^{-n-\delta} \int_{|x-y|=\frac{|x|+1}{2}} dS_y \\
&\le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)(|x|+1)^{-n-\delta}.
\end{aligned} \tag{13}
$$

The estimates (12) and (13) yield

$$
\begin{aligned}
|I_2| &\le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)\Big\{(|x|+1)^{-n} + (|x|+1)^{-n-\delta}\Big\} \\
&\le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\Big)(|x|+1)^{-n}.
\end{aligned} \tag{14}
$$

It follows from (11) and (14) that

$$
\begin{aligned}
|\nabla(K_s u)(x)| \le C\Big(&\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_{\infty,X_{n-1}} \\
&+ \|u\|_{\infty,X_{n-1}}\|\nabla u\|_{\infty,X_n}\Big)(|x|+1)^{-n}.
\end{aligned}
$$

Therefore, we conclude

$$
\nabla(K_s u) \in X_n
$$

with the estimate

$$
\|\nabla(K_s u)\|_{X_n} \le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_Y\Big).
$$

This completes the proof of Lemma 1.

We also define the operator K_p on Y by

$$
(K_p u)(t,x) := \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T \nabla E_p(t-s, x-y)(F - u\otimes u)(s,y)\, ds dy,
$$

and we use Proposition 1 to obtain the following estimates.

Lemma 2 *Let* $0<\delta<1$ *and suppose* F *satisfies* (6). *For* $u \in Y$ *we have*

$$
K_p u \in L^\infty_{per}(\mathbb{R}; X_{n+\delta}) \quad \textit{and} \quad \nabla(K_p u) \in L^\infty_{per}(\mathbb{R}; X_{n+1}),
$$

and there exists a constant $C = C(n,\delta,T)$ *such that*

$$
\|K_p u\|_{\infty,X_{n+\delta}} + \|\nabla(K_p u)\|_{\infty,X_{n+1}} \le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_Y\Big).
$$

If F *satisfies* (7), *then we have*

$$
K_p u \in L^\infty_{per}(\mathbb{R}; X_{n+1}) \quad \textit{and} \quad \nabla(K_p u) \in L^\infty_{per}(\mathbb{R}; X_{n+2}). \tag{15}
$$

Proof The periodicity of $K_p u$ follows from that of E_p and we may assume $t \in [0, T)$. We write

$$\begin{aligned}(K_p u)(t,x) &= \int_{|x-y|<\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \nabla E_p(t-s,x-y)(F-u\otimes u)(s,y)\,dsdy \\ &\quad + \int_{|x-y|>\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \nabla E_p(t-s,x-y)(F-u\otimes u)(s,y)\,dsdy \\ &=: I_3 + I_4.\end{aligned}$$

According to Proposition 1, we have $\nabla E_p \in L^1_{per}(\mathbb{R}\times\mathbb{R}^n)$ and thus

$$\begin{aligned}|I_3| &\le \int_{|x-y|<\frac{|x|+1}{2}} \left\{ \frac{\|F\|_{\infty,X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty,X_{n-1}}}{(|y|+1)^{2n-2}} \right\} \frac{1}{T}\int_0^T \left|\nabla E_p(t-s,x-y)\right| dsdy \\ &\le \frac{C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right)}{(|x|+1)^{n+\delta}} \int_{\mathbb{R}^n}\int_0^T \left|\nabla E_p(t-s,x-y)\right| dsdy \\ &\le C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right)(|x|+1)^{-n-\delta}.\end{aligned} \tag{16}$$

Furthermore, we use the estimate in Proposition 1 to get

$$\begin{aligned}|I_4| &\le \int_{|x-y|>\frac{|x|+1}{2}} \left\{ \frac{\|F\|_{\infty,X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty,X_{n-1}}}{(|y|+1)^{2n-2}} \right\} \frac{1}{T}\int_0^T \left|\nabla E_p(t-s,x-y)\right| dsdy \\ &\le C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right) \int_{|x-y|>\frac{|x|+1}{2}} \frac{dy}{|x-y|^{n+1}(|y|+1)^{n+\delta}} \\ &\le \frac{C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right)}{(|x|+1)^{n+1}} \int_{\mathbb{R}^n} \frac{dy}{(|y|+1)^{n+\delta}} \\ &\le C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right)(|x|+1)^{-n-1}.\end{aligned} \tag{17}$$

It follows from (16) and (17) that

$$\left|(K_p u)(t,x)\right| \le C\left(\|F\|_{\infty,X_{n+\delta}} + \|u\|^2_{\infty,X_{n-1}}\right)(|x|+1)^{-n-\delta}$$

uniformly in $t \in [0, T)$, and we derive

$$K_p u \in L^\infty_{per}(\mathbb{R}; X_{n+\delta})$$

with the estimate

$$\|K_p u\|_{\infty, X_{n+\delta}} \le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|_Y^2\Big).$$

Concerning the estimate for derivatives, we write

$$\begin{aligned}
\nabla(K_p u)(t,x) &= \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T \nabla E_p(t-s, x-y)(\operatorname{div}(F - u\otimes u))(s,y)\, dsdy \\
&= \int_{|x-y|<\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \nabla E_p(t-s, x-y)(\operatorname{div}(F - u\otimes u))(s,y)\, dsdy \\
&\quad + \int_{|x-y|\,>\,\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \nabla E_p(t-s, x-y)(\operatorname{div}(F - u\otimes u))(s,y)\, dsdy \\
&=: I_5 + I_6.
\end{aligned}$$

In addition, we integrate I_6 by parts to obtain

$$|I_6| \le I_{61} + I_{62},$$

where

$$\begin{aligned}
I_{61} &:= \int_{|x-y|\,>\,\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \big|\nabla^2 E_p(t-s, x-y)\big| |(F - u\otimes u)(s,y)|\, dsdy, \\
I_{62} &:= \int_{|x-y|=\frac{|x|+1}{2}} \frac{1}{T}\int_0^T \big|\nabla E_p(t-s, x-y)\big| |(F - u\otimes u)(s,y)|\, dsdS_y.
\end{aligned}$$

We estimate I_5 and I_{61} in the same way as (16) and (17), respectively, to get

$$\begin{aligned}
|I_5| &\le C\Big(\|\operatorname{div} F\|_{\infty, X_{n+1}} + \|u\|_{\infty, X_{n-1}} \|\nabla u\|_{\infty, X_n}\Big)(|x|+1)^{-n-1}, \\
I_{61} &\le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big)(|x|+1)^{-n-2}.
\end{aligned}$$

Furthermore, we see that

$$\begin{aligned}
I_{62} &\le \int_{|x-y|=\frac{|x|+1}{2}} \left\{ \frac{\|F\|_{\infty, X_{n+\delta}}}{(|y|+1)^{n+\delta}} + \frac{\|u\|^2_{\infty, X_{n-1}}}{(|y|+1)^{2n-2}} \right\} \frac{1}{T}\int_0^T \big|\nabla E_p(t-s, x-y)\big|\, dsdS_y \\
&\le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big) \int_{|x-y|=\frac{|x|+1}{2}} \frac{dS_y}{|x-y|^{n+1}(|y|+1)^{n+\delta}} \\
&\le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big)(|x|+1)^{-n-\delta-2}.
\end{aligned}$$

Hence

$$|I_6| \le C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big)(|x|+1)^{-n-2}$$

and the estimates for I_5 and I_6 lead us to

$$|\nabla(K_p u)(t,x)| \le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_{\infty,X_{n-1}} + \|u\|_{\infty,X_{n-1}}\|\nabla u\|_{\infty,X_n}\Big)(|x|+1)^{-n-1}$$

uniformly in $t \in [0,T)$. It follows that

$$\nabla(K_p u) \in L^\infty_{per}(\mathbb{R}; X_{n+1})$$

with the estimate

$$\|\nabla(K_p u)\|_{\infty,X_{n+1}} \le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_Y\Big).$$

We can easily verify (15) by using (7), instead of (6), in the estimates for I_3 and I_5.

Remark 4 Let $0<\gamma\le 1$ and set $\tilde{Y} := \{u \in Y; \partial_t u \in L^\infty_{per}(\mathbb{R}; X_{n+\gamma})\}$. Assuming that $\partial_t F \in L^\infty_{per}(\mathbb{R}; X_{n+\gamma})$, we can observe that $\partial_t(K_p u) \in L^\infty_{per}(\mathbb{R}; X_{n+\gamma})$ $(u \in \tilde{Y})$ with the estimate $\|\partial_t(K_p u)\|_{\infty,X_{n+\gamma}} \le C(\|\partial_t F\|_{\infty,X_{n+\gamma}} + \|u\|_{\infty,X_{n-1}}\|\partial_t u\|_{\infty,X_{n+\gamma}})$. This observation, together with suitable modifications of the proof of Theorem 1 below, yields Remark 3.

Set

$$K := K_s + K_p,$$

that is,

$$(Ku)(t,x) = \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T \nabla E(t-s, x-y)(F - u\otimes u)(s,y)\, dsdy.$$

Since the embedding $L^\infty_{per}(\mathbb{R}; X_{\mu_1}) \subset L^\infty_{per}(\mathbb{R}; X_{\mu_2})$ for $\mu_2 < \mu_1$ is continuous, Lemmas 1 and 2 yield the following lemma.

Lemma 3 *Let $0<\delta<1$ and suppose F satisfies (6). The operator K maps Y to itself and there exists a constant $C = C(n,\delta,T)$ such that*

$$\|Ku\|_Y \le C\Big(\|F\|_{\infty,X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty,X_{n+1}} + \|u\|^2_Y\Big) \quad (u \in Y).$$

Remark 5 The constant C in Lemma 3 is determined as follows. By Lemmas 1 and 2, there exists a constant $C_1 = C_1(n,\delta,T)$ such that the estimate for Ku above holds. On the other hand, in the proof of Theorem 1 below, we also need the estimate

$$\left\|\int_{\mathbb{R}^n}\int_0^T \nabla E(\cdot - s, \cdot - y)(u\otimes v)(s,y)\, dsdy\right\|_Y \le C_2\|u\|_Y\|v\|_Y \quad (u,v \in Y), \quad (18)$$

which follows immediately from the proofs of Lemmas 1 and 2. Here C_2 is a constant depending only on n and T. It is not clear from the proofs of Lemmas 1 and

2 whether C_1 is larger than C_2. The proof of Theorem 1 shall require the condition $C \geq \max\{C_1, C_2\}$, and hence we take $C = \max\{C_1, C_2\}$.

Now we follow the standard argument via the contraction mapping principle to construct a solution with desired decay properties of (4). We prove only Theorem 1, since Theorem 2 follows from Lemmas 1 and 2 together with the proof of Theorem 1 below. Indeed, the argument in the third paragraph of the proof of Theorem 1 is applicable to arbitrary solutions $u \in Y$ of (4) and the pointwise decay properties stated in Theorem 2 follow immediately from Lemmas 1 and 2.

Proof of Theorem 1 We employ the successive approximation

$$v_0(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \nabla E(t-s, x-y) F(s,y)\, dsdy,$$
$$v_{k+1}(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \nabla E(t-s, x-y)(F - v_k \otimes v_k)(s,y)\, dsdy.$$

According to Lemma 3, we have $v_k \in Y$ for all $k = 0, 1, \ldots$ with the estimate

$$\|v_{k+1}\|_Y \leq M_0 \Big(\|F\|_{\infty, X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty, X_{n+1}} + \|v_k\|_Y^2 \Big),$$

where M_0 is the constant in the lemma and is independent of k. We assume

$$\|F\|_{\infty, X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty, X_{n+1}} < \frac{1}{4M_0^2}$$

to deduce for all $k \geq 1$ that

$$\begin{aligned} \|v_k\|_Y \leq M_1 &:= \frac{1 - \sqrt{1 - 4M_0^2 (\|F\|_{\infty, X_{n+\delta}} + \|\mathrm{div}\, F\|_{\infty, X_{n+1}})}}{2M_0} \\ &< \frac{1}{2M_0}. \end{aligned}$$

We put

$$\begin{aligned} w_k &:= v_{k+1} - v_k \\ &= -\int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \nabla E(t-s, x-y)(w_{k-1} \otimes v_k + v_{k-1} \otimes w_{k-1})(s,y)\, dsdy. \end{aligned}$$

In view of Remark 5, we have

$$\|w_k\|_Y \leq M_0 (\|v_k\|_Y + \|v_{k-1}\|_Y) \|w_{k-1}\|_Y \leq 2M_0 M_1 \|w_{k-1}\|_Y,$$

so that

$$\|w_k\|_Y \leq (2M_0 M_1)^k \|w_0\|_Y.$$

Since $2M_0 M_1 < 1$, we see that $\{v_k\}$ converges in Y to a function u satisfying

$$u(t,x) = \int_{\mathbb{R}^n} \frac{1}{T} \int_0^T \nabla E(t-s, x-y)(F - u \otimes u)(s,y)\, dsdy.$$

Noting that this solution $u \in Y$ of (4) satisfies the estimate $\|u\|_Y \leq C(\|F\|_{\infty, X_{n+\delta}} + \|\operatorname{div} F\|_{\infty, X_{n+1}})$ with $C = C(n, \delta, T)$ and that F and $\operatorname{div} F$ are sufficiently small in $L^\infty_{per}(\mathbb{R}; X_{n+\delta})$ and $L^\infty_{per}(\mathbb{R}; X_{n+1})$ respectively, we can easily verify that u is unique in the class of small solutions in Y by applying the estimate (18).

Next, we prove the decay property of the associated pressure p. By the integration by parts, we get

$$\begin{aligned} p(t,x) &= \int_{\mathbb{R}^n} Q_s(x-y) \cdot (\operatorname{div} F - u \cdot \nabla u)(t,y)\, dy \\ &\leq I_7 + I_8 + I_9, \end{aligned}$$

where

$$\begin{aligned} I_7 &:= \int_{|x-y| < \frac{|x|+1}{2}} |Q_s(x-y)| |(\operatorname{div} F - u \cdot \nabla u)(t,y)|\, dy, \\ I_8 &:= \int_{|x-y| > \frac{|x|+1}{2}} |\nabla Q_s(x-y)| |(F - u \otimes u)(t,y)|\, dy, \\ I_9 &:= \int_{|x-y| = \frac{|x|+1}{2}} |Q_s(x-y)| |(F - u \otimes u)(t,y)|\, dS_y. \end{aligned}$$

It is clear that p is T-periodic and we may assume $t \in [0, T)$. Recalling the estimate $|\nabla^j Q_s(x-y)| \leq C|x-y|^{1-n-j}$ $(j = 0, 1, \ldots)$, we calculate I_7, I_8 and I_9 in the same way as (11), (12) and (13), respectively, to deduce

$$\begin{aligned} I_7 &\leq C\Big(\|\operatorname{div} F\|_{\infty, X_{n+1}} + \|u\|_{\infty, X_{n-1}} \|\nabla u\|_{\infty, X_n}\Big)(|x|+1)^{-n}, \\ I_8 &\leq C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big)(|x|+1)^{-n}, \\ I_9 &\leq C\Big(\|F\|_{\infty, X_{n+\delta}} + \|u\|^2_{\infty, X_{n-1}}\Big)(|x|+1)^{-n-\delta}. \end{aligned}$$

Consequently,

$$|p(t,x)| \leq C\Big(\|F\|_{\infty, X_{n+\delta}} + \|\operatorname{div} F\|_{\infty, X_{n+1}} + \|u\|_Y^2\Big)(|x|+1)^{-n}$$

uniformly in $t \in [0, T)$ and we conclude

$$p \in L^\infty_{per}(\mathbb{R}; X_n).$$

Finally, let u_s and u_p be the steady and time-periodic parts of the solution u obtained above. By Lemma 1, we have $u_s \in X_{n-1}$ and $\nabla u_s \in X_n$. Calculations similar to those of the estimate for the associated pressure p above yield

$$p_s(x) := \int_{\mathbb{R}^n} Q_s(x-y) \cdot \frac{1}{T}\int_0^T (\operatorname{div} F - u\cdot\nabla u)(s,y)\,dsdy \in X_n.$$

The pair $\{u_s, p_s\}$ is a solution of the stationary Stokes equation

$$\begin{cases} -\Delta u_s + \nabla p_s = \dfrac{1}{T}\displaystyle\int_0^T (\operatorname{div} F - u\cdot\nabla u)(s,\cdot)\,ds & \text{in } \mathbb{R}^n, \\ \operatorname{div} u_s = 0 & \text{in } \mathbb{R}^n, \end{cases}$$

in the sense of distributions. By the class of $\{u_s, p_s\}$ and $\int_0^T (\operatorname{div} F - u\cdot\nabla u)(s,\cdot)\,ds \in X_{n+1}$, we can apply the theory for the existence and uniqueness of strong solutions to the stationary Stokes equation ([6, Proposition 2.9]) to deduce that

$$\nabla^2 u_s \in L^q(\mathbb{R}^n) \quad \text{forall } 1<q<\infty.$$

Also, we write

$$\nabla^2 u_p(t,x) = \nabla^2 \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T E_p(t-s, x-y)(\operatorname{div} F - u\cdot\nabla u)(s,y)\,dsdy,$$

and the estimate for convolution in Proposition 1, together with $\operatorname{div} F - u\cdot\nabla u \in L^\infty_{per}(\mathbb{R}; X_{n+1})$, implies

$$\nabla^2 u_p \in L^q_{per}(\mathbb{R}\times\mathbb{R}^n) \quad \text{forall } 1<q<\infty.$$

Consequently, we derive

$$\nabla^2 u = \nabla^2 u_s + \nabla^2 u_p \in L^q_{per}(\mathbb{R}\times\mathbb{R}^n) \quad \text{forall } 1<q<\infty.$$

Similarly, the property $\partial_t u \in L^q_{per}(\mathbb{R}\times\mathbb{R}^n)$ $(1<q<\infty)$ follows from the representation

$$\partial_t u_p(t,x) = \partial_t \int_{\mathbb{R}^n} \frac{1}{T}\int_0^T E_p(t-s, x-y)(\operatorname{div} F - u\cdot\nabla u)(s,y)\,dsdy$$

and Proposition 1. The proof of Theorem 1 is complete.

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