J Elliptic Parabol Equ (2017) 3:127–134
https://doi.org/10.1007/s41808-017-0008-8

Hardy–Sobolev–Maz'ya and related inequalities in the half-space

Georgios Psaradakis[1]

Received: 11 September 2017 / Accepted: 16 November 2017 / Published online: 29 November 2017
© Orthogonal Publishing and Springer International Publishing AG, part of Springer Nature 2017

Abstract Starting from the Hardy–Sobolev–Maz'ya inequality, we present all known Hardy–Sobolev-type inequalities involving the distance to the boundary of a half space. We give the simpler proofs known in this particular case. Related inequalities are discussed and two open questions are stated.

Keywords Hardy–Sobolev–Maz'ya inequality · modulus of continuity

Mathematics Subject Classification 35A23 · 46E35 · 30H35

1 Introduction

Let $\mathbb{R}^n_+$ stand for the following half-space $\mathbb{R}^n_+ := \{(x', x_n) \mid x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}, x_n > 0\}$, $n \in \mathbb{N}$. The Hardy inequality in $\mathbb{R}^n_+$ asserts that i f $p > 1$ then

$$\int_{\mathbb{R}^n_+} |\nabla u|^p \mathrm{d}x \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} \mathrm{d}x \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+), \tag{1}$$

with the best possible constant. In particular, an integration by parts shows that

Note based on the talk given by the author at the International Conference on Elliptic and Parabolic Problems held in Gaeta on 22–26 May 2017.

✉ Georgios Psaradakis
gpsarada@mail.uni-mannheim.de

[1] University of Mannheim, Math IV, Mannheim 68131, Germany

$$\int_{\mathbb{R}^n_+} \frac{|u|^{p-1}|u|_{x_n}}{x_n^{p-1}}\mathrm{d}x = \frac{1}{p}\int_{\mathbb{R}^n_+} \frac{(|u|^p)_{x_n}}{x_n^{p-1}}\mathrm{d}x = \frac{p-1}{p}\int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p}\mathrm{d}x.$$

An application of Hölder's inequality with conjugate exponents p and $p/(p-1)$ on the left term gives the stronger form of (1) with $|u_{x_n}|$ in place of $|\nabla u|$.[1]

The Hardy–Sobolev–Maz'ya inequality: For $p = 2$, the critical Sobolev norm can be added on the right hand side of (1). More precisely, Maz'ya in his treatise [10] proved that for $n \geq 3$ there exists a positive constant C such that

$$\left(\int_{\mathbb{R}^n_+} |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_n^2}\mathrm{d}x\right)^{1/2} \geq C\left(\int_{\mathbb{R}^n_+} |u|^{2^*}\mathrm{d}x\right)^{1/2^*} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+), \tag{2}$$

where $2^* := 2n/(n-2)$. In [2] the optimal constant C in three dimensions is found to be the same with the best constant in the Sobolev inequality, while in [13] it has been shown that this fails in higher dimensions.

The Hardy–Sobolev inequality: The p-version of (2) for $2 \leq p < n$ is

$$\left(\int_{\mathbb{R}^n_+} |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{u^p}{x_n^p}\mathrm{d}x\right)^{1/p} \geq C\left(\int_{\mathbb{R}^n_+} |u|^{p^*}\mathrm{d}x\right)^{1/p^*} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+), \tag{3}$$

where $p^\star = np/(n-p)$. This has been established in [4] (see §3 of this note) and later on with a different method in [6]. However, both approaches seem to fail giving (3) for $1 < p < 2$.

Question 1: Is (3) true for $1 < p < 2$?

Having in mind the Sobolev embedding theorem, it is natural to ask for the corresponding inequalities when $p \geq n$.

The Hardy–Morrey inequality: In [5] (see §4 of this note) the complete answer in the case $p > n$ was given. More precisely, if $p > n \geq 2$ there exists a positive constant C such that

$$\sup_{\substack{x,y \in \mathbb{R}^n_+ \\ x \neq y}} \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \leq C\left(\int_{\mathbb{R}^n_+} |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p}\mathrm{d}x\right)^{1/p} \quad \text{for all } u \in C_c^\infty(\mathbb{R}^n_+). \tag{4}$$

[1] This observation is well known and applies to the whole note and also to all known Hardy inequalities obtained by integration by parts and Hölder's inequality, even with remainder terms. For example, it follows from the proof in [1], that when the weight in the Hardy inequality involves the distance to the boundary $d_{\partial\Omega}$ of a weakly mean convex domain Ω, or the distance to a point x_0 of any subset of $\mathbb{R}^n$, then one can replace $|\nabla u(x)|$ by $|\nabla u(x) \cdot \nabla d_{\partial\Omega}(x)|$, or $|\nabla u(x) \cdot \frac{x-x_0}{|x-x_0|}|$, respectively.

Moreover, (4) fails for $n = 1$ (see [5, §7] for a sharp substitute in this case).

The Hardy–Moser–Trudinger inequality: In the case $p = n = 2$, the following sharp result has been established in [9]: There exists a positive constant C such that

$$\int_{\mathbb{R}^2_+} \frac{e^{4\pi u^2} - 1 - 4\pi u^2}{x_2^2}\,\mathrm{d}x \le C,$$

$$\text{for all } u \in C_c^\infty(\mathbb{R}^2_+) \text{ satisfying } \int_{\mathbb{R}^2_+} |\nabla u|^2 \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^2_+} \frac{u^2}{x_2^2}\,\mathrm{d}x \le 1.$$

The proof uses the Riemann mapping theorem and it is natural to ask for a dimensional free proof and the following generalization

Question 2: Let $n \in \mathbb{N}\backslash\{1\}$. Does there exists a positive constant $C > 0$ such that

$$\int_{\mathbb{R}^n_+} \frac{\exp\left\{(n\omega_n^{1/n}|u|)^{n/(n-1)}\right\} - \sum_{j=0}^{n-1} \frac{(n\omega_n^{1/n}|u|)^{jn/(n-1)}}{j!}}{x_n^n}\,\mathrm{d}x \le C,$$

$$\text{for all } u \in C_c^\infty(\mathbb{R}^n_+) \text{satisfying } \int_{\mathbb{R}^n_+} |\nabla u|^n \mathrm{d}x - \left(\frac{n-1}{n}\right)^n \int_{\mathbb{R}^n_+} \frac{|u|^n}{x_n^n}\,\mathrm{d}x \le 1?$$

Here we have denoted by ω_n the volume of the unit ball in $\mathbb{R}^n$. Some subcritical results have been obtained in [5]: a Hardy–John–Nirenberg inequality and also Theorem B for $p = n$ there, which in this note is the outcome of (6) applied to (7) and taking $p = n$.

2 Two lower estimates on the Hardy difference

We recall here two estimates that we are going to use in the proofs of (3) and (4).

2.1 A lower estimate from the ground state transform

In [1] the authors obtained various auxiliary lower bounds for the *Hardy difference*:

$$I_p[u; \mathbb{R}^n_+] := \int_{\mathbb{R}^n_+} |\nabla u|^p \mathrm{d}x - \left(\frac{p-1}{p}\right)^p \int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p}\,\mathrm{d}x, \quad u \in C_c^\infty(\mathbb{R}^n_+).$$

In particular, the ground state transform

$$u = x_n^{1-1/p} v, \tag{5}$$

implies

$$I_p[u; \mathbb{R}^n_+] = \int_{\mathbb{R}^n_+} \left\{ \left| \frac{p-1}{p} x_n^{-1/p} v \mathbf{e}_n + x_n^{1-1/p} \nabla v \right|^p - \left| \frac{p-1}{p} x_n^{-1/p} v \mathbf{e}_n \right|^p \right\} \mathrm{d}x.$$

This together with the vectorial inequality (see [8])

$$|a+b|^p - |a|^p \geq (2^{p-1}-1)^{-1}|b|^p + p|a|^{p-2}a\cdot b, \quad \text{for all } a,b\in\mathbb{R}^n \text{and}\, p\geq 2,$$

gives the following lower estimate on $I_p[u;\mathbb{R}^n_+]$ (see [1, Lemma 3.3])

$$\begin{aligned} I_p[u;\mathbb{R}^n_+] &\geq c_p\int_{\mathbb{R}^n_+} x_n^{p-1}|\nabla v|^p \mathrm{d}x + \left(\frac{p-1}{p}\right)^{p-1}\int_{\mathbb{R}^n_+}\nabla|v|^p\cdot\mathbf{e}_n\mathrm{d}x \\ &= c_p\int_{\mathbb{R}^n_+} x_n^{p-1}|\nabla v|^p\mathrm{d}x, \end{aligned} \tag{6}$$

where $c_p = (2^{p-1}-1)^{-1}$ and $p\geq 2$.

2.2 A lower estimate from an inequality by Cabré and Ros-Oton

In [5] the following sharp lower estimate of the functional that appears on the right hand side of (6) is given: Let b, p, q satisfy

$$-1 < b \leq 0, 1\leq p < \frac{n}{b+1}, \quad \text{and } q := \frac{np}{n-p(b+1)}.$$

There exists a positive constant C such that

$$\int_{\mathbb{R}^n_+} x_n^{p-1}|\nabla v|^p\mathrm{d}x \geq C\left(\int_{\mathbb{R}^n_+}\left(x_n^{b+1/p'}|v|\right)^q\mathrm{d}x\right)^{p/q} \quad \text{for all } v\in C_c^\infty(\mathbb{R}^n_+). \tag{7}$$

This is to be compared with the case where the monomial weight in [3, Theorem 1.3], degenerates to the distance from the boundary of the half-space. In particular, by the choice $A_i = 0$ for all $i = 1,\ldots,n-1$ and $A_n = p-1$ in [3], one deduces the following weighted Sobolev inequality

$$\int_{\mathbb{R}^n_+} x_n^{p-1}|\nabla v|^p\mathrm{d}x \geq C\left(\int_{\mathbb{R}^n_+} x_n^{p-1}|v|^{p(p+n-1)/(n-1)}\mathrm{d}x\right)^{(n-1)/(p+n-1)} \quad \text{for all } v\in C_c^\infty(\mathbb{R}^n), \tag{8}$$

which for $v\in C_c^\infty(\mathbb{R}^n_+)$ is a special case of (7), as one can easily check by taking $b = -(p-1)/(p+n-1)$. Let us mention that the best constant C in the above inequality is obtained in [3], and that for $p=2$ this inequality (with its sharp constant) was known before by a result of Maz'ya and Shaposhnikova (see [11, §6]).

3 Proof of the Hardy–Sobolev inequality

Let $u\in C_c^\infty(\mathbb{R}^n_+)$. Following [4], we start from the Gagliardo-Nirenberg inequality

$$n\omega_n^{1/n}\left(\int_{\mathbb{R}^n}|f|^{n/(n-1)}\mathrm{d}x\right)^{1-1/n} \leq \int_{\mathbb{R}^n}|\nabla f|\mathrm{d}x \quad \text{for all} f\in W^{1,1}(\mathbb{R}^n),$$

and setting $f = |u|^{p^\star(1-1/n)}$ we get

$$n\omega_n^{1/n}\frac{n-p}{p(n-1)}\Big(\int_{\mathbb{R}^n_+}|u|^{p^\star}\mathrm{d}x\Big)^{1-1/n} \le \int_{\mathbb{R}^n_+}|u|^{p^\star(1-1/p)}|\nabla u|\mathrm{d}x. \tag{9}$$

To estimate the term of the right hand side of (9), we set $u = x_n^{1-1/p}v$ to obtain

$$\begin{aligned}\int_{\mathbb{R}^n_+}|u|^{p^*(1-1/p)}|\nabla u|\mathrm{d}x &= \int_{\mathbb{R}^n_+}|u|^{p^*(1-1/p)}\Big|x_n^{1-1/p}\nabla v + \frac{p-1}{p}x_n^{-1/p}v\mathbf{e}_n\Big|\mathrm{d}x\\ &\le \underbrace{\int_{\mathbb{R}^n_+}|u|^{p^*(1-1/p)}x_n^{1-1/p}|\nabla v|\mathrm{d}x}_{=:A}\\ &\quad + \frac{p-1}{p}\underbrace{\int_{\mathbb{R}^n_+}x_n^{p^*(1-1/p)^2-1/p}|v|^{p^*(1-1/p)+1}\mathrm{d}x.}_{=:B}\end{aligned} \tag{10}$$

To ease the computation, we set

$$\beta := p^*(1-1/p)^2 + 1 - 1/p,$$

so that

$$B = \int_{\mathbb{R}^n_+}x_n^{\beta-1}|v|^{\beta p/(p-1)}\mathrm{d}x.$$

To estimate B we integrate by parts as follows

$$\begin{aligned}B =& \frac{1}{\beta}\int_{\mathbb{R}^n_+}\nabla x_n^\beta\cdot\mathbf{e}_n\ |v|^{\beta p/(p-1)}\mathrm{d}x,\\ =& -\frac{p}{p-1}\int_{\mathbb{R}^n_+}x_n^\beta|v|^{\beta p/(p-1)-1}\nabla|v|\cdot\mathbf{e}_n\ \mathrm{d}x\\ =& -\frac{p}{p-1}\int_{\mathbb{R}^n_+}|u|^{p^*(1-1/p)}x_n^{1-1/p}\nabla|v|\cdot\mathbf{e}_n\ \mathrm{d}x\\ \le& \frac{p}{p-1}A.\end{aligned}$$

Inserting this into (10), we obtain

$$\int_{\mathbb{R}^n_+}|u|^{p^*(1-1/p)}|\nabla u|\mathrm{d}x \le 2A. \tag{11}$$

Now we estimate A using Hölder's inequality as follows

$$
\begin{aligned}
A &= \int_{\mathbb{R}^n_+} \left\{|u|^{p^*(1-1/p)}\right\}\left\{x_n^{1-1/p}|\nabla v|\right\}\mathrm{d}x \\
&\le \|u\|^{p^*(1-1/p)}_{L^{p^*}(\mathbb{R}^n_+)}\left(\int_{\mathbb{R}^n_+} x_n^{p-1}|\nabla v|^p \mathrm{d}x\right)^{1/p} \\
&\le c_p^{-1/p}\|u\|^{p^*(1-1/p)}_{L^{p^*}(\mathbb{R}^n_+)}(I_p[u;\mathbb{R}^n_+])^{1/p},
\end{aligned}
$$

where we have used (6). Inserting the above estimate of A in (11), we get

$$
\int_{\mathbb{R}^n_+} |u|^{p^*(1-1/p)}|\nabla u|\mathrm{d}x \le 2c_p^{-1/p}\|u\|^{p^*(1-1/p)}_{L^{p^*}(\mathbb{R}^n_+)}(I_p[u;\mathbb{R}^n_+])^{1/p}.
$$

Coupling this with (9) we deduce (3).

4 Proof of the Hardy–Morrey inequality

We first recall Morrey's "Dirichlet growth" theorem (see [12, Theorem 3.5.2] or [7, Theorem 7.19]).

Theorem 4.1 *Let Ω be a domain in $\mathbb{R}^n, n \ge 1$. Let $u \in C_c^\infty(\Omega)$ and suppose that for some $M > 0$ and $\alpha \in (0,1]$ the following estimate is true for all $B_r \subset \mathbb{R}^n$*

$$
\int_{B_r} |\nabla u|\mathrm{d}x \le M r^{n-1+\alpha}. \tag{12}
$$

Then there exists $c(n,\alpha) > 0$ such that for all $B_r \subset \mathbb{R}^n$

$$
\sup_{x,y\in B_r} |u(x) - u(y)| \le cMr^\alpha,
$$

or, equivalently (since u is compactly supported)

$$
\sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \le cM.
$$

In view of the above theorem, for (4) to be true, it is enough to establish the following estimate

$$
\int_{B_r} |\nabla u|\mathrm{d}x \le c\big(I[u;\mathbb{R}^n_+]\big)^{1/p} r^{n(1-1/p)}, \tag{13}
$$

for all $r > 0$ and for some positive constant c that depends only on n. To this end, let $B_r \subset \mathbb{R}^n$ such that $B_r \cap \mathbb{R}^n_+ \neq \emptyset$. Setting $u = x_n^{1-1/p}v$ we have

$$\int_{B_r} |\nabla u| \mathrm{d}x \leq \underbrace{\int_{B_r} x_n^{1-1/p} |\nabla v| \mathrm{d}x}_{=:K_r} + \frac{p-1}{p} \underbrace{\int_{B_r} x_n^{-1/p} |v| \mathrm{d}x}_{=:L_r} .$$

Using first Hölder's inequality and then (6) we get

$$\begin{aligned} K_r &\leq \left(\int_{B_r} x_n^{p-1} |\nabla v|^p \mathrm{d}x \right)^{1/p} (\omega_n r^n)^{1-1/p} \\ &\leq C(n,p) \left(I_p[u; \mathbb{R}^n_+] \right)^{1/p} r^{n(1-1/p)}. \end{aligned} \tag{14}$$

We will next estimate L_r. Setting $q := p(p+n-1)/(n-1)$ we apply Holder's inequality as follows

$$\begin{aligned} L_r &= \int_{B_r} \{x_n^{(p-1)/q} |v|\} \{x_n^{-1/p-(p-1)/q}\} \mathrm{d}x \\ &\leq \left(\int_{B_r} x_n^{p-1} |v|^q \mathrm{d}x \right)^{1/q} \left(\int_{B_r \cap \mathbb{R}^n_+} x_n^{-\theta} \mathrm{d}x \right)^{1-1/q} ; \theta := \left(\frac{1}{p} + \frac{p-1}{q} \right) \frac{q}{q-1} . \end{aligned} \tag{15}$$

To estimate the right factor in (15), let Q_{2r} be the cube with the same center as B_r and edges of length $2r$ that are parallel to the coordinate axes. Then

$$\begin{aligned} \int_{B_r \cap \mathbb{R}^n_+} x_n^{-\theta} \mathrm{d}x &\leq \int_{Q_{2r} \cap \mathbb{R}^n_+} x_n^{-\theta} \mathrm{d}x \\ &= (2r)^{n-1} \int_{\max\{0, y_n - r\}}^{y_n + r} x_n^{-\theta} \mathrm{d}x_n \\ &= \frac{1}{1-\theta} (2r)^{n-1} \left((y_n + r)^{1-\theta} - \max\{0, y_n - r\}^{1-\theta} \right), \end{aligned}$$

where y_n is the n-th coordinate of the center of B_r. If $y_n \leq r$ then

$$\int_{B_r \cap \mathbb{R}^n_+} x_n^{-\theta} \mathrm{d}x \leq \frac{1}{1-\theta} (2r)^{n-\theta}. \tag{16}$$

If $y_n > r$, then since $\theta \in (0,1)$ there holds $\alpha^{1-\theta} - \beta^{1-\theta} \leq (\alpha - \beta)^{1-\theta}$, for all $\alpha \geq \beta \geq 0$, and thus (16) again holds true.

The left factor in (15) increases if we integrate in the whole $\mathbb{R}^n_+$ and we may use first (8) and then (6) to estimate it by the Hardy difference. Altogether, we arrive at

$$L_r \leq C(n,p) \left(I_p[u; \Omega] \right)^{1/p} r^{(n-\theta)(q-1)/q}.$$

This is the desired estimate (13), since

$$(n - \theta) \frac{q-1}{q} = n \frac{p-1}{p} .$$

The proof is complete. □

Acknowledgements I would like to thank Giuseppina di Blasio and Giovanni Pisante for the invitation to talk in Gaeta. I would also like to thank the referee for spotting several typos. Partially supported by DFG, CH 955/4-1.

References

1. Barbatis, G., Filippas, S., Tertikas, A.: A unified approach to improved L^p Hardy inequalities with best constants. Trans. Am. Math. Soc. **356**(6), 2169–2196 (2004)
2. Benguria, R.D., Frank, R.L., Loss, M.: The sharp constant in the Hardy–Sobolev–Maz'ya inequality in the three dimensional upper half space. Math. Res. Lett. **15**(4), 613–622 (2008)
3. Cabré, X., Ros-Oton, X.: Sobolev and isoperimetric inequalities with monomial weights. J. Differ. Equ. **255**(11), 4312–4336 (2013)
4. Filippas, S., Maz'ya, V.G., Tertikas, A.: Critical Hardy–Sobolev inequalities. J. Math. Pures Appl. **87**(1), 37–56 (2007)
5. Filippas, S., Psaradakis, G.: The Hardy–Morrey & Hardy–John–Nirenberg inequalities involving distance to the boundary. J. Differ. Equ. **261**(6), 3107–3136 (2016)
6. Frank, R.L., Loss, M.: Hardy–Sobolev–Maz'ya inequalities for arbitrary domains. J. Math. Pures Appl. **97**(1), 39–54 (2012)
7. Gilbarg D., Trudinger, N.S.: Elliptic partial differential equations of second order. 2nd ed. (revised 3rd printing). In: Grundlehren der mathematischen Wissenschaften, vol 224, Springer (1998)
8. Lindqvist, P.: On the equation $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$.. Proc. Am. Math. Soc. **109**, 157–164 (1990)
9. Lu, G., Yang, Q.: A sharp Trudinger–Moser inequality on any bounded and convex planar domain. Calc. Var. Partial Differ. Equ. **55**(6), 153 (2016)
10. Maz'ya, V.G.: Sobolev Spaces. Translated from Russian by Tatyana Shaposhnikova. Springer Ser. Soviet Math., Springer, Berlin (1985)
11. Maz'ya, V, Shaposhnikova, T.: A collection of sharp dilation invariant integral inequalities for differentiable functions. In: Maz'ya, V. (ed.) Sobolev Spaces in Mathematics I. International Mathematical Series. Springer Science + Business Media, LLC (2009)
12. Morrey, C.B.: Multiple integrals in the calculus of variations. In: Grundlehren der mathematischen Wissenschaften, vol. 130. Springer (1966)
13. Tertikas, A., Tintarev, K.: On existence of minimizers for the Hardy–Sobolev–Maz'ya inequality. Ann. Mat. Pura Appl. **186**(4), 645–662 (2007)