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A note about weak * lower semicontinuity for functionals with linear growth in $W^{1,1} \times L^1$

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Abstract It is investigated the lower semicontinuity of functionals of the type $\int_\Omega W(x, u, \nabla u, v)dx$ with respect to the $L^1_{\text{strong}} \times \mathcal{M}_{\text{weak}*}$ topology, when the target fields (u, v) are in $W^{1,1} \times L^1$.

Keywords Relaxation · Quasiconvexity-convexity · Linear growth

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1 Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$ with Lipschitz boundary. We consider the integral functional of the form

$$E(u, v) = \int_\Omega W(x, u, \nabla u, v)dx, \tag{1.1}$$

where $u \in W^{1,1}(\Omega; \mathbb{R}^m)$, ∇u is its gradient and $v \in L^1(\Omega; \mathbb{R}^l)$.

Energies as in (1.1), which generalize those considered by [9, 10], have been introduced to deal with equilibrium states for systems depending on elastic strain

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and chemical composition. Lower semicontinuity and relaxation have been obtained in [9, 10] respectively when the target fields $(u, v) \in W^{1,p} \times L^q$, $p, q > 1$ and $(u, v) \in BV \times L^\infty$. In this context a multiphase alloy is described by the set Ω, the deformation gradient is represented by ∇u, and v (when l = 1) denotes the chemical composition of the system. We underline that this type of integrals may be regarded also in the framework of Elasticity, when dealing with Cosserat's theory in thin structures, also for the description of bending moment effects, see [5, 14] in the Sobolev setting. In particular, in [4] a 3D–2D dimension reduction was elaborated when the density W has linear growth, thus the limit energy obtained by Γ convergence techniques involves a BV deformation and a bending moment represented by a measure.

In [4, 5, 9, 10, 14] densities of the type $W(\nabla u, v)$ have been taken into account, while in the present paper we deal with heterogeneities and deformation as in [16, 17] but also allowing for autonomous and heterogeneity in the density W. We focus on the lower-semicontinuity of (1.1) with respect to L^1-strong $\times\mathcal{M}$-weak * convergence, where $\mathcal{M}(\Omega; \mathbb{R}^l)$ represents the set of bounded $\mathbb{R}^l$-valued Radon measures on Ω and $\mathcal{M}$- weak * denotes the weak * convergence in the sense of measures.

Observe that bounded sequences (u_n, v_n) in $W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^l)$ may converge in L^1-strong $\times\mathcal{M}$-weak $*$, up to a subsequence, to $(u, v) \in BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$.

In this paper we limit our analysis to (u, v) in $W^{1,1}(\Omega; \mathbb{R}^m) \times L^1(\Omega; \mathbb{R}^l)$, thus our result has to be considered as a first step, in the same spirit of the results contained in [7, 8], toward the study of relaxation in $BV \times \mathcal{M}$, when the energy density W has explicit dependence on x and u.

The present lower semicontinuity result, relies on the blow-up method introduced in [13] and extend the results obtained in [6, 10, 12, 16, 17].

Let us denote by $\tilde{E}$ the relaxed functional of E in (1.1).

$$\begin{aligned} \tilde{E}(u, v) = \inf\Bigg\{ \liminf_{n\to+\infty} \int_\Omega W(x, u_n, \nabla u_n, v_n)dx,\ u_n \in W^{1,1}(\Omega; \mathbb{R}^m), \\ u_n \to u \text{ in } L^1(\Omega; \mathbb{R}^m), v_n \in L^1(\Omega; \mathbb{R}^l), v_n \overset{*}{\rightharpoonup} v \text{ in } \mathcal{M}(\Omega; \mathbb{R}^l) \Bigg\}, \end{aligned} \tag{1.2}$$

for every $u \in W^{1,1}(\Omega; \mathbb{R}^m)$ and $v \in L^1(\Omega; \mathbb{R}^l)$. In the special case when $W(x, u, \nabla u, v) \equiv W(\nabla u, v)$, the relaxed energy in (1.2) can be deduced with arguments similar to those developed in [4] in the context of dimensional reduction. Here the presence of x and u requires more technicalities.

The integral representation of (1.2) will be achieved in Theorem 1 under the following hypotheses.

We assume that $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0, +\infty)$ is a continuous function verifying the following linear growth and coercivity conditions, i.e., there exist constants $0 < \gamma, \beta' \leq \beta < +\infty$, and a nonnegative, bounded, continuous function $g : \Omega \times \mathbb{R}^m \to [\gamma, +\infty)$ such that

(H1) $\beta' g(x,u)(|\xi| + |b|) - \beta \le W(x,u,\xi,b) \le \beta g(x,u)(1 + |\xi| + |b|)$ for all $(x,u,\xi,b) \in \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \times \mathbb{R}^l$.

(H2) For every compact $K \subset \Omega \times \mathbb{R}^m$ there exists a continuous function $\omega_K : [0,+\infty) \to [0,+\infty)$ with $\omega_K(0) = 0$ such that

$$|W(x,u,\xi,b) - W(x',u,\xi,b)| \le \omega_K(|x-x'|)(1 + |\xi| + |b|)$$

and

$$|W(x,u,\xi,b) - W(x,u',\xi,b)| \le \omega_K(|u-u'|),$$

for all $(x,u,\xi,b), (x',u',\xi,b) \in K \times \mathbb{R}^{N\times m} \times \mathbb{R}^l$.

We say that a Borel function $W : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0,+\infty)$ (satisfying $(H1)$) is quasiconvex-convex if for all $(x,u,\xi,v) \in \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l$ we have

$$\int_D W(x,u,\xi,v) \le \frac{1}{|D|}\int_D W(x,u,\xi + \nabla\theta(y), v + \eta(y))\,dy,$$

for all $\theta \in W_0^{1,\infty}(D;\mathbb{R}^m)$ and $\eta \in L^\infty(D;\mathbb{R}^l)$ such that $\int_D \eta(x)dx = 0$, where D is any bounded open set of $\mathbb{R}^N$.

For every $W : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0+\infty)$ satisfying $(H1)$, denoting by $QCW : \mathbb{R}^N \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0,+\infty)$ the quasiconvex-convex envelope of W, which is the largest quasiconvex-convex function smaller or equal than W, it admits the following representation obtained in [14]

$$QCW(x,u,\xi,b) := \inf\left\{\frac{1}{|D|}\int_D W(x,u,\xi + \nabla\theta(y), b + \eta(y))\,dy; \theta \in W_0^{1,\infty}(D;\mathbb{R}^m),\right.$$
$$\left.\eta \in L^\infty(D;\mathbb{R}^l) \text{ such that } \int_D \eta(x)dx = 0\right\}, \tag{1.3}$$

where D is any bounded domain with regular boundary in $\mathbb{R}^N$.

Our result is the following

Theorem 1 *Let $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0,+\infty)$ be a continuous function verifying (H1) and (H2). Let E and $\tilde{E}$ be given by (1.1) and (1.2) respectively, and let $J : W^{1,1}(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^l)$ be defined by*

$$J(u,v) = \int_\Omega QCW(x,u,\nabla u, v)dx. \tag{1.4}$$

Then for every $(u,v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^l)$ we have

$$\tilde{E}(u,v) = J(u,v).$$

Section 2 is devoted to the proof of Theorem 1, after the establishment of some notation.

2 *Proof of Theorem* 1

Let Ω be a generic open subset of $\mathbb{R}^N$, we denote by $\mathcal{M}(\Omega)$ the space of all Radon measures in Ω with bounded total variation. By the Riesz Representation Theorem, $\mathcal{M}(\Omega)$ can be identified with the dual of the separable space $\mathcal{C}_0(\Omega)$ of continuous functions on Ω vanishing on the boundary $\partial\Omega$. The N-dimensional Lebesgue measure in $\mathbb{R}^N$ is denoted by $\mathcal{L}^N$.

If $\mu, \lambda \in \mathcal{M}(\Omega)$ are nonnegative Radon measures, we denote by $\frac{d\mu}{d\lambda}$ the Radon-Nikodým derivative of μ with respect to λ.

We denote by $\mathcal{M}(\Omega;\mathbb{R}^l)$ the set of $\mathbb{R}^l$ valued Radon measures, namely vectors in $\mathbb{R}^l$ whose l components belong to $\mathcal{M}(\Omega)$. By a generalization of the Besicovich Differentiation Theorem (see [2, Proposition 2.2]), it can be proved that there exists a Borel set $N \subset \Omega$ such that $\lambda(N) = 0$ and

$$\frac{d\mu}{d\lambda}(x) = \lim_{\rho\to 0^+} \frac{\mu(x+\rho C)}{\lambda(x+\rho C)}$$

for all $x \in \operatorname{Supp}\mu/N$ and any open convex set C containing the origin. We recall that the exceptional set N above does not depend on C.

The achievement of Theorem 1 relies essentially on the proof of the lower bound inequality, presented in subsect. 2.1. In fact, in light of Proposition 1, there is no loss of generality assuming that W is quasiconvex-convex, i.e. $W = QCW$. Thus the upper bound inequality easily follows by the definition of $\tilde{E}$, the constant sequence $(u_n, v_n) \equiv (u, v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^l)$ being admissible for (1.2).

2.1 Lower bound

Lemma 1 *Let* $W : \Omega \times \mathbb{R}^m \times \mathbb{R}^{N\times m} \times \mathbb{R}^l \to [0,+\infty)$ *be a continuous function satisfying* $(H1)-(H2)$ *and let* E, $\tilde{E}$ *and* J *be the functionals defined in* (1.1), (1.2) *and* (1.4) *respectively. For every* $(u, v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^l)$ *we have that*

$$\tilde{E}(u, v) \geq J(u, v). \tag{2.1}$$

Remark 1 We observe that the result of Proposition 1 is not relevant for the proof of Lemma 1, since $W \geq QCW$.

Proof As in [1, Proof of Theorem II.4] we can reduce to the case where $u_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^m)$ and $v_n \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^l)$. Let $(u, v) \in W^{1,1}(\Omega;\mathbb{R}^m) \times L^1(\Omega;\mathbb{R}^l)$.

Let $(u_n, v_n) \in C_0^\infty(\mathbb{R}^N;\mathbb{R}^m) \times C_0^\infty(\mathbb{R}^N;\mathbb{R}^l)$ such that $u_n \to u$ in $L^1(\Omega;\mathbb{R}^m)$ and $v_n \overset{*}{\rightharpoonup} v$ in $\mathcal{M}(\Omega;\mathbb{R}^l)$ and assume, up to a not relabeled subsequence, that

$$\tilde{E}(u,v) = \lim_{n\to+\infty} \int_\Omega W(x,u_n,\nabla u_n,v_n)dx.$$

For every Borel set $B \subset \Omega$, define

$$\mu_n(B) = \int_B W(x,u_n,\nabla u_n,v_n)dx.$$

Since W is nonnegative and using (H1), we have that (μ_n) is a sequence of non-negative Radon measures uniformly bounded in $\mathcal{M}(\Omega)$ and thus, there exists a subsequence, still labeled (μ_n), and a nonnegative finite Radon measure μ such that $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$. Then, we decompose μ as the sum of two mutually singular measures, $\mu = \mu^a + \mu^s$, such that $\mu^a \ll \mathcal{L}^N$. Thus, (2.1) will be achieved once we prove that for every $u \in W^{1,1}(\Omega;\mathbb{R}^m)$ and $v \in L^1(\Omega;\mathbb{R}^l)$

$$\frac{d\mu^a}{d\mathcal{L}^N}(x_0) \geq W(x_0,u(x_0),\nabla u(x_0),v(x_0)) \quad \text{for } \mathcal{L}^N - \text{a.e. } x_0 \in \Omega. \tag{2.2}$$

Indeed, since $\mu_n \overset{*}{\rightharpoonup} \mu$ in $\mathcal{M}(\Omega)$, by the lower semicontinuity and the fact that μ^s is positive, we obtain that

$$\lim_{n\to+\infty} \int_\Omega W(x,u_n(x),\nabla u_n(x),v_n(x))dx \geq \int_\Omega d\mu(x)$$
$$\geq \int_\Omega d\mu^a(x) \geq \int_\Omega W(x,u,\nabla u,v)dx.$$

We prove assertion (2.2) using the blow-up method introduced in [12].

Let x_0 be a Lebesgue point for u, ∇u and v verifying the following properties (which hold $\mathcal{L}^N$-a.e. in Ω)

$$\frac{d\mu^a}{d\mathcal{L}^N}(x_0) = \frac{d\mu}{d\mathcal{L}^N}(x_0) = \lim_{\varepsilon\to 0} \frac{\mu(B(x_0,\varepsilon))}{\mathcal{L}^N(B(x_0,\varepsilon))} < +\infty,$$

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \left(\int_{B(x_0,\varepsilon)} |u(x) - u(x_0) - \nabla u(x_0)(x-x_0)|^{\frac{N}{N-1}} dx \right)^{\frac{N-1}{N}} = 0$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^N} \int_{B(x_0,\varepsilon)} |v(x) - v(x_0)|dx = 0.$$

Observe that we can choose a sequence $\varepsilon \to 0^+$ such that $\mu(\partial B(x_0,\varepsilon)) = 0$. Let $B := B(0,1)$. Applying Proposition 1.203 *iii*) in [11],

$$\begin{aligned}
\frac{d\mu^a}{d\mathcal{L}^N}(x_0) &= \lim_{\varepsilon\to 0}\frac{\mu(B(x_0,\varepsilon))}{\varepsilon^N\mathcal{L}^N(B)}\\
&= \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}\frac{1}{\varepsilon^N\mathcal{L}^N(B)}\int_{B(x_0,\varepsilon)} W(y,u_n(y),\nabla u_n(y),v_n(y))\,dy\\
&= \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}\int_B W(x_0+\varepsilon x,u_n(x_0+\varepsilon x),\nabla u_n(x_0+\varepsilon x),v_n(x_0+\varepsilon x))\,dx\\
&\geq \limsup_{\varepsilon\to 0}\lim_{n\to+\infty}\int_B W(x_0+\varepsilon x,u(x_0)+\varepsilon w_{n,\varepsilon}(x),\nabla w_{n,\varepsilon}(x),\eta_{n,\varepsilon}(x))\,dx
\end{aligned}$$

where $w_{n,\varepsilon}(x)=\dfrac{u_n(x_0+\varepsilon x)-u(x_0)}{\varepsilon}$ and $\eta_{n,\varepsilon}(x)=v_n(x_0+\varepsilon x)$.

Setting $w_0(x)=\nabla u(x_0)x$ we have that

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}||w_{n,\varepsilon}-w_0||_{L^1}=0.$$

On the other hand, since $v_n\mathcal{L}^N\lfloor\Omega\overset{*}{\rightharpoonup}v$ in $\mathcal{M}(\Omega;\mathbb{R}^l)$ and x_0 is a Lebesgue point for v, for every $\varphi\in C_0(B;\mathbb{R}^l)$ we have that

$$\lim_{\varepsilon\to 0}\lim_{n\to+\infty}\int_B\eta_{n,\varepsilon}(y)\varphi(y)dy=v(x_0)\int_B\varphi(y)dy.$$

Using a classical diagonalization process and the separability of $C_0(B;\mathbb{R}^l)$, we choose sequences $r_k\to 0$ and $n_k\to+\infty$ such that

$$||w_{n_k,r_k}-w_0||_{L^1}<\frac{1}{k}+\lim_{n\to+\infty}||w_{n,r_k}-w_0||_{L^1},$$

$$\left|\int_B\eta_{n_k,r_k}(y)\varphi_l(y)dy-v(x_0)\int_B\varphi_l(y)dy\right|<\frac{1}{k}+\lim_{n\to+\infty}\left|\int_B\eta_{n,r_k}(y)\varphi_l(y)dy-v(x_0)\int_B\varphi_l(y)dy\right|,$$

for every $l\in\{1,..,k\}$, where $\varphi_l\in\{\varphi_i\}_{i=1}^{+\infty}$ and the latter set is dense in $C_0(B;\mathbb{R}^l)$, and

$$\begin{aligned}
&\int_B W(x_0+r_kx,u(x_0)+r_kw_{n_k,r_k}(x),\nabla w_{n_k,r_k}(x),\eta_{n_k,r_k}(x))dx\\
&\leq\frac{1}{k}+\limsup_{n\to+\infty}\int_B W(x_0+r_kx,u(x_0)+r_kw_{n,r_k}(x),\nabla w_{n,r_k}(x),\eta_{n,r_k}(x))dx.
\end{aligned}$$

Exploiting the lower bound in $(H1)$ and the separability of $\{\varphi_i\}_{i=1}^{\infty}$, we obtain that the sequence $\eta_k:=\eta_{n_k,r_k}\in L^\infty(\mathbb{R}^N;\mathbb{R}^l)$, is such that $\eta_k\overset{*}{\rightharpoonup}v(x_0)$ in $\mathcal{M}(B;\mathbb{R}^l)$. Indeed let $\varphi\in C_0(B;\mathbb{R}^l)$ and let $\delta>0$. Take $\varphi_l\in\{\varphi_i\}_{i=1}^{\infty}$ such that $\|\varphi_l-\varphi\|_{L^\infty}\leq\delta$. Then, the uniform L^1 bound of (η_k) in B entails that

$$
\begin{aligned}
&\left|\int_B (\eta_k(y) - v(x_0))\varphi(y)dy\right| \\
&\leq \left|\int_B (\eta_k(y) - v(x_0))\varphi_l(y)dy\right| + \left|\int_B (\eta_k(y) - v(x_0))(\varphi_l(y) - \varphi(y))dy\right| \\
&\leq \left|\int_B (\eta_k(y) - v(x_0))\varphi_l(y)dy\right| + C\delta \leq (C+1)\delta,
\end{aligned}
$$

for sufficiently large k. Moreover, setting $w_k := w_{n_k, r_k}$ it results that $w_k \in W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^m)$, $w_k \to w_0$ in $L^1(B; \mathbb{R}^m)$ and,

$$
\frac{d\mu^a}{d\mathcal{L}^N}(x_0) \geq \lim_{k \to +\infty} \int_B W(x_0 + r_k x, u(x_0) + r_k w_k(x), \nabla w_k(x), \eta_k(x))dx.
$$

Then, using (H_2), we obtain that

$$
\begin{aligned}
\frac{d\mu^a}{d\mathcal{L}^N}(x_0) \geq \lim_{k \to +\infty} \int_B & W(x_0, u(x_0), \nabla w_k(x), \eta_k(x)) - \omega_{\overline{B}}(|r_k w_k(x)|) \\
& - \omega_{\overline{B}}(|r_k x|)(1 + |\nabla w_k(x)| + |\eta_k(x)|)dx \\
= \int_B & W(x_0, u(x_0), \nabla w_k(x), \eta_k(x))dx.
\end{aligned}
$$

At this point, in order to get the desired inequality, we use a slicing method as in [9] in order to modify η_k and w_k, exploiting the quasiconvexity-convexity of W, by new sequences denoted by $\{\bar{\eta}_k\} \subset L^1(B; \mathbb{R}^l) \cap C_0^\infty(\mathbb{R}^N; \mathbb{R}^l)$ and $\bar{w}_k$ such that

$$
\int_B \bar{\eta}_k(z)\, dz = v(x_0) \text{ and } \bar{w}_k \in w_0 + W_0^{1,\infty}(B; \mathbb{R}^m).
$$

Indeed, for each $k \in \mathbb{N}$ define a layer $L_k = \{z \in B : \mathrm{dist(z, \partial B) < 1/k}\}$. Consider the layer L_2 and recall that, by construction of η_j and w_j, there exists $c \in \mathbb{R}^+$ such that $\sup_{j \in \mathbb{N}} \|\eta_j\|_{L^1(B)} + \sup_{j \in \mathbb{N}} ||\nabla w_j||_{L^1(B)} \leq c$. If we divide L_2 in two sublayers, say S_2^1 and S_2^2, we have

$$
\forall\, j \in \mathbb{N} \quad \int_{S_2^1} (|\eta_j(z)| + |\nabla w_j(z)|)\, dz \leq \frac{c}{2} \quad \text{or} \quad \int_{S_2^2} (|\eta_\mathrm{j}(\mathrm{z})| + |\nabla \mathrm{w_j(z)}|)\, \mathrm{dz} \leq \frac{\mathrm{c}}{2}.
$$

Thus for some subsequences of $\{\eta_j\}$ and $\{w_j\}$, say $\{\eta_{j_2}\}$ and $\{w_{j_2}\}$, and one of the sublayers S_2^1 or S_2^2, say S_2, we have

$$
\forall\, j_2 \in \mathbb{N} \quad \int_{S_2} (|\eta_{j_2}(z)| + |\nabla w_{j_2}(z)|)\, dz \leq \frac{c}{2}.
$$

Note that for some $0 \leq \alpha_2 < \beta_2 \leq 1/2$ we can write

$$
S_2 = \{z \in B : \ \alpha_2 < \mathrm{dist(z, \partial B)} < \beta_2\}.
$$

Define then a cutoff function $\xi_2 : B \to [0,1]$ such that $\xi_2 = 0$ in $\partial B \cup \{z \in B : \mathrm{dist(z, \partial B)} \leq \alpha_2\}$, and $\xi_2 = 1$ in $\{z \in B : \mathrm{dist(z, \partial B)} \geq \beta_2\}$ and $||\nabla \xi_2|| \leq \frac{c}{\beta_2 - \alpha_2}$.

Since

$$\lim_{j_2 \to +\infty} \left| v(x_0) - \int_B \xi_2(z) \eta_{j_2}(z)\, dz \right| = |v(x_0)| \left| 1 - \int_B \xi_2(z)\, dz \right|,$$

then for $j(2)$ sufficiently large

$$\frac{\left| v(x_0) - \int_B \xi_2(z) \eta_{j(2)}(z)\, dz \right|}{\left| 1 - \int_B \xi_2(z)\, dz \right|} \leq |v(x_0)| + 1$$

and

$$\frac{1}{|S_2|} \int_{S_2} |w_{j(2)}(z) - w_0(z)|\, dz \leq \frac{1}{2}.$$

Repeating the procedure in the layer L_3 (now working with three sublayers) and so on for the next layers, we get $j(k) \in \mathbb{N}$ increasing with k, $S_k := \{z \in B : \alpha_k < \mathrm{dist(z, \partial B)} < \beta_\mathrm{k}\}$ layer of diameter $\frac{1}{k^2}$, and ξ_k a cutoff function on B verifying $\xi_k = 0$ in $\partial B \cup \{z \in B : \mathrm{dist(z, \partial B)} \leq \alpha_\mathrm{k}\}$, and $\xi_k = 1$ in $\{z \in B : \mathrm{dist(z, \partial B)} \geq \beta_\mathrm{k}\}$ such that

$$\frac{\left| v(x_0) - \int_B \xi_k(z) \eta_{j(k)}(z)\, dz \right|}{\left| 1 - \int_B \xi_k(z)\, dz \right|} \leq |v(x_0)| + 1, \tag{2.3}$$

$$\int_{S_k} (|\eta_{j(k))}(z)| + |\nabla w_{j(k)}(z)|)\, dz \leq \frac{c}{k} \tag{2.4}$$

and

$$\frac{1}{|S_k|} \int_{S_k} |w_{j(k)}(z) - w_0(z)|\, dz \leq \frac{1}{k}. \tag{2.5}$$

Then, defining

$$\bar{\eta}_k(z) := (1 - \xi_k(z)) \frac{v(x_0) - \int_B \xi_k(z) \eta_{j(k)}(z)\, dz}{1 - \int_B \xi_k(z)\, dz} + \xi_k(z) \eta_{j(k)}(z) \tag{2.6}$$

and

$$\bar{w}_k(z) := (1 - \xi_k(z)) w_0(z) + \xi_k(z) w_{j(k)}(z) \tag{2.7}$$

we have $\bar{\eta}_k \in L^1(B; \mathbb{R}^l) \cap C_0^\infty(B; \mathbb{R}^l)$ with

$$\int_B \bar{\eta}_k(z)\, dz = v(x_0)$$

and $\bar{w}_k \in w_0 + W_0^{1,\infty}(B;\mathbb{R}^n)$. Therefore, since

$$\int_B W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz = \int_{\{z\in B:\ \mathrm{dist}(z,\partial B)\geq\beta_k\}} W(x_0,u(x_0),\nabla w_{j(k)}(z),\eta_{j(k)}(z))\,dz$$
$$+\int_{S_k} W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz + \int_{\{z\in B:\ \mathrm{dist}(z,\partial B)\leq\alpha_k\}} W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz$$

Using the quasiconvexity-convexity of W in the last two variables, hypothesis $(H1)$, the definition of ξ_k, (2.3), (2.4), (2.5), (2.6) and (2.7), we obtain that

$$\begin{aligned}\frac{d\mu^a}{d\mathcal{L}^N}(x_0) &\geq \limsup_{k\to+\infty}\int_B W(x_0,u(x_0),\nabla w_{j(k)}(z),\eta_{j(k)}(z))\,dz \geq \limsup_{k\to+\infty}\\ &\times\left[\int_B W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz - \int_{S_k} W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz\right.\\ &\left.-\int_{\{z\in B:\ \mathrm{dist}(z,\partial B)\leq\alpha_k\}} W(x_0,u(x_0),\nabla\bar{w}_k(z),\bar{\eta}_k(z))\,dz\right]\\ &\geq \limsup_{k\to+\infty}\left[W(x_0,u(x_0),\nabla u(x_0),v(x_0)) - \int_{S_k} C(1+|\tilde{v}_k|+|v(x_0)|+|\nabla\bar{w}_k(z)|)\,dz\right.\\ &\left.-\int_{\{z\in B:\ \mathrm{dist}(z,\partial B)\leq\alpha_k\}} C(1+v(x_0)|+|\nabla u(x_0)|)\,dz\right] = W(x_0,u(x_0),\nabla u(x_0),v(x_0)),\end{aligned}$$

which gives the desired inequality. □

2.2 Upper bound

In order to prove the opposite inequality to (2.1), by the very definition of $\tilde{E}$, it is enough to consider the constant sequence $(u,v)\in W^{1,1}(\Omega;\mathbb{R}^m)\times L^1(\Omega;\mathbb{R}^l)$ as a test sequence in (1.2). Indeed one can prove that

$$\begin{aligned}\tilde{E}(u,v) = \quad &\inf\left\{\liminf_{n\to\infty}\int_\Omega QCW(x,u_n,\nabla u_n,v_n)dx :\right.\\ &\left. u_n\to u \text{ in } L^1(\Omega;\mathbb{R}^m), v_n\overset{*}{\rightharpoonup} v \text{ in } \mathcal{M}(\Omega;\mathbb{R}^l)\right\}.\end{aligned} \tag{2.8}$$

In fact, arguing as in [17], we can prove the following result, that allows us to replace W by its quasiconvex-convex envelope QCW in definition (1.2).

Proposition 1 *Let $W:\Omega\times\mathbb{R}^m\times\mathbb{R}^{N\times m}\times\mathbb{R}^l\to[0,+\infty)$ be a continuous function satisfying conditions (H_1) and (H_2), then (2.8) holds for every $(u,v)\in BV(\Omega;\mathbb{R}^m)\times\mathcal{M}(\Omega;\mathbb{R}^l)$.*

Proof Define for every $(u,v)\in BV(\Omega;\mathbb{R}^m)\times\mathcal{M}(\Omega;\mathbb{R}^l)$, the functional

$$\begin{aligned}\bar{E}(u,v) := \inf&\left\{\liminf_{n\to\infty}\int_\Omega QCW(x,u_n,\nabla u_n,v_n)dx :\right.\\ &\left. u_n\to u \text{ in } L^1(\Omega;\mathbb{R}^m), v_n\overset{*}{\rightharpoonup} v \text{ in } \mathcal{M}(\Omega;\mathbb{R}^l)\right\}.\end{aligned} \tag{2.9}$$

We first recall that $QCW \leq W$ and it satisfies (H_1) and (H_2). Consequently $\bar{E} \leq \tilde{E}$, for every $(u, v) \in BV(\Omega; \mathbb{R}^m) \times \mathcal{M}(\Omega; \mathbb{R}^l)$. For what concerns the opposite inequality, without loss of generality we assume that $\bar{E}(u, v) < +\infty$. Then for fixed $\delta > 0$, we can consider $u_n \in W^{1,1}(\Omega; \mathbb{R}^m)$ with $u_n \to u$ strongly in $L^1(\Omega; \mathbb{R}^m)$ and $v_n \in L^1(\Omega; \mathbb{R}^l)$, with $v_n \overset{*}{\rightharpoonup} v$ weakly in $\mathcal{M}(\Omega; \mathbb{R}^l)$ and such that

$$\bar{E}(u, v) \geq \lim_{n \to +\infty} \int_\Omega QCW(x, u_n, \nabla u_n, v_n) dx - \delta.$$

Applying the relaxation result in [6], we know that for each n there exists a sequence $(u_{n,k})$ converging to u_n weakly in $W^{1,1}(\Omega; \mathbb{R}^m)$ and $(v_{n,k})$ converging to v_n weakly in $L^1(\Omega; \mathbb{R}^l)$ as $k \to +\infty$, such that

$$\int_\Omega QCW(x, u_n(x), \nabla u_n(x), v_n(x)) dx = \lim_{k \to +\infty} \int_\Omega W(x, u_{n,k}(x), \nabla u_{n,k}(x), v_{n,k}(x)) dx.$$

Consequently

$$\bar{E}(u, v) \geq \lim_{n \to +\infty} \lim_{k \to +\infty} \int_\Omega W((x, u_{n,k}(x), \nabla u_{n,k}(x), v_{n,k}(x)) dx - \delta, \tag{2.10}$$

$$\lim_{n \to +\infty} \lim_{k \to +\infty} \|u_{n,k} - u\|_{L^1} = 0,$$

and for every $\varphi \in C_0(\Omega; \mathbb{R}^l)$,

$$\lim_{n \to \infty} \lim_{k \to \infty} \left| \int_\Omega (v_{n,k}(x) - v(x)) \varphi(x) dx \right| = 0.$$

Thus, taken (φ_j) a dense sequence in the separable space $C_0(\Omega; \mathbb{R}^l)$, it results that, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ and $k(n) \in \mathbb{N}$ increasing in n such that

$$\left| \int_\Omega (v_{n,k(n)} - v) \varphi_j dx \right| \leq \varepsilon, \quad \text{for every } j = 1, \ldots, n. \tag{2.11}$$

On the other hand the coercivity assumption (H_1) and (2.10) guarantee that $(v_{n,k(n)})$ is bounded in $L^1(\Omega; \mathbb{R}^l)$. Thus via a diagonal argument as that in [16, Remark 9] (see also the proof of Lemma 1 at page 5 herein) we can conclude that there exists a sequence $(u_{n,k(n)}, v_{n,k(n)})$ satisfying $u_{n,k(n)} \to u$ in $L^1(\Omega; \mathbb{R}^m)$, $v_{n,k(n)} \overset{*}{\rightharpoonup} v$ in $\mathcal{M}(\Omega; \mathbb{R}^l)$ and realizing the double limit in the right hand side of (2.10). Thus, it results

$$\bar{E}(u, v) \geq \lim_{n \to +\infty} \int_\Omega W(x, u_{n,k(n)}(x), \nabla u_{n,k(n)}(x), v_{n,k(n)}(x)) dx - \delta \geq \tilde{E}(u, v) - \delta.$$

Letting δ go to 0 the conclusion follows. □

Proof (of Theorem 1) The thesis is a consequence of Lemma 1 and Proposition 1. □

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