



The Duflo–Serganova Functor, Vingt Ans Après

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Abstract | We review old and new results concerning the DS functor and associated varieties for Lie superalgebras. These notions were introduced in the unpublished manuscript (Duflo and Serganova in On associated variety for Lie super-algebras, 2005) by Michel Duflo and the third author. This paper includes the results and proofs of the original manuscript, as well as a survey of more recent results.

1 Introduction

The DS functor was introduced by Michel Duflo and the third author approximately 20 years ago, but the original manuscript¹³ was never published. Since then much progress has been made in the study of the DS functor. This paper includes the results of the original manuscript, as well as a survey of more recent results obtained by different authors.

The DS functor has a large and growing list of applications throughout the literature. It was used in⁵⁵ to prove the Kac–Wakimoto conjecture (see Sect. 7); in⁴² to describe the supercharacter ring for $\mathfrak{p}(n)$ (see Sect. 8); in⁴¹ to study important $\mathfrak{sl}(\infty)$ -modules (see Sect. 9); in¹⁷ to give a formula for the superdimension of $\mathfrak{p}(n)$ -modules; in³⁸ to give a new proof of the superdimension formula for $GL(m|n)$ -modules; in³⁷ to obtain reductive envelopes of certain supergroups; in³ to compute complexity of certain modules over $\mathfrak{gl}(m|n)$; in⁹ to classify the indecomposable summands of tensor powers of the standard representation of $OSP(m|2n)$; and in²⁰ to construct universal tensor categories. The DS functor has been applied to study Deligne categories in numerous papers (see, e.g.,^{9,16,20}).

The associated variety of a module over a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a subvariety of the cone $X \subset \mathfrak{g}_1$ of self-commuting odd elements. The cone X was studied in^{32–34}, where geometric properties of X were used to obtain important results about the cohomology of Lie superalgebras.

Now if $x \in X$ and M is a \mathfrak{g} -module, then $x^2(M) = 0$ and hence we can take the cohomology $M_x = \text{Ker } x_M / \text{Im } x_M$. The assignment $M \mapsto M_x$ defines the Duflo–Serganova functor $DS_x : \text{mod}(\mathfrak{g}) \rightarrow \text{mod}(\mathfrak{g}_x)$, where $\mathfrak{g}_x = \text{Ker } ad_x / \text{Im } ad_x$ is a Lie algebra. It is easy

to see that this functor is symmetric monoidal. This obvious but remarkable fact does not have an analogue in the theory of Harish-Chandra modules or in the theory of restricted Lie algebras.

For the basic classical Lie superalgebras, $DS_x(L)$ has been computed for every simple finite-dimensional module L . These computations show that $DS_x(L)$ is semisimple and “pure” in the following sense: for every simple \mathfrak{g}_x -module L' one has $[DS_x(L) : L'] \cdot [DS_x(L) : \Pi L'] = 0$. It would be interesting to find a conceptual proof of these facts, see Sect. 12 for details.

The associated variety X_M for a \mathfrak{g} -module M is the closure in X of the subset consisting of all elements $x \in X$ for which M_x is nonzero. The associated variety for a module over a Lie superalgebra can be seen as an analogue of the associated varieties for Harish-Chandra modules, if we think about a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ as a symmetric pair. Associated varieties for Harish-Chandra modules have many interesting applications in the classical representation theory (see, for example,^{46,50,61}). While the associated variety in the theory of Harish-Chandra modules is trivial if a module is finite-dimensional, finite-dimensional modules over Lie superalgebras have interesting associated varieties. Some applications of these associated varieties are given in Sects. 7 and 11.

On the other hand, the associated variety for a module over a Lie superalgebra is also an analogue of the rank variety for restricted Lie algebras in positive characteristic, see²¹. For example, in many cases, these associated varieties for Lie superalgebras detect projectivity in the category of finite-dimensional \mathfrak{g} -modules. This is proven in Sect. 10 of the present paper; the original proof in the preprint¹³ had a mistake.

To Michel Duflo, with admiration.

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In the category of finite-dimensional \mathfrak{g} -modules, associated varieties are closely related to blocks and central characters, see Theorems 6.3 and 6.4. In the original preprint¹³, Theorem 6.4 was proven for $\mathfrak{gl}(m|n)$, now it is known for all basic classical superalgebras^{48,55}. It also seems that associated varieties can be used to study category \mathcal{O} for Kac–Moody superalgebras. Some results in this direction are obtained in^{11,29}.

Finally, let us mention that in contrast with restricted Lie algebras,²¹ the cohomological support varieties defined and studied in^{2,3} are quite different from the varieties studied in this paper. This may indicate existence of a third definition which interpolates these two constructions.

The Duflo–Serganova functor has also appeared under different guises in theoretical physics. Odd operators Q in a supersymmetric field theory that satisfy $Q^2 = 0$ are examples of BRST operators, and allow one to employ the so-called BRST formalism, which includes taking cohomology in Q . Such a situation arises in several places in the literature. In⁶², it was used as a key part in topological and holomorphic twists of supersymmetric field theories. Twisting a supersymmetric field theory gives rise to simpler field theories that can either be topological (giving a TQFT), holomorphic, or something in between, depending on properties of the chosen Q . A mathematically rigorous approach to the twisting of supersymmetric field theories is explained in¹⁰. In⁷, the authors apply DS functors (referred to as cohomological reductions in their paper) to the algebra of smooth functions as well as certain vector bundles on the target spaces of sigma models, which are homogeneous superspaces. Like in the mathematical setting, this reduction produces simpler theories which then in turn can give information about the original theory.

1.1 Notation

Throughout this paper (and in particular in Sects. 4–9 and 12), we will primarily focus on the following important list of Lie superalgebras, which by slight abuse of terminology we will refer to as *classical Lie superalgebras*:

$$\begin{aligned} & \mathfrak{sl}(m|n), m \neq n, \mathfrak{gl}(m|n), \mathfrak{osp}(m|2n), \\ & D(2|1; a), F(4), G(3), \mathfrak{p}(n), \mathfrak{q}(n). \end{aligned} \tag{1}$$

Note that each superalgebra appearing in this list has a “cousin” that is a classical Lie superalgebra in the sense of⁴³. Additionally, by *basic classical Lie superalgebra*, we will mean a superalgebra from the above list, excluding $\mathfrak{p}(n)$ and $\mathfrak{q}(n)$. We will sometimes refer to the superalgebras $D(2|1; a)$,

$G(3)$ and $F(4)$ as “exceptional” and to other superalgebras from our list as “non-exceptional”.

1.1.1 List of Notations

We present below a table of the commonly used notations in the article:

- $\text{mod}(\mathfrak{g})$ the category of \mathfrak{g} -modules.
- $\mathcal{F}in(\mathfrak{g})$ the category of finite-dimensional \mathfrak{g} -modules.
- $\mathcal{F}(\mathfrak{g})$ the category of finite-dimensional \mathfrak{g} -modules semisimple over \mathfrak{g}_0 .
- p a parity function on weights.
- $\text{mod}_\chi^r(\mathfrak{g})$ the category of \mathfrak{g} -modules with generalized central character χ of order r .
- $\text{mod}_\chi(\mathfrak{g})$ category of \mathfrak{g} -modules admitting generalized central character χ .
- $\text{Res}_{\mathfrak{k}}^{\mathfrak{g}}$ the restriction functor from \mathfrak{g} -modules to \mathfrak{k} -modules.
- $\text{Ind}_{\mathfrak{k}_0}^{\mathfrak{g}}$ the induction functor.
- X the self-commuting cone of \mathfrak{g} .
- X_k the set of rank k elements in X .
- \mathcal{O}_X the structure sheaf on X .
- ∂ the differential on $\mathcal{O}_X \otimes M$ in Sect. 11.
- X_M the associated variety of M .
- DS_x the Duflo–Serganova functor determined by $x \in X$.
- DS^r the Duflo–Serganova functor determined by a rank r element.
- $\eta_x : \mathcal{Z}(\mathfrak{g}) \rightarrow \mathcal{Z}(\mathfrak{g}_x)$ the induced map on center.
- σ_x the involution of \mathfrak{g}_x for classical Lie superalgebras (Sect. 3.1).
- $\mathcal{K}(\mathcal{C})$ the Grothendieck group of a full abelian subcategory \mathcal{C} of $\text{mod}(\mathfrak{g})$.
- M_{gr} the image of a module M in $\mathcal{K}(\mathcal{C})$.
- $\mathcal{K}_-(\mathcal{C})$ the reduced Grothendieck group (quotient by $M_{gr} = -(\Pi M)_{gr}$).
- $[M]$ the image of a module M in $\mathcal{K}_-(\mathcal{C})$.
- $\mathcal{K}_+(\mathcal{C})$ the character group (quotient by $M_{gr} = (\Pi M)_{gr}$).
- $\text{sch}M$ the supercharacter of M .
- ds_x the map induced by the functor DS_x on reduced Grothendieck groups.
- ds^r the map ds_x for a rank r element x .
- G_0 the simply connected, connected Lie group corresponding to \mathfrak{g}_0 .
- Δ the roots of \mathfrak{g} with respect to a Cartan subalgebra of \mathfrak{g}_0 .
- W the Weyl group.
- ρ the Weyl vector.
- $L(\lambda) = L_{\mathfrak{g}}(\lambda)$ is the irreducible \mathfrak{g} -module of highest weight λ with respect to a chosen Borel subalgebra.

- $\text{atyp}\chi, \text{atyp}\lambda$ the degree of atypicality of χ, λ resp.
- $[M : L]$ the multiplicity of a simple module L in a finite-length module M .
- $[M : L]_{\text{non}}$ the non-graded multiplicity of a simple module L in M , meaning the number of times both L and ΠL appear.
- R the super Weyl denominator.
- $k(\lambda)$ a virtual Kac module.

2 Definitions and Basic Properties

Our ground field is \mathbb{C} , and by \bar{Y} we denote the Zariski closure of a subset Y of an affine space. By Π we denote the change of parity functor in the category of vector superspaces.

Throughout this paper, we assume that the Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is finite dimensional. Let G_0 denote a simply-connected connected algebraic group with Lie algebra \mathfrak{g}_0 . We will write $\mathcal{F}in(\mathfrak{g})$ for the category of finite-dimensional \mathfrak{g} -modules, and $\mathcal{F}(\mathfrak{g})$ for the full subcategory of $\mathcal{F}in(\mathfrak{g})$ consisting of modules which are semisimple over \mathfrak{g}_0 . The category $\mathcal{F}(\mathfrak{g})$ will be the main object of study after Sect. 2.

2.1 The Associated Variety X_M

Let $X = \{x \in \mathfrak{g}_1 \mid [x, x] = 0\}$.

It is clear that X is a G_0 -invariant Zariski closed cone in \mathfrak{g}_1 .

Let M be a \mathfrak{g} -module. For every $x \in X$, the corresponding element $x_M \in \text{End}_{\mathbb{C}}(M)$ satisfies $x_M^2 = 0$. Set

$$M_x := \text{Ker } x_M/xM$$

and define

$$X_M := \{x \in X \mid M_x \neq 0\}.$$

We call X_M the associated variety of M .

Lemma 2.1 *If M is a finite-dimensional (\mathfrak{g}, G_0) -module, then X_M is Zariski closed G_0 -invariant subvariety.*

Proof For a finite-dimensional M ,

$$X \setminus X_M = \{x \in X \mid \text{rank } x_M = \dim M_0 = \dim M_1\}.$$

Hence, X_M is Zariski closed. Now M is a G_0 -module. For each $g \in G_0$ and $x \in M$ one has

$$M_{\text{Ad}_g(x)} = gM_x,$$

which implies the lemma. □

2.2 The Lie Superalgebra \mathfrak{g}_x

For $x \in X$, we define

$$\mathfrak{g}_x := \mathfrak{g}^x/[x, \mathfrak{g}],$$

where $\mathfrak{g}^x := \text{Ker } \text{ad}_x$ and $[x, \mathfrak{g}] := \text{Im } \text{ad}_x$.

The next lemma follows from the definitions.

Lemma 2.2 *Let \mathfrak{g} be a finite-dimensional Lie superalgebra and $x \in X$.*

1. *Then $[x, \mathfrak{g}]$ is an ideal in \mathfrak{g}^x , and hence \mathfrak{g}_x has the natural structure of a Lie superalgebra.*
2. *If M is a \mathfrak{g} -module, then M_x is a \mathfrak{g}_x -module.*

Now we observe that for each $x \in X$, the correspondence $M \mapsto M_x$ is functorial. Let $\text{mod}(\mathfrak{g})$ (respectively, $\text{mod}(\mathfrak{g}_x)$) denote the category of all \mathfrak{g} -modules (respectively, \mathfrak{g}_x -modules).

Definition 2.3 The Duflo–Serganova functor $DS_x : \text{mod}(\mathfrak{g}) \rightarrow \text{mod}(\mathfrak{g}_x)$ is defined by $DS_x(M) := M_x$.

The functor DS_x has many nice properties. The following lemma shows that DS_x is a symmetric monoidal tensor functor.

Lemma 2.4 *Let \mathfrak{g} be a finite-dimensional Lie superalgebra, let $x \in X$, and let M, N be \mathfrak{g}_x -modules.*

1. *We have a canonical isomorphism $(M \otimes N)_x \simeq M_x \otimes N_x$ of \mathfrak{g}_x -modules.*
2. *For any \mathfrak{g} -module M we have a canonical isomorphism $(M^*)_x \rightarrow (M_x)^*$ of \mathfrak{g}_x -modules.*

Hence, $DS_x : \text{mod}(\mathfrak{g}) \rightarrow \text{mod}(\mathfrak{g}_x)$ is a symmetric monoidal tensor functor.

Proof For (1), we have the natural morphism of \mathfrak{g}_x -modules $M_x \otimes N_x \rightarrow (M \otimes N)_x$. We have to check that this is an isomorphism. This follows from the fact that over the $(0|1)$ -dimensional superalgebra $\mathbb{C}x$, we have $M = M_x \oplus F$, $N = N_x \oplus F'$ for some free $\mathbb{C}x$ -modules F and F' , and we have

$$(M \otimes N) = M_x \otimes N_x \oplus (F \otimes N \oplus M \otimes F'),$$

where $F \otimes N \oplus M \otimes F'$ is a free $\mathbb{C}x$ -module.

For (2), we have a natural map $(M^*)_x \rightarrow (M_x)^*$ given by $\varphi \mapsto \varphi|_{\text{Ker } x}$, using the fact that $\varphi(\text{im } x) = 0$. In the other direction: given $\varphi : M_x \rightarrow \mathbb{C}$, write $\tilde{\varphi}$ for the lift of φ to $\text{Ker } x$ and choose a splitting $M = \text{Ker } x \oplus V$. Then $\phi = \tilde{\varphi} \oplus 0$ is annihilated by x , and this defines a morphism $(M_x)^* \rightarrow (M^*)_x$ inverse to our previous map. □

The next lemma shows that the functor DS_x preserves the superdimension of a finite-dimensional module M , where the superdimension of $M = M_0 \oplus M_1$ is given by $\text{sdim}M := \dim M_0 - \dim M_1$.

Lemma 2.5 For any finite-dimensional \mathfrak{g} -module M and $x \in X$, $\text{sdim}M = \text{sdim}M_x$.

Proof Let $\Pi(N)$ stand for the superspace isomorphic to N with switched parity. Since $M/\text{Ker } x_M$ is isomorphic to $\Pi(xM)$, we have

$$\begin{aligned} \text{sdim}M &= \text{sdim}(\text{Ker } x_M) - \text{sdim}(xM) \\ &= \text{sdim}(\text{Ker } x_M/xM) = \text{sdim}M_x. \end{aligned}$$

□

In fact, Lemma 2.5 has a natural generalization, as we will see in the next section.

2.3 Reduced Grothendieck Groups and ds_x

Let \mathcal{C} be a full abelian subcategory of $\text{mod}(\mathfrak{g})$ such that

(*) ΠM is an object of \mathcal{C} whenever M is.

We define the Grothendieck group $\mathcal{K}(\mathcal{C})$ in the usual way as the quotient of the free \mathbb{Z} -module with basis M_{gr} , for each object M in \mathcal{C} , with relations $M_{gr} = M'_{gr} + M''_{gr}$ for every short exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ in \mathcal{C} .

The reduced Grothendieck group $\mathcal{K}_-(\mathcal{C})$ of the category \mathcal{C} is the quotient $\mathcal{K}(\mathcal{C})$ by the relation $M_{gr} = -(\Pi M)_{gr}$ for all objects $M \in \mathcal{C}$. We define the character group $\mathcal{K}_+(\mathcal{C})$ to be the quotient of $\mathcal{K}(\mathcal{C})$ by $M_{gr} = (\Pi M)_{gr}$. Write $(-)_{\mathbb{Q}}$ for the extension of scalars from \mathbb{Z} to \mathbb{Q} . Then by the Chinese Remainder Theorem we have

$$\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathcal{K}_-(\mathcal{C})_{\mathbb{Q}} \times \mathcal{K}_+(\mathcal{C})_{\mathbb{Q}}. \tag{2}$$

If \mathcal{C} is closed under \otimes , then its tensor structure provides $\mathcal{K}(\mathcal{C})$ with a ring structure such that $\mathcal{K}_-(\mathcal{C})$ and $\mathcal{K}_+(\mathcal{C})$ are quotient rings, and (2) becomes an isomorphism of rings.

Remark 2.6 Since we work over the integers, observe that if M is any module in \mathcal{C} with $M \cong \Pi M$, its image in $\mathcal{K}_-(\mathcal{C})$ will be 2-torsion (although it need not be 0).

Lemma 2.7 (V. Hinich) *If*

$$0 \rightarrow N \xrightarrow{\psi} M \xrightarrow{\varphi} L \rightarrow 0$$

is an exact sequence of \mathfrak{g} -modules, then there exists an exact sequence

$$0 \rightarrow E \rightarrow N_x \rightarrow M_x \rightarrow L_x \rightarrow \Pi E \rightarrow 0$$

for some \mathfrak{g}_x -module E .

Proof Set E be the kernel of the induced map $\psi : N_x \rightarrow M_x$ and E' be the quotient $L_x/\varphi(M_x)$. Then we have the exact sequence

$$0 \rightarrow E \rightarrow N_x \rightarrow M_x \rightarrow L_x \rightarrow E' \rightarrow 0.$$

The odd map $\psi^{-1}x\varphi^{-1} : L_x \rightarrow N_x$ induces an isomorphism $E' \rightarrow \Pi E$. □

Lemmas 2.4 and 2.7 imply the following.

Corollary 2.8 The functor DS_x is a middle exact tensor functor and satisfies $DS_x(\Pi M) = \Pi DS_x(M)$.

Corollary 2.9 Let \mathcal{C} , (resp. \mathcal{C}_x) be full abelian subcategories of $\text{mod}(\mathfrak{g})$ (resp. $\text{mod}(\mathfrak{g}_x)$) satisfying (*). Suppose that $DS_x(M)$ lies in \mathcal{C}_x whenever M lies in \mathcal{C} . Then the functor $DS_x : \mathcal{C} \rightarrow \mathcal{C}_x$ induces a homomorphism on the corresponding reduced Grothendieck groups

$$ds_x : \mathcal{K}_-(\mathcal{C}) \rightarrow \mathcal{K}_-(\mathcal{C}_x).$$

We now focus in particular on the case when $\mathcal{C} = \text{Fin}(\mathfrak{g})$. Then Lemma 2.7 in particular implies that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{K}_-(\text{Fin}(\mathfrak{g})) & \longrightarrow & \mathcal{K}_-(\text{Fin}(\mathfrak{g}^x)) \\ & \searrow ds_x & \uparrow \\ & & \mathcal{K}_-(\text{Fin}(\mathfrak{g}_x)) \end{array} \tag{3}$$

where the horizontal arrow is induced by the restriction functor $\text{Res}_{\mathfrak{g}^x}^{\mathfrak{g}}$, and the up arrow is induced by $\text{Res}_{\mathfrak{g}_x}^{\mathfrak{g}^x}$, where $\mathfrak{g}^x \rightarrow \mathfrak{g}_x$ is the canonical surjection.

Remark 2.10 The map $ds_x : \mathcal{K}_-(\text{Fin}(\mathfrak{g})) \rightarrow \mathcal{K}_-(\text{Fin}(\mathfrak{g}_x))$ is a ring homomorphism compatible with duality. This follows from the fact that $\text{Fin}(\mathfrak{g})$ and $\text{Fin}(\mathfrak{g}_x)$ are tensor categories, since DS_x is a tensor functor that preserves the duality.

Remark 2.11 The existence of a homomorphism between reduced Grothendieck rings was first observed when $\mathfrak{g} = \mathfrak{gl}(m|n)$ in³⁸. In⁴⁰, this homomorphism was introduced in the language of supercharacters for the category of finite-dimensional modules of finite-dimensional Kac–Moody superalgebras, and its kernel and image were described.

2.4 ds_x as Restriction

For a module M , we write $[M]$ for its image in the reduced Grothendieck ring.

Lemma 2.12 *Suppose that we have a splitting $\mathfrak{g}_x \subseteq \mathfrak{g}^x$ so that $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$. Then for a finite-dimensional \mathfrak{g} -module M we have*

$$ds_x[M] = [\text{Res}_{\mathfrak{g}_x}^{\mathfrak{g}} M].$$

Proof This follows immediately by applying the restriction $\mathcal{K}_-(\text{Fin}(\mathfrak{g}^x)) \rightarrow \mathcal{K}_-(\text{Fin}(\mathfrak{g}_x))$ to the equality $[DS_x M] = [\text{Res}_{\mathfrak{g}_x}^{\mathfrak{g}} M]$ coming from Lemma 2.7. \square

Lemma 2.13 *Let $x, y \in X$ such that $[x, y] = 0$, and suppose that we have splittings*

$$\mathfrak{g}^y = \mathfrak{g}_y \ltimes [y, \mathfrak{g}], \quad \mathfrak{g}^{x+y} = \mathfrak{g}_{x+y} \ltimes [x + y, \mathfrak{g}],$$

Furthermore suppose that under these splittings, $x \in \mathfrak{g}_y$ and

$$(\mathfrak{g}_y)^x = \mathfrak{g}_{x+y} \ltimes [x, \mathfrak{g}_y].$$

Then we have

$$ds_{x+y} = ds_x \circ ds_y : \mathcal{K}_-(\mathfrak{g}) \rightarrow \mathcal{K}_-(\mathfrak{g}_{x+y})$$

Proof This follows immediately from Lemma 2.12 and the corresponding statement for restriction. \square

Lemma 2.14 *Suppose that $x, y \in X$ and that there exists $g \in G_0$ such that $gx = y$. Then we have a commutative diagram*

$$\begin{array}{ccc} \mathcal{K}_-(\text{Fin}(\mathfrak{g})) & \xrightarrow{ds_x} & \mathcal{K}_-(\text{Fin}(\mathfrak{g}_x)) \\ & \searrow ds_y & \downarrow \sim \\ & & \mathcal{K}_-(\text{Fin}(\mathfrak{g}_y)) \end{array}$$

where the downward arrow is an isomorphism and is induced by the action of g . In particular:

$$\text{Ker}(ds_x|_{\mathcal{K}_-(\text{Fin}(\mathfrak{g}))}) = \text{Ker}(ds_y|_{\mathcal{K}_-(\text{Fin}(\mathfrak{g}))})$$

Proof We have a commutative diagram

$$\begin{array}{ccc} \text{Fin}(\mathfrak{g}) & \xrightarrow{DS_x} & \text{Fin}(\mathfrak{g}_x) \\ & \searrow DS_y & \downarrow \sim \\ & & \text{Fin}(\mathfrak{g}_y) \end{array}$$

where the downward arrow is induced by the action of g , and is an equivalence. Passing to the reduced Grothendieck ring completes the argument. \square

2.5 Supermultiplicity

Let \mathfrak{g} be a finite-dimensional Lie superalgebra, and let \mathfrak{a} be any subalgebra of \mathfrak{g} . We will view

$$\mathfrak{a}_x := \mathfrak{a}^x / ([\mathfrak{g}, x] \cap \mathfrak{a})$$

as a subalgebra of \mathfrak{g}_x .

In addition to preserving superdimension, the DS functor also preserves the supermultiplicity of \mathfrak{g}^x -modules, when this notion is well-defined. We continue to work just with finite-dimensional modules. The *multiplicity* of a simple module L in a finite-length module M , denoted $[M : L]$, is the number of factors in the Jordan–Hölder series of M which are isomorphic to L . If M is a finite-dimensional module and L is a finite-dimensional simple module, then we can define the *supermultiplicity* of L in M to be:

$$\text{smult}(M; L) := \begin{cases} [M : L] - [M : \Pi L] & \text{if } L \not\cong \Pi L \\ [M : L] \bmod 2 & \text{if } L \cong \Pi L \end{cases} \quad (4)$$

The following lemma is immediate.

Lemma 2.15 *Let L be simple, finite-dimensional \mathfrak{g} -module. Then $\text{smult}(-; L)$ defines a homomorphism*

$$\mathcal{K}_-(\text{Fin}(\mathfrak{g})) \rightarrow \begin{cases} \mathbb{Z} & \text{if } L \not\cong \Pi L \\ \mathbb{Z}_2 & \text{if } L \cong \Pi L \end{cases}$$

The following proposition is from²⁵.

Proposition 2.16 *Let M be in $\text{Fin}(\mathfrak{g})$, and let L be a simple finite-dimensional \mathfrak{g}^x -module. Then one has*

$$\text{smult}(\text{Res}_{\mathfrak{g}_x}^{\mathfrak{g}} M; L) = \text{smult}(DS_x(M); L),$$

where $DS_x(M)$ is viewed as a \mathfrak{g}^x -module.

Proof This follows immediately from (3). \square

Remark 2.17 In many cases \mathfrak{g}_x can be viewed as a subalgebra of \mathfrak{g} in a way that $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$, and in these cases, the above formula also holds for each simple \mathfrak{g}_x -module L . In particular, the claim holds if \mathfrak{g} is a classical Lie superalgebra (see Proposition 4.5).

Proposition 2.18 We have the following commutative diagram

$$\begin{CD} \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) @>>> \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{a}^x)) \\ @V ds_x VV @A \text{res}_{\mathfrak{a}^x}^{\mathfrak{a}^x} AA \\ \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}_x)) @>>> \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{a}_x)) \end{CD}$$

where the horizontal arrows are induced by the corresponding restriction functors and $\text{res}_{\mathfrak{a}^x}^{\mathfrak{a}^x}$ is induced by the functor $\text{Res}_{\mathfrak{a}^x}^{\mathfrak{a}^x}$ for the canonical surjection $\mathfrak{a}^x \rightarrow \mathfrak{a}_x$.

Proof The restriction functors give the commutative diagram

$$\begin{CD} \mathcal{F}\text{in}(\mathfrak{g}^x) @>>> \mathcal{F}\text{in}(\mathfrak{a}^x) \\ @. @. \\ @. @. \\ @. @. \\ \mathcal{F}\text{in}(\mathfrak{g}_x) @>>> \mathcal{F}\text{in}(\mathfrak{a}_x) \end{CD}$$

which allows us to rewrite the original diagram in the form

$$\begin{CD} \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) @>>> \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{a}^x)) \\ @V ds_x VV @A \uparrow A \\ \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}_x)) @>>> \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g}^x)) \end{CD}$$

where all arrows except ds_x are induced by the restriction functors. By (3), the above diagram is commutative. \square

Example 2.19 Suppose \mathfrak{g} is a classical Lie superalgebra in the sense of ⁴³, and let $\mathfrak{a} := \mathfrak{h}$ be a Cartan subalgebra of \mathfrak{g}_0 . Restriction induces a map $\mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) \rightarrow \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{h}))$ which we write as $[N] \mapsto \text{sch}N$, where $\text{sch}N$ denotes the supercharacter of N (see (12)). If \mathfrak{h}_x is a Cartan subalgebra of $(\mathfrak{g}_x)_0$, then the composed map $ds_x : \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{g})) \rightarrow \mathcal{K}_-(\mathcal{F}\text{in}(\mathfrak{h}_x))$ is given by $[N] \mapsto \text{sch}DS_x(N)$. If we fix an embedding $\mathfrak{h}_x \rightarrow \mathfrak{h}$, then Proposition 2.18 gives the Hoyt-Reif formula⁴⁰

$$\text{sch}DS_x(N) = (\text{sch}N)|_{\mathfrak{h}_x}. \tag{5}$$

2.6 Properties of Associated Varieties

Here we list a few basic properties of associated varieties for a finite-dimensional Lie superalgebra \mathfrak{g} . Let $\mathcal{U}(\mathfrak{g})$ denote the universal enveloping algebra of \mathfrak{g} .

We have the following.

Lemma 2.20 Let \mathfrak{g} be a finite-dimensional Lie superalgebra.

1. If $M = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} M'$ for some \mathfrak{g}_0 -module M' , then $X_M = \{0\}$;
2. If $M = \mathbb{C}$ is trivial, then $X_M = X$;
3. For any \mathfrak{g} -modules M and N , one has $X_{M \oplus N} = X_M \cup X_N$;
4. For any \mathfrak{g} -modules M and N , one has $X_{M \otimes N} = X_M \cap X_N$;
5. For any \mathfrak{g} -module M , $X_{M^*} = X_M$;

Proof (2) and (3) follow directly from the definition, while (4) and (5) follow from Lemma 2.4.

To prove (1), let $x \in X$ and $x \neq 0$. Let $\{v_j\}_{j \in J}$

be a basis of M' and x_1, \dots, x_m be a basis of \mathfrak{g}_1 such that $x = x_1$. Then by the PBW Theorem for Lie superalgebras, the elements $x_{i_1}x_{i_2} \dots x_{i_k} \otimes v_j$ for all $1 \leq i_1 < i_2 < \dots < i_k \leq m, j \in J$ form a basis of M . The action of

$x = x_1$ in this basis is easy to write, and it is clear that $\text{Ker } x = xM$ is spanned by the vectors $x_1x_{i_2} \dots x_{i_k} \otimes v_j$. \square

The following lemma is the premise of Sect. 10, where the relationship between projectivity of module and its associated variety will be studied more in depth.

Lemma 2.21 Suppose that \mathfrak{g}_0 is reductive (i.e., \mathfrak{g} is quasireductive). If M is projective in $\mathcal{F}(\mathfrak{g})$ then we have $X_M = \{0\}$.

Proof This follows from (1) of Lemma 2.20 using that M will be a direct summand of $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} \text{Res}_{\mathfrak{g}_0}^{\mathfrak{g}} M$. \square

Remark 2.22 There is a natural action of $G_0 \times \mathbb{G}_m$ on the associated variety X of \mathfrak{g} , where the one-dimensional torus \mathbb{G}_m acts by scaling.

For $\lambda \in \mathbb{G}_m$, it is easy to check we have an equality of functors $DS_{\lambda x} = DS_x$. For $g \in G_0$, the functors DS_{gx} and DS_x are isomorphic, in a suitable sense, when we restrict to finite-dimensional modules.

3 The Universal Enveloping Algebra and Central Characters

In this section, \mathfrak{g} denotes a finite-dimensional Lie superalgebra. Let $Z(\mathfrak{g})$ (respectively, $Z(\mathfrak{g}_x)$) denote the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ (respectively, $\mathcal{U}(\mathfrak{g}_x)$).

For each central character $\chi : Z(\mathfrak{g}) \rightarrow \mathbb{C}$, we denote by $\text{mod}_{\chi}^r(\mathfrak{g})$ the full subcategory of

$\text{mod}(\mathfrak{g})$ consisting of the modules that are annihilated by $(z - \chi(z))^r$ for every $z \in Z(\mathfrak{g})$. We set

$$\text{mod}_\chi(\mathfrak{g}) = \bigcup_{r=1}^\infty \text{mod}_\chi^r(\mathfrak{g}),$$

and we say that a \mathfrak{g} -module M admits central character χ if M lies in $\text{mod}_\chi(\mathfrak{g})$. By Dixmier’s generalization of Schur’s Lemma (see¹⁴), each simple module lies in $\text{mod}_\chi^1(\mathfrak{g})$ for a suitable central character χ .

Symmetrization gives an isomorphism $\mathcal{U}(\mathfrak{g}) \simeq S(\mathfrak{g})$ as $\text{ad}_\mathfrak{g}$ -modules. Then since $M \mapsto M_x$ is a tensor functor we have

$$\mathcal{U}(\mathfrak{g})_x \simeq \mathcal{U}(\mathfrak{g}_x).$$

Observe that $\text{ad}_x(\mathcal{U}(\mathfrak{g}))$ is an ideal in $\mathcal{U}(\mathfrak{g})^{\text{ad}_x}$ and consider the canonical homomorphism of algebras $\pi : \mathcal{U}(\mathfrak{g})^{\text{ad}_x} \rightarrow \mathcal{U}(\mathfrak{g}_x)$. Then we have a homomorphism

$$Z(\mathfrak{g}) \hookrightarrow \mathcal{U}(\mathfrak{g})^{\text{ad}_x} \xrightarrow{\pi} \mathcal{U}(\mathfrak{g}_x)$$

and since $Z(\mathfrak{g}_x) = \mathcal{U}(\mathfrak{g}_x)^{\text{ad}_{\mathfrak{g}_x}}$ we have a well-defined homomorphism

$$\eta_x : Z(\mathfrak{g}) \rightarrow Z(\mathfrak{g}_x).$$

The dual map of central characters

$$\eta_x^* : \text{Hom}(Z(\mathfrak{g}_x), \mathbb{C}) \rightarrow \text{Hom}(Z(\mathfrak{g}), \mathbb{C}) \quad (6)$$

is very important due to the following statement.

Proposition 3.1 Take $M \in \text{mod}_\chi^r(\mathfrak{g})$.

1. If η_x is surjective, then $DS_x(M)$ lies in $\text{mod}_{(\eta_x^*)^{-1}\chi}^r(\mathfrak{g}_x)$.
2. For each simple subquotient L' of $DS_x(M)$ there exists $\chi' \in (\eta_x^*)^{-1}(\chi)$ such that $L' \in \text{mod}_{\chi'}^1(\mathfrak{g}_x)$. In particular, $DS_x(M) = 0$ if $\chi \notin \text{Im}(\eta_x^*)$.

Proof View M as a $\mathcal{Z}(\mathfrak{g})$ -module. Note that $DS_x(M)$ can be viewed as a subquotient of this module. Take $z \in \text{Ker}\chi$ and set $z_x := \eta_x(z)$. Since $M \in \text{mod}_\chi^r(\mathfrak{g})$ one has $z^r M = 0$, so $z_x^r DS_x(M) = 0$. This gives (i). For (ii) take $\chi' : \mathcal{Z}(\mathfrak{g}_x) \rightarrow \mathbb{C}$ such that $\text{Ker}\chi' L' = 0$. Then $(z_x - \chi'(z_x))L' = 0$, so $\chi'(z_x) = 0$. Hence $\eta_x(\text{Ker}\chi) \subset \text{Ker}\chi'$ which implies $\eta_x(\chi') = \chi$ as required. \square

Corollary 3.2 If M admits a central character χ and M_x has a subquotient admitting a central character ζ , then $\chi = \eta^*(\zeta)$.

Using Proposition 2.16, we obtain the following interesting corollary.

Corollary 3.3 Assume that \mathfrak{g}_x can be embedded into \mathfrak{g} (i.e., $\mathfrak{g}^x = \mathfrak{g}_x \rtimes [x, \mathfrak{g}]$). If $M \in \text{mod}_\chi(\mathfrak{g})$ and $L' \in \text{mod}_{\chi'}(\mathfrak{g}_x)$ is a simple \mathfrak{g}_x -module such that $[\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M : L'] < \infty$ and $[\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M : \Pi(L')] < \infty$, then

$$[\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M : L'] = [\text{Res}_{\mathfrak{g}_x}^\mathfrak{g} M : \Pi(L')] \text{ if } \chi' \notin (\eta_x^*)^{-1}(\chi).$$

3.1 The Involutions σ_x for Classical Lie Superalgebras

The maps η_x for classical Lie superalgebras were described in²⁵ using the results of^{44,56,57}. There is a nice uniform description of the image, which requires us to introduce an involution σ_x on \mathfrak{g}_x . We note that for $\mathfrak{p}(n)$ the center is always trivial, however we will introduce an involution σ_x for later use.

- For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2m+1|2n), \mathfrak{q}(n), \mathfrak{p}(n)$ or $G(3)$, we declare the involution σ_x on \mathfrak{g}_x to be trivial, i.e., $\sigma_x = \text{Id}$.
- For $\mathfrak{g} = D(2|1; a)$ and $x \neq 0$, one has $\mathfrak{g}_x = \mathbb{C}z$, and we set $\sigma_x = -\text{Id}$.
- For $\mathfrak{g} = F(4)$ and $x \neq 0$, one has $\mathfrak{g}_x \cong \mathfrak{sl}_3$, and σ_x is induced by the involution of the Dynkin diagram of \mathfrak{sl}_3 .
- For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ one has $\mathfrak{g}_x = \mathfrak{osp}(2(m-s)|2(n-s))$; we set $\sigma_x = \text{Id}$ if $m-s = 0$, and if $m-s > 0$, σ_x is induced by the involution of one of the Dynkin diagrams of \mathfrak{g}_x .

Remark 3.4 Let us give another description of the involution σ_x . Consider an embedding $x \in \mathfrak{sl}(1|1) \subset \mathfrak{g}$ as in the proof of Theorem 5.11. Let K denote the normalizer of $\mathfrak{sl}(1|1)$ inside the adjoint supergroup of \mathfrak{g} . Then K has a normal subgroup with the Lie superalgebra \mathfrak{g}_x . The image of the natural homomorphism $K \rightarrow \text{Aut}\mathfrak{g}_x$ is disconnected if $\mathfrak{g} = \mathfrak{osp}(2m|2n)$, $m-s > 0$, $D(2, 1; a)$ or F_4 . In these cases, the image is a semidirect product of the adjoint group of \mathfrak{g}_x and \mathbb{Z}_2 . The involution σ_x is a generator of \mathbb{Z}_2 .

Proposition 3.5 For \mathfrak{g} a classical Lie superalgebra, $x \in X$, and involution σ_x on \mathfrak{g}_x as above, we have

$$\eta_x(\mathcal{Z}(\mathfrak{g})) = \mathcal{Z}(\mathfrak{g}_x)^{\sigma_x}.$$

Remark 3.6 Although we do not give a proof of Proposition 3.5, it can be used to give another proof of Theorem 6.12, using Proposition 3.7 below.

In the following lemma \mathfrak{g} is general, but with the obvious view toward the cases of interest above.

Proposition 3.7 *Assume that $\text{Im}\eta_x = \mathcal{Z}(\mathfrak{g}_x)^{\sigma_x}$, where σ_x is an involutive automorphism of \mathfrak{g}_x . For any $\chi \in \text{Im}\eta_x^*$ we have*

1. *the set $(\eta_x^*)^{-1}(\chi)$ is of the form $\{\chi', \sigma_x(\chi')\}$;*
2. *if $\sigma_x(\chi') \neq \chi'$, then $DS_x(\text{mod}_{\chi'}^1(\mathfrak{g})) \subseteq \text{mod}_{\chi'}^1(\mathfrak{g}_x) \oplus \text{mod}_{\sigma_x(\chi')}^1(\mathfrak{g}_x)$;*
3. *if $\sigma_x(\chi') = \chi'$, then $DS_x(\text{mod}_{\chi'}^1(\mathfrak{g})) \subseteq \text{mod}_{\chi'}^2(\mathfrak{g}_x)$.*

Proof We set

$$\mathfrak{m} := \text{Ker } \chi, \quad I := \mathcal{Z}(\mathfrak{g}_x)\eta_x(\mathfrak{m}), \quad A := \mathcal{Z}(\mathfrak{g}_x)/I$$

and denote by ψ the canonical map $\mathcal{Z}(\mathfrak{g}_x) \rightarrow A$. The algebra A inherits the action of σ_x and $A^{\sigma_x} = \mathcal{Z}(\mathfrak{g}_x)^{\sigma_x}/\eta_x(\mathfrak{m}) \cong \mathbb{C}$, so $A = \mathbb{C} \oplus A_-$, where $A_- := \{a \in A \mid \sigma_x(a) = -a\}$.

The central characters in $(\eta_x^*)^{-1}(\chi)$ correspond to the maximal ideals of A : for each $\chi'' \in (\eta_x^*)^{-1}(\chi)$ the ideal $\text{Ker}\chi''$ is a maximal ideal of $\mathcal{Z}(\mathfrak{g}_x)$; this ideal contains I and $\psi(\text{Ker } \chi'')$ is a maximal ideal in A . For each $N \in \text{mod}_{\chi}^1(\mathfrak{g})$ one has $\mathfrak{m}N = 0$, so $I(DS_x(N)) = 0$. Hence, $DS_x(N)$ has a structure of an A -module.

If $A_- = \mathbb{C}a$ with $a^2 = 1$, then A has two maximal ideals $\mathbb{C}(1 \pm a)$ (one has $1 - a = \sigma(1 + a)$). Taking $\mathfrak{m}' := \psi^{-1}(\mathbb{C}(1 + a))$ we get $(\eta_x^*)^{-1}(\chi) = \{\chi', \sigma_x(\chi')\}$, where $\text{Ker } \chi' = \mathfrak{m}'$. One has $DS_x(N) = N'_+ \oplus N'_-$, where $N'_\pm = \{v \in DS_x(N) \mid (a \pm 1)v = 0\}$. Therefore, $\mathfrak{m}'N'_+ = 0$ and $\sigma(\mathfrak{m}')N'_- = 0$, so $DS_x(N)$ lies in $\text{mod}_{\chi'}^1(\mathfrak{g}_x) + \text{mod}_{\sigma_x(\chi')}^1(\mathfrak{g}_x)$.

Consider the case when $A_- \neq \mathbb{C}a$ with $a^2 = 1$. For any $a_1, a_2 \in A_-$ one has $a_1a_2 \in A^{\sigma_x} = \mathbb{C}$. If $a_1a_2 = 1$ for some $a_1, a_2 \in A_-$, then for each $a \in A_-$ one has $a_2a \in \mathbb{C}$, so $a = a_1a_2a \in \mathbb{C}a_1$ that is $A_- \neq \mathbb{C}a_1$, a contradiction. Therefore, $a_1a_2 = 0$ for all $a_1, a_2 \in A_-$, so $(A_-)^2 = 0$ and A_- is the unique maximal ideal in A . Then $(\eta_x^*)^{-1}(\chi) = \chi'$ where $\text{Ker } \chi' = \psi^{-1}(A_-)$. Since $(A_-)^2DS_x(N) = 0$ we have $DS_x(\text{mod}_{\chi'}^1(\mathfrak{g})) \subset \text{mod}_{\chi'}^2(\mathfrak{g}_x)$. \square

4 Description of \mathfrak{g}_x for Classical Lie Superalgebras \mathfrak{g}

In this section, we describe \mathfrak{g}_x and realize \mathfrak{g}_x as a subalgebra of \mathfrak{g} for classical Lie superalgebras.

4.1 Iso-sets and Defect

Now we assume that \mathfrak{g}_0 is a reductive Lie algebra and \mathfrak{g}_1 is a semisimple \mathfrak{g}_0 -module. Such Lie superalgebras are called quasireductive.

For a quasireductive Lie superalgebra \mathfrak{g} , we may choose a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}_0$ and obtain roots $\Delta \subseteq \mathfrak{t}^* \setminus \{0\}$ by considering its adjoint action on \mathfrak{g} . We have subsets $\Delta_i \subseteq \Delta$ for $i = 0, 1$ consisting of roots which are either even or odd. In particular, we have a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where \mathfrak{h} denotes the centralizer of \mathfrak{t} in \mathfrak{g} . We write each $a \in \mathfrak{g}_i$ (for $i = 0, 1$) in the form

$$a = \sum_{\alpha \in \text{supp}(a)} a_\alpha, \quad \text{where } a_\alpha \in \mathfrak{g}_\alpha \setminus \{0\},$$

$$\text{supp}(a) \subset \Delta_i \cup \{0\}.$$

We say that $A \subset \Delta_1$ is an *iso-set* if the elements of A are linearly independent and if for each $\alpha, \beta \in \Delta_1 \cap (A \cup (-A))$ one has $\alpha + \beta \notin \Delta_0$. We call the maximal cardinality of an iso-set the *defect* of \mathfrak{g} . We let \mathcal{S} denote the set of iso-sets in Δ_1 . The Weyl group W of \mathfrak{g}_0 acts on \mathcal{S} in the obvious way. Put $\mathcal{S}_k = \{A \in \mathcal{S} \mid |A| = k\}$, with $\mathcal{S}_0 = \{\emptyset\}$.

4.2 Basic Classical Lie Superalgebras

Suppose \mathfrak{g} is a basic classical Lie superalgebra (see 1.1). If $\mathfrak{g} \neq \mathfrak{gl}(m|n)$, then \mathfrak{g} is a simple Kac–Moody superalgebra (see^{39,43}). The Lie superalgebra $\mathfrak{gl}(m|n)$ has as an ideal $\mathfrak{sl}(m|n)$, and when $m \neq n$, $\mathfrak{sl}(m|n)$ is simple and $\mathfrak{gl}(m|n) \cong \mathfrak{sl}(m|n) \oplus \mathbb{C}$. The Lie superalgebra $\mathfrak{gl}(n|n)$ has a unique simple subquotient $\mathfrak{psl}(n|n) := \mathfrak{sl}(n|n)/\text{span}\{\text{Id}\}$.

We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then \mathfrak{h} coincides with a Cartan subalgebra \mathfrak{t} of \mathfrak{g}_0 , and each root space \mathfrak{g}_α is one dimensional. In this case, the parity of $\alpha \in \Delta$ is by definition the parity of the root space \mathfrak{g}_α .

A finite-dimensional Kac–Moody superalgebra has a nondegenerate symmetric invariant bilinear form (\cdot, \cdot) . This form is not necessarily positive definite, and some roots can be isotropic. For a non-isotropic root β , we denote by β^\vee the element of \mathfrak{t} such that $\alpha(\beta^\vee) = \frac{2(\alpha, \beta)}{(\beta, \beta)}$. For an isotropic root β , set $\beta^\vee \in \mathfrak{t}$ to be the element of \mathfrak{t} corresponding to β under the isomorphism $\mathfrak{t} \rightarrow \mathfrak{t}^*$ induced by the form.

Remark 4.1 The notion of defect was originally introduced in⁴⁵ for Kac–Moody superalgebras. Finite-dimensional Kac–Moody superalgebras are quasireductive, and in this case, the notion of iso-set corresponds to the well-known notion of isotropic set: a set of mutually orthogonal linearly independent isotropic roots in Δ_1 . One can see that the defect of \mathfrak{g} in these cases is equal to the dimension of maximal isotropic subspace in $\text{span}_{\mathbb{R}} \Delta$.

For finite-dimensional Kac–Moody Lie superalgebras, the defect has the following geometric interpretation: it is given by the dimension of the geometric quotient of \mathfrak{g}_1 by G_0 . In fact, in these cases, $S(\mathfrak{g}_1)^{G_0}$ is a polynomial algebra, and the number of generators is given by the defect. For $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{p}(n)$, $S(\mathfrak{g}_1)^{G_0}$ is again a polynomial algebra, except the number of generators for $\mathfrak{q}(n)$ is n while the number of generators of $\mathfrak{p}(n)$ is $\lfloor \frac{n}{2} \rfloor$.

In², a cohomological definition of support varieties was given using the relative Ext functor. There, they define the defect of a \mathfrak{g} to be the dimension of $\text{Ext}_{\mathcal{F}(\mathfrak{g})}^{\bullet}(\mathbb{C}, \mathbb{C}) \cong S(\mathfrak{g}_1)^{G_0}$.

4.3 The Lie Superalgebras $\mathfrak{p}(n)$ and $\mathfrak{q}(n)$

4.3.1 $\mathfrak{p}(n)$

The periplectic Lie superalgebra $\mathfrak{p}(n)$ and the queer Lie superalgebra $\mathfrak{q}(n)$ are quasireductive Lie superalgebras and can be realized as subalgebras of $\mathfrak{gl}(n|n)$.

With respect to a suitable basis, the periplectic Lie superalgebra $\mathfrak{g} = \mathfrak{p}(n)$ consists of block matrices of the form

$$\left(\begin{array}{c|c} A & B \\ \hline C & -A^t \end{array} \right),$$

where B is symmetric, C is skew-symmetric, and $\mathfrak{t} := \mathfrak{h}_0$ is the diagonal Cartan subalgebra of $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$. Then \mathfrak{g} has a \mathbb{Z} -grading $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$ such that $\mathfrak{g}^0 = \mathfrak{g}_0$, $\mathfrak{g}_1 = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$, and corresponding sets of roots $\Delta_0 = \Delta(\mathfrak{g}^0) = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}$, $\Delta(\mathfrak{g}^{-1}) = \{-\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq n\}$, and $\Delta(\mathfrak{g}^1) = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i \leq j \leq n\}$. Imposing the additional condition that $\text{tr}A = 0$ defines the Lie superalgebra $\mathfrak{sp}(n)$, which is simple when $n \geq 3$; however, $\mathfrak{sp}(n)$ does not admit a nondegenerate (even or odd) invariant bilinear form.

4.3.2 $\mathfrak{q}(n)$

With respect to an appropriate basis of $\mathbb{C}^{n|n}$, the queer Lie superalgebra $\mathfrak{g} = \mathfrak{q}(n)$ consists of block matrices of the form

$$T_{A,B} := \left(\begin{array}{c|c} A & B \\ \hline B & A \end{array} \right), \tag{7}$$

such that $\mathfrak{t} := \mathfrak{h}_0$ is the diagonal Cartan subalgebra of $\mathfrak{g}_0 \cong \mathfrak{gl}(n)$. The set of roots for $\mathfrak{q}(n)$ is $\Delta = \{\pm(\varepsilon_i - \varepsilon_j) \mid 1 \leq i < j \leq n\}$, and each root $\alpha \in \Delta$ is both even and odd since $\dim(\mathfrak{g}_\alpha)_0 = \dim(\mathfrak{g}_\alpha)_1 = 1$. Imposing the additional condition $\text{tr}B = 0$ defines the Lie superalgebra $\mathfrak{sq}(n)$, and since $\text{Id} \in \mathfrak{sq}(n)$ we can also define the Lie superalgebra $\mathfrak{psq}(n) := \mathfrak{sq}(n)/\langle \text{Id} \rangle$, which is simple for $n \geq 3$.

4.4 Table of Defects

The defect of a classical Lie superalgebra is given in the following table.

\mathfrak{g}	Defect
$\mathfrak{gl}(m n)$	$\min\{m, n\}$
$\mathfrak{sl}(m n), m \neq n$	$\min\{m, n\}$
$\mathfrak{osp}(m 2n)$	$\min\{\lfloor m/2 \rfloor, n\}$
$\mathfrak{p}(n)$	n
$\mathfrak{q}(n)$	$\lfloor n/2 \rfloor$
$D(2 1; a)$	1
$F(4)$	1
$G(3)$	1

4.5 Description and Realization of \mathfrak{g}_x in \mathfrak{g}

Suppose \mathfrak{g} is classical Lie superalgebra. Let $A = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}$ be non-empty, and take $x = x_{\beta_1} + \dots + x_{\beta_k}$, where each $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$ is nonzero. If $\mathfrak{g} = \mathfrak{p}(n)$, let s denote the number of roots $\beta \in A$ of the form $2\varepsilon_j$. Note that by Sect. 5.1, all elements of X are G_0 -conjugate to an element of this form.

The following table describes \mathfrak{g}_x .

\mathfrak{g}	\mathfrak{g}_x
$\mathfrak{gl}(m n)$	$\mathfrak{gl}(m-k n-k)$
$\mathfrak{sl}(m n), m \neq n$	$\mathfrak{sl}(m-k n-k)$
$\mathfrak{osp}(m 2n)$	$\mathfrak{osp}(m-2k 2n-2k)$
$\mathfrak{p}(n)$	$\mathfrak{p}(n-(2k-s))$
$\mathfrak{q}(n)$	$\mathfrak{q}(n-2k)$
$D(2 1; a)$	\mathbb{C}
$F(4)$	$\mathfrak{sl}(3)$
$G(3)$	$\mathfrak{sl}(2)$

Note that in the last three rows the defect of \mathfrak{g} is 1, so $k = 1$.

The functor DS_x reduces the defect of \mathfrak{g} by a non-negative integer which is called the *rank of x* , that is, $\text{rk}x := \text{def} \mathfrak{g} - \text{def} \mathfrak{g}_x$.

Definition 4.2 Let \mathfrak{g} be one of the Lie superalgebras listed in the above table, and let $x \in X$. Then the rank of x is as follows:

- if $x = 0$, then $\text{rk}x = 0$;
- if $x \neq 0$ and \mathfrak{g} is an exceptional Lie superalgebra, then $\text{rk}x = k = 1$;
- if \mathfrak{g} is not exceptional and $\mathfrak{g} \neq \mathfrak{p}(n)$, then $\text{rk}x = k$;
- if $\mathfrak{g} = \mathfrak{p}(n)$, then $\text{rk}x = 2k - s$.

Remark 4.3 Note that if $\mathfrak{g} \neq \mathfrak{p}(n)$ then $\text{rk}x = k$, the size of \mathcal{S} .

Remark 4.4 For $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$ and $\mathfrak{p}(n)$, we observe that $\text{rk}x$ is given by the rank of x as a linear operator acting on the standard representation. For $\mathfrak{osp}(m|2n)$ and $\mathfrak{q}(n)$, we have that $\text{rk}x$ is half the rank of x as linear operator in the standard representation.

Let \mathfrak{g} be a classical Lie superalgebra with $\mathfrak{g} \neq \mathfrak{p}(n)$, and let $x \in X$. We now explain how to realize \mathfrak{g}_x as a subalgebra of \mathfrak{g} , in such a way that $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$. For $\mathfrak{p}(n)$ we will also have such an embedding, but the construction is more ad hoc, so we state it separately.

Thus, we assume $\mathfrak{g} \neq \mathfrak{p}(n)$, and as above, we let $A = \{\beta_1, \dots, \beta_k\} \in \mathcal{S}$ and $x = x_{\beta_1} + \dots + x_{\beta_k}$ for some nonzero $x_{\beta_i} \in (\mathfrak{g}_{\beta_i})_1$. Let $y = y_{\beta_1} + \dots + y_{\beta_k}$ for some nonzero $y_{\beta_i} \in (\mathfrak{g}_{-\beta_i})_1$, and set $h_{\beta_i} = [x_{\beta_i}, y_{\beta_i}]$, $h = [x, y]$. Clearly, $h = h_{\beta_1} + \dots + h_{\beta_k}$, and h, x, y generate an $\mathfrak{sl}(1|1)$ -subalgebra in \mathfrak{g} . We choose the y_{β_i} such that h is generic, meaning that it satisfies

$$\ker \text{ad}_h = \ker \text{ad}_{h_{\beta_1}} \cap \dots \cap \ker \text{ad}_{h_{\beta_k}}.$$

We may decompose \mathfrak{g} with respect to the adjoint action of h giving $\mathfrak{g} = \bigoplus_{\mu} \mathfrak{g}^{\mu}$, where $\mathfrak{g}^{\mu} = \{g \in \mathfrak{g} \mid [h, g] = \mu(h)g\}$. In particular, $\mathfrak{g}^0 = \text{Ker ad}_h$, and this is in fact a decomposition of $\mathfrak{sl}(1|1)$ -modules.

For each $\beta_i \in A$, set $\mathfrak{h}_{\beta_i} = [(\mathfrak{g}_{\beta_i})_1, (\mathfrak{g}_{-\beta_i})_1]$. Set

$$A^{\perp} = \text{Ker} \beta_1^{\vee} \cap \dots \cap \text{Ker} \beta_k^{\vee} \subset \mathfrak{h}^*. \tag{8}$$

Define

$$\begin{aligned} \mathfrak{h}_A &:= \mathfrak{h}_{\beta_1} \oplus \dots \oplus \mathfrak{h}_{\beta_k}, \\ \mathfrak{h}_A^{\perp} &:= \text{Ker} \beta_1 \cap \dots \cap \text{Ker} \beta_k, \\ \Delta_x &:= A^{\perp} \cap (\Delta \setminus (A \cup -A)). \end{aligned}$$

We have the following.

Proposition 4.5 Suppose $\mathfrak{g} \neq \mathfrak{p}(n)$ is a classical Lie superalgebra, and let $A \in \mathcal{S}$ with corresponding $x \in X$. Then \mathfrak{g}_x is isomorphic to the root subalgebra generated by $\{\mathfrak{g}_{\alpha}\}_{\alpha \in \Delta_x}$ and a splitting of $\mathfrak{h}_x \cong \mathfrak{h}_A^{\perp} / \mathfrak{h}_A$ of \mathfrak{h}_A , and this splitting \mathfrak{h}_x will define a Cartan subalgebra of \mathfrak{g}_x . If we identify \mathfrak{g}_x with its image in \mathfrak{g} we have $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$.

Proof First, note that there is an isomorphism

$$\mathfrak{g}_x \cong (\mathfrak{g}^0 \cap \mathfrak{g}^x) / (\mathfrak{g}^0 \cap [x, \mathfrak{g}]).$$

We observe that

$$\begin{aligned} \mathfrak{g}^0 \cap \mathfrak{g}^x &= \mathfrak{h}_A^{\perp} \oplus \bigoplus_{\alpha \in A^{\perp} \cap (\Delta \setminus -A)} \mathfrak{g}_{\alpha}, \mathfrak{g}^0 \cap [x, \mathfrak{g}] \\ &= \mathfrak{h}_A \oplus \mathfrak{g}_{\beta_1} \oplus \dots \oplus \mathfrak{g}_{\beta_k}. \end{aligned}$$

The above equalities follow from the representation theory of $\mathfrak{sl}(1|1)$. Now it is clear that one can choose \mathfrak{h}_x in such a way that $\mathfrak{g}_x = (\mathfrak{h}_x \oplus \bigoplus_{\alpha \in \Delta_x} \mathfrak{g}_{\alpha})$ is a subalgebra of \mathfrak{g} . \square

Separately, we have

Proposition 4.6 Let $\mathfrak{g} = \mathfrak{p}(n)$ and choose $x \in X$ of rank r arising from a subset $A \in \mathcal{S}$. Then A lies in the span of $\epsilon_{i_1}, \dots, \epsilon_{i_r}$ for a unique set of indices $I = \{i_1, \dots, i_r\} \subseteq [n] := \{1, \dots, n\}$. Write $\mathfrak{p}(n-r)$ for the root subalgebra corresponding to the weights ϵ_i for $i \in [n] \setminus I$. Then we have a natural isomorphism $\mathfrak{g}_x \cong \mathfrak{p}(n-r)$, and $\mathfrak{g}^x = \mathfrak{p}(n-r) \ltimes [x, \mathfrak{g}]$.

Proof Straightforward check. \square

Remark 4.7 Propositions 4.5 and 4.6 have the following useful application: if \mathfrak{t} acts diagonally

on N and $\Omega(N) = \{\nu \in \mathfrak{t}^* \mid N_\nu \neq 0\}$, then for x as in the propositions one has

$$\Omega(DS_x(N)) \subset (\Omega(DS_x(N)))|_{\mathfrak{t}_x}. \tag{9}$$

Remark 4.8 By Theorem 5.1 and Proposition 5.2, if \mathfrak{g} is classical then all G_0 -orbits on X contain an element x arising from an iso set $A \in \mathcal{S}$. Thus Propositions 4.5 and 4.6 imply that our table in Sect. 4.5 is accurate.

5 Geometry of X for Classical Lie Superalgebras

In this section, we study the G_0 -orbits on X , and for $\mathfrak{g} \neq \mathfrak{p}(n)$, we prove an important bijection between the G_0 -orbits on X and the W -orbits on \mathcal{S} . We also describe the orbits in the $\mathfrak{p}(n)$ case. Next, we study the stabilizer and normalizer of x in G_0 . Finally, we give a dimension formula for the G_0 -orbits on X for basic classical Lie superalgebras.

5.1 G_0 -Orbits on X

Theorem 5.1 *Suppose \mathfrak{g} is a basic classical Lie superalgebra or $\mathfrak{g} = \mathfrak{q}(n)$. Then there are finitely many G_0 -orbits on X , and these orbits are in one-to-one correspondence with W -orbits in \mathcal{S} via the map*

$$\Phi : \mathcal{S} \rightarrow X/G_0 \tag{10}$$

defined by

$$A = \{\beta_1, \dots, \beta_k\} \mapsto G_0x,$$

where $x = x_{\beta_1} + \dots + x_{\beta_k}$ for some nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$.

Proof To see that $\Phi(A)$ does not depend on a choice of x_{β_i} note that since β_1, \dots, β_k are linearly independent, for any other choice

$$x' = \sum x'_{\beta_i} = \sum c_i x_{\beta_i}$$

there is $h \in \mathfrak{h}$ such that $c_i = e^{\beta_i(h)}$ and, therefore,

$$x' = \exp(\text{ad}_h)(x).$$

If $B = w(A)$ for some $w \in W$, then clearly $\Phi(B)$ and $\Phi(A)$ belong to the same orbit. Therefore, Φ induces the map $\overline{\Phi} : \mathcal{S}/W \rightarrow X/G_0$. We check case by case that $\overline{\Phi}$ is injective and surjective.

If \mathfrak{g} is $\mathfrak{sl}(m|n)$ or $\mathfrak{gl}(n|n)$, \mathfrak{g} has a natural \mathbb{Z} grading $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$ such that

$\mathfrak{g}_0 = \mathfrak{g}^0$, $\mathfrak{g}_1 = \mathfrak{g}^1 \oplus \mathfrak{g}^{-1}$. The orbits of W on \mathcal{S} are enumerated by the pairs of numbers (p, q) , where $p = |A \cap \Delta(\mathfrak{g}^1)|$, $q = |A \cap \Delta(\mathfrak{g}^{-1})|$. The orbits of G_0 on X are enumerated by the same pairs of numbers (p, q) in the following way. If $x = x^+ + x^-$, where $x^\pm \in \mathfrak{g}^{(\pm 1)}$, then $p = \text{rank}(x^+)$, $q = \text{rank}(x^-)$. We can see by the construction of $\overline{\Phi}$, that $\overline{\Phi}$ maps (p, q) -orbit on \mathcal{S} to the (p, q) -orbit on X .

Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$. If $m = 2l + 1$ or $m = 2l$ with $l > n$, then the W -orbits on \mathcal{S} are in one-to-one correspondence with $\{0, 1, 2, \dots, \min(l, n)\}$. Namely, A and B are on the same orbit if they have the same number of elements. As it was shown in³³, X can be identified with the set of all linear maps $x : \mathbb{C}^m \rightarrow \mathbb{C}^{2n}$, such that $\text{Im}x$ is an isotropic subspace in \mathbb{C}^{2n} and $\text{Im}x^*$ is an isotropic subspace in \mathbb{C}^m . Furthermore, $x, y \in X$ belong to the same G_0 -orbit iff $\text{rank}(x) = \text{rank}(y)$. One can see that $\text{rank} \Phi(A) = |A|$.

Now let $\mathfrak{g} = \mathfrak{osp}(2l|2n)$, where $l \leq n$. If $A, B \in \mathcal{S}$ and $|A| = |B| < l$, then A and B are on the same W -orbit. In the same way if $\text{rank}(x) = \text{rank}(y) < l$, then x and y are on the same G_0 -orbit. However, the set of all $x \in \mathfrak{g}_1$ of maximal rank splits into two orbits, since the Grassmannian of maximal isotropic subspaces in \mathbb{C}^{2l} has two connected components. In the same way, \mathcal{S}_l splits into two W -orbits. Hence, in this case again $\overline{\Phi}$ is a bijection.

If \mathfrak{g} is one of exceptional Lie superalgebras $D(2|1; a)$, $G(3)$ or $F(4)$, then the direct calculation shows that X has two G_0 -orbits: $\{0\}$ and the orbit of a highest vector in \mathfrak{g}_1 . The set \mathcal{S} also consists of two W -orbits: \emptyset and the set of all isotropic roots in Δ .

Finally, let $\mathfrak{g} = \mathfrak{q}(n)$. Then \mathfrak{g} is isomorphic to the subalgebra of $\mathfrak{gl}(n|n)$ consisting of block matrices of the form $T_{A,B}$ in (7) and $X = \{T_{0,B} \mid B^2 = 0\}$. So G_0 is isomorphic to $GL(n)$ and acts by conjugation on B . Thus the G_0 -orbits correspond to Jordan forms. If for $r = 0, 1, \dots, [\frac{n}{2}]$, we set $\mathcal{S}_r := \{\beta_{n-1-2i}\}_{i=0}^{r-1}$ and fix an element $x_r \in \mathcal{S}_r$ with $\text{supp}(x_r) = \mathcal{S}_r$ ($x_0 := 0$), then the elements $x_0, x_1, \dots, x_{[\frac{n}{2}]}$ form a set of representatives for G_0 -orbits in X . \square

Theorem 5.1 does not hold for $\mathfrak{p}(n)$; however, we have the following proposition, whose proof is an exercise in linear algebra which we omit.

Proposition 5.2 *For $\mathfrak{g} = \mathfrak{p}(n)$, the G_0 -orbits on X are indexed by a pair of nonnegative integers (r, s) such that $r + 2s \leq n$. An explicit representative of the orbit labeled by (r, s) is given by*

$$x = x_{2\epsilon_1} + \cdots + x_{2\epsilon_r} + x_{-\epsilon_{r+1}-\epsilon_{r+2}} + \cdots + x_{-\epsilon_{r+2s-1}-\epsilon_{r+2s}}$$

Remark 5.3 Note that for a finite-dimensional Kac–Moody superalgebra \mathfrak{g} the representation of G_0 in \mathfrak{g}_1 is symplectic and multiplicity free (see⁴⁷). The cone X is the preimage of 0 under the moment map $\mathfrak{g}_1 \rightarrow \mathfrak{g}_0^*$.

5.2 The Stabilizer and Normalizer of x in G_0

Lemma 5.4 *Let \mathfrak{g} be a basic classical Lie superalgebra. Let $x \in X$. The stabilizer G_0^x of x in G_0 is connected. Furthermore, G_0^x is a semidirect product of a reductive group K and the normal unipotents subgroup U with Lie algebras $(\mathfrak{g}_x)_0$ and $[x, \mathfrak{g}_1]$, respectively.*

Proof The second assertion follows from Proposition 4.5. To check the connectedness we use the explicit construction of orbits given in the proof of Theorem 5.1.

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$. Then $G_0 = GL(m) \times GL(n)$, consider the parabolic subgroups $P_1 \subset GL(m)$ which stabilizes the flag $\text{Im}x^+ \subset \text{Ker}x^-$ and $P_2 \subset GL(n)$ which stabilizes the flag $\text{Im}x^- \subset \text{Ker}x^+$. The Levi subgroup of K_1 of P_1 is isomorphic to $GL(p) \times GL(q) \times GL(m - k)$ and the Levi subgroup of K_2 of P_2 is isomorphic to $GL(p) \times GL(q) \times GL(n - k)$. Let $K \simeq K_1 \times K_2$ is the subgroup isomorphic to $GL(p) \times GL(q) \times GL(m - k) \times GL(n - k)$, where $GL(p)$ and $GL(q)$ are embedded diagonally. Then $G_0^x = K \rtimes U$, where U is the unipotent normal subgroup of $P_1 \times P_2$.

Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$. Then $G_0 = SO(m) \times SP(2n)$, consider the parabolic subgroups $P_1 \subset SO(m)$ which stabilizes $\text{Im}x^*$ and $P_2 \subset SP(n)$ which stabilizes $\text{Im}x$. The Levi subgroup of K_1 of P_1 is isomorphic to $GL(k) \times SO(m - 2k)$ and the Levi subgroup of K_2 of P_2 is isomorphic to $GL(k) \times SP(2n - 2k)$. Let $K \simeq K_1 \times K_2$ is the subgroup isomorphic to $GL(k) \times SO(m - 2k) \times SP(2n - 2k)$ where $GL(k)$ is embedded diagonally. Then $G_0^x = K \rtimes U$, where U is the unipotent normal subgroup of $P_1 \times P_2$.

In all exceptional cases x is a highest weight vector in \mathfrak{g}_1 and G_0^x is a subgroup of codimension 1 in the parabolic subgroup $P \subset G_0$, the

latter is the stabilizer of $\mathbb{C}x$ in the projectivization of \mathfrak{g}_1 . \square

Remark 5.5 It follows from above proof that there exists a parabolic subgroup $P \subset G_0$ such that G_0^x is a subgroup of P and the maximal normal unipotent subgroup of G_0^x is equal to that of P .

Remark 5.6 For the $\mathfrak{q}(n)$ it remains true that G_0^x is connected; this follows from a result of Springer and Steinberg (see Chpt. IV, Sec. 1.7 of⁵⁹).

We write N_0^x for the normalizer of x in G_0 .

Corollary 5.7 *For \mathfrak{g} basic classical and for $\mathfrak{g} = \mathfrak{q}(n)$, N_0^x is connected.*

Proof We have an exact sequence

$$1 \rightarrow G_0^x \rightarrow N_0^x \xrightarrow{\alpha} \mathbb{G}_m \rightarrow 1,$$

where \mathbb{G}_m is the one-dimensional torus and $\alpha(g) = (g \cdot x)/x$, where $g \cdot x$ is the action of g on x . By a case-by-case check, the map α is always surjective and split. We may now use Lemma 5.4. \square

Remark 5.8 For $\lambda \in \mathbb{C}^\times$, we have an equality of functors $DS_x = DS_{\lambda x}$. It follows that N_0^x acts naturally on the functor DS_x . We have shown that

$$N_0^x = G_0^x \rtimes \mathbb{G}_m.$$

For \mathfrak{g} basic classical or $\mathfrak{q}(n)$, we have shown G_0^x is connected, and thus its symmetries are encompassed in $(\mathfrak{g}_x)_0$. It follows that the only additional symmetries we obtain in this fashion are from the extra action of \mathbb{G}_m .

Remark 5.9 We note that for the $\mathfrak{g} = \mathfrak{p}(n)$, Lemma 5.4 and Corollary 5.7 are not true for all choices of x . Issues arise due to the orthogonal group being disconnected and the lack of a splitting for α in general.

5.3 Dimension of the G_0 -Orbits on X

Throughout the rest of Sect. 5, we assume that \mathfrak{g} is a basic classical Lie superalgebra. We recall the notation $\Phi : \mathcal{S} \rightarrow X/G_0$ introduced in (10).

Using the explicit description of G_0 -orbits on X and the description of root systems, which can be found in⁴³, one can check the following statements case by case. We omit this computation here.

Lemma 5.10 *Let $A, B \in \mathcal{S}$.*

1. *If $\alpha \in \Delta$ is a linear combination of roots from A , then $\alpha \in A \cup -A$;*
2. *If $|A| \leq |B|$, then there exists $w \in W$ such that $w(A) \subset B \cup -B$;*
3. *$\Phi(A)$ lies in the closure of $\Phi(B)$ iff $w(A) \subset B$ for some $w \in W$.*

Recall the definition of A^\perp from (8). In the basic classical case, A^\perp is the set of all weights orthogonal to A with respect to the standard form on \mathfrak{h}^* .

Theorem 5.11 *Let $A \in \mathcal{S}$. Then $\dim\Phi(A) = \frac{|\Delta_1 \setminus A^\perp|}{2} + |A|$.*

Proof Let $A = \{\beta_1, \dots, \beta_k\}$, $x = x_{\beta_1} + \dots + x_{\beta_k}$ for some choice of $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, $y = y_{\beta_1} + \dots + y_{\beta_k}$ for some $y_{\beta_i} \in \mathfrak{g}_{-\beta_i}$. Let $h = [x, y]$, $h_{\beta_i} = [x_{\beta_i}, y_{\beta_i}]$. Clearly, $h = h_{\beta_1} + \dots + h_{\beta_k}$ and h, x, y generate an $\mathfrak{sl}(1|1)$ -subalgebra in \mathfrak{g} . As a module over this subalgebra \mathfrak{g} has a decomposition

$$\mathfrak{g} = \bigoplus_{\mu} \mathfrak{g}^{\mu},$$

where

$$\mathfrak{g}^{\mu} = \{g \in \mathfrak{g} \mid [h, g] = \mu(h)g\}.$$

Note that

$$\dim[\mathfrak{g}, x] = \sum_{\mu} \dim[\mathfrak{g}^{\mu}, x],$$

and from the description of irreducible $\mathfrak{sl}(1|1)$ -modules for $\mu \neq 0$

$$\dim[\mathfrak{g}^{\mu}, x] = \frac{\dim \mathfrak{g}^{\mu}}{2}.$$

On the other hand, for $\mu \neq 0$, $\text{sdim} \mathfrak{g}^{\mu} = 0$ and, therefore,

$$\dim \mathfrak{g}^{\mu} = 2 \dim \mathfrak{g}_1^{\mu}.$$

Observe that for a generic choice of $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$, $\mathfrak{g}_{\alpha} \subset \mathfrak{g}^0$ iff $(\alpha, \beta_i) = 0$ for all $i \leq k$. Indeed, for generic choice of x_{β_i} the condition $\alpha(h) = 0$ implies $\alpha(h_{\beta_i}) = 0$ for all i , and therefore, $(\alpha, \beta_i) = 0$ for all i . Hence,

$$\bigoplus_{\mu \neq 0} \mathfrak{g}_1^{\mu} = \bigoplus_{\alpha \in \Delta_1 \setminus A^\perp} \mathfrak{g}_{\alpha}$$

and

$$\sum_{\mu \neq 0} \dim[\mathfrak{g}^{\mu}, x] = \sum_{\mu \neq 0} \dim \mathfrak{g}_1^{\mu} = |\Delta_1 \setminus A^\perp|.$$

To calculate $\dim[\mathfrak{g}^0, x]$ note that

$$\mathfrak{g}^0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \cap A^\perp} \mathfrak{g}_{\alpha}.$$

We claim that

$$[\mathfrak{g}^0, x] = \bigoplus_{i=1}^k \mathbb{C}h_{\beta_i} \oplus \bigoplus_{i=1}^k \mathfrak{g}_{\beta_i};$$

hence, $\dim[\mathfrak{g}^0, x] = 2k$. Indeed, if $(\alpha, \beta_i) = 0, \alpha \neq \pm \beta_i$ then $\alpha \pm \beta_i \notin \Delta$. Therefore, $[x, \mathfrak{g}_{\alpha}] = 0$ for any $\alpha \in \Delta \cap A^\perp, \alpha \neq -\beta_i$. Furthermore, $[x, \mathfrak{g}_{-\beta_i}] = \mathbb{C}h_{\beta_i}$ and $[x, \mathfrak{h}] = \bigoplus_{i=1}^k \mathfrak{g}_{\beta_i}$. Thus, we obtain

$$\dim[\mathfrak{g}, x] = |\Delta_1 \setminus A^\perp| + 2k. \tag{11}$$

Now the statement will follow from the lemma.

Lemma 5.12 $\text{sdim}[\mathfrak{g}, x] = 0$.

Proof Define an odd skew-symmetric form ω on \mathfrak{g} by

$$\omega(y, z) := (x, [y, z]).$$

Obviously the kernel of ω coincides with the centralizer \mathfrak{g}^x . Thus, ω induces a non-degenerate odd skew-symmetric form on the quotient $\mathfrak{g}/\mathfrak{g}^x$. Hence, $\text{sdim} \mathfrak{g}/\mathfrak{g}^x = 0$. But $[\mathfrak{g}, x] \cong \Pi(\mathfrak{g}/\mathfrak{g}^x)$, which implies the lemma. \square

Now Lemma 5.12 implies $\dim[\mathfrak{g}_0, x] = \frac{1}{2} \dim[\mathfrak{g}, x]$. Since $\dim G_0 x = \dim[\mathfrak{g}_0, x]$, Theorem 5.11 follows from (11). \square

Theorem 5.11 has the following corollaries.

Corollary 5.13 *If $|A| = |B|$, then $\dim\Phi(A) = \dim\Phi(B)$.*

Proof Follows from Theorem 5.1 and Lemma 5.10 (2). \square

Corollary 5.14 *Let d be the defect of \mathfrak{g} . Then the irreducible components of X are in bijection with W -orbits on $\mathcal{S}_d := \{A \in \mathcal{S} \mid |A| = d\}$. If all odd roots of \mathfrak{g} are isotropic, then the dimension of each component is equal to $\frac{\dim \mathfrak{g}_1}{2} = \frac{|\Delta_1|}{2}$.*

Proof As follows from Theorem 5.1 and Lemma 5.10 (3), each irreducible component is the closure of $\Phi(A)$ for a maximal $A \in \mathcal{S}$. By Lemma 5.10 (2) $|A| = d$. Hence the first statement. Theorem 5.11 immediately implies the statement about dimension. \square

Corollary 5.15 *If all odd roots of \mathfrak{g} are isotropic, then the codimension of $\Phi(A)$ in X equals $\frac{|\Delta_1 \cap A^\perp|}{2} - |A|$.*

Proof The codimension of $\Phi(A)$ in X equals $\dim X - \dim \Phi(A)$. Using Theorem 5.11 and Corollary 5.14 we obtain

$$\begin{aligned} \text{codim} \Phi(A) &= \frac{|\Delta_1| - |\Delta_1 \setminus A^\perp|}{2} - |A| \\ &= \frac{|\Delta_1 \cap A^\perp|}{2} - |A|. \end{aligned}$$

□

Recall that $\mathfrak{g}_x = \mathfrak{g}^x/[x, \mathfrak{g}]$ and M_x is a \mathfrak{g}_x -module, see Lemma 2.2.

Lemma 5.16 *Let \mathfrak{m}^\perp denote the orthogonal complement to \mathfrak{m} with respect to the invariant form on \mathfrak{g} . Then $[x, \mathfrak{g}]^\perp = \mathfrak{g}^x$.*

Proof Let $u \in \mathfrak{g}^x, v \in [x, \mathfrak{g}]$. Then $v = [x, z]$ and

$$[u, [x, z]] = (-1)^{p(u)} [x, [u, z]] \in [x, \mathfrak{g}].$$

Now the statement follows from the identity

$$(u, [x, z]) = ([u, x], z).$$

□

6 Central Characters and Atypicality for Classical $\mathfrak{g} \neq \mathfrak{p}(n)$

Throughout this section, \mathfrak{g} denotes a basic classical Lie superalgebra or $\mathfrak{g} = \mathfrak{q}(n)$. We define the notion of atypicality for a central character, and see how it is affected by the DS functor. Furthermore, we describe the associated variety of an irreducible module in terms of its degree of atypicality.

6.1 The Weyl Group and Weyl Vector

Let us fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ by choosing a decomposition $\Delta = \Delta^+ \cup \Delta^-$. Note that in general this choice is not unique but our consideration will not depend on it. Later we will use different Borel subalgebras in some proofs. Set

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha - \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

Define the shifted action of W on \mathfrak{h}^* by

$$w \cdot \lambda := w(\lambda + \rho) - \rho.$$

Note that for $\mathfrak{g} = \mathfrak{q}(n), \rho = 0$, so there is no shift in the W -action.

For \mathfrak{g} basic classical, recall that in Sect. 4.2 we defined for each $\alpha \in \Delta_1$ a coroot $\alpha^\vee \in \mathfrak{t} = \mathfrak{h}_0$. In this section, for $\mathfrak{g} = \mathfrak{q}(n)$ we will denote by α^\vee a

non-zero element of $[(\mathfrak{g}_\alpha)_1, (\mathfrak{g}_{-\alpha})_1]$. Notice that in the basic classical cases we have $w\alpha^\vee = (w\alpha)^\vee$ for every $w \in W$ (under the non-shifted action); for $\mathfrak{g} = \mathfrak{q}(n)$ we impose this condition on the set of α^\vee . We say that an iso-set $A \subset \Delta_1$ is orthogonal to μ if $\mu(\alpha^\vee) = 0$ for each $\alpha \in A$, and we write $A \subset \mu^\perp$ and $\mu \in A^\perp$. Note that this is compatible with our definition of A^\perp in (8).

6.2 Central Characters

Recall that $Z(\mathfrak{g})$ denotes the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. For $\lambda \in \mathfrak{t}^*$ and chosen, fixed Borel subalgebra \mathfrak{b} , we denote by $M(\lambda)$ the Verma module of highest λ and by $L(\lambda)$ the unique irreducible quotient of $M(\lambda)$. We say that $\lambda \in \mathfrak{t}^*$ is *dominant* if $L(\lambda)$ is finite-dimensional. One can see that any $z \in Z(\mathfrak{g})$ acts as a scalar $\chi_\lambda(z)$ on $M(\lambda)$ and $L(\lambda)$. Therefore $\lambda \in \mathfrak{t}^*$ defines a central character $\chi_\lambda : Z(\mathfrak{g}) \rightarrow \mathbb{C}$. We emphasize that χ_λ depends also on the choice of Borel subalgebra. For a central character χ , let

$$\mathfrak{t}_\chi^* = \{\mu \in \mathfrak{t}^* \mid \chi_\mu = \chi\}.$$

For every $\lambda \in \mathfrak{t}^*$, set

$$\mathcal{S}_\lambda := \{A \in \mathcal{S} \mid A \subset (\lambda + \rho)^\perp\}.$$

Lemma 6.1 *Let $\chi = \chi_\lambda, A \in \mathcal{S}_\lambda$ be maximal. Then*

$$\mathfrak{t}_\chi^* = \bigcup_{w \in W} w \cdot (\lambda + \text{span}A).$$

Proof This easily follows from the description of $Z(\mathfrak{g})$ formulated in⁴⁴ and proven in^{23,57}. □

We call λ *regular* if there is a unique maximal $A \in \mathcal{S}_\lambda$. For every χ , there exists a regular $\lambda \in \mathfrak{t}_\chi^*$.

6.3 Degree of Atypicality

We define the degree of atypicality following⁴⁵. For a central character χ set

$$\mathcal{S}_\chi = \bigcup_{\lambda \in \mathfrak{t}_\chi^*} \mathcal{S}_\lambda.$$

Lemma 6.2 *There exists a number k such that $\mathcal{S}_\chi = \bigcup_{i \leq k} \mathcal{S}_i$.*

Proof It follows easily from Lemma 6.1 that \mathcal{S}_χ is W -invariant. Furthermore, if $A \in \mathcal{S}_\chi$ and A' is obtained from A by multiplication of some roots in A by -1 , then $A' \in \mathcal{S}_\chi$. Hence, the statement follows from Lemma 5.10 (1) and (2), which also holds for $q(n)$. \square

The number k is called the *degree of atypicality* of χ . It is clear that the degree of atypicality of χ is not bigger than the defect of \mathfrak{g} . The degree of atypicality of a weight λ is by definition the degree of atypicality of χ_λ . If $k = 0$, then χ is called *typical*. We say a module is typical if it lies in $\bigoplus_{\chi \text{ typical}} \text{mod}_\chi(\mathfrak{g})$.

Let $X_k = \Phi(\mathcal{S}_k)$, \bar{X}_k denote the closure of X_k . Lemma 5.10 (3) implies that

$$\bar{X}_k = \bigcup_{i=0}^k X_i.$$

Theorem 6.3 *Let \mathfrak{g} be a basic classical Lie superalgebra or $q(n)$, and let M be a \mathfrak{g} -module which admits central character χ , with degree of atypicality of χ equal to k . Then we have $X_M \subset \bar{X}_k$.*

The proof of Theorem 6.3 will be given in Sect. 6.4.

Theorem 6.4 *Let \mathfrak{g} be a basic classical Lie superalgebra. For any integral dominant $\lambda \in \mathfrak{t}^*$ with degree of atypicality k , $X_{L(\lambda)} = \bar{X}_k$.*

This theorem is proven in⁵⁵ for the Lie superalgebras $\mathfrak{osp}(m|2n)$ and $\mathfrak{gl}(m|n)$. For exceptional Lie superalgebras it is a consequence of results in⁴⁸.

Remark 6.5 Theorem 6.4 is easy for typical λ since in this case $L(\lambda)$ is projective.

Remark 6.6 Theorem 6.4 fails for $\mathfrak{g} = q(n)$; indeed if we consider the irreducible $q(3)$ -module $L = \mathfrak{psq}(3)$, we have $X_L = \{0\}$. However, L has atypicality 1.

6.4 Proof of Theorem 6.3

We assume that \mathfrak{g} is as in Theorem 6.3. Recall that up to conjugacy, we may present $x \in X_k$ as $x = x_1 + \dots + x_k$, where x_i is a non-zero element of $(\mathfrak{g}_{\beta_i})_1$ for an odd root β_i . Here $A = \{\beta_1, \dots, \beta_k\}$ will be an iso-set. We begin with a lemma.

Lemma 6.7 *For suitable choices of Borel subalgebras $\mathfrak{b} \subseteq \mathfrak{g}$ and $\mathfrak{b}_x \subseteq \mathfrak{g}_x$, for each $\lambda' \in \mathfrak{t}_x^*$ there exists $\lambda \in \mathfrak{t}^*$ such that*

- $\lambda|_{\mathfrak{t}_x} = \lambda'$;
- $\text{atyp}\lambda = \text{atyp}\lambda' + k$;
- $[DS_x(L_{\mathfrak{g}}(\lambda)) : L_{\mathfrak{g}_x}(\lambda')] = 1$.

In particular, $\eta^(\chi_{\lambda'}) = \chi_\lambda$ and thus $\text{atyp}\chi_{\lambda'} = \text{atyp}\eta^*(\chi_{\lambda'}) - k$.*

Proof We can always choose a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ such that β_1, \dots, β_k are simple roots. Note that in this case $(\rho, \beta_i) = 0$ for all $i = 1, \dots, k$. Further recall from Proposition 4.5 that we may realize \mathfrak{g}_x in \mathfrak{g} such that \mathfrak{h}_x will be a subalgebra of \mathfrak{h} , and $\mathfrak{t}_x := (\mathfrak{h}_x)_0$ will lie in $\ker \beta_1 \cap \dots \cap \ker \beta_k$. Moreover, \mathfrak{g}_x will admit a natural Borel subalgebra $\mathfrak{b}_x \subseteq \mathfrak{b}$ containing \mathfrak{h}_x .

Let $\lambda \in \mathfrak{t}^*$ be a weight such that $\lambda|_{\mathfrak{t}_x} = \lambda'$ and $\lambda(\beta_i^\vee) = 0$ for all i . For $q(n)$, we strengthen our assumption on λ : we further require that $(\lambda, \epsilon_{i_1}) = (\lambda, \epsilon_{i_2}) = 0$, where $\beta_i = \epsilon_{i_1} - \epsilon_{i_2}$, for all i .

Now to prove our statement with this choice of λ , we observe that a nonzero highest weight vector $v_\lambda \in L_{\mathfrak{g}}(\lambda)_\lambda$ satisfies $xv_\lambda = 0$ and $v_\lambda \notin xL_{\mathfrak{g}}(\lambda)$. The former statement is obvious because it is a highest weight vector. For the latter statement, we show that $L_{\mathfrak{g}}(\lambda)_{\lambda-\beta_i} = 0$ for all i , which clearly is sufficient. In the basic classical case this follows from the fact that $(\lambda, \beta_i) = 0$ and β_i is simple. In the $q(n)$ case, the statement follows from the representation theory of the $q(2)$ -subalgebra associated with each simple root β_i , again using that the β_i are simple.

Now to finish the proof of Lemma 6.7, we observe that $\eta^*(\chi_{\lambda'}) = \chi_\lambda$ by Proposition 3.1, and that $\text{atyp}\lambda' = \text{atyp}\lambda - k$. \square

Lemma 6.7 implies Theorem 6.3 and the following.

Theorem 6.8 *Let M be a \mathfrak{g} -module that admits a central character with degree of atypicality k , and $x \in X_k$. Then M_x is a typical module. In particular, if M is semisimple over \mathfrak{g}_0 and M_x is finite dimensional, then M_x semi-simple over \mathfrak{g}_x .*

Proof We only need to prove the last assertion. For this, we use that $(\mathfrak{g}_x)_0$ is reductive, so it acts semisimply on M_x if and only if its centre does. But its centre lies in the even part of any Cartan subalgebra, whose action is induced by the action of the Cartan subalgebra of \mathfrak{g} on M . Thus, the

condition that \mathfrak{g}_0 acts semisimply on M , along with the typicality of M_x , ensures the semisimplicity of M_x . \square

Recall, from the notation of Sect. 4.5, the isomorphism $\mathfrak{h}_x \simeq \mathfrak{h}_A^\perp/\mathfrak{h}_A$, and set $\mathfrak{t}_x := (\mathfrak{h}_x)_0$. Then this isomorphism induces a canonical isomorphism of dual spaces $\mathfrak{t}_x^* \simeq A^\perp/\text{span}A$. Consider the natural projection $p_A : A^\perp \rightarrow \mathfrak{t}_x^*$. It follows immediately from Lemma 6.1 that $p_A(\lambda) = p_A(\nu)$ implies $\chi_\lambda = \chi_\nu$. Proposition 3.1 and Lemma 6.7 imply the following

Corollary 6.9 *If $\lambda \in A^\perp$ then $\chi_\lambda = \eta^*(\chi_{p_A(\lambda)})$.*

6.5 The Preimage of η^*

Now we will compute the preimage $(\eta^*)^{-1}(\chi)$ for any χ , showing in particular it is always finite of size one or two. Our description will use the involutions σ_x described in Sect. 3.1.

Define the following subgroup of the Weyl group W :

$$W_A = \{w \in W \mid w(A) \subset A \cup -A\}.$$

It is clear that A^\perp and \mathfrak{h}_A are W_A -stable. Hence, W_A acts in the natural way on \mathfrak{h}_x and \mathfrak{h}_x^* . By $W_{\mathfrak{g}_x}$ we denote the Weyl group of \mathfrak{g}_x viewed as a subalgebra of \mathfrak{g} . Obviously, $W_{\mathfrak{g}_x} \subset W_A$.

Lemma 6.10 *Fix $A \in \mathcal{S}_k$. Let $\chi = \chi_\lambda$ be a central character of atypicality degree k where $\lambda \in A^\perp$ is some regular weight. Then*

$$|(\eta^*)^{-1}(\chi)| = \frac{|W_A \cdot p_A(\lambda)|}{|W_{\mathfrak{g}_x} \cdot p_A(\lambda)|}.$$

Proof Lemma 6.1 implies the following equality

$$(\mathfrak{h}_x^*)_{reg} \cap A^\perp = W_A \cdot \lambda + \text{span}A.$$

The condition that $\eta^*(\chi_{p_A(\lambda)}) = \eta^*(\chi_{p_A(\nu)})$ for $\lambda, \nu \in (\mathfrak{h}_x^*)_{reg}$ is equivalent to $p_A(\nu) \in W_{\mathfrak{g}_x} \cdot p_A(\lambda)$. Hence, the statement. \square

Lemma 6.11 *Consider the action homomorphism $a : W_A \rightarrow \text{Aut}(\mathfrak{t}_x)$. Let $k = \text{rk}x$. If $\sigma_x = \text{id}$ (i.e., $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{q}(n), \mathfrak{osp}(2m+1|2n), G(3)$, or $\mathfrak{osp}(2m|2n)$ with $k = m$) then $a(W_A) = a(W_{\mathfrak{g}_x})$. If $\sigma_x \neq \text{Id}$ (i.e., $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ with $k < m, D(2|1; a)$ or $F(4)$), then $[a(W_A) : a(W_{\mathfrak{g}_x})] = 2$, and we have*

$$a(W_A) = a(W_{\mathfrak{g}_x}) \sqcup \sigma_x a(W_{\mathfrak{g}_x})$$

where by abuse of notation we also write σ_x for the involution induced on \mathfrak{t} by σ_x .

Proof If $\mathfrak{osp}(2m+1|2n)$ or $G(3)$, then $\mathfrak{g}_x = \mathfrak{osp}(2m+1-2k|2n-2k)$ or \mathfrak{sl}_2 , respectively. In both cases the automorphism group of the root system $\Delta(\mathfrak{g}_x)$ coincides with the Weyl group $W_{\mathfrak{g}_x}$. Since $a(W_A) \subset \text{Aut}\Delta(\mathfrak{g}_x)$, the statement follows. Similarly in the case $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and $k = m, \mathfrak{g}_x = \mathfrak{sp}(2(n-m))$ and $\text{Aut}\Delta(\mathfrak{g}_x) = W_{\mathfrak{g}_x}$.

For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{g}_x = \mathfrak{gl}(m-k|n-k)$, we get

$$W_A = S_k \times S_{m-k} \times S_{n-k}, \quad W_{\mathfrak{g}_x} = S_{m-k} \times S_{n-k}$$

and $\ker a = S_k$.

For $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{g}_x = \mathfrak{q}(n-2k)$ and we get

$$W_A = S_2^k \times S_{n-2k}, \quad W_{\mathfrak{g}_x} = S_{n-2k}$$

and $\ker a = S_2^k$.

If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$, with $k < m, D(2|1; a)$ or $F(4)$, then $\mathfrak{g}_x = \mathfrak{osp}((2m-2k)|(2n-2k)), \mathfrak{o}(2)$ or \mathfrak{sl}_3 . Then by direct computation $\text{Aut}\Delta(\mathfrak{g}_x)/W_{\mathfrak{g}_x} \cong \langle \sigma_x \rangle$, where for $\mathfrak{g} = D(2|1; a)$ we set $\text{Aut}\Delta(\mathfrak{g}_x) = \{\pm \text{Id}\}$. Further, by direct computation

$$a(W_A) = \text{Aut}\Delta(\mathfrak{g}_x).$$

\square

Theorem 6.12 *If $\mathfrak{g} \neq \mathfrak{osp}(2l|2n), F(4)$ or $D(2|1; a)$, then η^* is injective. If $\mathfrak{g} = \mathfrak{osp}(2l|2n), F(4)$, or $D(2|1; a)$, then a preimage of η^* has at most two elements.*

Proof In the case when $\text{rk}(x) = |A| = k$ equals the atypicality degree of χ , we have $(\eta^*)^{-1}(\chi)$ has at most two elements by Lemmas 6.10 and 6.11. If k is less than the atypicality degree of χ , then consider the embedding $A \subset B$ with $|B|$ equal to the degree of atypicality. Let $z = x + y$ with $y = \sum_{\beta \in B \setminus A} x_\beta$. Then we have $(\mathfrak{g}_x)_y = \mathfrak{g}_z$. The composed map

$$\eta_z^* : \text{Hom}(Z(\mathfrak{g}_z), \mathbb{C}) \xrightarrow{\eta_{z,x}^*} \text{Hom}(Z(\mathfrak{g}_x), \mathbb{C}) \xrightarrow{\eta^*} \text{Hom}(Z(\mathfrak{g}), \mathbb{C}).$$

Since the statement holds for η_z^* and for $\eta_{z,x}^*$, it holds for η^* . \square

7 Superdimensions and Supercharacters for Basic Classical Lie Superalgebras

In this section, \mathfrak{g} denotes a basic classical Lie superalgebra. We explore connections between the superdimension and supercharacter of a \mathfrak{g} -module M and of the corresponding associated variety X_M .

7.1 Superdimensions

Recall that $\text{sdim}M := \dim M_0 - \dim M_1$, and that by Lemma 2.5, $\text{sdim}M = \text{sdim}M_x$. So we have the following.

Lemma 7.1 *If $X_M \neq X$, then $\text{sdim}M = 0$. In particular, if a finite-dimensional module M admits a central character whose degree of atypicality is less than the defect of \mathfrak{g} , then $\text{sdim}M = 0$.*

Remark 7.2 In fact, Serganova proved a stronger statement, namely the Kac–Wakimoto conjecture: a simple finite-dimensional module over a basic classical Lie superalgebra has nonzero superdimension if and only if it has maximal degree of atypicality (i.e., equal to the defect of \mathfrak{g}). For a proof, see⁵⁵. A version of the Kac–Wakimoto conjecture for periplectic Lie superalgebras $\mathfrak{p}(n)$ was proven in¹⁸. Meanwhile, for $\mathfrak{q}(n)$ it is known that $\text{sdim}L = 0$ for all nontrivial finite-dimensional simple modules L , see⁸.

7.2 Supercharacters

For a finite-dimensional \mathfrak{g} -module M with weight decomposition $M = \bigoplus_{\mu \in \mathfrak{h}^*} M^\mu$, the supercharacter of M is defined to be

$$\text{sch}M = \sum_{\mu \in \mathfrak{h}^*} (\text{sdim} M^\mu) e^\mu. \tag{12}$$

The supercharacter $\text{sch}M$ is a W -invariant analytic function on \mathfrak{h} , so we will also write it as $\text{sch}_M(h)$, for $h \in \mathfrak{h}$. Then $\text{sch}_M(h) = \text{str}(e^h)$, and the Taylor series for sch_M at $h = 0$ is

$$\text{sch}_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

where $p_i(h)$ is a homogeneous polynomial of degree i on \mathfrak{h} . The order of sch_M at zero is by definition the minimal i such that $p_i \neq 0$.

Theorem 7.3 *Assume that all odd roots of \mathfrak{g} are isotropic. Let M be a finite-dimensional \mathfrak{g} -module, s be the codimension of X_M in X . The order of sch_M at zero is not less than s . Moreover, the polynomial*

$p_s(h)$ in Taylor series for sch_M is determined uniquely up to proportionality.

Proof The proof is based on the following lemma, the proof of this lemma is similar to the proof of Lemma 2.5 (6). We leave it to the reader.

Lemma 7.4 *Let $x \in X$, $h \in \mathfrak{h}_0$ and $[h, x] = 0$. Then $\text{Ker}x$ and xM are h -invariant and $\text{str}_M e^h = \text{str}_{M_x} e^h$.*

Now we proceed to the proof of the Theorem 7.3. If X_M contains an irreducible component of X , the statement is trivial since $s = 0$. Otherwise there exists k smaller than the defect of \mathfrak{g} such that

$$X_M \subset \bigcup_{A \in \mathcal{S}, |A| \leq k} \Phi(A). \tag{13}$$

Let $A = \{\beta_1, \dots, \beta_{k+1}\} \in \mathcal{S}$, $x = x_{\beta_1} + \dots + x_{\beta_{k+1}}$ for some nonzero $x_{\beta_i} \in \mathfrak{g}_{\beta_i}$. Then $M_x = \{0\}$. If $h \in \mathfrak{h}$ satisfies $\beta_1(h) = \dots = \beta_{k+1}(h) = 0$, then $[h, x] = 0$. Hence, by Lemma 7.4 $\text{str}_M e^h = \text{str}_{M_x} e^h = 0$. This we have proved the following property:

$$\text{sch}_M \left(\mathfrak{h}_A^\perp \right) = 0 \text{ for all } A \in \mathcal{S}, |A| = k + 1. \tag{14}$$

Let p_i be the first nonzero polynomial in the Taylor series for sch_M at zero. Then p_i also satisfies (14). Let $B = \{\alpha_1, \dots, \alpha_k\} \in \mathcal{S}$ and \bar{p}_i be the restriction of p_i to \mathfrak{h}_B^\perp . If $\bar{p}_i \neq 0$, then degree of \bar{p}_i is i . Since $p_i(\mathfrak{h}_{B \cup \alpha}^\perp) = 0$ for any $\alpha \neq \pm \alpha_i$, $\alpha \in B^\perp$, then α divides \bar{p}_i . That gives the estimate on i . Indeed, i is not less than the number of all possible α , i.e., $\frac{|\Delta_1 \cap B^\perp|}{2} - |B|$. By Corollary 5.15 the latter number is the codimension s of X_M in X . Hence, $i \geq s$.

To prove the second assertion we need to show that if two homogeneous W -invariant polynomials p and q of degree s satisfy (14), then $p = cq$ for some $c \in \mathbb{C}$. After restriction to \mathfrak{h}_B^\perp

$$\bar{p} = a \prod_{\alpha \in (\Delta_1^+ \cap B^\perp) \setminus \pm B} \alpha, \bar{q} = b \prod_{\alpha \in (\Delta_1^+ \cap B^\perp) \setminus \pm B} \alpha$$

for some constants a and b . Therefore, there exists $f = p - cq$ such that $f(\mathfrak{h}_B^\perp) = 0$. Thus, f satisfies (14) for k instead of $k + 1$. Then $\text{deg}f > s$, which implies $f = 0$. \square

8 Reduced Grothendieck Rings and ds_x

We retain the notation of Sect. 2. In this section we discuss the homomorphism $ds_x : \mathcal{K}_-(\mathfrak{g}) \rightarrow \mathcal{K}_-(\mathfrak{g}_x)$, for classical Lie superalgebras \mathfrak{g} , where $\mathcal{K}_-(\mathfrak{g})$ and $\mathcal{K}_-(\mathfrak{g}_x)$ stand for

the reduced Grothendieck rings of $\mathcal{F}(\mathfrak{g})$ and $\mathcal{F}(\mathfrak{g}_x)$, respectively. The study of ds_x was initiated by Hoyt and Reif in⁴⁰ for the basic classical Lie superalgebras. We consider \mathfrak{g}_x as a subalgebra of \mathfrak{g} using the splitting $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$ as in Propositions 4.5 and 4.6.

8.1 Subcategories of $\mathcal{F}(\mathfrak{g})$ and the DS Functor

Before discussing ds_x , we describe certain subcategories of $\mathcal{F}(\mathfrak{g})$ and their relation to the DS functor. Let \mathfrak{g} be one of basic classical superalgebras and $\Lambda_{\mathcal{F}(\mathfrak{g})}$ denote the abelian subgroup of \mathfrak{t}^* consisting of weights of $M \in \mathcal{F}(\mathfrak{g})$. For any subset $\Lambda \subset \Lambda_{\mathcal{F}(\mathfrak{g})}$ we denote by $\mathcal{F}^\Lambda(\mathfrak{g})$ the full subcategory of $\mathcal{F}(\mathfrak{g})$ consisting of modules with weights in Λ .

Let G be an algebraic supergroup with Lie superalgebra \mathfrak{g} . Then G is determined by the lattice $\Lambda_G \subset \Lambda_{\mathcal{F}(\mathfrak{g})}$ and the category $\mathcal{F}(G)$ of finite-dimensional representations of G is equivalent to $\mathcal{F}^{\Lambda_G}(\mathfrak{g})$.

If $x \in \mathfrak{g}_1$ is a self-commuting element and G^x is the centralizer of x in G then the Lie algebra of G^x is the kernel of ad_x . We denote by G_x the quotient of G^x such that $\text{Ker } G_x = \mathfrak{g}_x$. It is clear that DS_x induces the functor $\mathcal{F}(G) \rightarrow \mathcal{F}(G_x)$.

We denote by G the particular supergroup for every basic classical superalgebra:

1. If $\mathfrak{g} = \mathfrak{gl}(m|n)$ we set $G := GL(m|n)$, $\Lambda_G := \sum_{i=1}^m \mathbb{Z}\varepsilon_i + \sum_{j=1}^n \mathbb{Z}\delta_j$.
2. If $\mathfrak{g} = \mathfrak{sl}(m|n)$ we set $G := SL(m|n)$, $\Lambda_G := \Lambda_{GL(m|n)} \cap \mathfrak{h}^*$.
3. If $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ or $\mathfrak{osp}(2m+1|2n)$ we set $G := SOSp(2m|2n)$ (respectively, $SOSp(2m+1|2n)$), $\Lambda_G := \sum_{i=1}^m \mathbb{Z}\varepsilon_i + \sum_{j=1}^n \mathbb{Z}\delta_j$.
4. If $\mathfrak{g} = \mathfrak{p}(n)$ we set $G := P(n)$, $\Lambda_G := \sum_{i=1}^n \mathbb{Z}\varepsilon_i$.
5. If $\mathfrak{g} = \mathfrak{q}(n)$ we set $G := Q(n)$, $\Lambda_G := \sum_{i=1}^n \mathbb{Z}\varepsilon_i$.
6. If \mathfrak{g} is an exceptional Lie algebra of type $G(3)$ and $F(4)$ then G is the adjoint group with Λ_G being the root lattice. One has $\Lambda_{\mathcal{F}(\mathfrak{g})} = \Lambda_G$ for $G(3)$ and $\Lambda_{\mathcal{F}(\mathfrak{g})}/\Lambda_G = \mathbb{Z}_2$ for $F(4)$, see⁴⁸.
7. For $\mathfrak{g} = D(2, 1, a)$ we consider the algebraic group G with $\Lambda_G := \Lambda_{\mathcal{F}(\mathfrak{g})}$.

Next we set $\Lambda_{\mathfrak{g}} := \Lambda_G$ in all cases when $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. In the remaining cases, we set

$$\begin{aligned} \Lambda_{\mathfrak{gl}(m|n)} &:= \Lambda_{GL(m|n)} + \mathbb{C}\text{str} \\ &\text{where str} := \varepsilon_1 + \dots + \varepsilon_m + \delta_1 + \dots + \delta_n, \\ \Lambda_{\mathfrak{p}(m)} &:= \Lambda_{P(m)} + \mathbb{C}\text{str}, \\ &\text{where str} := \varepsilon_1 + \dots + \varepsilon_m, \\ \Lambda_{\mathfrak{q}(m)} &:= \Lambda_{Q(m)} + \mathbb{Z}\frac{\text{str}}{2}, \\ &\text{where str} := \varepsilon_1 + \dots + \varepsilon_m. \end{aligned}$$

8.1.1 Consider the case when \mathfrak{g} is non-exceptional. Take $x \neq 0$ as in Prop. 4.5. Let $\Lambda'_{\mathfrak{g}} = \Lambda_{\mathcal{F}(\mathfrak{g})} \setminus \Lambda_{\mathfrak{g}}$. Then we have a decomposition

$$\mathcal{F}(\mathfrak{g}) = \mathcal{F}^{\Lambda_{\mathfrak{g}}}(\mathfrak{g}) \oplus \mathcal{F}^{\Lambda'_{\mathfrak{g}}}(\mathfrak{g}).$$

Every module $M \in \mathcal{F}^{\Lambda'_{\mathfrak{g}}}(\mathfrak{g})$ is projective and hence $DS_x(M) = 0$. Furthermore, DS_x induces the functor $\mathcal{F}^{\Lambda_G}(\mathfrak{g}) \rightarrow \mathcal{F}^{\Lambda_{G_x}}(\mathfrak{g}_x)$.

In the cases when $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ we can be more precise. Namely, when $\mathfrak{g}_x \neq 0$, DS_x restricts to the functor:

$$\mathcal{F}^{\Lambda_G + \text{cstr}}(\mathfrak{g}) \rightarrow \mathcal{F}^{\Lambda_{G_x} + \text{cstr}}(\mathfrak{g}_x),$$

where $c \in \mathbb{C}$ for $\mathfrak{gl}(m|n)$ and $\mathfrak{p}(n)$, $c = 0, \frac{1}{2}$ for $\mathfrak{q}(m)$. To see this for $\mathfrak{gl}(m|n)$ and $\mathfrak{p}(n)$ we just note that every $M \in \mathcal{F}^{\Lambda_G + \text{cstr}}$ can be obtained from $M_0 \in \mathcal{F}(G)$ by tensoring with one dimensional character in $\chi_c \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$ and therefore it suffices to compute DS_x in the case $c = 0$ and then use $DS_x(M \otimes \chi_c) = DS_x(M) \otimes \chi_c$. The case of $\mathfrak{q}(n)$ is straightforward.

8.1.2 Exceptional Algebras

Take $x \neq 0$. All such x are conjugate by the adjoint action of G_0 .

For $D(2, 1, a)$, $G_x = \mathbb{C}^*$. Therefore $DS_x : \mathcal{F}(D(2, 1, a)) \rightarrow \mathcal{F}(\mathbb{C}^*)$. By [Germoni] for $\mathfrak{g} = D(2, 1, a)$ with $a \notin \mathbb{Q}$ all atypical modules in $\mathcal{F}(\mathfrak{g})$ have zero central character. Using the filtration of projective modules obtained by Germoni, one can show (see²⁶) that $DS_x(L)$ is a trivial \mathbb{C}^* -module for any simple atypical \mathfrak{g} -module L and therefore for any \mathfrak{g} -module. In other words, the image of DS_x lies in the category of vector superspaces equipped with the trivial action of \mathbb{C}^* .

Combining the description of dominant weights (see⁴⁸) and (9) we obtain the following results for $G(3)$ (with $\mathfrak{g}_x = \mathfrak{sl}_2$) and $F(4)$ (with $\mathfrak{g}_x = \mathfrak{sl}_3$):

$$\begin{aligned} DS_x(\mathcal{F}(G(3))) &\subset \mathcal{F}(PSL(2)) \\ DS_x(\mathcal{F}(F(4))) &\subset \mathcal{F}(PSL(3)), \end{aligned}$$

where $\mathcal{F}(G)$ denotes the category of finite-dimensional representations of the algebraic group G . In fact, from²⁵ and⁴⁸, we obtain that

$$DS_x(\mathcal{F}(F(4))) \subset \mathcal{F}(G'_x),$$

where G'_x is a non-connected algebraic group fitting into a non-split exact sequence

$$1 \rightarrow PSL(3) \rightarrow G'_x \rightarrow \mathbb{Z}_2 \rightarrow 1, \tag{15}$$

compare to Remark 3.4.

8.2 Properties of ds_x

We now begin the discussion of ds_x with general remarks which are valid for classical Lie superalgebras \mathfrak{g} satisfying $\mathfrak{h} = \mathfrak{t}$, i.e., classical $\mathfrak{g} \neq \mathfrak{q}(n)$. (The case of general \mathfrak{g} is considered in³¹). As already noted, by Sect. 4.5, we have splittings $\mathfrak{g}^x = \mathfrak{g}_x \ltimes [x, \mathfrak{g}]$. Further we have $\mathfrak{t} = \mathfrak{t}_x \oplus \mathfrak{t}'$, where $\mathfrak{t}_x \subseteq \mathfrak{g}_x$ is a Cartan subalgebra.

In these cases, the ring $\mathcal{K}_-(\mathfrak{g})$ is spanned by the images of the simple finite-dimensional modules. Since these modules are highest weight modules, the map $[N] \mapsto [\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} N]$ gives an embedding of $\mathcal{K}_-(\mathfrak{g}) \rightarrow \mathcal{K}_-(\mathfrak{h})$ and we identify $\mathcal{K}_-(\mathfrak{g})$ with this ring. The image is called the *ring of supercharacters*, since $[\text{Res}_{\mathfrak{h}}^{\mathfrak{g}} N] = \text{sch} N$. By Lemma 2.12

$$\text{sdim} DS_x(N)_{\mu'} = \sum_{\mu \in \mathfrak{t}^*: \mu|_{\mathfrak{t}_x} = \mu'} \text{sdim} N_{\mu}$$

for each $N \in \mathcal{F}(\mathfrak{g})$ and $\mu' \in \mathfrak{t}_x^*$. Thus ds_x written for the supercharacter rings takes the form

$$ds_x \left(\sum_{\nu \in \mathfrak{t}^*} m_{\nu} e^{\nu} \right) = \sum_{\nu \in \mathfrak{t}^*} m_{\nu} e^{\nu|_{\mathfrak{t}_x}}. \tag{16}$$

coincides with the restriction of the map $f \mapsto f|_{\mathfrak{t}_x}$. Using the representation theory of $\mathfrak{sl}(1|1)$ it is easy to see that if x can be embedded into an $\mathfrak{sl}(1|1)$ -triple x, y, α^{\vee} with $\alpha^{\vee} \in \mathfrak{t}$, then for

$$ds_x \left(\sum_{\nu \in \mathfrak{t}^*} m_{\nu} e^{\nu} \right) = \sum_{\nu \in \mathfrak{t}^*: \nu(\alpha^{\vee})=0} m_{\nu} e^{\nu|_{\mathfrak{t}_x}}. \tag{17}$$

These formulas appeared in⁴⁰; for $\mathfrak{q}(n)$ a similar formula is given in³¹.

8.2.1 The case with \mathfrak{g} from the list (1.1)

By Lemma 2.12 one has $ds_x = ds_y$ if $\mathfrak{g}_x = \mathfrak{g}_y$ as embedded subalgebras. For x of rank r we denote the map $ds_x : \mathcal{K}_-(\mathfrak{g}) \rightarrow \mathcal{K}_-(\mathfrak{g}_x)$ by ds^r . By Lemma 2.13 one has $ds^i = ds^1 \circ ds^1 \circ \dots \circ ds^1$ if $DS^1(DS^1 \dots (\mathfrak{g})) = DS^i(\mathfrak{g})$. By Lemma 2.14

we have $\ker ds_x = \ker ds_y$ if x, y are conjugated by an inner automorphism. Note that $\text{rk} x = \text{rk} y$ implies the existence of $x' \in X$ such that $\mathfrak{g}_{x'} = \mathfrak{g}_y$ and x, x' are conjugated by an inner automorphism. Hence, the ideal $\ker ds^r \subset \mathcal{K}_-(\mathfrak{g})$ does not depend on the choice of x of rank r .

8.3 The Ring $\mathcal{K}_-(\mathfrak{g})$

For $\mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n)$, Sergeev and Veselov interpreted the supercharacter ring as a ring of functions admitting certain supersymmetry conditions, see⁵⁸. For example, the supercharacter ring for the category $\mathcal{F}^{\mathbb{Z}}(\mathfrak{gl}(m|n))$ can be realized as

$$\mathcal{J} \cong \left\{ f \in \mathbb{Z} [x_1^{\pm 1}, \dots, x_m^{\pm 1}, y_1^{\pm 1}, \dots, y_n^{\pm 1}]^{S_m \times S_n} \mid f|_{x_1=y_1=t} \text{ is independent of } t \right\},$$

and in this case, if $\text{rk} x = 1$, then $ds_x(f) = f|_{x_1=y_1}$ (see⁴⁰). The supercharacter ring for $\mathfrak{p}(n)$ was described in⁴² using an inductive argument with the help of ds^2 . Using a similar method Reif described the ring $\mathcal{K}_+(\mathfrak{q}_n)$ in⁵³. Note that for $\mathfrak{q}(n)$ the supercharacter of a finite-dimensional non-trivial simple $\mathfrak{q}(n)$ -module is always zero⁸.

8.4 The Image of ds_x

Let \mathfrak{g} be one of the superalgebras from the list (1). The categories $\mathcal{F}^{\Lambda \mathfrak{g}}(\mathfrak{g}) \cong \mathcal{F}(G)$ were introduced in Sect. 8.1. Let $\sigma_x \in \text{Aut}(\mathfrak{g}_x)$ be the involution introduced in Sect. 3.1. Note that $\sigma_x = \text{Id}$ except for the cases $D(2, 1, a), F(4)$ and $\mathfrak{osp}(2m|2n)$ with $\text{rk} x < m$. We also denote by σ_x the induced involution of the ring $\mathcal{K}_-(\mathfrak{g}_x)$.

Theorem 8.1 *Take $x \neq 0$. For non-exceptional \mathfrak{g} from the list (1), one has*

$$ds_x(\mathcal{K}_-(\mathcal{F}(G))) = \mathcal{K}_-(\mathcal{F}(G_x))^{\sigma_x}.$$

For $\mathfrak{g} = D(2, 1, a)$ with $a \in \mathbb{Q}$, $G(3)$ and $F(4)$ one has

$$ds_x(\mathcal{K}_-(\mathcal{F}(G))) = \mathcal{K}_-(\mathcal{F}(G'_x))^{\sigma_x}$$

where $G'_x = \mathbb{C}^*, PSL(2), PSL(3)$ for respectively. For $\mathfrak{g} = D(2, 1, a)$ with $a \notin \mathbb{Q}$ one has $ds_x(\mathcal{K}_-(\mathcal{F}(G))) = \mathbb{Z}$.

Recall that $\mathcal{F}(\mathfrak{g})$ is equivalent to $\mathcal{F}(G)$ for $\mathfrak{g} = G(3)$ and $D(2, 1, a)$. We have

$$ds_x(\mathcal{K}_-(\mathfrak{g})) = \begin{cases} \mathcal{K}_-(\mathfrak{g}_x)^{\sigma_x} & \text{for } \mathfrak{gl}(m|n), \mathfrak{osp}(m|n), \mathfrak{p}(n), F(4) \\ \mathcal{K}_-(\mathcal{F}(G_x))^{\sigma_x} & \text{for } \mathfrak{q}(n). \end{cases}$$

Note that for $\mathfrak{g} = F(4)$ the categories $\mathcal{F}(G'_x)^{\sigma_x}$ and $\mathcal{F}(G''_x)$ are equivalent, see (15). For $\mathfrak{g} \neq \mathfrak{q}(n), \mathfrak{p}(n)$ the statement was established in⁴⁰ and for $\mathfrak{p}(n)$ with $x \in \mathfrak{g}^{-1}$ the assertion was proven in⁴²; the proofs are based on the evaluation (17) (in these case x can be embedded in an $\mathfrak{sl}(1|1)$ -triple); for $\mathfrak{q}(n)$ the assertion is proven in³¹. For $\mathfrak{p}(n)$ with x of rank 1 we prove the assertion in Corollary 12.25; since $ds^r = ds^1 \circ \dots \circ ds^1$, this implies the assertion for general x .

Remark 8.2 For a precise description of $ds_x(\mathcal{K}_-(\mathcal{F}(\mathfrak{g})))$ when $\mathfrak{g} = D(2, 1; a)$ with $a \in \mathbb{Q}$, see Sect. 12.4.1.

Remark 8.3 For $\mathfrak{q}(n)$ -case $ds_x(\mathcal{K}_-(\mathcal{F}(\mathfrak{g}))) = \mathcal{K}_-(\mathcal{F}(G))^{\sigma_x}$, but it can be easily seen that if $M \in \mathcal{F}(\mathfrak{g})$ then $DS_x M$ need not have the structure of a G -module.

8.5 The Kernel of ds_x

8.5.1 Notation

Take $\mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n)$. For a fixed choice of negative roots $\Delta^- = \Delta_0^- \sqcup \Delta_1^-$, we denote the super Weyl denominator by $R = \frac{R_0}{R_1}$, where $R_0 = \prod_{\alpha \in \Delta_0^-} (1 - e^\alpha)$ and $R_1 = \prod_{\alpha \in \Delta_1^-} (1 - e^\alpha)$.

For $\lambda \in \mathfrak{t}^*$ set

$$k(\lambda) := R^{-1} \cdot \sum_{w \in W} (-1)^{l(w)+p(w(\rho)-\rho)} e^{w(\lambda+\rho)-\rho},$$

where $l(w)$ denotes the length of w as a product of simple reflections with respect to a set of simple roots for \mathfrak{g}_0 . Let $P^+(\mathfrak{g}_0)$ be the set of dominant weights of \mathfrak{g}_0 . For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n)$ with the distinguished choice of simple roots, $k(\lambda)$ is the supercharacter of a Kac module $K(\lambda) = \text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}^1}^{\mathfrak{g}} L_{\mathfrak{g}_0}(\lambda)$

when $\lambda \in P^+(\mathfrak{g}_0)$, and we have

$$k(\lambda) = \text{sch} L_{\mathfrak{g}_0}(\lambda) \prod_{\alpha \in \Delta_1^-} (1 - e^\alpha).$$

However in Type II, $k(\lambda)$ is a virtual supercharacter.

For $\mathfrak{p}(n)$ we set $K(\lambda) = \text{Ind}_{\mathfrak{g}_0 \oplus \mathfrak{g}^1}^{\mathfrak{g}} L_{\mathfrak{g}_0}(\lambda)$; then

$K(\lambda)$ is a “thin” Kac module. For $\lambda \in P^+(\mathfrak{g}_0)$ the expression $k(\lambda) := \text{sch} K(\lambda)$ is given by the above formulas. Finally we need one more virtual supercharacter for $\mathfrak{p}(n)$ given by

$$k'(\lambda) := k(\lambda) \text{sch} S^\bullet \Pi(\mathbb{C}^n)^*$$

Here \mathbb{C}^n denotes the standard representation of $\mathfrak{gl}(n)$. In coordinates $\epsilon_1, \dots, \epsilon_n$, this is given explicitly by

$$k'(\lambda) = \text{sch} L_{\mathfrak{g}_0}(\lambda) \prod_{i=1}^n (1 - e^{-\epsilon_i}) \prod_{\alpha \in \Delta_{-1}} (1 - e^\alpha).$$

Let $\rho_{iso} := \frac{1}{2} \sum_{\alpha \in \Delta_{iso}^+} \alpha$.

Theorem 8.4 Let $\ker_1 \subset \ker_2 \subset \dots$ be the kernels for $ds^r : \mathcal{K}_-(\mathfrak{g}) \rightarrow \mathcal{K}_-(\mathfrak{g}_x)$.

1. For $\mathfrak{g} \neq \mathfrak{p}(n), \mathfrak{q}(n), \mathfrak{gl}(1|1)$ the set $\{k(\lambda) \mid \lambda \in P^+(\mathfrak{g}_0) + \rho_{iso}\}$ forms a basis of \ker_1 .
2. For $\mathfrak{p}(n)$ with $n > 1$ the set $\{k'(\lambda) \mid \lambda \in P^+(\mathfrak{g}_0)\}$ forms a basis of \ker_1 .
3. For $\mathfrak{p}(n)$ with $n > 2$ the supercharacters of the thin Kac modules form a basis of \ker_2 .

Hoyt and Reif proved (1) in⁴⁰, and (3) was proven in⁴². We will give a proof of (2) in Sect. 8.5.1 below.

- *Remark 8.5* The ring $\mathcal{K}_-(\mathfrak{p}(1))$ is the group ring of \mathbb{C} and ds^1 acts as $ds^1(e^c) = 1$.
- The ring $\mathcal{K}_-(\mathfrak{gl}(1|1))$ lies in the group ring of $\mathbb{C} \times \mathbb{C}$: this ring is spanned by the elements $e^{(0,a)}$ and $e^{(b,a)} - e^{(b,a-1)}$ for $a, b \in \mathbb{C}$ with $b \neq 0$. The map ds^1 is the restriction of the algebra map given by $ds^1(e^c) = 1$ for any $c \in \mathbb{C} \times \mathbb{C}$.
- The kernel of ds^1 for $\mathfrak{g} = \mathfrak{q}(n)$ is computed in³¹. The result is in terms of an explicit basis of $\mathcal{K}_-(Q(n))$.

Proof (Proof of (ii)) Retain the notation of Sect. 8.1. Clearly, $K_-(\lambda + \text{cstr}) = K_-(\lambda) \otimes \chi_c$, where χ_c is a one-dimensional \mathfrak{p}_n -module corresponding to cstr . Using Sect. 8.1, we can reduce the statement to $\mathcal{F}^{\Lambda_G}(\mathfrak{g})$.

We utilize methods of⁴⁰. Let h_1, \dots, h_n be the standard basis of \mathfrak{t} (which is dual to $\epsilon_1, \dots, \epsilon_n \in \mathfrak{t}^*$). Take $x \in \mathfrak{g}_{2\epsilon_n}$ and identify \mathfrak{g}_x with the “natural copy” of $\mathfrak{p}(n-1)$ in $\mathfrak{p}(n)$;

in this case \mathfrak{t}_x is spanned by h_1, \dots, h_{n-1} . Take $f \in \mathcal{K}_-(P(n))$ such that $ds^1(f) = 0$.

Identify \mathfrak{t}_x^* with the span of $\varepsilon_1, \dots, \varepsilon_{n-1}$ and write

$$f = \sum m_\nu e^\nu = \sum_{\mu' \in \mathfrak{t}_x^*} f_{\mu'},$$

$$\text{where } f_{\mu'} := e^{\mu'} \sum_{a \in \mathbb{C}} m_{\mu' + a\varepsilon_n} e^{a\varepsilon_n}.$$

By (16) $ds^1(f) = 0$ is equivalent to $\sum_{a \in \mathbb{C}} m_{\mu' + a\varepsilon_n} = 0$ for each $\mu' \in \mathfrak{t}_x^*$.

Therefore, $\sum_{a \in \mathbb{C}} m_{\mu' + a\varepsilon_n} = 0$ means that $f_{\mu'}$ is divisible by $1 - e^{-\varepsilon_n}$. Hence, f is divisible by $1 - e^{-\varepsilon_n}$.

Recall that $\ker ds^1 \subset \ker ds^2$, so $ds^2(f) = 0$. Using the above argument for $x \in \mathfrak{g}_{\varepsilon_{n-1} + \varepsilon_n}$, we obtain that f is divisible by $1 - e^{-\varepsilon_{n-1} - \varepsilon_n}$.

The restriction $\text{Res}_{\mathfrak{g}_0}^{\mathfrak{g}}$ gives an embedding of the supercharacter ring of $\mathfrak{p}(n)$ to the supercharacter ring of $\mathfrak{p}(n)_0 = \mathfrak{gl}_n$. In particular, f is W -invariant and thus f is divisible by the element

$$R' := \prod_{i=1}^n (1 - e^{-\varepsilon_i}) \prod_{\alpha \in \Delta_{-1}} (1 - e^\alpha).$$

Since R' is W -invariant one has $f = R'f'$, where f' is a W -invariant element in $\mathbb{Z}[e^\nu, \nu \in \mathfrak{t}^*]$. The ring $\mathbb{Z}[e^\nu, \nu \in \mathfrak{t}^*]^W$ is the character ring of \mathfrak{gl}_n , so f' can be written as $f' = \sum_j m_j \text{sch} L_{\mathfrak{gl}_n}(v_j)$. This gives $f = \sum_j m_j R' \text{sch} L_{\mathfrak{gl}_n}(v_j) = \sum_j m_j k(v_j)$ as required.

Finally, one can use a standard highest weight argument to show that the $k(\lambda)$ are linearly independent. \square

Remark 8.6 Take $n > 0$. One has $\Lambda_G/\mathbb{Z}\Delta \cong \mathbb{Z}_2$. Writing $\Lambda_G = \mathbb{Z}\Delta + (\varepsilon_1 + \mathbb{Z}\Delta)$ we have

$$\mathcal{F}(P(n)) = \mathcal{F}_0(\mathfrak{g}) \oplus \mathcal{F}_1(\mathfrak{g}),$$

where $\mathcal{F}_0(\mathfrak{g}) := \mathcal{F}^{\mathbb{Z}\Delta}(\mathfrak{g})$, $\mathcal{F}_1(\mathfrak{g}) := \mathcal{F}^{\mathbb{Z}\Delta + \varepsilon_1}(\mathfrak{g})$.

Thus, we have

$$\mathcal{K}_-(\mathcal{F}(P(n))) = \mathcal{K}_-(\mathcal{F}_0(\mathfrak{g})) \oplus \mathcal{K}_-(\mathcal{F}_1(\mathfrak{g})).$$

If $f \in \mathcal{K}_-(\mathcal{F}(P(n))) = \mathcal{K}_-(\mathcal{F}_0(\mathfrak{g}))$ has $ds^1(f) = 0$, then the above argument will imply that f is divisible by $(1 - e^{2\varepsilon_n})$ and $(1 - e^{\varepsilon_{n-1} + \varepsilon_n})$. Applying W -invariance of f , we learn that it is the subspace of $\mathcal{K}_-(\mathcal{F}(P(n)))$ spanned by the supercharacters $k_+(\lambda)$, i.e., the supercharacters of thick Kac modules $\text{Ind}_{\mathfrak{g} \oplus \mathfrak{g}^{-1}}^{\mathfrak{g}} L_{\mathfrak{g}_0}(\lambda)$.

However, thick Kac modules do not span the kernel of ds^1 ; for instance when $n = 2$, we have

$$ds^1([L(\varepsilon_1)] - [\mathbb{C}_{\text{str}}] + [\mathbb{C}_{-\text{str}}]) = 0.$$

However, one can show (using evaluation arguments) that $[L(\varepsilon_1)] - [\mathbb{C}_{\text{str}}] + [\mathbb{C}_{-\text{str}}]$ is not in the span of supercharacters of thick Kac modules.

9 The DS Functor and $\mathfrak{sl}(\infty)$ -Modules

In this section, we discuss a connection between the DS functor and $\mathfrak{sl}(\infty)$ -modules arising from $\mathfrak{gl}(m|n)$ -representation theory, which was discovered and studied by Hoyt, Penkov and Serganova in⁴¹. We will recall some basic facts for $\mathfrak{sl}(\infty)$, and refer the reader to the book “Classical Lie algebras at infinity” by Penkov and Hoyt for an in-depth treatment of the Lie algebra $\mathfrak{sl}(\infty)$ and other locally finite Lie algebras⁵¹.

In the pioneering paper⁴, Brundan showed that the complexification of the Grothendieck group for the categories $\mathcal{F}(GL(m|n))$ and the integral BGG category $\mathcal{O}_{m|n}^{\mathbb{Z}}$ inherit a natural $\mathfrak{sl}(\infty)$ -module structure from the action of translation functors E_i, F_i . This action and general categorification methods were used by Brundan, Losev and Webster in⁶ to develop Kazhdan–Lusztig theory for $\mathfrak{gl}(m|n)$.

Now since the DS functor commutes with translation functors, the induced homomorphism ds of reduced Grothendieck groups is, in fact, a homomorphism of $\mathfrak{sl}(\infty)$ -modules⁴¹. This homomorphism ds was used in⁴¹ to help obtain a description of the $\mathfrak{sl}(\infty)$ -module structure of the reduced Grothendieck groups for both of the categories $\mathcal{F}_{m|n}^{\mathbb{Z}}$ and $\mathcal{O}_{m|n}^{\mathbb{Z}}$ of integral $\mathfrak{gl}(m|n)$ -modules.

9.1 The Lie Algebra $\mathfrak{sl}(\infty)$

The Lie algebra $\mathfrak{gl}(\infty)$ can be defined by taking countable-dimensional vector spaces \mathbf{V}, \mathbf{V}_* with bases $\{v_i\}_{i \in \mathbb{Z}}, \{v_j^*\}_{j \in \mathbb{Z}}$, and letting $\mathfrak{gl}(\infty) = \mathbf{V} \otimes \mathbf{V}_*$ with bracket (extended linearly) given by

$$[v_i \otimes v_j^*, v_k \otimes v_l^*] = \langle v_k, v_j^* \rangle v_i \otimes v_l^* - \langle v_i, v_l^* \rangle v_k \otimes v_j^*,$$

where $\langle \cdot, \cdot \rangle : \mathbf{V} \otimes \mathbf{V}_* \rightarrow \mathbb{C}$ is the nondegenerate pairing defined by $\langle v_i, v_j^* \rangle = \delta_{ij}$.

We can identify $\mathfrak{gl}(\infty)$ with the space of infinite matrices $(a_{ij})_{i,j \in \mathbb{Z}}$ which have only finitely

many nonzero entries, using the correspondence $v_i \otimes v_j^* \mapsto E_{ij}$, where E_{ij} is the matrix with 1 in the i, j -position and zeros elsewhere. Under this identification, $\langle \cdot, \cdot \rangle$ is the trace map on $\mathfrak{gl}(\infty)$, and the kernel of $\langle \cdot, \cdot \rangle$ is the Lie algebra

$\mathfrak{sl}(\infty)$. The center of $\mathfrak{gl}(\infty)$ is trivial, and the following exact sequence does not split:

$$0 \rightarrow \mathfrak{sl}(\infty) \rightarrow \mathfrak{gl}(\infty) \rightarrow \mathbb{C} \rightarrow 0.$$

The Lie algebra $\mathfrak{sl}(\infty)$ is generated by the elements $e_i := E_{i,i+1}, f_i := E_{i+1,i}$ for $i \in \mathbb{Z}$.

We can realize $\mathfrak{sl}(\infty)$ as a direct limit of finite-dimensional Lie algebras $\varinjlim \mathfrak{sl}(n)$, that is, $\mathfrak{sl}(\infty)$ is isomorphic to a union $\bigcup_{n \in \mathbb{Z}_{\geq 2}} \mathfrak{sl}(n)$, of nested Lie algebras

$$\mathfrak{sl}(2) \subset \mathfrak{sl}(3) \subset \dots \subset \mathfrak{sl}(n) \subset \mathfrak{sl}(n+1) \subset \dots$$

The Lie algebra obtained from this union is independent, up to isomorphism, of the choice of the inclusions $\mathfrak{sl}(n) \hookrightarrow \mathfrak{sl}(n+1)$.

9.2 Modules Over $\mathfrak{sl}(\infty)$

The modules \mathbf{V}, \mathbf{V}_* are the defining representations of $\mathfrak{sl}(\infty)$, and for $p, q > 0$, the tensor modules $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$, $p, q \in \mathbb{Z}_{\geq 0}$ are not semisimple. Schur-Weyl duality for $\mathfrak{sl}(\infty)$ implies that the module $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ decomposes as

$$\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q} = \bigoplus_{|\lambda|=p, |\mu|=q} (\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) \otimes (Y_\lambda \otimes Y_\mu),$$

where Y_λ and Y_μ are irreducible S_p - and S_q -modules, \mathbb{S}_λ denotes the Schur functor corresponding to the Young diagram λ , and $|\lambda|$ is the size of λ .

The $\mathfrak{sl}(\infty)$ -modules $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$ are indecomposable, and their socle filtration was described by Penkov and Styrcas in⁵². We recall that the *socle* of a module \mathbf{M} , denoted $\text{soc}\mathbf{M}$, is the largest semisimple submodule of \mathbf{M} , and that the *socle filtration* of \mathbf{M} is defined inductively by $\text{soc}^0\mathbf{M} := \text{soc}\mathbf{M}$ and $\text{soc}^i\mathbf{M} := p_i^{-1}(\text{soc}(\mathbf{M}/(\text{soc}^{i-1}\mathbf{M})))$, where $p_i: \mathbf{M} \rightarrow \mathbf{M}/(\text{soc}^{i-1}\mathbf{M})$ is the natural projection. We denote the layers of the socle filtration by $\overline{\text{soc}}^i\mathbf{M} := \text{soc}^i\mathbf{M}/\text{soc}^{i-1}\mathbf{M}$. From Thm. 2.3 of⁵², we have that the layers of $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$ are

$$\overline{\text{soc}}^k(\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)) \cong \bigoplus_{\lambda', \mu', |\gamma|=k} N_{\lambda', \gamma}^\lambda N_{\mu', \gamma}^\mu \mathbf{V}^{\lambda', \mu'}$$

where $N_{\lambda', \gamma}^\lambda$ are the standard Littlewood-Richardson coefficients. In particular, the indecomposable module $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$ has a simple socle, denoted by $\mathbf{V}^{\lambda, \mu}$. For example, the layers of $\Lambda^m\mathbf{V} \otimes \Lambda^n\mathbf{V}_*$ are given by $\overline{\text{soc}}^i(\Lambda^m\mathbf{V} \otimes \Lambda^n\mathbf{V}_*) \cong \mathbf{V}^{(m-i)^\perp, (n-i)^\perp}$, where \perp indicates the conjugate Young diagram.

An $\mathfrak{sl}(\infty)$ -module is called a *tensor module* if it is isomorphic to a submodule of a finite direct sum of modules of the form $\mathbf{V}^{\otimes p_i} \otimes \mathbf{V}_*^{\otimes q_i}$

for $p_i, q_i \in \mathbb{Z}_{\geq 0}$. The category of tensor modules $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is by definition the full subcategory of $\mathfrak{sl}(\infty)$ -mod consisting of tensor modules¹². The modules $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$, $p, q \in \mathbb{Z}_{\geq 0}$ are injective in $\mathbb{T}_{\mathfrak{sl}(\infty)}$. Moreover, every indecomposable injective object of $\mathbb{T}_{\mathfrak{sl}(\infty)}$ is isomorphic to an indecomposable direct summand of $\mathbf{V}^{\otimes p} \otimes \mathbf{V}_*^{\otimes q}$ for some $p, q \in \mathbb{Z}_{\geq 0}$, which means, it is isomorphic to $\mathbb{S}_\lambda(\mathbf{V}) \otimes \mathbb{S}_\mu(\mathbf{V}_*)$ for some λ, μ ¹².

9.3 Representation Theory of $\mathfrak{gl}(m|n)$

Let $\mathcal{O}_{m|n}$ denote the category of \mathbb{Z}_2 -graded modules over $\mathfrak{gl}(m|n)$ which when restricted to $\mathfrak{gl}(m|n)_0$, belong to the BGG category $\mathcal{O}_{\mathfrak{gl}(m|n)_0}$ (see⁴⁹, Sec. 8.2.3). This category depends only on a choice of Borel subalgebra for $\mathfrak{gl}(m|n)_0$, and not for $\mathfrak{gl}(m|n)$. We denote by $\mathcal{O}_{m|n}^{\mathbb{Z}}$ the Serre subcategory of $\mathcal{O}_{m|n}$ consisting of modules with integral weights. Any simple object in $\mathcal{O}_{m|n}^{\mathbb{Z}}$ is isomorphic to $L(\lambda)$ for some integral weight λ (for a fixed Borel subalgebra of $\mathfrak{gl}(m|n)$). The objects of the category $\mathcal{O}_{m|n}^{\mathbb{Z}}$ have finite length. We denote by $\mathcal{F}_{m|n}^{\mathbb{Z}}$ the Serre subcategory of $\mathcal{O}_{m|n}^{\mathbb{Z}}$ consisting of finite-dimensional modules. (Note that $\mathcal{F}_{m|n}^{\mathbb{Z}} = \mathcal{F}^{\Lambda_G}(\mathfrak{gl}(m|n))$ as defined in Sect. 8.1.) Each simple object of $\mathcal{F}_{m|n}^{\mathbb{Z}}$ is isomorphic to $L(\lambda)$ for some dominant integral weight λ .

We define the translation functors E_i, F_i on the category $\mathcal{O}_{m|n}^{\mathbb{Z}}$ as follows. Let X_j, Y_j be a pair of \mathbb{Z}_2 -homogeneous dual bases of $\mathfrak{gl}(m|n)$ with respect to the $\mathfrak{gl}(m|n)$ -invariant form $\text{str}(XY)$. For a pair of $\mathfrak{gl}(m|n)$ -modules V, W , we define the Casimir operator $\Omega \in \text{End}_{\mathfrak{gl}(m|n)}(V \otimes W)$ on homogeneous vectors by setting

$$\Omega(v \otimes w) := \sum_j (-1)^{p(X_j)(p(v)+1)} X_j v \otimes Y_j w,$$

where $p(\cdot)$ denotes the parity function. Let U, U^* be the defining $\mathfrak{gl}(m|n)$ -modules. Then for every $M \in \mathcal{O}_{m|n}^{\mathbb{Z}}$, we let $E_i(M)$ (respectively, $F_i(M)$) be the generalized eigenspace of Ω in $M \otimes U^*$ (respectively, $M \otimes U$) with eigenvalue i . Then, as it follows from⁶, the functor $\cdot \otimes U^*$ (respectively, $\cdot \otimes U$) decomposes into the direct sum of functors $\bigoplus_{i \in \mathbb{Z}} E_i(\cdot)$ (respectively, $\bigoplus_{i \in \mathbb{Z}} F_i(\cdot)$). Moreover, the functors E_i and F_i are adjoint functors on $\mathcal{O}_{m|n}^{\mathbb{Z}}$.

9.4 Grothendieck Groups and the $\mathfrak{sl}(\infty)$ -Modules $\mathbf{K}_{m|n}, \mathbf{J}_{m|n}$

We let $\mathbf{K}_{m|n}$ (respectively, $\mathbf{J}_{m|n}$) denote the complexification of the reduced Grothendieck group of $\mathcal{O}_{m|n}^{\mathbb{Z}}$ (respectively, of $\mathcal{F}_{m|n}^{\mathbb{Z}}$), that is,

$$\mathbf{K}_{m|n} := \mathcal{K}_-(\mathcal{O}_{m|n}^{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}, \mathbf{J}_{m|n} := \mathcal{K}_-(\mathcal{F}_{m|n}^{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{C}.$$

We will denote by e_i, f_i the linear operators that the translation functors E_i, F_i induce on $\mathbf{K}_{m|n}$ and $\mathbf{J}_{m|n}$. Brundan showed in⁴ that if we identify e_i, f_i with the Chevalley generators $E_{i,i+1}, E_{i+1,i}$ of $\mathfrak{sl}(\infty)$, then we obtain an $\mathfrak{sl}(\infty)$ -module structure on $\mathbf{J}_{m|n}$ and $\mathbf{K}_{m|n}$.

Let $\mathbf{T}_{m|n} \subset \mathbf{K}_{m|n}$ denote the subspace generated by the classes $[M(\lambda)]$ of all Verma modules $M(\lambda)$ for $\lambda \in \Phi$. Let furthermore $\mathbf{W}_{m|n} \subset \mathbf{J}_{m|n}$ denote the subspace generated by the classes $[K(\lambda)]$ of all Kac modules $K(\lambda)$ for $\lambda \in \Phi^+$. Then $\mathbf{W}_{m|n}$ and $\mathbf{T}_{m|n}$ are $\mathfrak{sl}(\infty)$ -modules under the action defined above, and $\mathbf{W}_{m|n} \cong \Lambda^m \mathbf{V} \otimes \Lambda^n \mathbf{V}_*$ and $\mathbf{T}_{m|n} \cong \mathbf{V}^{\otimes m} \otimes \mathbf{V}_*^{\otimes n}$. The modules $\mathbf{T}_{m,n}$ and $\mathbf{W}_{m|n}$ are injective in the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$, and $\mathbf{W}_{m|n}$ is an indecomposable summand of $\mathbf{T}_{m|n}$. Now let $\mathbf{P}_{m,n} := \mathcal{K}_-(\mathcal{P}_{m|n}) \otimes_{\mathbb{Z}} \mathbb{C}$ (respectively, $\mathbf{Q}_{m|n} := \mathcal{K}_-(\mathcal{Q}_{m|n}) \otimes_{\mathbb{Z}} \mathbb{C}$), where $\mathcal{P}_{m|n}$ (respectively, $\mathcal{Q}_{m|n}$) is the semi-simple subcategory of $\mathcal{O}_{m|n}^{\mathbb{Z}}$ (respectively, of $\mathcal{F}_{m|n}^{\mathbb{Z}}$) consisting of projective modules. Then we have $\text{soc} \mathbf{K}_{m,n} = \text{soc} \mathbf{T}_{m,n} = \mathbf{P}_{m,n}$ and $\text{soc} \mathbf{J}_{m,n} = \text{soc} \mathbf{W}_{m,n} = \mathbf{Q}_{m,n}$ ^{11,41}. Consequently, $\mathbf{T}_{m|n}$ (respectively, $\mathbf{W}_{m|n}$) is the maximal submodule of $\mathbf{K}_{m|n}$ (respectively, of $\mathbf{J}_{m|n}$) lying in the category $\mathbb{T}_{\mathfrak{sl}(\infty)}$ and in particular, $\mathbf{K}_{m,n}$ and $\mathbf{J}_{m|n}$ are not objects of $\mathbb{T}_{\mathfrak{sl}(\infty)}$. A new category $\mathbb{T}_{\mathfrak{sl}(\infty),2}$ of $\mathfrak{sl}(\infty)$ -modules was introduced in⁴¹ (wherein it is denoted $\mathbb{T}_{\mathfrak{g},\mathfrak{k}}$) for which $\mathbf{K}_{m,n}$ and $\mathbf{J}_{m|n}$ are injective objects.

9.5 The DS Functor on $\mathcal{O}_{m|n}^{\mathbb{Z}}$

Let X be the associated variety for $\mathfrak{gl}(m|n)$, and let $x \in X_k = \Phi(S_k)$. By Prop. 33 of⁴¹, the restriction of the functor DS_x to $\mathcal{O}_{m|n}$ is a well-defined functor to $\mathcal{O}_{m-k|n-k}$, and it follows that the further restriction to $\mathcal{O}_{m|n}^{\mathbb{Z}}$ gives a well-defined functor

$$DS_x : \mathcal{O}_{m|n}^{\mathbb{Z}} \rightarrow \mathcal{O}_{m-k|n-k}^{\mathbb{Z}}.$$

Using the naturality of the Casimir (see^{29,41}) and that DS_x is a tensor functor, we obtain that it commutes with translation functors.

The following propositions are proven in⁴¹.

Proposition 9.1 *The map $ds_x : \mathbf{K}_{m|n} \rightarrow \mathbf{K}_{m-k|n-k}$ is a homomorphism of $\mathfrak{sl}(\infty)$ -modules, and so is its restriction $ds_x : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-k|n-k}$.*

The map $ds_x : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-k|n-k}$ depends only on $k = |S|$ and not on x , so we will simply denote it by ds when $k = 1$. (Note that this does not hold for $\mathbf{K}_{m|n}$.)

The next proposition follows from Theorem 8.4 (1).

Proposition 9.2 *The kernel of $ds : \mathbf{J}_{m|n} \rightarrow \mathbf{J}_{m-1|n-1}$ is*

$$\text{Ker } ds = \mathbf{W}_{m|n}.$$

The following result is from⁴¹, Prop. 43.

Proposition 9.3 *Fix a nonzero $x \in \mathfrak{g}_{\delta_j - \varepsilon_i}$, and denote by $ds_{i,j} : \mathbf{K}_{m|n} \rightarrow \mathbf{K}_{m-1|n-1}$ the $\mathfrak{sl}(\infty)$ -module homomorphism ds_x . We have*

$$\bigcap_{i,j} \text{Ker } ds_{i,j} = \mathbf{T}_{m|n}.$$

9.6 The Socle Filtration

Here is a description of the $\mathfrak{sl}(\infty)$ -module $\mathbf{J}_{m|n}$ (see⁴¹, Cor. 29).

Theorem 9.4 *The module $\mathbf{J}_{m|n}$ is an injective hull of the simple module $\mathbf{Q}_{m|n}$, and the socle filtration of $\mathbf{J}_{m|n}$ has layers*

$$\overline{\text{soc}}^i \mathbf{J}_{m|n} \cong \left(\mathbf{V}^{(m-i)^\perp (n-i)^\perp} \right)^{\oplus (i+1)}.$$

For a proof of the following theorem, see⁴¹, Thm. 24.

Theorem 9.5 *The $\mathfrak{sl}(\infty)$ -module $\mathbf{K}_{m|n}$ is an injective hull in the category $\mathbb{T}_{\mathfrak{sl}(\infty),2}$ of the semi-simple module $\mathbf{P}_{m|n}$. Furthermore, there is an isomorphism*

$$\mathbf{K}_{m|n} \cong \bigoplus_{|\lambda|=m, |\mu|=n} \mathbf{I}^{\lambda, \mu} \otimes (Y_\lambda \otimes Y_\mu)$$

where Y_λ and Y_μ are irreducible modules over S_m and S_n , respectively, and $\mathbf{I}^{\lambda, \mu}$ is an injective hull of the simple module $\mathbf{V}^{\lambda, \mu}$ in $\mathbb{T}_{\mathfrak{sl}(\infty),2}$. The layers of the socle filtration of $\mathbf{K}_{m|n}$ are given by

$$\overline{\text{soc}}^k \mathbf{K}_{m|n} \cong \bigoplus_{|\lambda|=m, |\mu|=n} (\overline{\text{soc}}^k \mathbf{I}^{\lambda, \mu})^{\oplus (\dim Y_\lambda \dim Y_\mu)},$$

where

$$\overline{\text{soc}}^k \mathbf{I}^{\lambda, \mu} \cong \bigoplus_{\lambda', \mu' \mid |\lambda_1| + |\lambda_2| = k} \bigoplus_{\lambda', \mu'} N_{\lambda_1, \lambda_2, \lambda'}^\lambda N_{\mu_1, \mu_2, \mu'}^\mu \mathbf{V}^{\lambda', \mu'}.$$

10 Projectivity Criteria for Quasireductive Lie Superalgebras

In this section, we assume that \mathfrak{g} is a quasireductive Lie superalgebra, that is, \mathfrak{g}_0 is reductive and acts semisimply on \mathfrak{g}_1 . We discuss to what extent the associated varieties for Lie superalgebras can be used to detect projectivity in the category of finite-dimensional \mathfrak{g} -modules.

10.1 Projectivity and the Associated Variety

Let $\mathcal{F}(\mathfrak{g})$ be the category of finite-dimensional \mathfrak{g} -modules which are semisimple over \mathfrak{g}_0 . The latter condition is automatic if \mathfrak{g}_0 is semisimple. The category $\mathcal{F}(\mathfrak{g})$ has enough projective modules and injective modules. By duality, every injective object is projective and vice versa. Moreover, every indecomposable projective module is a direct summand of $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L$ for some simple \mathfrak{g}_0 -module L . We say that a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a quasireductive subalgebra if \mathfrak{k}_0 is reductive and \mathfrak{g} is a semisimple \mathfrak{k}_0 -module.

The following fact is useful.

Proposition 10.1 *Let \mathfrak{k} be a quasireductive subalgebra of \mathfrak{g} . If P is projective in $\mathcal{F}(\mathfrak{g})$ then $\text{Res}_{\mathfrak{k}} P$ is projective in $\mathcal{F}(\mathfrak{k})$.*

Proof If P is projective then it is a direct summand of the induced module $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ for some semisimple \mathfrak{g}_0 -module N . Furthermore, we have an isomorphism of \mathfrak{g}_0 -modules:

$$\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N \simeq N \otimes S^*(\mathfrak{g}_1)$$

and an isomorphism of \mathfrak{k}_0 -modules

$$\mathfrak{g}_1 = \mathfrak{k}_1 \oplus (\mathfrak{g}_1/\mathfrak{k}_1).$$

By Frobenius reciprocity the homomorphism of \mathfrak{k}_0 -modules

$$N \otimes S^*(\mathfrak{g}_1/\mathfrak{k}_1) \rightarrow \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$$

induces an isomorphism

$$\text{Ind}_{\mathfrak{k}_0}^{\mathfrak{k}} (N \otimes S^*(\mathfrak{g}_1/\mathfrak{k}_1)) \simeq \text{Res}_{\mathfrak{k}} \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N.$$

We obtain that $\text{Res}_{\mathfrak{k}} P$ is a direct summand of some module induced from a semisimple \mathfrak{k}_0 -module. Therefore, P is projective in $\mathcal{F}(\mathfrak{k})$. \square

We can now give another proof of Lemma 2.21.

Theorem 10.2 *Suppose \mathfrak{g} is quasireductive. If $M \in \mathcal{F}(\mathfrak{g})$ is projective, then $X_M = \{0\}$.*

Proof Let $x \in X$ be nonzero, and consider the subalgebra $\mathfrak{k} = k\langle x \rangle$ generated by x . Since \mathfrak{k} is quasireductive, $\text{Res}_{\mathfrak{k}} M$ is projective over \mathfrak{k} , which implies that $M_x = 0$. \square

10.2 Criteria for Type I Lie Superalgebras

In this section, we prove that for certain quasireductive Lie superalgebras the converse of Theorem 10.2 holds. We start with the following.

Lemma 10.3 *Let \mathfrak{g} be quasireductive and $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$. If $X_M = \{0\}$ then M is projective in $\mathcal{F}(\mathfrak{g})$.*

Proof Since $\mathcal{U}(\mathfrak{g}_1)$ is isomorphic to the exterior algebra $\Lambda(\mathfrak{g}_1)$ we have that $X_M = \{0\}$ implies that M is free over $\mathcal{U}(\mathfrak{g}_1)$, see¹. Then an embedding of \mathfrak{g}_0 -modules $M/\mathfrak{g}_1 M \hookrightarrow M$ induces an isomorphism $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} (M/\mathfrak{g}_1 M) \simeq M$. Therefore, M is projective. \square

Theorem 10.4 *Assume that \mathfrak{g}_0 is reductive and there exists an element h in the center of \mathfrak{g}_0 such that ad_h acts diagonally on \mathfrak{g}_1 with nonzero real eigenvalues. If $X_M = \{0\}$ then M is projective in $\mathcal{F}(\mathfrak{g})$.*

Proof Write down $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$, where \mathfrak{g}_+ (respectively, \mathfrak{g}_-) denote the span of ad_h -eigenvectors with positive (respectively, negative) eigenvalues. Since \mathfrak{g}_{\pm} are purely odd subalgebras, they are commutative; hence, $\mathfrak{p}_+ := \mathfrak{g}_0 \oplus \mathfrak{g}_+$ and $\mathfrak{p}_- := \mathfrak{g}_0 \oplus \mathfrak{g}_-$ are subalgebras satisfying the condition of Lemma 10.3. In particular, if $X_M = \{0\}$, then M is projective in $\mathcal{F}(\mathfrak{p}_{\pm})$. For a \mathfrak{g}_0 -module L set $K^{\pm}(L) := \text{Ind}_{\mathfrak{p}_{\pm}}^{\mathfrak{g}} L$. We claim that there exists a finite filtration

$$0 = M_0 \subset M_1 \subset \dots \subset M_k$$

such that $M_i/M_{i-1} \simeq K^-(L_i)$. Indeed, let L_1 be h -eigenspace with maximal eigenvalue. Then $\mathfrak{g}_+ L_1 = 0$ and we have an embedding $K^-(L_1) \subset M$. The quotient $M/K^-(L_1)$ is again free over $\mathcal{U}(\mathfrak{g}_+)$ and projective in $\mathcal{F}(\mathfrak{p}_+)$. Hence we can finish the proof by induction on dimension of M . Similarly M^* has a finite filtration with quotients isomorphic to $K^+(N_j)$. Therefore, $M \otimes M^*$ has a filtration with quotients isomorphic to $K^+(L_i) \otimes K^-(N_j) \simeq \text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} (L_i \otimes N_j)$. In other words $M \otimes M^*$ has a filtration by projective modules. Therefore $M \otimes M^*$ is projective in $\mathcal{F}(\mathfrak{g})$. Then $M \otimes M^* \otimes M$ is also projective. In

any symmetric monoidal rigid category M is a direct summand of $M \otimes M^* \otimes M$. Therefore, M is projective. \square

Corollary 10.5 Theorem 10.4 holds for $\mathfrak{gl}(m|n)$, $\mathfrak{sl}(m|n)$, $m \neq n$, $\mathfrak{osp}(2|2n)$ and $\mathfrak{p}(n)$.

Remark 10.6 Let $\mathfrak{g} = \mathfrak{sl}(1|1)$. Then it is easy to construct a \mathfrak{g} -module M such that $X_M = 0$ and M is not projective. Recall that \mathfrak{g}_1 has a basis $\{x, y\}$ and $\mathfrak{g}_0 = \mathbb{C}z$ with $[x, y] = z$, $[z, x] = [z, y] = 0$. Then $X = \mathbb{C}x \cup \mathbb{C}y$. Let $M = \mathbb{C}^{1|1}$, z acts trivially on M , and both x and y act via the same matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly M is not projective. Note that the $\mathfrak{sl}(1|1)$ -module M is not the restriction of a $\mathfrak{gl}(1|1)$ -module.

10.3 Rank Varieties

From reductivity, we have a well-defined notion of semisimple elements in \mathfrak{g}_0 , so the following definition makes sense.

Definition 10.7 We set

$$\mathfrak{g}_1^{hom} := \{x \in \mathfrak{g}_1 : [x, x] \text{ is semisimple}\}.$$

We refer to elements of \mathfrak{g}_1^{hom} as homological elements.

Remark 10.8 Clearly we have $X \subseteq \mathfrak{g}_1^{hom}$. Further \mathfrak{g}_1^{hom} is G_0 -stable, just like X . However \mathfrak{g}_1^{hom} is no longer closed in \mathfrak{g}_1 , and its geometric structure is much more complicated.

Let $x \in \mathfrak{g}_1^{hom}$ and write $h = [x, x]$. Then for M in $\mathcal{F}(\mathfrak{g})$, if we consider M^h , the fixed points of h on M , it is x -stable and further x defines a square-zero endomorphism on it. Thus we may define

$$M_x := (\text{Ker } x|_{M^h}) / (\text{Im } x|_{M^h}).$$

This defines a functor which we continue to call DS_x , the Duflo–Serganova functor for the element x . Note that the Duflo–Serganova functor as we defined it in Sect. 2 comes from the case when $h = 0$.

Remark 10.9 It is easy to check that Lemmas 2.2 and 2.4 hold for this generalization of the DS functor.

The following space was considered in¹⁹.

Definition 10.10 Let M be in $\mathcal{F}(\mathfrak{g})$ and define the rank variety of M to be

$$X_M^{rk} := \{x \in \mathfrak{g}_1^{hom} : M_x \neq 0\}.$$

Again we have that $X_M^{rk} \subseteq \mathfrak{g}_1^{hom}$ is G_0 -stable; however as is hinted in Remark 10.8, the geometric structure of X_M^{rk} can be quite complicated.

We note that rank varieties share many of the same properties as associated varieties; in particular, all properties from Lemma 2.20 continue to hold. In particular, we can use an analogous proof as in Theorem 10.2 to show that:

Proposition 10.11 Let P be projective in $\mathcal{F}(\mathfrak{g})$. Then $X_P^{rk} = \{0\}$.

We make the following conjecture:

Conjecture 10.12 Let \mathfrak{g} be quasireductive and suppose that M is in $\mathcal{F}(\mathfrak{g})$ with $X_M^{rk} = \{0\}$. Then M is projective.

A proof of this conjecture is currently forthcoming, and is being worked on by the third and fourth authors with other collaborators.

Example 10.13 Consider the example given in Remark 10.6. For $\mathfrak{g} = \mathfrak{sl}(1|1)$ we have $\mathfrak{g}_1^{hom} = \mathfrak{g}_1$. Clearly for the module M considered there, $X_M^{rk} = \{c(x - y) : c \in \mathbb{C}\}$.

11 Localization of the DS Functor

In this section, we associate to every finite-dimensional \mathfrak{g} -module a vector bundle on X with a square-zero \mathcal{O}_X -module endomorphism, which interpolates the actions of the elements of X . We relate the cohomology of this operator to the associated variety of M , and apply it to a cohomology computation for $\mathfrak{gl}(m|n)$.

11.1 Localization

Let \mathfrak{g} be a finite-dimensional Lie superalgebra, and let M be a \mathfrak{g} -module. Let \mathcal{O}_X denote the structure sheaf of X . Then $\mathcal{O}_X \otimes M$ is the sheaf of sections of the trivial vector bundle with fiber isomorphic to M . Let $\partial : \mathcal{O}_X \otimes M \rightarrow \mathcal{O}_X \otimes M$ be the map defined by

$$\partial\varphi(x) = x\varphi(x)$$

for any $x \in X$, $\varphi \in \mathcal{O}_X \otimes M$. Clearly $\partial^2 = 0$ and the cohomology \mathcal{M} of ∂ is a quasi-coherent sheaf on X . If M is finite-dimensional, then \mathcal{M} is coherent.

For any $x \in X$ denote by \mathcal{O}_x the local ring at x , by \mathcal{I}_x the maximal ideal. Let $\tilde{\mathcal{M}}_x$ be the cohomology of $\partial : \mathcal{O}_x \otimes M \rightarrow \mathcal{O}_x \otimes M$ and $\mathcal{M}_x := \tilde{\mathcal{M}}_x / \mathcal{I}_x \tilde{\mathcal{M}}_x$. The evaluation map $j_x : \mathcal{O}_x \otimes M \rightarrow M$ satisfies $j_x \circ \partial = x \circ j_x$. Hence, we have the maps

$$j_x : \text{Ker} \partial \rightarrow \text{Ker} x, j_x : \text{Im} \partial \rightarrow xM.$$

The embedding $M \hookrightarrow \mathcal{O}_x \otimes M$ ensures the surjectivity of the latter map. Thus, j_x induces the map $\bar{j}_x : \tilde{\mathcal{M}}_x \rightarrow M_x$, and $\text{Im} \bar{j}_x \cong \mathcal{M}_x$.

Remark 11.1 It is easy to see that \mathcal{M}_x is a $(\mathfrak{g}_x)_0$ -module and \bar{j}_x is a homomorphism of $(\mathfrak{g}_x)_0$ -modules.

Lemma 11.2 *Let M be a finite-dimensional \mathfrak{g} -module.*

1. *The support of \mathcal{M} is contained in X_M .*
2. *The map \bar{j}_x is surjective for a generic point $x \in X$. In particular, if $X_M = X$, then $\text{supp} \mathcal{M} = X$.*

Proof First, we will show that for any $x \in X \setminus X_M$ there exists a neighborhood U of x such that $\mathcal{M}(U) = 0$. Indeed, there exists $h_x \in \text{End}_{\mathbb{C}}(M)$ such that $x \circ h_x + h_x \circ x = id_M$. Therefore, in some neighborhood U of x there exists an $\mathcal{O}(U)$ -morphism $h : \mathcal{O}(U) \otimes M \rightarrow \mathcal{O}(U) \otimes M$ such that $\partial \circ h + h \circ \partial$ is invertible and $h(x) = h_x$. Hence the cohomology of $\partial : \mathcal{O}(U) \otimes M \rightarrow \mathcal{O}(U) \otimes M$ are trivial. In other words, $\mathcal{M}(U) = 0$. Thus, x does not belong to the support of \mathcal{M} and we have obtained that $\text{supp} \mathcal{M} \subset X_M$.

To prove (2) let $x \in X$ be a non-singular point such that $\dim M_x$ is minimal. Let $m \in \text{Ker} x_M$. Then there exists some neighborhood U of x and $\varphi \in \mathcal{O}(U) \otimes M$ such that $\partial \varphi = 0$ and $\varphi(x) = m$. By definition $\varphi \in \mathcal{M}_x$ and $\bar{j}_x(\varphi) = m$. \square

Corollary 11.3 *Let $x \in X$ be a generic point, then in some neighborhood U of x , the sheaf \mathcal{M}_U coincides with the sheaf of sections of a vector bundle with fiber M_x .*

Let $X_M \neq X$. Then \mathcal{M} is the extension by zero of the sheaf \mathcal{M}_{X_M} and \mathcal{M}_{X_M} locally is the sheaf of sections of a vector bundle with fiber $\bar{j}_x(\mathcal{M}_x)$ for a generic $x \in X_M$. Note that $\bar{j}_x(\mathcal{M}_x) \subset M_x$, but usually this is a strict embedding, as one can see from the following example.

Example 11.4 Let $\mathfrak{g} = \mathfrak{gl}(1|1)$ and M be the standard \mathfrak{g} -module. Then

$$X = \left\{ \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix} \mid uv = 0 \right\}.$$

Therefore, $X_M = \{0\}$, but a simple computation shows that $\mathcal{M} = 0$, and in particular, the support of \mathcal{M} is empty.

For $x \in X$, let G_0^x denote the stabilizer of x in G_0 . The following statement illustrates a geometric meaning of $(\mathfrak{g}_x)_1$.

Proposition 11.5 *Let $x \in X$. Then the G_0 -vector bundle $G_0 \times_{G_0^x} (\mathfrak{g}_x)_1$ is canonically isomorphic to the normal bundle to G_0x in X .*

Proof First, we compute the tangent space $T_x X$. The condition

$$= 0 \pmod{\epsilon^2}$$

is equivalent to $y \in \text{Ker} \text{ad}_x$. Therefore, $T_x X \cong (\text{Ker} \text{ad}_x)_1$. On the other hand, the tangent space $T_x(G_0x)$ to the orbit is canonically isomorphic to $[\mathfrak{g}_0, x] = (\text{Im} \text{ad}_x)_1$. Hence the normal space to G_0x in X at the point x is isomorphic to $(\mathfrak{g}_x)_1$. Using G_0 -action we obtain

$$\mathcal{N}_{G_0x} X \cong G_0 \times_{G_0^x} (\mathfrak{g}_x)_1. \quad \square$$

11.2 A Special G_0 -Invariant Subset for Basic Classical Lie Superalgebras

For this subsection, let \mathfrak{g} be a basic classical Lie superalgebra.

Let $x \in \bar{X}_k$ and

$$Y_x := \{y \in (\mathfrak{g}_x)_1 \mid [y, y] = 0\}.$$

Then

$$x + Y_x \subset X \tag{18}$$

The following is a consequence of Theorem 5.1 and Lemma 5.10.

Corollary 11.6 *Let $x \in X$ and denote by $X' \subset X$ the union of all G_0 -orbits O such that $x \in \bar{O}$. Then $X' = G_0(x + Y_x)$.*

Lemma 11.7 *Let $x \in X$. There exists a subgroup $Q \subset G_0$ satisfying the following properties*

1. $G_0^{x+y} \cap Q = \{1\}$ for any $y \in Y_x$,
2. $Q(G_0^x)$ is Zariski dense in G_0 .

Proof First, we check the statement for classical \mathfrak{g} . We denote by V the defining representation of \mathfrak{g} . Then for some subspace $V' \subset V$ we have a

decomposition $V = xV' \oplus V_x \oplus V'$, and we may assume that V' , xV' are isotropic subspaces and orthogonal to V_x in the orthosymplectic case. We set

$$Q := \{g \in G_0 \mid g|_W = \text{id}_{V'}\},$$

$$g(V_x) \subset W \oplus V_x, g|_{V_x} \equiv \text{id}_{V_x} \pmod{V'}.$$

Now let \mathfrak{g} be exceptional and $x \neq 0$. Then G_0^x is a subgroup of codimension 1 in some parabolic P with maximal normal unipotent subgroup U . We set $Q = \mathbb{C}^* \rtimes U^-$ where U^- is the opposite (complementary) to U and \mathbb{C}^* be a one-parameter subgroup in the maximal torus of G_0 which acts freely on \mathbb{C}^*x . □

Lemma 11.8 *Let $N := M_x$ and \mathcal{N}_0 denote the fiber at 0 of the sheaf \mathcal{N} on Y_x . Then there exists an injective morphism $\mathcal{N}_0 \hookrightarrow \mathcal{M}_x$.*

Proof The action map $Q \times (x + Y_x) \rightarrow X$ defines an isomorphism $a : \mathcal{U} \rightarrow Q \times Y_x$ for some Zariski dense open $\mathcal{U} \subset G'$. Denote by p the composition of a with the projection. Then $p^* : \mathcal{N} \rightarrow \mathcal{M}(\mathcal{U})$ is injective and hence induces an injection of fibers. □

Lemma 11.9 *Let $x \in X$ and K is the algebraic subgroup of G_0 with the Lie algebra $(\mathfrak{g}_x)_0$. For every $x' \in X'$ we have $G_0x' \cap (x + Y_x) = K(x + y)$ for some $y \in Y_x$. Thus we have a bijection between G_0 -orbits in X' and K -orbits in Y_x .*

Proof Let $x = \sum_{\alpha \in A} x_\alpha$ and $S' = \{B \in \mathcal{S} \mid A \subset B\}$. Then G_0 -orbits in X are in bijection with S'/W and K -orbits in Y_x are in bijection with $\mathcal{S}_{\mathfrak{g}_x}/W_{\mathfrak{g}_x}$, where $\mathcal{S}_{\mathfrak{g}_x}$ and $W_{\mathfrak{g}_x}$ are analogues of \mathcal{S} and W for \mathfrak{g}_x . The map $S' \rightarrow \mathcal{S}_{\mathfrak{g}_x}$ defined by $B \rightarrow B \setminus A$ induces the bijection $S'/W \rightarrow \mathcal{S}_{\mathfrak{g}_x}/W_{\mathfrak{g}_x}$. Hence the statement. □

11.3 Application to Cohomology of Finite-Dimensional $\mathfrak{gl}(m|n)$ -Modules

For the rest of this section, $\mathfrak{g} = \mathfrak{gl}(m|n)$. Recall the grading $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^{-1}$ and observe that the abelian subalgebra \mathfrak{g}^1 is an irreducible component of X . We can identify \mathfrak{g}^1 with $\text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m)$. Then

$$\mathfrak{g}_k^1 := X_k \cap \mathfrak{g}^1 = \{\varphi \in \text{Hom}_{\mathbb{C}}(\mathbb{C}^n, \mathbb{C}^m) \mid \text{rk} \varphi = k\},$$

\mathfrak{g}_k^1 is a single G_0 -orbit.

Let M be a \mathfrak{g} -module. The restriction \mathcal{M}^h of M to \mathfrak{g}^1 is given by the cohomology

$$\partial : \mathcal{O}_{\mathfrak{g}^1} \otimes M \rightarrow \mathcal{O}_{\mathfrak{g}^1} \otimes M,$$

where ∂ is the same as for the sheaf \mathcal{M} . The complex of global section equipped with the standard grading

$$\cdots \rightarrow S^p(\mathfrak{g}^1)^* \otimes M \rightarrow \cdots \rightarrow (\mathfrak{g}^1)^* \otimes M \rightarrow M \rightarrow 0$$

is nothing else but the Koszul complex computing the cohomology $H^\bullet(\mathfrak{g}^1, M)$. These cohomology groups are important since they are used in the Kazhdan–Lusztig theory for $\mathcal{F}(GL(m|n))$,^{4,54}. The sheaf \mathcal{M}^h can be considered as the localization of $H^\bullet(\mathfrak{g}^1, M)$ in the sense of Beilinson–Bernstein. It is clear that

$$\text{supp} \mathcal{M}^h \subset X_M \cap \mathfrak{g}^1. \tag{19}$$

Lemma 11.10 *If M admits a typical central character, then $\text{supp} \mathcal{M}^h = \{0\}$ and the fiber of \mathcal{M}^h at 0 equals $H^0(\mathfrak{g}^1, M)$.*

Proof Follows from the fact that M is a free $\mathcal{U}(\mathfrak{g}^1)$ -module and Koszul duality between $\mathcal{U}(\mathfrak{g}^1)$ and $S(\mathfrak{g}^1)$. □

Theorem 11.11 *Let M be an irreducible finite-dimensional \mathfrak{g} -module with atypicality degree k . Then $\text{supp} \mathcal{M}^h = \overline{X}_k \cap \mathfrak{g}^1$.*

Proof The inclusion $\text{supp} \mathcal{M}^h \subset \overline{X}_k \cap \mathfrak{g}^1$ follows from Theorem 6.4 and (19). To prove the equality consider $x \in \mathfrak{g}_k^1$. The fiber $\mathcal{M}_x^h \neq 0$ by Lemmas 11.10 and 11.8. □

Consider the Hilbert–Poincaré series

$$H_M(t) := \sum_{i=1}^{\infty} \dim H^i(\mathfrak{g}^1, M) t^i.$$

The Hilbert–Serre Theorem and Theorem 11.11 imply

Corollary 11.12 *Let M be an irreducible finite-dimensional \mathfrak{g} -module with atypicality degree k . Then*

$$H_M(t) = \frac{q(t)}{(1-t)^{k(m+n-k)}}$$

for some polynomial $q(t)$.

Proof The degree in the denominator equals $\dim \mathfrak{g}_k^1 = k(m+n-k)$. □

12 The DS Functor on Simple Modules

In this section, we discuss what is known about the action of DS_x on simple modules for classical Lie superalgebras. Serganova originally conjectured that these functors are semisimple when \mathfrak{g} is basic classical, meaning that they takes semisimple modules to semisimple modules. Following the work of^{27,38} this is now a theorem. For $\mathfrak{p}(n)$ these functors are known not to be semisimple; however, by the work of¹⁷, the composition factors of $DS_x(L)$ for x of rank one when L is a simple module.

The case of $\mathfrak{q}(n)$ was studied in³⁰; in this case, it is interesting to study DS_x on simple modules for all $x \in \mathfrak{g}_1^{rk}$ (see Sect. 10.3), not only for those x with $[x, x] = 0$. Note, however, that \mathfrak{g}_1^{rk} has infinitely many G_0 -orbits. In³⁰, DS_x is computed on simple modules and shown to be semisimple for x of rank 1. In general it is expected that DS_x will be always be semisimple. We will not discuss this case here any further.

12.1 General Results

We begin with a general statement. Recall that if N is a \mathfrak{g} -module and L is a simple \mathfrak{g} -module, we write $[N : L]_{non}$ for the ungraded Jordan-Hölder multiplicity of L in N , meaning for the number of times both L and ΠL appear as Jordan-Hölder factors of N .

The following result is a compilation of results from^{17,27,38,48}.

Theorem 12.1 *Let \mathfrak{g} be one of the Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, $\mathfrak{p}(n)$, or a simple exceptional Lie superalgebra. Let L be a simple \mathfrak{g} -module, x a rank one odd root vector (see Definition 4.2), and L' a simple \mathfrak{g}_x -module.*

1. If $\mathfrak{g} \neq \mathfrak{p}(n)$, then $DS_x(L)$ is a semisimple \mathfrak{g}_x -module.
2. We have $[DS_x(L) : L']_{non} \leq 2$; if $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{p}(n)$ then $DS_x(L)$ is multiplicity free.
3. (Purity) If \mathfrak{g} is basic classical ($\mathfrak{g} \neq \mathfrak{p}(n)$) then we have

$$[DS_x(L) : L'] [DS_x(L) : \Pi L'] = 0.$$

4. For $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, and $\mathfrak{p}(n)$, the composition factors of $DS_x(L)$ are determined by removing maximal arcs from the arc diagram associated with L (see the subsections below for explanations on the arc diagrams for each case).

Remark 12.2 In (2) of Theorem 12.1 the multiplicity bound states that $[DS_x L : L']_{non} \leq 1$ when \mathfrak{g} is of type I, i.e., when it has a \mathbb{Z} -grading (i.e., $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2|2n)$, or $\mathfrak{p}(n)$).

The proof of these bounds (and the rest of the results above) are still case-dependent; general proofs are unknown but would be of great interest.

Remark 12.3 There is an elegant explanation of the purity property, i.e., part (3) of Theorem 12.1, which is explained by Gorelik in²⁶. For \mathfrak{g} basic classical there exists a semisimple subcategory $\mathcal{C}(\mathfrak{g})$ of the category of finite-dimensional modules such that for any simple module L of \mathfrak{g} , exactly one of L or ΠL lies in $\mathcal{C}(\mathfrak{g})$. These semisimple subcategories can be chosen so that if L lies in $\mathcal{C}(\mathfrak{g})$ then $DS_x L$ lies in $\mathcal{C}(\mathfrak{g}_x)$, which of course implies part (3) of Theorem 12.1.

Further, it is possible to choose these semisimple subcategories so that when \mathfrak{g} is reductive (i.e., $\mathfrak{g} = \mathfrak{g}_0$ or $\mathfrak{g} = \mathfrak{osp}(1|2n)$), $\mathcal{C}(\mathfrak{g})$ contains all simple modules of positive superdimension. Using this and the fact that the DS functor preserves superdimension, one can obtain combinatorial formulas given by a sum of non-negative numbers for the superdimension of any simple module. This was done in³⁸ for $\mathfrak{gl}(m|n)$. For $\mathfrak{p}(n)$ the superdimension was computed in¹⁷.

Before explaining part (4) of Theorem 12.1 and beginning the discussion of arc diagrams, we state a result from which we compute the value of any DS functor for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$ on any simple module.

Let $x_r \in X$ be a rank r element of the associated variety of \mathfrak{g} . Write DS^1 for the functor obtained by applying DS_x for a rank one vector x .

Theorem 12.4 (See^{27,38}) *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ or $\mathfrak{osp}(m|2n)$. For a simple \mathfrak{g} -module L we have an isomorphism of $DS_{x_r}(\mathfrak{g}) \cong DS^1(DS^1(\dots DS^1(\mathfrak{g}) \dots))$ -modules:*

$$DS_{x_r}(L) \cong DS^1(DS^1(\dots DS^1(L) \dots)).$$

We will now discuss arc diagrams and part (4) of Theorem 12.1, after which we will explain the case of the exceptional superalgebras.

Remark 12.5 As we will see, the calculus of arc diagrams below will explain how to compute DS^1 on simple modules with integral weights. By Sect. 8.1, this is enough to compute DS^1 on all of $\mathcal{F}(\mathfrak{g})$.

12.2 An Overview of Arc Diagrams

We now begin the explanation of part (4) of Theorem 12.1, which will consume the rest of this section. For the classical series $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(m|2n)$, and $\mathfrak{p}(n)$, there is a remarkable thread which links the computations of the composition factors of $DS_x L$ for a simple module L , namely arc diagrams. (For $\mathfrak{gl}(m|n)$ and $\mathfrak{p}(n)$ these diagrams have also been called cap/cup diagrams, in³⁸ for $\mathfrak{gl}(m|n)$ and in¹⁷ for $\mathfrak{p}(n)$; we have changed the name for the sake of consistency.) These arc diagrams are defined individually for each superalgebra and provide a combinatorial tool to study this question.

We summarize the situation as follows. For each of the four Lie superalgebras listed above, we explain a procedure which associates to each simple module L an arc diagram, which consists of symbols lying on (half)-integer points on the real line, along with arcs which connect them. These arcs sometimes are nested within one another, giving rise to the notion of maximal arcs, those which do not lie beneath another arc. Then, as is stated in Theorem 12.1, the composition factors of $DS_x L$ are given by the simple modules whose associated arc diagram is obtained by removing one maximal arc from the arc diagram of L . As will be seen, the procedure for defining arc diagrams is different for each superalgebra.

The idea of using arc diagrams to study the representations of Lie superalgebras goes back to the work of Brundan and Stroppel, where they realized the category $\text{Rep}GL(m|n)$ as a certain diagram algebra of Khovanov type (see⁵). Their arc diagrams are, notation aside, in essence the same as what will define for $\mathfrak{gl}(m|n)$ below.

For the orthosymplectic supergroup, Gruson and Serganova used arc diagrams for ‘tailless’ dominant weights in³⁶. More recently, Ehrig and Stroppel have done similar work on realizing $\text{Rep}OSp(m|2n)$ as a certain diagram algebra, (see¹⁵). Their diagram algebra is related to type D Khovanov algebras; however, their arc diagrams differ from those used in²⁷ to study the action of DS_x on simple modules. A dictionary to go between them is described in Appendix A of²⁷.

Remark 12.6 There is an interesting link between arc diagrams and the computations of character formulas for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$ (see²⁸) as well as for $\mathfrak{q}(n)$ (see⁶⁰). A similar connection is expected for $\mathfrak{p}(n)$ as well.

We now begin our case by case explanations of arc diagrams. We will write $\Lambda_{m|n}$ for the free \mathbb{Z} -module with basis $\epsilon_1, \dots, \epsilon_m, \delta_1, \dots, \delta_n$ which will be used for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m|2n)$, and

$\mathfrak{osp}(2m + 1|2n)$. For these superalgebras we define a parity homomorphism $p : \Lambda_{\mathfrak{g}} \rightarrow \mathbb{Z}_2 = \{0, 1\}$ by $p(\epsilon_i) = 0$ and $p(\delta_j) = 1$ for all i, j , and extending linearly. For $\mathfrak{g} = \mathfrak{p}(n)$ we will use that lattice Λ_n , which is the free \mathbb{Z} -module with basis $\epsilon_1, \dots, \epsilon_n$.

12.3 $\mathfrak{gl}(m|n)$ Case

The $\mathfrak{g} = \mathfrak{gl}(m|n)$ case is due to³⁸, and we refer the reader there for full details and more in-depth results.

We take the Borel subalgebra corresponding to the simple roots

$$\epsilon_1 - \epsilon_2, \dots, \epsilon_m - \delta_1, \dots, \delta_{n-1} - \delta_n.$$

Let

$$\rho = -\epsilon_2 - 2\epsilon_3 - \dots - (m-1)\epsilon_m + (m-1)\delta_1 + (m-2)\delta_2 + \dots + (m-n)\delta_n.$$

We identify $\lambda \in \Lambda_{m|n}$ with the $(m|n)$ -tuple of integers

$$(a_1, \dots, a_m | b_1, \dots, b_n),$$

where

$$\lambda + \rho = a_1\epsilon_1 + \dots + a_m\epsilon_m - b_1\delta_1 - \dots - b_n\delta_n.$$

We write $\Lambda^+(\mathfrak{gl}(m|n))$ for the set of dominant weights in $\Lambda_{m|n}$ with respect to this Borel. Then λ is dominant if and only if $a_1 > \dots > a_m$ and $b_1 > \dots > b_n$.

12.3.1 Weight and Arc Diagrams

Write $I_<(\lambda) = \{a_1, \dots, a_m\}$ and $I_>(\lambda) = \{b_1, \dots, b_n\}$. Then define the weight diagram associated with λ to be the following labelling: $f_\lambda : \mathbb{Z} \rightarrow \{\times, \circ, <, >\}$:

$$f_\lambda(k) = \begin{cases} \times & k \in I_<(\lambda) \cap I_>(\lambda); \\ \circ & k \notin I_<(\lambda) \cup I_>(\lambda); \\ < & k \in I_<(\lambda) \setminus (I_>(\lambda) \cap I_<(\lambda)); \\ > & k \in I_>(\lambda) \setminus (I_<(\lambda) \cap I_>(\lambda)); \end{cases}$$

The correspondence $\lambda \mapsto f_\lambda$ defines a bijection between $\Lambda^+(\mathfrak{gl}(m|n))$ and the labellings of \mathbb{Z} by the appropriate number of the symbols $\times, \circ, <, >$.

Remark 12.7 It is not hard to check that the atypicality of a dominant weight λ is equal to the number of symbols \times in its weight diagram.

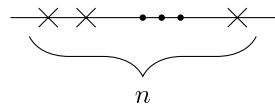
Given a weight diagram f_λ we associate an arc diagram via the following inductive construction. Connect an arc between $i < j$ if:

1. $f(i) = \times, f(j) = \circ$; and

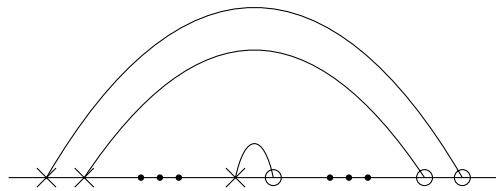
- for all k satisfying $i < k < j$ and $f_{\bar{\lambda}}(k) = \circ$, k already lies on a previously drawn arc.

In other words, if $f(i) = \times$, $f(j) = \circ$, and for all $i < k < j$ we have $f(k) \notin \{\times, \circ\}$, then we connect i and j by an arc. Then we continue drawing arcs inductively according to the above procedure.

Example 12.8 For $\mathfrak{g} = \mathfrak{gl}(n|n)$, the trivial weight $\lambda = 0$ has the following weight diagram:



The arc diagram is given by

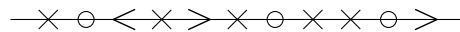


Clearly $DS_x \mathbb{C} = \mathbb{C}$, and from the diagram we also see that when removing the only maximal arc we obtain the arc diagram of the trivial module for $\mathfrak{gl}(n-1|n-1)$.

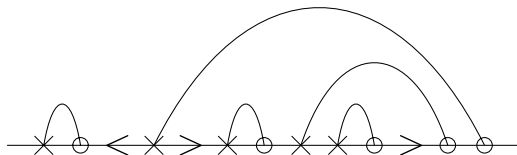
Example 12.9 For $\mathfrak{gl}(6|7)$ consider the weight

$$\lambda = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 + \epsilon_5 - 2\delta_2 - 2\delta_3 - 2\delta_4 - 3\delta_5 - 3\delta_6 - 6\delta_7.$$

Its weight diagram looks as follows:



Its arc diagram is given by



12.3.2 dex and Simple Modules

For $\lambda \in \Lambda^+(\mathfrak{gl}(m|n))$, to properly specify the parity of $L(\lambda)$ we need to briefly explain the equivalences of blocks for $\mathfrak{gl}(m|n)$. Namely, every block of atypicality k for $\mathfrak{gl}(m|n)$ is equivalent to the principal block of $\mathfrak{gl}(k|k)$.

This equivalence defines a correspondence on simple modules, and thus on dominant weights, which we denote by $\lambda \mapsto \bar{\lambda}$, and it works as follows.

In the weight diagram of λ , move all core symbols (i.e. $>$, $<$) to the right of the symbols \times

by simply swapping adjacent symbols one at a time. This pictorial procedure corresponds to applying translation functors between different blocks of the same atypicality. After moving all core symbols to the right, we simply remove them from the diagram, leaving us with a diagram only with the symbols \times , and thus it will correspond to a dominant weight $\bar{\lambda}$ in the principal block of $\mathfrak{gl}(k|k)$.

For example, for the simple module of Example 12.9, the atypicality is 5 and the weight diagram of $\bar{\lambda}$ is given by



Remark 12.10 The equivalence of categories here described commutes with the application of DS , and thus it in fact suffices to understand how DS acts on the principal block of $\mathfrak{gl}(k|k)$, although we will explain the general case for $\mathfrak{gl}(m|n)$. However, for $\mathfrak{osp}(m|2n)$ we will use this principal and thus only explain in full how DS acts on the principal blocks of certain superalgebras.

Definition 12.11 For $\lambda \in \Lambda^+(\mathfrak{gl}(m|n))$, we define

$$\text{dex} \lambda := p(\bar{\lambda}).$$

Then we set $L(\lambda)$ to be the simple module of highest weight λ such that the parity of the highest weight vector is $\text{dex} \lambda$.

Example 12.12 Consider an integer multiple of the Berezinian weight of $\mathfrak{gl}(n|n)$, that is, for $k \in \mathbb{Z}$,

$$\lambda = k(\epsilon_1 + \dots + \epsilon_n - \delta_1 - \dots - \delta_n).$$

Its weight diagram is a translation of the weight diagram of the trivial module. We have $\text{dex}(\lambda) = kn \pmod 2$.

Example 12.13 For λ as in Example 12.9, we have $\text{dex} \lambda = 1$.

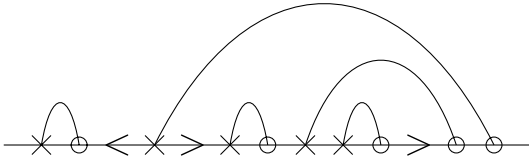
Theorem 12.14 [38] For $\lambda \in \Lambda^+(\mathfrak{gl}(m|n))$, we have

$$DS_x L(\lambda) = \bigoplus_i \Pi^{n_i} L(\lambda_i),$$

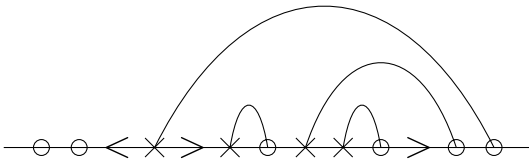
where λ_i are the weights which correspond to the arc diagrams obtained by removing a single maximal arc from the arc diagram of λ , and $n_i = \text{dex} \lambda - \text{dex} \lambda_i$.

Remark 12.15 For $\mathfrak{gl}(m|n)$ there are two conjugacy classes of rank one odd root vectors, but as is explained in³⁸ the action of the corresponding Duflo–Serganova functors on simple modules is the same up to isomorphism.

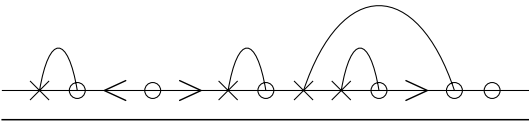
Example 12.16 For $\mathfrak{gl}(6|7)$ consider the weight introduced in Example 12.9. We recall its arc diagram is given by



To apply DS_x to $L(\lambda)$ we remove the maximal arcs from the diagram to obtain two new arc diagrams:



which corresponds to the weight $\lambda_1 = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 + \epsilon_4 + \epsilon_5 - 2\delta_1 - 2\delta_2 - 2\delta_3 - 3\delta_4 - 3\delta_5 - 4\delta_6$ which has $\text{dex}\lambda_1 = 0$.



$$\Sigma := \begin{cases} \epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \dots, \epsilon_k - \delta_k, \delta_k & \text{for } \mathfrak{osp}(2k + 1|2k) \\ \delta_1 - \epsilon_1, \epsilon_1 - \delta_2, \dots, \epsilon_{k-1} - \delta_k, \delta_k \pm \epsilon_k & \text{for } \mathfrak{osp}(2k|2k) \\ \epsilon_1 - \delta_1, \delta_1 - \epsilon_2, \dots, \epsilon_k - \delta_k, \delta_k \pm \epsilon_{k+1} & \text{for } \mathfrak{osp}(2k + 2|2k). \end{cases}$$

which corresponds to the weight $\lambda_2 = 3\epsilon_1 + 3\epsilon_2 + 2\epsilon_3 - \epsilon_5 + \delta_1 - 2\delta_2 - 2\delta_3 - 3\delta_4 - 3\delta_5 - 6\delta_6$ which has $\text{dex}\lambda_2 = 0$. Thus, we have

$$DS_x L(\lambda) = L(\lambda_1) \oplus L(\lambda_2).$$

12.4 $\mathfrak{osp}(m|2n)$ Case

For a full explanation of the \mathfrak{osp} case with many examples, see²⁷. Below we closely follow the treatment given in²⁶.

We have the following equivalences of categories that are obtained via a functor which respects the action of DS .

- A block of atypicality k for $\mathfrak{osp}(2m + 1|2n)$ is equivalent to the principal block for $\mathfrak{osp}(2k + 1|2k)$.

- For $\mathfrak{osp}(2m|2n)$ with $m, n > 0$, a block of atypicality k is equivalent to the principal block of either $\mathfrak{osp}(2k|2k)$ or $\mathfrak{osp}(2k + 2|2k)$.

In this way, we obtain that every block for $\mathfrak{osp}(m|2n)$ is equivalent to the principal block of $\mathfrak{osp}(2k + t|2k)$ for some k and some $t = 0, 1, 2$, and so it suffices to understand how DS_x acts on modules in these blocks. We will deal with these three cases individually, and refer to them according to the value of t .

Remark 12.17 There are a number of parallels between the principal blocks of $\mathfrak{osp}(2k + 1|2k)$ and $\mathfrak{osp}(2k + 2|2k)$. In particular, in²⁷, they find an explicit bijection τ between simple modules such that it respects the action of the DS functor, meaning we have an equality of multiplicity numbers $[DS_x(\tau(L)) : \tau(L)] = [DS_x L : L]$.

It is an important open question whether there is an equivalence of categories between these principal blocks, and in particular if there is one which commutes with the DS functor.

12.4.1 The weight lattice of $\mathfrak{osp}(2k + t|2k)$ is given by $\Lambda_{k+\ell|k}$, where we set $\ell = 0$ for $t = 0, 1$ and $\ell = 1$ for $t = 2$

We fix triangular decompositions corresponding to the “mixed” bases:

We have $\rho = 0$ for $t = 0, 2$ and $\rho = \frac{1}{2} \sum_{i=1}^k (\delta_i - \epsilon_i)$ for $t = 1$.

12.4.2 Highest Weights in the Principal Block

For $\lambda \in \Lambda_{k+\ell|k}$ we set

$$a_i := -(\lambda|\delta_i)$$

. Write $\Lambda^0(\mathfrak{osp}(2k + t|2k))$ for the dominant weights of $\mathfrak{osp}(2k + t|2k)$ which lie in the principal block. By³⁵, $\lambda \in \Lambda^0(\mathfrak{osp}(2k + t|2k))$ if and only if a_1, \dots, a_k are non-negative integers with $a_{i+1} > a_i$ or $a_i = a_{i+1} = 0$, and

$$\lambda + \rho = \begin{cases} \sum_{i=1}^{k-1} a_i(\varepsilon_i + \delta_i) + a_k(\delta_k + \xi \varepsilon_k) & \text{for } t = 0 \\ \sum_{i=1}^k a_i(\varepsilon_i + \delta_i) & \text{for } t = 2 \\ \sum_{i=1}^{s-1} (a_i + \frac{1}{2})(\varepsilon_i + \delta_i) + \frac{1}{2}(\delta_s + \xi \varepsilon_s) + \sum_{i=s+1}^k \frac{1}{2}(\delta_i - \varepsilon_i) & \text{for } t = 1 \end{cases}$$

for $\xi \in \{\pm 1\}$. For $t = 1$ we have $1 \leq s \leq k + 1$ and $a_s = a_{s+1} = \dots = a_k = 0$ if $s \leq k$ (for $s = k + 1$ we have $\lambda + \rho = \sum_{i=1}^k (a_i + \frac{1}{2})(\varepsilon_i + \delta_i)$).

12.4.3 Weight Diagrams

Take $\lambda \in \Lambda^0(\mathfrak{osp}(2k + t|2k))$ and define a_i for $i = 1, \dots, k$ as above. We assign to λ a weight diagram f_λ , which is a number line with one or several symbols drawn at each position with non-negative integral coordinate:

- we put the sign \times at each position with the coordinate a_i ;
- for $t = 2$ we add $>$ at the zero position;
- we add the “empty symbol” \circ to all empty positions.

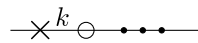
For $t \neq 2$ a weight $\lambda \in \Lambda^0(\mathfrak{osp}(2k + t|2k))$ is not uniquely determined by the weight diagram constructed by the above procedure. Therefore, for $t = 0$ with $a_k \neq 0$ and for $t = 1$ with $s \leq k$, we write the sign of ξ before the diagram ($+$ if $\xi = 1$ and $-$ if $\xi = -1$).

Notice that each position with a nonzero coordinate contains either \times or \circ . For $t = 0, 1$ the zero position is occupied either by \circ or by several symbols \times ; we write this as \times^i for $i \geq 0$. Similarly, for $t = 2$ the zero position is occupied by \times^i with $i \geq 0$.

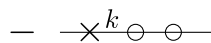
Remark 12.18 The weight diagrams we have defined are essentially the same as those defined in³⁵, except that when $t = 1$ we shift by $-1/2$.

12.4.4 Examples

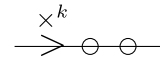
The weight diagram of 0 is



where the three small \bullet s together are an ellipsis, indicating the diagram continues with \circ s. We omit these in future diagrams. For $t = 0$,

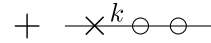


for $t = 1$, and



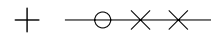
for $t = 2$.

The diagram



corresponds to $\mathfrak{osp}(2k + 1|2k)$ -weight $\lambda = \varepsilon_1$.

The diagram



corresponds to the $\mathfrak{osp}(4|4)$ -weight $\lambda = \lambda + \rho = (\varepsilon_2 + \delta_2) + 2(\varepsilon_1 + \delta_1)$.

The empty diagram correspond to $\mathfrak{osp}(0|0) = \mathfrak{osp}(1|0) = 0$; the diagram $>$ corresponds to the weight 0 for $\mathfrak{osp}(2|0) = \mathbb{C}$.

12.4.5

The definition of weight diagrams defines a one to one correspondence between dominant weights for $\mathfrak{osp}(2k + t|2k)$ and (some-time signed) weight diagrams with $n \times$ symbols and one symbol $>$ if $t = 2$, satisfying certain conditions.

For $t = 0$ (respectively, $t = 1$) a diagram f_λ in has a sign if and only if $f_\lambda(0) = \circ$ (respectively, $f_\lambda(0) \neq \circ$).

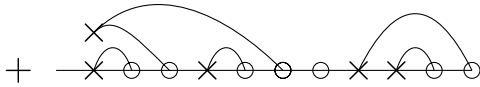
12.4.5 Arc Diagrams

We associate an arc diagram to each weight diagram constructed according to the following steps:

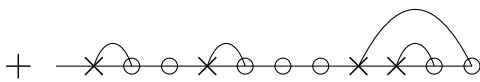
1. For $0 < i < j$ with $f_\lambda(i) = \times$ and $f_\lambda(j) = \circ$, connect an arc from i to j if for all k with $i < k < j$ and $f_\lambda(k), k$ already lies on an arc.
2. If there is at least one \times at 0, order them from top to bottom. If $t = 0, 1$ then draw a single arc from the bottom \times to the nearest position with \circ . If $t = 2$, draw two arcs emanating from the bottom \times to the two nearest positions with \circ not already lying on an arc. Then for any t , for each \times at 0 above the bottom one (and working from bottom to top), draw two arcs from the \times to the two nearest positions \circ not already on an arc.

In what follows we refer to the arcs (either one or two) which lie on a

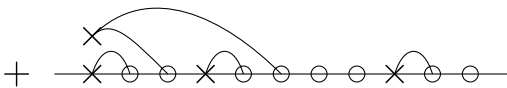
single \times as just an arc. For example, consider the arc diagram associated with the weight $\lambda = 9\epsilon_1 + 8\epsilon_2 + 4\epsilon_3 + \epsilon_4 + 8\delta_1 + 7\delta_2 + 3\delta_3$ for $\mathfrak{osp}(11|10)$; in this case $\lambda + \rho = (8 + 1/2)(\epsilon_1 + \delta_1) + (7 + 1/2)(\epsilon_2 + \delta_2) + (3 + 1/2)(\epsilon_3 + \delta_3) + 1/2(\epsilon_4 + \delta_4) + 1/2(\delta_5 - \epsilon_5)$:



There are two maximal arcs in the above diagram: one which consists of the two arcs emanating from the top \times at 0. If we remove this top arc we obtain the diagram corresponding to the weight $\lambda_1 = 9\epsilon_1 + 8\epsilon_2 + 4\epsilon_3 + \epsilon_4 + 8\delta_1 + 7\delta_2 + 3\delta_3$ for $\mathfrak{osp}(9|8)$:



The other maximal arc is the one emanating from the \times lying at 7. If we remove it, we obtain the diagram corresponding to the weight $\lambda_2 = 9\epsilon_1 + 4\epsilon_2 + \epsilon_3 + 8\delta_1 + 3\delta_2$ of $\mathfrak{osp}(9|8)$:



12.4.6 dex and Simple Modules

Given a dominant weight λ , we define

$$\|\lambda\| = \sum_{i=1}^k a_i - \ell(k - \text{tail}(\lambda))$$

where $\text{tail}(\lambda)$ denotes the number of symbols \times at 0 when $\ell = 1$ (we omit the definition for other cases since we do not need it). The we define

$$\text{dex}\lambda := \|\lambda\| \bmod 2.$$

We will write $L(\lambda)$ for the simple module of highest weight λ , where the parity of the highest weight vector is given by $\text{dex}(\lambda)$.

Theorem 12.19 ([²⁷, Thm. 8.2)

- (i) Let $\lambda \in \Lambda^0(\mathfrak{osp}(2k + t|2k))$ and $v \in \Lambda^0(\mathfrak{osp}(2k + t - 2|2k - 2))$. Then $[DS_x L(\lambda) : L(v)]_{\text{non}} \neq 0$ if and only if the arc diagram of v can be obtained from the arc diagram of λ by removing a maximal arc. If $t \neq 1$, then the sign of v and λ (if relevant)

need not agree, while if $t = 1$ then if v has sign it must agree with the sign of λ .

- (ii) Let e denote the number of free positions (i.e., those with \circ and not attached to any arc) to the left of the maximal arc removed to obtain v . For $t = 1, 2$ we have:

$$[DS_x(L(\lambda)) : L(v)] = \begin{cases} (1|0) & e = 0; \\ (2|0) & e > 0 \text{ and even}; \\ (0|2) & e \text{ odd}. \end{cases}$$

For $t = 0$ we have

$$[DS_x(L(\lambda)) : L(v)] = \begin{cases} (1|0) & e \text{ even}; \\ (0|1) & e \text{ odd}. \end{cases}$$

Remark 12.20 For every m, n with $m > 0$, the Lie superalgebra $\mathfrak{osp}(2m|2n)$ admits an involution σ which comes from a reflection of its Kac–Dynkin diagram. This involution is the same as the involution σ_x defined in Sect. 3.1.

In²⁷ it is proven via a general argument that for a simple $\mathfrak{osp}(2m|2n)$ -module L , we have $DS_x(L^\sigma) \cong DS_x L$, and $(DS_x L)^{\sigma_x} \cong DS_x L$, where σ_x is the corresponding involution for $\mathfrak{osp}(2m - 2|2n - 2)$.

It would be interesting to know if these isomorphisms hold for all finite-dimensional $\mathfrak{osp}(2m|2n)$ -modules.

We now give an example of the above theorem for each case $t = 0, 1, 2$:

Example 12.21 For $t = 1$, consider the weight $\lambda = 9\epsilon_1 + 8\epsilon_2 + 4\epsilon_3 + \epsilon_4 + 8\delta_1 + 7\delta_2 + 2\delta_3$. We have $\text{dex}\lambda = 0$, and we looked at the weight diagram already in Sect. 12.4.6. By Theorem 12.19 we have

$$DS_x L(\lambda) = L(\lambda_1) \oplus \Pi L(\lambda_2)^{\oplus 2}.$$

Note for λ_1 we have $e = 0$ and $\lambda_2, e = 1$; hence, the parities are as shown.

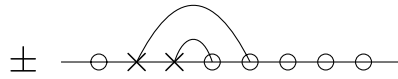
Example 12.22 For $t = 0$ consider the weight $\lambda = 6(\delta_1 + \epsilon_1) + 2(\delta_2 + \epsilon_2) + (\delta_3 - \epsilon_3)$ for $\mathfrak{osp}(6|6)$, which has $\text{dex}\lambda = 1$ and arc diagram:



There are two maximal arcs. Removing the arc starting at position one gives the arc diagram associated with $\lambda_1^\pm = 6(\delta_1 + \epsilon_1) + 2(\delta_2 \pm \epsilon_2)$ with $e = 1$:



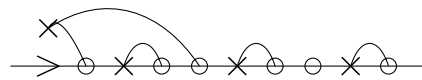
Removing the maximal arc starting at position 6 gives the arc diagrams associated with $\lambda_2^\pm = 2(\delta_1 + \epsilon_1) + (\delta_2 \pm \epsilon_2)$ with $e = 2$:



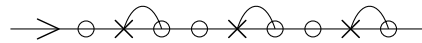
Thus we have that

$$DS_x L(\lambda) = \Pi L(\lambda_1^+) \oplus \Pi L(\lambda_1^-) \oplus L(\lambda_2^+) \oplus L(\lambda_2^-).$$

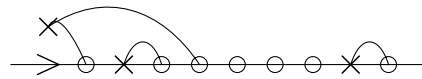
Example 12.23 In the $t = 2$ case consider the weight $\lambda = 8(\epsilon_1 + \delta_1) + 5(\epsilon_2 + \delta_2) + 8(\epsilon_3 + \delta_3)$ for $\mathfrak{osp}(6|4)$ with $\text{dex}\lambda = 0$ and arc diagram:



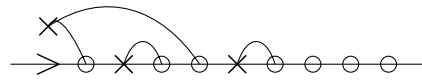
There are three maximal arcs. If we remove the one emanating from 0, we obtain the arc diagram associated with the weight $\lambda_1 = 8(\epsilon_1 + \delta_1) + 5(\epsilon_2 + \delta_2) + 2(\epsilon_3 + \delta_3)$ with $e = 0$:



If we remove the maximal arc emanating from 5 we obtain the arc diagram associated with the weight $\lambda_2 = 8(\epsilon_1 + \delta_1) + 2(\epsilon_2 + \delta_2)$ with $e = 0$:



Finally, if we remove the arc emanating from 8 we obtain the arc diagram associated with the weight $\lambda_3 = 5(\epsilon_1 + \delta_1) + 2(\epsilon_2 + \delta_2)$ with $e = 1$:



It follows that we have

$$DS_x L(\lambda) = L(\lambda_1) \oplus L(\lambda_2) \oplus \Pi L(\lambda_3)^{\oplus 2}.$$

12.5 $\mathfrak{p}(n)$ Case

We refer to¹⁷ for a full explanation of the $\mathfrak{p}(n)$ case with examples.

We write Λ_n for the \mathbb{Z} -module spanned by $\epsilon_1, \dots, \epsilon_n$. We fix the following simple roots for $\mathfrak{p}(n)$:

$$\pm\epsilon_n - \epsilon_{n-1}, \epsilon_{n-1} - \epsilon_{n-2}, \dots, \epsilon_2 - \epsilon_1.$$

Write $\Lambda^+(\mathfrak{p}(n))$ for the set of dominant integral weights with respect to the corresponding Borel subalgebra. Let

$$\rho = \epsilon_2 + 2\epsilon_3 + \dots + (n - 1)\epsilon_n.$$

For $\lambda \in \Lambda^{\mathfrak{p}(n)}$ we write

$$\lambda + \rho = a_1\epsilon + \dots + a_n\epsilon_n.$$

Then the dominance condition is precisely that $a_1 < \dots < a_n$.

Given $\lambda \in \Lambda^+(\mathfrak{p}(n))$ we write $L(\lambda)$ for the irreducible representation corresponding to λ such that the highest weight vector is even.

12.5.1 Weight and Arc Diagrams

To $\lambda \in \Lambda^+(\mathfrak{p}(n))$ we define the associated weight diagram $f_\lambda : \mathbb{Z} \rightarrow \{\circ, \bullet\}$ by $f_\lambda(a_i) = \bullet$ and $f_\lambda(j) = \circ$ if $n \neq a_i$ for all i .

Now we define an arc diagram associated with λ according to the same approach as for $\mathfrak{gl}(m|n)$, except we work from left to right now, i.e., from negative to positive integers. More explicitly, we draw an arc connecting i and j if $f_\lambda(i) = \circ$, $f_\lambda(j) = \bullet$ and all k with $i < k < j$ already lie on an arc.

Theorem 12.24 ([¹⁷]) *Let $\lambda \in \Lambda^+(\mathfrak{p}(n))$ and $\mu \in \Lambda^+(\mathfrak{p}(n - 1))$. Then $\Pi^z L(\mu)$ appears as a factor of $DS_x L(\lambda)$ for some z if and only if the arc diagram of μ can be obtained from the arc diagram of λ by removing a maximal arc. In this case, z is equal to the number of arcs to the right of the one removed. Further, $DS_x L(\lambda)$ is multiplicity-free.*

We can now give the proof of Theorem 8.1 for $\mathfrak{p}(n)$.

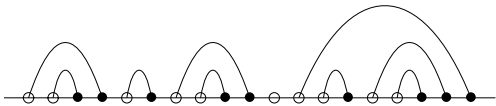
Corollary 12.25 *The map $ds^1 : \mathcal{K}_-(\mathfrak{p}(n)) \rightarrow \mathcal{K}_-(\mathfrak{p}(n - 1))$ is surjective.*

Proof In particular $DS_x L$ is multiplicity-free for a simple \mathfrak{g} -module L , and the factors are obtained by removing maximal arcs. So let L' be a simple \mathfrak{g}_x -module, with arc diagram f' . Let f_1 be the arc diagram obtained by adding a symbol \bullet to the first free space to the right of all symbols \bullet of f' . Then this new symbol will give a maximal arc in f_1 . If we write L_1 for the irreducible representation corresponding to f_1 , then in $\mathcal{K}_-(\mathfrak{p}(n - 1))$ we will have a multiplicity-free sum

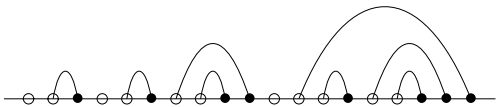
$$ds_x[L_1] = \pm[L'] + \sum_{L''} \pm[L''].$$

By switching the parity of L_1 , we can assume the sign in front of $[L^i]$ is positive. Now we can induct on the length of the furthest most right string of symbols \bullet in the arc diagram, for which all L'' 's will have a longer length than L , giving surjectivity. □

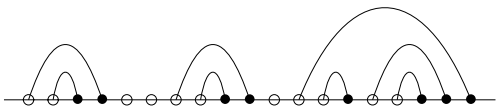
Example 12.26 (The following example is taken from¹⁷) Let $n = 9$, and consider the dominant weight $\lambda = \epsilon_3 + 3\epsilon_4 + 3\epsilon_5 + 6\epsilon_6 + 8\epsilon_7 + 8\epsilon_8 + 8\epsilon_9$. We draw the associated arc diagram below:



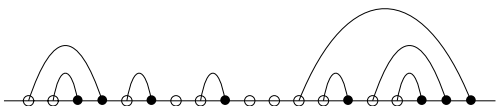
We see that there are 4 maximal arcs in the diagram, thus $DS_x L(\lambda)$ has 4 simple factors $\Pi L(\lambda_1)$, $L(\lambda_2)$, $L(\lambda_3)$, and $L(\lambda_4)$. They are listed with their arc diagrams below, along with the corresponding value of z : $\lambda_1 = 2\epsilon_2 + 4\epsilon_3 + 4\epsilon_4 + 7\epsilon_5 + 9\epsilon_6 + 9\epsilon_7 + 9\epsilon_8$, $z = 7$:



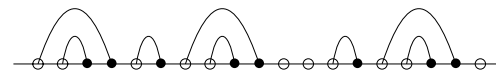
$\lambda_2 = 4\epsilon_3 + 4\epsilon_4 + 7\epsilon_5 + 9\epsilon_6 + 9\epsilon_7 + 9\epsilon_8, z = 6$:



$\lambda_3 = \epsilon_3 + 3\epsilon_4 + 7\epsilon_5 + 9\epsilon_6 + 9\epsilon_7 + 9\epsilon_8, z = 4$:



$\lambda_4 = \epsilon_3 + 3\epsilon_4 + 3\epsilon_5 + 6\epsilon_6 + 8\epsilon_7 + 8\epsilon_8, z = 0$:



12.6 Exceptional Cases

We now explain the case of the exceptional Lie superalgebras $G(3)$, $F(4)$, or $D(2|1; a)$. These all have defect one, and their atypical blocks have one of the following extension graphs:

$$A_\infty : \dots \text{---} L^{-1} \text{---} L^0 \text{---} L^1 \text{---} \dots$$

$$D_\infty : \begin{array}{ccccccc} L^1 & \text{---} & L^2 & \text{---} & L^3 & \text{---} & L^4 & \text{---} & \dots \\ & & | & & & & & & \\ & & L^0 & & & & & & \end{array}$$

Specifically, A_∞ will be the extension graph for certain blocks of $F(4)$ and for blocks of $D(2|1; a)$ when $a \in \mathbb{Q}$. On the other hand D_∞ will be the extension graph for all blocks of $G(3)$, along with certain blocks of $F(4)$ and $D(2|1; a)$.

Remark 12.27 Note that it is clear that the above extension graphs are bipartite. Extension graphs are in fact always bipartite for basic classical Lie superalgebras, as was hinted in Remark 12.3 and is shown in²⁶.

The following lemma is determined from the full relations on the extension graphs for each block, which are described in^{22,48}.

Lemma 12.28 *Let $P(L^i)$ be the projective indecomposable cover of a simple, non-projective module L^i over an exceptional Lie superalgebra. Then the radical and socle filtrations of $P(L^i)$ coincide, with socle and cosocle isomorphic to L , and middle layer isomorphic to*

$$\bigoplus_{j \in \text{Adj}(i)} L^j$$

where $\text{Adj}(j)$ denotes the vertices adjacent to i in the extension graph containing L^i .

Using the above lemma, we get the following.

Proposition 12.29 *Let \mathcal{B} be an atypical block for an exceptional Lie superalgebra \mathfrak{g} , and let $x \in \mathfrak{g}_1$ be non-zero such that $[x, x] = 0$. Suppose that for some simple module L in \mathcal{B} , L_x is pure. Then L_x is pure for all simples L in \mathcal{B} . Further we have the following isomorphisms of \mathfrak{g}_x -modules:*

$$\text{Ext}(\mathcal{B}) = A_\infty : L_x^i \cong \Pi^i L_x^0$$

$$\text{Ext}(\mathcal{B}) = D_\infty : L_x^0 \cong L_x^1, L_x^i \cong \Pi^{i-1}(L_x^0)^{\oplus 2} \text{ for } i \geq 2.$$

By $\text{Ext}(\mathcal{B})$ we denote the extension graph of \mathcal{B} .

Proof For any simple module L^i in \mathcal{B} we have a short exact sequence

$$0 \rightarrow M \rightarrow P(L^i) \rightarrow L^i \rightarrow 0,$$

where $P(L^i)$ denotes the projective cover of L^i and M is its radical. By Lemma 2.7 and that $P(L^i)_x = 0$, we find that $M_x \cong \Pi L_x^i$. On the other hand, by Lemma 12.28, we have the short exact sequence

$$0 \rightarrow L^i \rightarrow M \rightarrow \bigoplus_{j \in \text{Adj}(i)} L^j \rightarrow 0$$

Suppose that L_x^i is pure, so that $\text{Hom}(L_x^i, M_x) = 0$. Then we obtain by Lemma 2.7 the short exact sequence (using that $M_x \cong \Pi L_x^i$):

$$0 \rightarrow \Pi L_x^i \rightarrow \bigoplus_{j \in \text{Adj}(i)} L_x^j \rightarrow \Pi L_x^i \rightarrow 0.$$

Using connectedness of the extension graph, purity of L_x^j for any j easily follows, along with the formulas in the case of each type of block. \square

To complete the description of the DS functor for exceptional Lie superalgebras, we need to compute the value of the DS functor on one module in every atypical block, and in particular check that it is pure so that Proposition 12.29 will apply.

Thus let \mathfrak{g} be one of the Lie superalgebras $D(2|1; a)$, $G(3)$, $F(4)$. Let \mathfrak{h} be a Cartan subalgebra of \mathfrak{g}_0 . We denote by W the Weyl group of \mathfrak{g}_0 and by $(-|-)$ the symmetric non-degenerate form on \mathfrak{h}^* which is induced by a non-degenerate invariant form on \mathfrak{g} .

Let Σ be a base of \mathfrak{g} which contains an isotropic root β . Fix a non-zero $x \in \mathfrak{g}_\beta$. Set $\Delta_x := (\beta^\perp \cap \Delta) \setminus \{\beta, -\beta\}$. By Proposition 4.5, \mathfrak{g}_x can be identified with a subalgebra of \mathfrak{g} generated by the root spaces \mathfrak{g}_α with $\alpha \in \Delta_x$ and a Cartan subalgebra $\mathfrak{h}_x \subset \mathfrak{h}$. If Δ_x is not empty, then Δ_x is the root system of the Lie superalgebra \mathfrak{g}_x and one can choose Σ_x in Δ_x such that $\Delta^+(\Sigma_x) = \Delta^+ \cap \Delta_x$. For $\mathfrak{g} = D(2|1; a)$, $G(3)$, $F(4)$ one has $\mathfrak{g}_x = \mathbb{C}$, \mathfrak{sl}_2 , \mathfrak{sl}_3 , respectively.

Lemma 12.30 *Let $L := L(\lambda)$ be a finite-dimensional module and $(\lambda|\beta) = 0$. Set $L' := L_{\mathfrak{g}_x}(\lambda|_{\mathfrak{h}_x})$. One has*

$$DS_x(L) \cong \begin{cases} L' & \text{for } G(3) \\ L' & \text{for } D(2|1; a), F(4) \text{ if } L' \cong (L')^* \\ L' \oplus (L')^* & \text{for } D(2|1; a), F(4) \text{ if } L' \not\cong (L')^*. \end{cases}$$

Proof It is easy to see that $[DS_x(L) : L'] = 1$. Set $\lambda' := \lambda|_{\mathfrak{h}_x}$. By Sect. 6.5, $DS_x(L)$ is a typical module and each simple subquotient of $DS_x(L)$ is of the form $L_{\mathfrak{g}_x}(v)$ with $v \in \{\lambda', \sigma(\lambda')\}$, where $\sigma = \text{Id}$ for $\mathfrak{g} = G(3)$, $\sigma = -\text{Id}$ for $D(2|1; a)$, and σ is the Dynkin diagram automorphism of $\mathfrak{g}_x = \mathfrak{sl}_3$ in $F(4)$ -case. This gives the first formula. For $D(2|1; a)$, $F(4)$ one has

$L_{\mathfrak{g}_x}(v)^* \cong L_{\mathfrak{g}_x}(\sigma(v))$; giving the second formula. Finally in $D(2|1; a)$, $F(4)$ the Weyl group contains $-\text{Id}$, so $L \cong L^*$ and thus $DS_x(L) \cong DS_x(L^*)$ by Lemma 2.4, implying the third formula. \square

We fix a triangular decomposition of \mathfrak{g}_0 and denote by Δ_0^+ the corresponding set of positive roots. We consider all bases Σ for Δ which satisfy $\Delta_0^+ \subset \Delta^+(\Sigma)$. We say that an isotropic root β is of the *first type* if β lies in a base Σ with $\Delta_0^+ \subset \Delta^+(\Sigma)$.

Take any base Σ as above and denote by ρ the corresponding Weyl vector. It is easy to see that a simple atypical module $L = L(v)$ satisfies the assumptions of Lemma 12.30 for some Σ' and $\beta \in \Sigma'$ if and only if $v + \rho$ is orthogonal to an isotropic root of the first type.

Let \mathcal{B} be an atypical block of \mathfrak{g} . We call the block containing the trivial module $L(0)$ a *principal block*. Clearly, $DS_x(L(0))$ is the trivial \mathfrak{g}_x -module, so Proposition 12.29 gives $DS_x(L)$ for each simple module L in \mathcal{B}_0 .

Combining Proposition 12.29 and Lemma 12.30, we see that to compute $DS_x(L)$ for each simple L in \mathcal{B} , it is enough to find $L(v) \in \text{Irr}(\mathcal{B})$ such that $v + \rho$ is orthogonal to an isotropic root of the first type. Below we will list such v for each non-principal atypical block for $D(2|1; a)$, $F(4)$ and $G(3)$.

12.6.1 Case $D(2|1; a)$

For $\mathfrak{g} := D(2|1; a)$ one has $\mathfrak{g}_x = \mathbb{C}$. The atypical blocks were described in²², Thm. 3.1.1.

The extension graph of the principal block \mathcal{B}_0 is D_∞ , so for a simple L^i in \mathcal{B}_0 we have $DS_x(L^i) = \mathbb{C}$ for $i = 0, 1$ and $DS_x(L^i) = \Pi^{i-1}(\mathbb{C})^{\oplus 2}$ for $i > 1$ (where \mathbb{C} stands for the trivial even \mathfrak{g}_x -module).

If a is irrational, the principal block is the only

atypical block in $\mathcal{F}(\mathfrak{g})$. Consider the case when a is rational. Recall that \mathfrak{h}^* has an orthogonal basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ with

$$(\varepsilon_1, \varepsilon_1) = -\frac{1+a}{2}, \quad (\varepsilon_2, \varepsilon_2) = \frac{1}{2}, \quad (\varepsilon_3, \varepsilon_3) = \frac{a}{2}.$$

One has

$$D(2|1; 1) = \mathfrak{osp}(4|2),$$

$$D(2|1; a) \cong D(2|1; -1 - a) \cong D(2|1; a^{-1})$$

so we can assume that $0 < a < 1$ and write $a = \frac{p}{q}$,

where p, q are relatively prime positive integers.

The atypical blocks are \mathcal{B}_k for $k \in \mathbb{N}$ (the principal block is \mathcal{B}_0). Consider the block \mathcal{B}_k with $k > 0$. The extension graph of \mathcal{B}_k is A_∞ . By²², Thm. 3.1.1, the block \mathcal{B}_k contains a simple module L with the highest weight $\lambda_{k;0}$ satisfying $(\lambda_{k;0} + \rho|\beta) = 0$ for

$$\beta := \varepsilon_1 + \varepsilon_2 - \varepsilon_3.$$

Taking $x \in \mathfrak{g}_\beta$ we can identify \mathfrak{g}_x with $\mathbb{C}h$ for $h := \varepsilon_1^* - \varepsilon_2^*$ (where $\varepsilon_1^*, \varepsilon_2^*, \varepsilon_3^*$ is the dual basis in \mathfrak{h}). By Lemma 12.30 we get

$$DS_x(L) = L_{\mathfrak{g}_x}(k) \oplus L_{\mathfrak{g}_x}(-k),$$

where $L_{\mathfrak{g}_x}(u)$ stands for the even one-dimensional \mathfrak{g}_x -module with h acting by $k(p + q)$. By Proposition 12.29, $DS_x(L^i) \cong \Pi^i(DS_x(L))$ for each $L^i \in \text{Irr}(\mathcal{B}_k)$ (for $k > 0$).

12.6.2 Case G(3)

For $\mathfrak{g} := G(3)$ the atypical blocks were described in Thm. 4.1.1 of²². The atypical blocks in $\mathcal{F}(\mathfrak{g})$ are \mathcal{B}_k for $k \in \mathbb{N}$; the extension graphs are D_∞ . The block \mathcal{B}_k contains a simple module with the highest weight $\lambda_{k;0}$ satisfying $(\lambda_{k;0} + \rho|\beta) = 0$ for

$$\beta := -\varepsilon_1 + \delta.$$

Taking $\Sigma := \{\delta - \varepsilon_1, \varepsilon_2 - \delta, \delta\}$ and $x \in \mathfrak{g}_\beta$ we can identify \mathfrak{g}_x with \mathfrak{sl}_2 -triple corresponding to the root $\alpha = \varepsilon_1 + 2\varepsilon_2$. One has $\lambda_{k;0} = k\alpha$. Combining Lemma 12.30 and Proposition 12.29 we get

$$DS_x(L^0) \cong DS_x(L^1) \cong L_{\mathfrak{sl}_2}(2k),$$

$$DS_x(L^i) = \Pi^{i-1}(L_{\mathfrak{sl}_2}(2k))^{\oplus 2} \text{ for } i > 1.$$

12.6.3 Case F(4)

For $\mathfrak{g} := F(4)$ we have $\mathfrak{g}_x \cong \mathfrak{sl}_3$. The integral weight lattice is spanned by $\varepsilon_1, \varepsilon_2, \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)$ and $\frac{1}{2}\delta$; the parity is given by $p(\frac{\varepsilon_i}{2}) = 0$ and $p(\frac{\delta}{2}) = 1$.

The atypical blocks are described in Thm. 2.1 of⁴⁸. These blocks are parametrized by the pairs (m_1, m_2) , where $m_1, m_2 \in \mathbb{N}$, $m_1 \geq m_2$, and $m_1 - m_2$ is divisible by 3. We denote the corresponding block by $\mathcal{B}_{(m_1; m_2)}$.

The extension graph of $\mathcal{B}_{(i; i)}$ is D_∞ ; the block $\mathcal{B}_{(0; 0)}$ is principal. For $i > 0$ the block $\mathcal{B}_{(i; i)}$ contains a simple module $L(\lambda)$ with

$$\lambda + \rho = (i + 1)(\varepsilon_1 + \varepsilon_2) - \beta_1,$$

$$\text{where } \beta_1 := \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 - \varepsilon_3 + \delta).$$

One has $(\lambda + \rho|\beta_1) = 0$. Take $x \in \mathfrak{g}_{\beta_1}$ and consider the base

$$\Sigma_1 := \{\beta_1; \frac{1}{2}(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \delta); \varepsilon_3; \varepsilon_1 - \varepsilon_2\}.$$

Then \mathfrak{g}_x can be identified with \mathfrak{sl}_3 corresponding to the set of simple roots $\{\varepsilon_2 + \varepsilon_3; \varepsilon_1 - \varepsilon_3\}$ and Lemma 12.30 gives

$$DS_x(L(\lambda)) = L_{\mathfrak{sl}_3}(i\omega_1 + i\omega_2),$$

where ω_1, ω_2 are the fundamental weights of \mathfrak{sl}_3 . By Proposition 12.29 we get for the simple module L^j in $\mathcal{B}_{(i; i)}$:

$$DS_x(L^0) \cong DS_x(L^1) \cong L_{\mathfrak{sl}_3}(i\omega_1 + i\omega_2),$$

$$DS_x(L^j) \cong \Pi^{j-1}(L_{\mathfrak{sl}_3}(i\omega_1 + i\omega_2))^{\oplus 2} \text{ for } j > 1.$$

Consider a block $\mathcal{B}_{(i_1; i_2)}$ for $i_1 \neq i_2$. The extension graph of this block is A_∞ and this block contains a simple module $L := L(\lambda')$ with

$$\lambda' + \rho = i_1\varepsilon_1 + i_2\varepsilon_2 + (i_1 - i_2)\varepsilon_3.$$

In particular, $(\lambda' + \rho|\beta_2) = 0$ for $\beta_2 := \frac{1}{2}(-\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \delta)$. Taking $x \in \mathfrak{g}_{\beta_1}$ and

$$\Sigma_2 := \{\beta_2; \varepsilon_2 - \varepsilon_3; -\beta_1; \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta)\}$$

we identify \mathfrak{g}_x with \mathfrak{sl}_3 corresponding to the set of simple roots $\{\varepsilon_2 - \varepsilon_3; \varepsilon_1 + \varepsilon_3\}$. Combining Lemma 12.30 and Proposition 12.29 we get

$$DS_x(L) = L_{\mathfrak{sl}_3}(i_1\omega_1 + i_2\omega_2) \oplus L_{\mathfrak{sl}_3}(i_2\omega_1 + i_1\omega_2),$$

$$DS_x(L^i) \cong \Pi^i(DS_x(L))$$

for each L^i in the block $\mathcal{B}_{(i_1; i_2)}$.

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Acknowledgements

We would like to thank Kevin Coulembier, Inna Entova-Aizenbud, Thorsten Heidersdorf, Vladimir Hinich, Victor Kac, Victor Ostrik, Ivan Penkov, Julia Pevtsova, Shifra Reif, and Ilya Zakharevich for helpful comments and suggestions. Needless to say, this paper would not have been possible without the original contribution of Michel Duflo. In addition, we thank the referees for very thorough reviews of an earlier version of this article.

Declarations

Funding

M.G. was supported by ISF Grant 1957/21. C.H. was supported by ISF Grant 1221/17. V.S. was supported by NSF Grant 2001191. A.S. was supported by ISF Grant 711/18 and NSF-BSF Grant 2019694.

Conflict of interest

The authors have no conflicts of interest.

Received: 28 April 2022 Accepted: 29 July 2022

Published online: 17 May 2022

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