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SPECIAL TOPICS

# Isomagnetovortical perturbations and wave energy of MHD flows

**Yasuhide Fukumoto**[1] · **Rong Zou**[2]

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## Abstract

The ideal magnetohydrodynamics (MHD) is a Hamiltonian system of infinite degrees of freedom and wave energy plays an important role in stability and bifurcation of a steady state. A formula of the energy of waves on a steady incompressible isentropic flow of the ideal MHD is established, which facilitates its calculation when the vorticity and the magnetic field is localized. For definition of the wave energy, the isovortical perturbation for a neutral fluid is extended to the isomagnetovortical perturbation for the MHD. Evolution equations of the two Lagrangian displacement fields included in it are derived from the two-fluid model. Equivalence of several energy formulas is proved.

**Keywords** Ideal MHD · Wave energy · Isomagnetovortical perturbation · Lagrangian displacement field

## 1 Introduction

Stability and bifurcation of MHD flows have broad application to both laboratory research as exemplified by the tokamak, the plasma confinement equipment, and to astrophysical phenomena as exemplified by the formation mechanism of a star from an accretion disk. According to Krein's theory of Hamiltonian spectra Krein (1950), the signature of wave energy plays a vital role for the stability criterion

Yasuhide Fukumoto and Rong Zou contributed equally to this work.

✉ Rong Zou
zourong@hawaii.edu

Yasuhide Fukumoto
yasuhide@imi.kyushu-u.ac.jp

[1] Institute of Mathematics for Industry, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

[2] Civil and Environmental Engineering Department, University of Hawaii, 2540 Dole Street, Honolulu, HI 96822, USA

Arnold (1989); Arnold and Khesin (1998); Morrison (1998). Coexistence of two modes with opposite signed energy or of zero-energy modes is necessary for triggering instability. For flows subject to three dimensional perturbations, negative-energy modes are ubiquitously excited together with positive ones Fukumoto (2003); Ilgisonis et al. (2009) and, therefore, generally speaking to maintain a flow stable is rather difficult.

Arnold's theorem for the hydrodynamics Arnold (1989) states that a steady Euler flow is the extremal of the kinetic energy with respect to the isovortical perturbations, which preserves the local circulation, and that the energy of perturbations, second order in amplitude, is expressible in terms solely of the first-order disturbance field. By taking advantage of this theorem, Arnold Arnold (1966) derived the following formula of the wave energy $\delta^2 H$ for the ideal incompressible flow (see also ref Kop'ev and Chernyshev (2000); Fukumoto and Hirota (2008); Fukumoto et al. (2011)):

$$\delta^2 H = \frac{1}{2} \int \boldsymbol{\omega} \cdot \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\xi} \right) d^3 x, \tag{1}$$

where $\boldsymbol{\omega}(\boldsymbol{x}, t)$ is the vorticity field of the basic flow and $\boldsymbol{\xi}(\boldsymbol{x}, t)$ is the Lagrangian displacement field, with infinitesimal amplitude, of fluid particles as functions of the position $\boldsymbol{x}$ and the time $t$ governed by the Frieman–Rotenberg equation Friemann and Rotenberg (1960); Goedbloed et al. (2010). There are several expressions of the formula for energy of waves Ilgisonis et al. (2009); Fukumoto et al. (2011) and even for continuous spectra as well Hirota and Fukumoto (2008a, 2008b). The above formula is, among others, useful when the vorticity is localized in a compact region, such as a slender tube and a thin layer. In Appendix 1, we show utility of (1) for calculating the wave energy for the Rankine vortex, a circular vortex with uniform vorticity in the core.

We intend to extend the energy formula (1) to the ideal MHD and the extended MHD. For the ideal MHD, the conservation law of the local circulation is destroyed by the Lorentz force, which requires the modification of the isovortical perturbations. The so called isomagnetovortical perturbations were heuristically constructed for incompressible Vladimirov et al. (1999) and compressible flows Isichenko (1998). The ideal MHD is described by a non-canonical Hamiltonian equation with Lie–Poisson bracket Morrison et al. (1980); Holm and Kupershmidt (1983), which admit several Casimir invariants Hameiri (2004). The isomagnetovortical perturbations preserve all the Casimirs and are automatically created by taking the Hamiltonian to be an arbitrary functional of the MHD variables Hameiri (2003). Alternatively, from the viewpoint of the Lagrangian description, these perturbations locally preserve the entropy of a fluid element, the mass of a material volume and the magnetic flux of a material surface, without specifying the advecting velocity field, and may be called the kinematically accessible perturbations. It should be born in mind that, in addition, the cross-helicity is among the Casimirs Hameiri (2003, 2004) and that it is characterized by the invariance with respect to particle relabeling as the variational symmetry Padhye and Morrison (1996); Webb and Zank (2007); Fukumoto and Sakuma (2013). Arnold's theorem carries over to the ideal MHD; a steady MHD flow is the extremal of the Hamiltonian of the MHD with respect to the isomagnetovortical perturbations Hameiri (2003); Vladimirov et al. (1999); Hirota and Fukumoto (2008a).

The energy of an incompressible MHD flow includes the magnetic energy in addition to the kinetic energy. Description of the variation of the magnetic field necessitates a second Lagrangian displacement field $\boldsymbol{\eta}$. Extension of the energy formula (1) to the MHD necessarily includes both $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. We are requested to find the relation of $\boldsymbol{\eta}$ to given perturbations of the magnetic field as well as of the hydrodynamic variables. This relation was sought from the MHD Vladimirov et al. (1999) and the Hall MHD equations Hirota et al. (2006). Recently, by an ingenious geometric analysis of the two-fluid model in the context of the extended MHD, the isomagnetovortical perturbations were derived from the extended Frieman–Rotenberg equations in the incompressible case and the formula for the wave energy in terms of the Lagrangian displacements was derived Hirota (2021). In this investigation, we establish a formula of the energy of isomagnetovortical waves on a steady MHD flow of an ideal incompressible fluid in the form extended from (1) in a straightforward and comprehensive manner, and then prove its equivalence to the known formulas. As a necessary ingredient, evolution equation of $\boldsymbol{\eta}$ is derived from those for the fundamental relations which hold individually for the Lagrangian displacement fields of the ions and the electrons.

In Sect. 2, we reproduce the non-canonical Hamiltonian structure of the ideal MHD system and then, in Sect. 3, deduce the isomagnetovortical perturbations. By introducing the two Lagrangian displacement fields and two scalar functions, of infinitesimal amplitude, which are left arbitrary, we express the state variables of the ideal MHD in such a way to preserve all the Casimirs, with the effects of compressibility and baroclinicity taken into account. Section 4 writes out the evolution equation of the two Lagrangian displacements $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, together with Appendix 2, where, given the isomagnetovortical perturbations, they are derived from the two-fluid model with allowance made for the Hall and the electron–inertia effects. In Sect. 5, an energy formula is derived in the form of an extension of (1) augmented with the counterpart of the magnetic field. The last section (Sect. 6) is devoted to summary and conclusions, where comments are given to possible relevance of (1) and the formula of Sect. 5 to the variational principle for the motion of vortex filaments and to future investigations. Proof of equivalence between the formulas of the wave energy is given in Appendices 3 and 4.

## 2 Ideal MHD system

Let us consider motion of an inviscid electrically conducting fluid of infinite conductivity, namely, the ideal magnetohydrodynamics (MHD). The basic equations governing the ideal MHD consist of the conservation laws of the momentum and the mass, the induction law and the assumptions of the adiabatic motion and solenoidal magnetic field:

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t}+\boldsymbol{u}\cdot\nabla\boldsymbol{u}\right)=-\nabla p+\boldsymbol{J}\times\boldsymbol{B}, \tag{2}$$

$$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{u})=0, \tag{3}$$

$$\frac{\partial s}{\partial t}+\boldsymbol{u}\cdot\nabla s=0, \tag{4}$$

$$\frac{\partial \boldsymbol{B}}{\partial t}=\nabla\times(\boldsymbol{u}\times\boldsymbol{B}), \tag{5}$$

$$\nabla\cdot\boldsymbol{B}=0, \tag{6}$$

where $\rho, p, s, \boldsymbol{u}$ and $\boldsymbol{B}$ represent the density, the pressure, the entropy per unit mass, the fluid velocity and the magnetic field, respectively, and $\boldsymbol{J}=\nabla\times\boldsymbol{B}$ is the electron current density. For the boundary conditions, the velocity and the magnetic fields are assumed not to penetrate through the boundary, that is

$$\boldsymbol{u}\cdot\boldsymbol{n}=0,\ \boldsymbol{B}\cdot\boldsymbol{n}=0 \quad \text{on the boundary}\ , \tag{7}$$

where $\boldsymbol{n}$ is the unit normal vector to the boundary. Let $\boldsymbol{A}$ be the vector potential for the magnetic field and it satisfies

$$\frac{\partial \boldsymbol{A}}{\partial t}=\boldsymbol{u}\times(\nabla\times\boldsymbol{A})+\nabla\phi. \tag{8}$$

For the coupled system of (2–7) of the MHD equations, the mass of a material volume, 3-form, the magnetic-flux through a material surface element, 2-form, and the magnetic–helicity density (with the gauge for A chosen, so that $\boldsymbol{\phi}=-\boldsymbol{u}\cdot\boldsymbol{A}$) and the specific entropy, both being 0-forms, are advected by the flow Webb, et al. (2014):

$$\begin{aligned}
&\frac{D}{Dt}(\rho \mathrm{d}V)=0,\\
&\frac{D}{Dt}(\boldsymbol{B}\cdot \mathrm{d}\boldsymbol{S})=0,\\
&\frac{D}{Dt}\left(\frac{\boldsymbol{A}\cdot\boldsymbol{B}}{\rho}\right)=0,\\
&\frac{Ds}{Dt}=0,
\end{aligned} \tag{9}$$

where

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla=\frac{\partial}{\partial t}+\mathcal{L}_{\boldsymbol{u}}, \tag{10}$$

is the Lagrangian derivative and $\mathcal{L}_{\boldsymbol{u}}$ is the Lie derivative with respect to the vector field $\boldsymbol{u}$. The ideal MHD equations (6) are a Hamiltonian system and are describable as the Poisson equation Morrison et al. (1980); Holm and Kupershmidt (1983); Morrison (1998); Hameiri (2003):

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \{F, H\}, \tag{11}$$

where $F(w,\ t)$ is a functional of the state variables and, in this paper, we take $w = (\boldsymbol{M}, \boldsymbol{B}, \rho, s)$, with $\boldsymbol{M} = \rho\boldsymbol{u}$ being the momentum density. The Hamiltonian $H$ is

$$H = \int \left\{ \frac{1}{2}\rho\boldsymbol{u}^2 + \frac{1}{2}\boldsymbol{B}^2 + \rho e(\rho, s) \right\} \mathrm{d}^3x, \tag{12}$$

where $e$ is the internal energy per unit mass. The Lie–Poisson bracket for any functional $F(w, t)$ and $G(w, t)$ is given by

$$\begin{aligned}\{F, G\} = \int \Big\{ &\boldsymbol{M} \cdot \left[ \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{M}} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{M}} \right] \\ &+ \boldsymbol{B} \cdot \left[ \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{B}} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{B}} + \left( \nabla \frac{\delta G}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} - \left( \nabla \frac{\delta F}{\delta \boldsymbol{M}} \right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} \right] \\ &+ \rho \left[ \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho} \right] + s\nabla \cdot \left[ \frac{\delta G}{\delta \boldsymbol{M}} \frac{\delta F}{\delta s} - \frac{\delta F}{\delta \boldsymbol{M}} \frac{\delta G}{\delta s} \right] \Big\} \mathrm{d}^3x.\end{aligned} \tag{13}$$

By partial integration, the Lie–Poisson bracket (13) can be rewritten in the form as

$$\{F, G\} = \int \frac{\delta F}{\delta w} \mathcal{J} \frac{\delta G}{\delta w} \mathrm{d}^3x, \tag{14}$$

where $\mathcal{J}$ is an antisymmetric operator and is written explicitly as

$$\mathcal{J} = \begin{bmatrix} -(\nabla\circ) \cdot M - \nabla \cdot (\circ M) & -(\nabla\circ) \cdot B + \nabla \cdot (B\circ) & -\rho\nabla & \nabla s \\ -\nabla \cdot (\circ B) + B \cdot \nabla & 0 & 0 & 0 \\ -\nabla \cdot (\rho\circ) & 0 & 0 & 0 \\ -(\circ \cdot \nabla)s & 0 & 0 & 0 \end{bmatrix}, \tag{15}$$

with $\circ$ denoting the position of the elements operated by $\mathcal{J}$ Morrison et al. (1980); Hirota and Fukumoto (2008a). The Poisson equation (11) yields equations for $w$ as

$$\frac{\partial w}{\partial t} = \mathcal{J} \frac{\delta H}{\delta w}, \tag{16}$$

which coincides with Eqs. (2–5).

An equilibrium state satisfies $\mathcal{J}\delta H/\delta w = 0$, and therefore, $\{F, H\} = 0$ for any $F(w,\ t)$. The degeneracy of the Poisson bracket admits Casimir invariants, constant functionals $C(w,\ t)$:

$$\{C, G\} = 0, \tag{17}$$

for any functional $G(w, t)$. Therefore, a Casimir satisfies

$$\mathcal{J}\frac{\delta C}{\delta w} = 0. \tag{18}$$

The total mass $\int \rho d^3x$ and the total entropy $\int \rho s d^3x$ are Casimirs by the mass conservation law and the assumption of adiabatic motion. The magnetic helicity $\int \boldsymbol{A}\cdot\boldsymbol{B}d^3x$ is the well-known Casimir. In addition, the cross-helicity is qualified as a Casimir, though there is some difficulty in showing that it satisfies (18) Hameiri (2004); Webb, et al. (2014).

## 3 Isomagnetovortical perturbations

By leaving the Hamiltonian $H$ an arbitrary functional, the Poisson equation (11) automatically generates perturbations, called the dynamically accessible variation, that preserve all the Casimirs Morrison (1998). This is applied to the MHD Hameiri (2003, 2004); Hirota and Fukumoto (2008a) and to the extended MHD Kaltsas et al. (2021). For a neutral fluid, such a perturbation is referred to as the isovortical perturbation Arnold (1989); Arnold and Khesin (1998). Following ref Vladimirov et al. (1999), we may call its MHD version the *isomagnetovortical perturbation*.[1]

Taking an arbitrary functional $K$ in place of $H$ and denoting the virtual time to be $\tau$, $\mathrm{d}F/\mathrm{d}\tau = \{F, K\}$ generates the isomagnetovortical perturbation:

$$\delta w = \frac{\partial w}{\partial \tau}\Big|_{\tau=0} = \mathcal{J}\frac{\delta K}{\delta w}. \tag{19}$$

The functional derivative $\delta K/\delta w$ is evaluated at $\tau = 0$. Denoting $\chi = (\boldsymbol{\xi}, \boldsymbol{\zeta}, \lambda, \sigma) = (\delta K/\delta \boldsymbol{M}, \delta K/\delta \boldsymbol{B}, \delta K/\delta\rho, \delta K/\delta s)$, a collection of arbitrary vector and scalar fields, the perturbation is written as

$$\frac{\partial \boldsymbol{u}}{\partial \tau} = \boldsymbol{\xi}\times(\nabla\times\boldsymbol{u}) + (\nabla\times\boldsymbol{\zeta})\times\boldsymbol{B}/\rho - \nabla(\lambda + \boldsymbol{\xi}\cdot\boldsymbol{u}) + \sigma\nabla s, \tag{20}$$

$$\frac{\partial \rho}{\partial \tau} = -\nabla\cdot(\rho\boldsymbol{\xi}), \tag{21}$$

$$\frac{\partial s}{\partial \tau} = -\boldsymbol{\xi}\cdot\nabla s, \tag{22}$$

$$\frac{\partial \boldsymbol{B}}{\partial \tau} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}), \tag{23}$$

where with abuse of notation, $\boldsymbol{u}, \boldsymbol{B}, \rho, s$ denote the basic or the unperturbed state. To be precise, for the variations of the state variables, we have to translate as

[1] This kind of perturbation may be alternatively called the kinematically accessible perturbations, since it is driven by arbitrary velocity field and arbitrary current field Hirota and Fukumoto (2008a, 2008b).

$(\partial w/\partial\tau|_{\tau=0})\tau \rightarrow \delta w$, together with $\chi \rightarrow \chi\tau$. Any functional $G$ can be expanded with respect to infinitesimal $\tau$ as

$$G = G|_{\tau=0} + \frac{\mathrm{d}G}{\mathrm{d}\tau}|_{\tau=0}\tau + \frac{1}{2}\frac{\mathrm{d}^2 G}{\mathrm{d}\tau^2}|_{\tau=0}\tau^2 + \cdots . \tag{24}$$

By virtue of (18), the following theorems hold Hameiri (2003).

**Theorem 1** *For any Hamiltonian K, the first variation of a Casimir C vanishes, that is*

$$\frac{\mathrm{d}C}{\mathrm{d}\tau} = \{C, K\} = -\{K, C\} = -\int \frac{\delta K}{\delta w}\mathcal{J}\frac{\delta C}{\delta w}\mathrm{d}x \equiv 0. \tag{25}$$

This is a strong conclusion and does not require the flow to be at an equilibrium. Otherwise stated, (25) is valid over the whole range of $\tau$, and therefore, the following is true.

**Theorem 2** *For kinematically accessible disturbance, any Casimir satisfies*

$$\frac{\mathrm{d}^n C}{\mathrm{d}\tau^n} = 0 \quad \text{for } \forall n. \tag{26}$$

Extension of Arnold's theorem to the ideal MHD, which states that an equilibrium attains the extremum, is represented compactly by vanishment of the first-order variation of the Hamiltonian at the equilibrium as Hameiri (2003)

$$\delta H = \frac{\mathrm{d}H}{\mathrm{d}\tau} = \{H, K\} = -\{K, H\} = -\int \frac{\delta K}{\delta w}\mathcal{J}\frac{\delta H}{\delta w}\mathrm{d}^3 x = 0. \tag{27}$$

The second-order variation furnishes us with the energy of waves on the equilibrium, and the rest of section is devoted to derivation of one of the energy formulas:

$$\delta^2 H = \frac{1}{2}\frac{\mathrm{d}^2 H}{\mathrm{d}\tau^2} = \frac{1}{2}\{\{H, K\}, K\} = \frac{1}{2}\int \frac{\delta\{H, K\}}{\delta w}\mathcal{J}\frac{\delta K}{\delta w}d^3 x. \tag{28}$$

From the structure of the Poisson bracket, the last term in (28) becomes

$$\frac{\delta\{H, K\}}{\delta w} = \frac{\delta^2 H}{\delta w^2}\mathcal{J}\frac{\delta K}{\delta w} - \frac{\delta^2 K}{\delta w^2}\mathcal{J}\frac{\delta H}{\delta w} + (H, K)_1, \tag{29}$$

where the last term is the vector originating from the derivatives of the Lie–Poisson structure in the bracket (13):

$$
(F,G)_1 = \begin{bmatrix} \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{M}} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{M}} \\ \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \boldsymbol{B}} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \boldsymbol{B}} + \left(\nabla \frac{\delta G}{\delta \boldsymbol{M}}\right) \cdot \frac{\delta F}{\delta \boldsymbol{B}} - \left(\nabla \frac{\delta F}{\delta \boldsymbol{M}}\right) \cdot \frac{\delta G}{\delta \boldsymbol{B}} \\ \frac{\delta G}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta F}{\delta \rho} - \frac{\delta F}{\delta \boldsymbol{M}} \cdot \nabla \frac{\delta G}{\delta \rho} \\ \nabla \cdot \left( \frac{\delta G}{\delta \boldsymbol{M}} \frac{\delta F}{\delta s} - \frac{\delta F}{\delta \boldsymbol{M}} \frac{\delta G}{\delta s} \right) \end{bmatrix}. \tag{30}
$$

The second term in (29) vanishes at the equilibrium, since $\mathcal{J}\delta H/\delta w = 0$, resulting in

$$
\frac{\mathrm{d}^2 H}{\mathrm{d}\tau^2} = \int \left\{ \left[ \left( \frac{\delta^2 H}{\delta w^2} \mathcal{J}\chi \right) \mathcal{J}\chi + (H,K)_1 \right] \mathcal{J}\chi \right\} \mathrm{d}^3 x. \tag{31}
$$

In (31), the second-order functional derivative of $H$ is

$$
\frac{\delta^2 H}{\delta w^2} = \begin{bmatrix} \frac{1}{\rho} & 0 & -\frac{\boldsymbol{M}}{\rho^2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\boldsymbol{M}}{\rho^2} & 0 & \frac{1}{\rho}\frac{\partial p}{\partial \rho} + \frac{\boldsymbol{M}^2}{\rho^3} & T + \rho \frac{\partial T}{\partial \rho} \\ 0 & 0 & T + \rho \frac{\partial T}{\partial \rho} & \rho \frac{\partial T}{\partial s} \end{bmatrix}, \tag{32}
$$

where the thermodynamic law $\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}v$, with $v = 1/\rho$, has been used to get $\partial T/\partial v = -\partial p/\partial s$ or $\partial T/\partial \rho = \rho^2 \partial p/\partial s$, and $\mathcal{J}\chi$ is

$$
\mathcal{J}\chi = \begin{bmatrix} -(\nabla \boldsymbol{\xi}) \cdot \boldsymbol{M} - \nabla \cdot (\boldsymbol{\xi} \boldsymbol{M}) - (\nabla \boldsymbol{\zeta}) \cdot \boldsymbol{B} + \nabla \cdot (\boldsymbol{B} \boldsymbol{\zeta}) - \rho \nabla \lambda + \sigma \nabla s \\ -\nabla \cdot (\boldsymbol{\xi} \boldsymbol{B}) + \boldsymbol{B} \cdot \nabla \boldsymbol{\xi} \\ -\nabla \cdot (\rho \boldsymbol{\xi}) \\ -\boldsymbol{\xi} \cdot \nabla s \end{bmatrix}. \tag{33}
$$

For the incompressible isentropic MHD flow, upon substitution from (32) and (33), the energy (31) simplifies to

$$
\begin{aligned} \frac{\mathrm{d}^2 H}{\mathrm{d}\tau^2}\Big|_{\tau=0} = \int & \left\{ \rho \left[ \boldsymbol{u}_\tau + (\boldsymbol{\xi} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\xi} \right] \cdot \boldsymbol{u}_\tau \right. \\ & \left. + \left[ \boldsymbol{B}_\tau + \boldsymbol{J} \times \boldsymbol{\xi} - (\nabla \times \boldsymbol{\zeta}) \times \boldsymbol{u} + \nabla(\boldsymbol{\xi} \cdot \boldsymbol{B} - \boldsymbol{\zeta} \cdot \boldsymbol{u}) \right] \cdot \boldsymbol{B}_\tau \right\} \mathrm{d}^3 x, \end{aligned} \tag{34}
$$

where subscript $\tau$ signifies the partial derivative with respect to $\tau$ and use has been made of $\nabla \cdot \boldsymbol{u} = \nabla \cdot \boldsymbol{\xi} = 0$.

For a flowing MHD, the second-order variation of the energy requires the knowledge of second-order variation of both the velocity and the magnetic field:

$$
\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \boldsymbol{u} + \delta \boldsymbol{u}(\boldsymbol{x},t) + \delta^2 \boldsymbol{u}(\boldsymbol{x},t) + \cdots, \\ \boldsymbol{B}(\boldsymbol{x},t) &= \boldsymbol{B} + \delta \boldsymbol{b}(\boldsymbol{x},t) + \delta^2 \boldsymbol{b}(\boldsymbol{x},t) + \cdots, \end{aligned} \tag{35}
$$

in which, for correctness, the Lagrangian displacement fields should include higher order terms as

$$\begin{aligned} \xi(\boldsymbol{x},t) &\to \xi(\boldsymbol{x},t) + \delta\xi_2(\boldsymbol{x},t) + \cdots, \\ \zeta(\boldsymbol{x},t) &\to \zeta(\boldsymbol{x},t) + \delta\zeta_2(\boldsymbol{x},t) + \cdots. \end{aligned} \tag{36}$$

The second-order variation of energy, or the wave energy, is calculated through

$$\delta^2 H = \frac{1}{2}\int \left\{ \delta\boldsymbol{u}\cdot\delta\boldsymbol{u} + 2\boldsymbol{u}\cdot\delta^2\boldsymbol{u} + \delta\boldsymbol{b}\cdot\delta\boldsymbol{b} + 2\boldsymbol{B}\cdot\delta^2\boldsymbol{b} \right\} \mathrm{d}^3x. \tag{37}$$

In Appendix 3, we show that, for a steady basic flow, the second order variations $\boldsymbol{\xi}_2$ and $\boldsymbol{\zeta}_2$ are ruled out as in the case of the neutral fluid Fukumoto and Hirota (2008), and a formidable mathematical manipulation is avoided. Notably, this result shares the concept of a *wave property* Bühler (2009). With cancellation of the terms including $\boldsymbol{\xi}_2$ and $\boldsymbol{\zeta}_2$, (37) is shown to reduce to (34). It is easy to confirm that (34) is the same as the correspondent $\mathrm{d}^2H/\mathrm{d}\tau^2|_{\tau=0}$ of ref Hameiri (2003).

The above formula (34) includes the two Lagrangian displacement fields, of first order, $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$. In the next section, we inquire into time evolution of these Lagrangian displacement fields.

## 4 Evolution of Lagrangian displacement

Given the isomagnetovortical perturbations (20–23), let us find the time evolution equation of $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$, which have so far been taken as arbitrary vector fields. To derive the formula of wave energy, (20–23) should be compatible with the linearized equations of the MHD.

Suppose that a trajectory of a fluid particle is given by $\boldsymbol{x}(t)$. Then, the velocity of the basic flow at $\boldsymbol{x}(t)$ is given by $\boldsymbol{u}(\boldsymbol{x}(t), t) = \mathrm{d}\boldsymbol{x}(t)/\mathrm{d}t$. When a infinitesimal perturbation is superimposed on the basic flow, the particle trajectory is shifted from $\boldsymbol{x}(t)$ to $\boldsymbol{x}(t) + \boldsymbol{\xi}(\boldsymbol{x}, t)$, where $\xi(\boldsymbol{x}, t)$ is the Lagrangian displacement. The velocity field $\boldsymbol{u}$ is perturbed to $\boldsymbol{u} + \delta\boldsymbol{u}$ by $\delta\boldsymbol{u}$ so as to satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\boldsymbol{x}(t) + \xi(\boldsymbol{x}(t), t)\Big) = \boldsymbol{u}\big(\boldsymbol{x}(t) + \xi(\boldsymbol{x}(t), t), t\big). \tag{38}$$

Retaining the terms linear in the amplitude $|\xi|$, we are led to the relation of the velocity variation $\delta\boldsymbol{u}$ to the Lagrangian displacement $\xi$ as Goedbloed et al. (2010)

$$\delta\boldsymbol{u} = \frac{\partial\xi}{\partial t} + (\boldsymbol{u}\cdot\nabla)\xi - (\xi\cdot\nabla)\boldsymbol{u}. \tag{39}$$

For the isomagnetovortical perturbation, the velocity variation is given by (20) and $\delta\boldsymbol{u} = \partial\boldsymbol{u}/\partial\tau$ is identified as the right-hand side of (20).

For the ideal MHD, $\boldsymbol{\zeta}$ or $\boldsymbol{\eta}(= \nabla\times\boldsymbol{\zeta})$ is expressed in terms of $\boldsymbol{\xi}$ and the system of the governing equations are closed by the second-order equation in $\boldsymbol{\xi}$ called the Frieman–Rotenberg equation is Friemann and Rotenberg (1960); Goedbloed et al. (2010):

$$\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2}+2\rho(\boldsymbol{u}\cdot\nabla)\frac{\partial\boldsymbol{\xi}}{\partial t}=\boldsymbol{F}(\boldsymbol{\xi}), \tag{40}$$

where $\boldsymbol{F}(\boldsymbol{\xi})$ is the force operator and is given, for the ideal gas, in the language of the dyadic notation by

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{\xi})=&\nabla\cdot(\rho\boldsymbol{\xi}\boldsymbol{u}\cdot\nabla\boldsymbol{u}-\rho\boldsymbol{u}\boldsymbol{u}\cdot\nabla\boldsymbol{\xi})+\nabla(\gamma p\nabla\cdot\boldsymbol{\xi})-\boldsymbol{B}\times(\nabla\times\delta\boldsymbol{b})\\&+\nabla(\boldsymbol{\xi}\cdot\nabla p)+\boldsymbol{J}\times\delta\boldsymbol{b}+(\nabla\Phi)\nabla\cdot(\rho\boldsymbol{\xi}),\end{aligned} \tag{41}$$

where $\gamma$ is the ratio of the specific heat of constant pressure to that of constant volume and $\Phi$ is the potential for the external body force. The boundary condition to be imposed is

$$\boldsymbol{\xi}\cdot\boldsymbol{n}=0 \quad \text{on the boundary}\,. \tag{42}$$

The Frieman–Rotenberg equation (40) is obtained by substituting into the linearized equation of (2), the disturbances of velocity, density, entropy and magnetic field as the right hand sides of (39), (21), (22) and (23), which are guaranteed by mass, magnetic-flux and specific entropy conservation laws in the perturbation process. Invoking the self-adjointness of the force operator Goedbloed et al. (2010) (see Hirota (2021) for the incompressible case), the second-order wave energy is directly deduced from (41) as Goedbloed et al. (2010); Ilgisonis et al. (2009)

$$\delta^2 H=\frac{1}{2}\int\left\{\rho\left|\frac{\partial\boldsymbol{\xi}}{\partial t}\right|^2-\boldsymbol{\xi}\cdot\boldsymbol{F}(\boldsymbol{\xi})\right\}\mathrm{d}^3x. \tag{43}$$

An alternative form of the wave energy is obtained by eliminating $\boldsymbol{F}$ from (43), by use of the Frieman–Rotenberg equation (40):

$$\delta^2 H=\int\rho\left\{\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}\right)-\frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\boldsymbol{\xi}\right)\right\}\mathrm{d}^3x. \tag{44}$$

For time-periodic waves, like purely oscillating modes $\boldsymbol{\xi}\propto Re[e^{-i\omega t}]$, the wave energy becomes Fukumoto et al. (2011)

$$\delta^2 H=\int\rho\frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\left(\frac{\partial\boldsymbol{\xi}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{\xi}\right)\mathrm{d}^3x. \tag{45}$$

This formula clearly tells that negative-energy waves do not exist for the static basic state or when the basic flow $\boldsymbol{u}$ is absent. In the presence of a steady basic flow, negative-energy waves are commonly excited and are not easy to be suppressed, particularly in three dimensions, for guaranteeing the positive or negative definiteness of the energy for an equilibrium state Ilgisonis et al. (2009). Equivalence of (43) to $\mathrm{d}^2H/\mathrm{d}\tau^2$ given by (34) was confirmed Hameiri (2003). For the case of an incompressible isentropic flow, equivalence of (44) to (34) is proved in Appendix 4.

The origin of the second Lagrangian displacement $\boldsymbol{\eta}(=\nabla\times\boldsymbol{\zeta})$ is the difference of the Lagrangian displacements of the ions $\boldsymbol{\xi}^i$ and the electrons $\boldsymbol{\xi}^e$. Its proper definition is

$$\boldsymbol{\eta} = \rho_q(\boldsymbol{\xi}^i - \boldsymbol{\xi}^e), \tag{46}$$

where $\rho_q$ is the charge density and is written, with use of the elementary charge $e$ and the charge number density $n_e$, as $\rho_q = en_e$. Neutrality in the local electron charge is presumed. Equation describing the time evolution was derived from the system of equations governing the incompressible MHD Vladimirov et al. (1999) and the extended MHD Hirota et al. (2006); Hirota (2021). Equations for the Lagrangian disturbance $(\boldsymbol{\xi}, \boldsymbol{\eta})$ was virtually derived for the compressible MHD in ref Hirota and Fukumoto (2008a), in which they are shown to be the adjoint problem of the equations of Eulerian disturbances $(\delta\boldsymbol{u}, \delta\boldsymbol{b})$. In Appendix 2, we derive it from the fundamental relation between the velocity perturbation and the time evolution of the Lagrangian displacement that holds individually for the ions and the electrons, allowing for the Hall and the electron–inertia effects. For the incompressible ideal MHD flow of uniform number density, the resulting relation (82) restores equation of $\boldsymbol{\eta}$, as an auxiliary vector field, derived from the MHD equations in ref Vladimirov et al. (1999):

$$\delta\boldsymbol{j} = \frac{\partial\boldsymbol{\eta}}{\partial t} + (\boldsymbol{J}\cdot\nabla)\boldsymbol{\xi} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{J} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\eta} - (\boldsymbol{\eta}\cdot\nabla)\boldsymbol{u}. \tag{47}$$

On the left-hand side, $\delta\boldsymbol{j}$ is provided by the curl of (23). Eq. (47) satisfies equations derived by linearizing (2) and (5) with the disturbances of velocity, density, entropy and magnetic field expressed by right hand sides of (20) to (23), which conversely verifies the ansatz (46).

## 5 Energy formula in terms of vorticity and magnetic field

The vorticity field $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$ have tendency to be localized in space, and the formula of wave energy expressed in terms of $\boldsymbol{\omega}$ and $\boldsymbol{B}$ is advantageous for efficient calculation. In this section, we extend (1) to include the counterpart of $\boldsymbol{B}$ by exploiting (34). This extension necessitates the evolution equation of the second Lagrangian displacement field $\boldsymbol{\eta}$.

We restrict our attention to the incompressible isentropic flow and take the density and the specific entropy to be uniform. Under these conditions, (20) and (23) become

$$\delta\boldsymbol{u} = \mathcal{P}[\boldsymbol{\xi}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\boldsymbol{B}], \tag{48}$$

$$\delta\boldsymbol{b} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}), \tag{49}$$

where $\mathcal{P}[\,\cdot\,]$ is the operator projecting a vector field to a solenoidal one and we set $\rho = 1$. Time evolution of the displacement field $\boldsymbol{\xi}$ is described by (39) with $\delta\boldsymbol{u}$ substituted from (48), and that of $\boldsymbol{\eta}$ by (47) with $\delta\boldsymbol{j}$ substituted from the curl of (49). For $\boldsymbol{\zeta} = (\nabla\times)^{-1}\boldsymbol{\eta}$, (47) reads

$$\nabla \times (\boldsymbol{\xi} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{\zeta}}{\partial t} + \mathcal{P}[\boldsymbol{\xi} \times \boldsymbol{J} + (\nabla \times \boldsymbol{\zeta}) \times \boldsymbol{u}]. \tag{50}$$

We assume $\nabla \cdot \boldsymbol{\zeta} = 0$, correspondingly to solenoidality of $\delta \boldsymbol{b}$.

With identification of $\boldsymbol{u}_\tau = \delta \boldsymbol{u}$ and $\boldsymbol{B}_\tau = \delta \boldsymbol{b}$, the wave energy (34) collapses to

$$\delta^2 H = \frac{1}{2} \frac{\mathrm{d}^2 H}{\mathrm{d}\tau^2} = \frac{1}{2} \int \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \delta \boldsymbol{u} + \frac{\partial \boldsymbol{\zeta}}{\partial t} \cdot \delta \boldsymbol{b} \right\} \mathrm{d}^3 x. \tag{51}$$

Further substituting (48) and (49) and taking partial integration, (51) becomes

$$\delta^2 H = \frac{1}{2} \int \left\{ \left[ \boldsymbol{\xi} \times \boldsymbol{\omega} + (\nabla \times \boldsymbol{\zeta}) \times \boldsymbol{B} \right] \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} + (\boldsymbol{\xi} \times \boldsymbol{B}) \cdot \frac{\partial (\nabla \times \boldsymbol{\zeta})}{\partial t} \right\} \mathrm{d}^3 x, \tag{52}$$

and we have thus reached a desired formula for the energy of a wave on a steady flow $\boldsymbol{u}$ subject to a steady magnetic field $\boldsymbol{B}$ of an incompressible and isentropic fluid:

$$\delta^2 H = \frac{1}{2} \int \left\{ \boldsymbol{\omega} \cdot \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\xi} \right) + \boldsymbol{B} \cdot \left( \frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\eta} - \boldsymbol{\xi} \times \frac{\partial \boldsymbol{\eta}}{\partial t} \right) \right\} \mathrm{d}^3 x. \tag{53}$$

Time average of (52) further simplifies (53) to

$$\overline{\delta^2 H} = \frac{1}{2} \int \left\{ \boldsymbol{\omega} \cdot \left( \overline{\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\xi}} \right) + 2\boldsymbol{B} \cdot \left( \overline{\frac{\partial \boldsymbol{\xi}}{\partial t} \times \boldsymbol{\eta}} \right) \right\} \mathrm{d}^3 x. \tag{54}$$

Equations (53) and (54) are desired formulas of the wave-energy, of second order in wave amplitude, on a steady state of an incompressible isentropic flow of the ideal MHD. By developing a refined geometric machinery, a formula generalizing (53) for the extended MHD was concisely derived Hirota (2021). Our step-by-step derivation admits a wider accessibility and provides a physical insight into the wave energy. Equivalence of (53) to the standard form (37) is proved in Appendix 3, and to (44), directly derived from the Frieman–Rotenberg equation (40) is proved in Appendix 4.

Formula (53) or (54) is particularly useful when the vorticity for the magnetic field is localized, as exemplified by a vortex tube and a vortex sheet or by a magnetic-flux tube. In case the electric current $\boldsymbol{J}$ and its perturbation $\delta \boldsymbol{j}$ are localized, Eq. (47) governing the time evolution of $\boldsymbol{\eta}$ may rather be easy to handle.

## 6 Conclusion

Arnold's theorem that a steady Euler flow of an incompressible fluid is the extremal of the kinetic energy with respect to the isovortical, or kinematically accessible, perturbations makes the energy of waves, of second order in amplitude, expressible solely in terms of the first-order Lagrangian displacement field, if perturbations are limited to isovortical ones. For an electrically conducting fluid, the local circulation is no longer conserved by the action of the Lorentz force, but Arnold's theorem is applicable as it is to the ideal MHD if the isovortical perturbations are replaced by the isomagnetovortical ones that preserve all the Casimirs of the ideal MHD. The velocity and the magnetic field, the density and the specific entropy belonging to the isomagnetovortical one are generated by the Lie–Poisson bracket. For the ideal MHD, the isomagnetovortical perturbation is expressed in terms of the two Lagrangian displacements $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, and accordingly the formula of wave energy may be represented in terms of the both.

To calculate the wave energy, we are requested to derive equations governing $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Equation for $\boldsymbol{\xi}$ is well-known as the Frieman–Rotenberg equation. That for $\boldsymbol{\eta}$ is less known and is derived in Appendix 2 from the two fluid model comprising the ions and the electrons. We have established a formula (53) of the wave energy including the vorticity and the magnetic field of the steady basic flow, which invites the both Lagrangian displacements. Equivalence of this formula to other ones (37) and (44) has been proved by invoking the evolution equations of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$.

We point out that the wave energy of an ideal incompressible fluid bears resemblance with the action for deriving motion of a vortex filament Rasetti and Regge (1975); Lund and Regge (1976). The kinetic part $L_{\text{kin}}$ of the Lagrangian of the variational principle for the position $\boldsymbol{\xi}(\sigma, t)$ on a vortex filament, as a vector-valued function of the arcwise parameter $\sigma$ and the time $t$, is given by a line integral along the filament:

$$L_{\text{kin}} = \frac{\Gamma\rho}{3}\int \frac{\partial \boldsymbol{\xi}}{\partial \sigma}\cdot\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\times\boldsymbol{\xi}\right)\mathrm{d}\sigma, \tag{55}$$

where $\Gamma$ is the circulation or the total vorticity over the core of a vortex filament, and the vorticity distributed in the infinitely thin core is represented by

$$\boldsymbol{\omega}(\boldsymbol{x}, t) = \Gamma\int \delta(\boldsymbol{x} - \boldsymbol{\xi})\frac{\partial \boldsymbol{\xi}}{\partial \sigma}\mathrm{d}\sigma, \tag{56}$$

where $\delta(\boldsymbol{x})$ is 3 dimensional Dirac's delta function. The form (53) of the energy formula of the MHD may give a hint for formulating a variational principle for the dynamics of a magnetic flux tube.

The formula (52) facilitates calculation of the energy of waves when the vorticity and/or the magnetic filed of the basic flow is localized in a thin region and its utility will be demonstrated in the future. These are several directions for extending the analysis developed here. The effects of compressibility and that of baroclinicity associated with the density stratification may have a substantial influence on the stability of MHD flows and are worth pursuing. The wave energy is indispensable for understanding the results. Influence of the Hall effect and of the electron–inertia effect on stability of the MHD flow attracts a broad attention, and the energy will play a crucial role.

For the Hall-MHD, the wave energy $H_2^{\text{h}}$ is manipulated as

$$H_2^{\text{h}} = \frac{1}{2}\int\left\{\boldsymbol{\omega}\cdot\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\times\boldsymbol{\xi}\right) + \boldsymbol{B}\cdot\left(\frac{\partial \boldsymbol{\xi}}{\partial t}\times\boldsymbol{\eta} - \boldsymbol{\xi}_e\times\frac{\partial \boldsymbol{\eta}}{\partial t}\right)\right\}\mathrm{d}^3x, \tag{57}$$

where the time evolution of $\boldsymbol{\eta}$, as shown in (81), gives way to

$$\frac{\partial \boldsymbol{\eta}}{\partial t} = \delta\boldsymbol{j} + \nabla\times(\boldsymbol{u}\times\boldsymbol{\eta} + \boldsymbol{J}\times\boldsymbol{\xi}_e), \tag{58}$$

with $\delta \boldsymbol{j} = \nabla \times \left(\nabla \times \left(\boldsymbol{\xi}_e \times \boldsymbol{B}\right)\right)$. The formula (57) is the special case of the one for the extended MHD Hirota (2021).

There are important issues, in connection with wave energy, which wait for future investigations.

## Appendix A Enegy of waves on a circular vortex patch

To illustrate the utility of the formula (1), we calculate the energy of the 2D Kelvin waves on the Rankine vortex, that is, waves on a circular vortex patch of uniform vorticity, embedded in an inviscid incompressible fluid. We begin with calculation based on the definition Fukumoto (2003), followed by calculation by use of (1).

As a basic flow, we consider a circular region, of radius $a$, endowed with uniform vorticity $\omega_0$, surrounded by an irrotational flow. We introduce cylindrical coordinates $(r, \theta, z)$, with $\boldsymbol{e}_i$ being the unit vector along the $i$ direction. The velocity and the vorticity fields of the basic flow is given by $\boldsymbol{U} = V(r)\boldsymbol{e}_\theta$ and $\boldsymbol{\omega} = \omega(r)\boldsymbol{e}_z$, where

$$V = \begin{cases} \dfrac{1}{2}\omega_0 r \\ \dfrac{\omega_0 a^2}{2r} \end{cases}, \quad \omega = \begin{cases} \omega_0 = \text{ const.} & (r \le a) \\ 0 & r > a \end{cases}. \tag{59}$$

Suppose that the boundary of the vortex patch is perturbed to $r = \eta(\theta, t) = a + D\cos(m\theta - \omega t)$, by a wave of infinitesimal amplitude $D$, with $m$ being an integer. The irrotational velocity field guarantees the isovorticity. The velocity field perturbed by an irrotational disturbance is given by $\boldsymbol{u} = \tilde{u}\boldsymbol{e}_r + (\tilde{V} + \tilde{v})\boldsymbol{e}_\theta$, where

$$\tilde{V} = \begin{cases} V^- = \dfrac{1}{2}\omega_0 r \ (r \le \eta(\theta, t)) \\ V^+ = \dfrac{\omega_0 a^2}{2r} \ (r > \eta(\theta, t)) \end{cases}, \tag{60}$$

and

$$\begin{aligned} \tilde{u} &= \begin{cases} -\dfrac{\omega_0}{2}\left(\dfrac{r}{a}\right)^{|m|-1} D\sin(m\theta - \omega t) \ (r \le \eta(\theta, t)) \\ -\dfrac{\omega_0}{2}\left(\dfrac{a}{r}\right)^{|m|+1} D\sin(m\theta - \omega t) \ (r > \eta(\theta, t)) \end{cases}, \\ \tilde{v} &= \begin{cases} -\dfrac{\omega_0}{2}\left(\dfrac{r}{a}\right)^{|m|-1} D\cos(m\theta - \omega t) \ (r \le \eta(\theta, t)) \\ \dfrac{\omega_0}{2}\left(\dfrac{a}{r}\right)^{|m|+1} D\cos(m\theta - \omega t) \quad (r > \eta(\theta, t)) \end{cases}, \end{aligned} \tag{61}$$

where the frequency $\omega$ satisfies the dispersion relation:

$$\omega = \frac{1}{2}\omega_0 m\left(1 - \frac{1}{|m|}\right). \tag{62}$$

We take the density of fluid to be $\rho = 1$. The wave energy is calculated to $O(D^2)$ as follows:

$$\delta^2 H = \frac{1}{2}\int \left(\boldsymbol{u}^2 - \boldsymbol{U}^2\right) dS = \delta^2 K + \delta^2 U, \tag{63}$$

where the 'kinetic energy' $\delta^2 K$ and the 'potential energy' $\delta^2 U$ are defined by

$$\delta^2 K = \frac{1}{2}\int_0^{2\pi} d\theta \left\{ \int_0^{\eta} \left[2V^- \tilde{v} + \tilde{\boldsymbol{u}}^2\right] r dr + \int_{\eta}^{\infty} \left[2V^+ \tilde{v} + \tilde{\boldsymbol{u}}^2\right] r dr \right\}, \tag{64}$$

$$\delta^2 U = \frac{1}{2}\int_0^{2\pi} d\theta \left\{ \int_a^{\eta} (V^-)^2 r dr + \int_{\eta}^{a} \left(V^+\right)^2 r dr \right\}. \tag{65}$$

By substitution from (60) and (61), we obtain, after some manipulation:

$$\delta^2 K = \frac{\pi}{4}\left(\frac{1}{|m|} - 2\right)\omega_0^2 a^2 D^2, \quad \delta^2 U = \frac{\pi}{4}\omega_0^2 a^2 D^2, \tag{66}$$

and thus we are led to the energy of the Kelvin wave:

$$\delta^2 H = \frac{\pi}{4}\left(\frac{1}{|m|} - 1\right)\omega_0^2 a^2 D^2. \tag{67}$$

Next, we calculate the energy of the Kelvin wave through (1). Since the vorticity $\boldsymbol{\omega}$ is confined in the core ($r < \eta$), we may dispense with the calculation of the displacement field $\boldsymbol{\xi}$ outside of the core. Equation (39) of $\boldsymbol{\xi} = \xi_r \boldsymbol{e}_r + \xi_\theta \boldsymbol{e}_\theta$ reads, for $r \le \eta$:

$$\begin{aligned} \frac{\partial \xi_r}{\partial t} + \frac{\omega_0}{2}\frac{\partial \xi_r}{\partial \theta} &= \tilde{u}, \\ \frac{\partial \xi_\theta}{\partial t} + \frac{\omega_0}{2}\frac{\partial \xi_\theta}{\partial \theta} &= \tilde{v}. \end{aligned} \tag{68}$$

For the perturbation velocity (61), (68) is easily integrated to give, for $r \le \eta$:

$$\begin{aligned} \xi_r &= \frac{m}{|m|}\left(\frac{r}{a}\right)^{|m|-1} D\cos(m\theta - \omega t), \\ \xi_\theta &= -\frac{m}{|m|}\left(\frac{r}{a}\right)^{|m|-1} D\sin(m\theta - \omega t), \end{aligned} \tag{69}$$

where use has been made of the dispersion relation (62). For the wave energy of second order in amplitude, it suffices to perform integration (1) in the circular region of radius $a$, resulting in (67). This example illustrates the efficiency of using the formula (1) for evaluating the wave energy for a slender vortex tube.

## Appendix B Equations of Lagrangian displacement fields from two fluid model

Given the disturbance field $\delta \boldsymbol{u}$ and $\delta \boldsymbol{J}$, we derive evolution equation of the Lagrangian displacement fields $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. The existing derivation Vladimirov et al. (1999); Hirota et al. (2006); Hirota (2021) starts with the momentum equations. We retain only the kinematic relation between the velocity perturbation and the Lagrangian displacement field of ions and electrons. We allow for the Hall and the electron inertia effects.

Let $\boldsymbol{x}^{i,e}$ denote to be trajectories of the ions and electrons of the basic flow and $\epsilon\boldsymbol{\xi}^{i,e}(\boldsymbol{x}^{i,e}(t), t)$ to be displacement from the basic trajectories with a small parameter $\epsilon$. Here, the superscript $i$, $e$ represent ion and electron, respectively. Then, the disturbed trajectories of the ions and electrons are, to first order in $\epsilon$:

$$\boldsymbol{x}_{\epsilon}^{i,e}(t) = \boldsymbol{x}^{i,e}(t) + \epsilon\boldsymbol{\xi}^{i,e}(\boldsymbol{x}^{i,e}(t), t), \tag{70}$$

The velocity at the displaced position reads

$$\begin{aligned} \boldsymbol{v}_{\epsilon}^{i,e}(\boldsymbol{x}_{\epsilon}^{i,e}(t), t) =& \frac{D}{Dt}(\boldsymbol{x}^{i,e}(t) + \epsilon\boldsymbol{\xi}^{i,e}(\boldsymbol{x}^{i,e}(t), t), t) \\ =& \boldsymbol{v}^{i,e}(\boldsymbol{x}^{i,e}(t), t) + \epsilon\left(\frac{\partial \boldsymbol{\xi}^{i,e}}{\partial t} + \boldsymbol{v}^{i,e}\cdot\nabla\boldsymbol{\xi}^{i,e}\right)(\boldsymbol{x}^{i,e}(t), t). \end{aligned} \tag{71}$$

Substitution from (70) on the left-hand side, the disturbance of the velocity reads, to $O(\epsilon)$:

$$\begin{aligned} \epsilon\delta\boldsymbol{v}^{i,e}(\boldsymbol{x}^{i,e}(t), t) =& \boldsymbol{v}_{\epsilon}^{i,e}(\boldsymbol{x}^{i,e}(t), t) - \boldsymbol{v}^{i,e}(\boldsymbol{x}^{i,e}(t), t) \\ =& \epsilon\left(\frac{\partial \boldsymbol{\xi}^{i,e}}{\partial t} + \boldsymbol{v}^{i,e}\cdot\nabla\boldsymbol{\xi}^{i,e} - \boldsymbol{\xi}^{i,e}\cdot\nabla\boldsymbol{v}^{i,e}\right)(\boldsymbol{x}^{i,e}(t), t). \end{aligned} \tag{72}$$

We impose the condition $n_i Z = n_e$ of quasi charge neutral, where $n_{i,e}$ are the number densities of ion and electron and $Z$ is the charge number of the ion. Let $m_{i,e}$ be the mass of the ion and electron, with $m_e \ll m_i$. The appropriate definition of the averaged velocity and the averaged displacement is

$$\boldsymbol{u} = \frac{n_i m_i \boldsymbol{v}^i + n_e m_e \boldsymbol{v}^e}{n_i m_i + n_e m_e}, \quad \boldsymbol{\xi} = \frac{n_i m_i \boldsymbol{\xi}^i + n_e m_e \boldsymbol{\xi}^e}{n_i m_i + n_e m_e}. \tag{73}$$

On the other hand, the electric current density $\boldsymbol{J}$ and the corresponding Lagrangian displacement field $\boldsymbol{\eta}$ are defined by difference, of the velocity and the Lagrangian displacement, between the ions and the electrons, with the charge density $\rho_q \equiv eZn_i = en_e$ for multiplication factor. Here, $e$ is the elementary charge:

$$\boldsymbol{J} = \rho_q(\boldsymbol{v}^i - \boldsymbol{v}^e), \ \boldsymbol{\eta} = \rho_q(\boldsymbol{\xi}^i - \boldsymbol{\xi}^e). \tag{74}$$

In the matrix form, (73) and (74) are written as

$$\begin{bmatrix} \boldsymbol{\xi}_i \\ \boldsymbol{\xi}_e \end{bmatrix} = T \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta}/\rho_q \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{v}_i \\ \boldsymbol{v}_e \end{bmatrix} = T \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{J}/\rho_q \end{bmatrix}, \tag{75}$$

where

$$T = \begin{bmatrix} 1 & \hat{m}_e \\ 1 & -\hat{m}_i \end{bmatrix}, \tag{76}$$

and $\hat{m}_i \equiv m_i/(Zm_e + m_i)$ and $\hat{m}_e \equiv Zm_e/(Zm_e + m_i)$.

By using (72) and (73), the disturbed velocity is expanded as

$$\begin{aligned} \boldsymbol{u}_\epsilon =& \hat{m}_i \boldsymbol{v}^i_\epsilon + \hat{m}_e \boldsymbol{v}^e_\epsilon \\ =& \hat{m}_i \boldsymbol{v}^i + \hat{m}_e \boldsymbol{v}^e + \epsilon \left( \frac{\partial}{\partial t} \left( \hat{m}_i \boldsymbol{\xi}^i + \hat{m}_e \boldsymbol{\xi}^e \right) + \left( \hat{m}_i (\boldsymbol{v}^i \cdot \nabla \boldsymbol{\xi}^i) \right.\right. \\ & \left.\left. - \hat{m}_i (\boldsymbol{\xi}^i \cdot \nabla \boldsymbol{v}^i) + \hat{m}_e (\boldsymbol{v}^e \cdot \nabla \boldsymbol{\xi}^e) - \hat{m}_e (\boldsymbol{\xi}^e \cdot \nabla \boldsymbol{v}^e) \right) \right) \\ =& \boldsymbol{u} + \epsilon \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} + \hat{m}_i \hat{m}_e \left( \frac{\boldsymbol{J}}{\rho_q} \cdot \nabla \frac{\boldsymbol{\eta}}{\rho_q} - \frac{\boldsymbol{\eta}}{\rho_q} \cdot \nabla \frac{\boldsymbol{J}}{\rho_q} \right) \right\}. \end{aligned} \tag{77}$$

We are thus led to the link between time evolution of $\boldsymbol{\xi}$ and the disturbance velocity $\delta \boldsymbol{u}$ as

$$\begin{aligned} \delta \boldsymbol{u} \equiv& \frac{\boldsymbol{u}_\epsilon - \boldsymbol{u}}{\epsilon} \\ =& \frac{\partial \boldsymbol{\xi}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\xi} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} + \frac{Zm_i m_e}{(m_i + Zm_e)^2} \left( \frac{\boldsymbol{J}}{\rho_q} \cdot \nabla \frac{\boldsymbol{\eta}}{\rho_q} - \frac{\boldsymbol{\eta}}{\rho_q} \cdot \nabla \frac{\boldsymbol{J}}{\rho_q} \right). \end{aligned} \tag{78}$$

The last terms including $\boldsymbol{J}$ express the electron inertia effect. In the limit of $m_e/m_i \to 0$, the condition applicable to the Hall-MHD as well as the MHD, these terms vanish and (78) reduces to (39).

Similarly, the disturbed electric current density becomes

$$\begin{aligned} \boldsymbol{J}_\epsilon =& \rho_q (\boldsymbol{v}^i_\epsilon - \boldsymbol{v}^e_\epsilon) - \epsilon \nabla \cdot (\rho_q \boldsymbol{\xi}) \frac{\boldsymbol{J}}{\rho_q} \\ =& \boldsymbol{J} + \epsilon \left\{ \rho_q \frac{\partial}{\partial t} \left( \frac{\boldsymbol{\eta}}{\rho_q} \right) + \boldsymbol{J} \cdot \nabla \boldsymbol{\xi} - \rho_q \boldsymbol{\xi} \cdot \nabla \frac{\boldsymbol{J}}{\rho_q} + \rho_q \boldsymbol{u} \cdot \nabla \frac{\boldsymbol{\eta}}{\rho_q} - \boldsymbol{\eta} \cdot \nabla \boldsymbol{u} \right. \\ & \left. + (\hat{m}_i - \hat{m}_e) \left( \boldsymbol{\eta} \cdot \nabla \frac{\boldsymbol{J}}{\rho_q} - \boldsymbol{J} \cdot \nabla \frac{\boldsymbol{\eta}}{\rho_q} \right) - \nabla \cdot (\rho_q \boldsymbol{\xi}) \frac{\boldsymbol{J}}{\rho_q} \right\}, \end{aligned} \tag{79}$$

where mass conservation is used as $(n_e dx_1 \wedge dx_2 \wedge dx_3)_\epsilon = n_e dx_1 \wedge dx_2 \wedge dx_3$. It follows that evolution of $\boldsymbol{\eta}$ is related to the disturbance electric current via

$$
\begin{aligned}
\delta \boldsymbol{j} \equiv & \frac{\boldsymbol{J}_\epsilon - \boldsymbol{J}}{\epsilon} \\
= & \rho_q \frac{\partial}{\partial t}\left(\frac{\boldsymbol{\eta}}{\rho_q}\right) + \boldsymbol{J}\cdot\nabla\boldsymbol{\xi} - \rho_q \boldsymbol{\xi}\cdot\nabla\frac{\boldsymbol{J}}{\rho_q} + \rho_q \boldsymbol{u}\cdot\nabla\frac{\boldsymbol{\eta}}{\rho_q} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{u} \\
& + \frac{m_i - Zm_e}{m_i + Zm_e}\left(\boldsymbol{\eta}\cdot\nabla\frac{\boldsymbol{J}}{\rho_q} - \boldsymbol{J}\cdot\nabla\frac{\boldsymbol{\eta}}{\rho_q}\right) - \nabla\cdot(\rho_q\boldsymbol{\xi})\frac{\boldsymbol{J}}{\rho_q}.
\end{aligned}
\tag{80}
$$

The last two terms reflect the Hall effect. In the limit of $m_e/m_i \to 0$, (80) recovers (27) in ref Hirota et al. (2006) for compressible Hall-MHD and for an incompressible Hall-MHD flow of uniform number density, (80) further becomes

$$
\delta \boldsymbol{j} = \frac{\partial \boldsymbol{\eta}}{\partial t} + \boldsymbol{J}\cdot\nabla\boldsymbol{\xi}_e - \boldsymbol{\xi}_e\cdot\nabla\boldsymbol{J} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{u}. \tag{81}
$$

If we assume $|\boldsymbol{\eta}/\rho_q| = |\boldsymbol{\xi}_i - \boldsymbol{\xi}_e| \ll |\boldsymbol{\xi}|$ and $|\boldsymbol{J}/\rho_q| = |\boldsymbol{v}_i - \boldsymbol{v}_e| \ll |\boldsymbol{u}|$, then the last two terms in (80) are neglected and

$$
\delta \boldsymbol{j} = \rho_q \frac{\partial}{\partial t}\left(\frac{\boldsymbol{\eta}}{\rho_q}\right) + \boldsymbol{J}\cdot\nabla\boldsymbol{\xi} - \rho_q \boldsymbol{\xi}\cdot\nabla\frac{\boldsymbol{J}}{\rho_q} + \rho_q \boldsymbol{u}\cdot\nabla\frac{\boldsymbol{\eta}}{\rho_q} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{u}. \tag{82}
$$

For an incompressible ideal MHD flow of uniform number density, (82) further simplifies to

$$
\delta \boldsymbol{j} = \frac{\partial \boldsymbol{\eta}}{\partial t} + \boldsymbol{J}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{J} + \boldsymbol{u}\cdot\nabla\boldsymbol{\eta} - \boldsymbol{\eta}\cdot\nabla\boldsymbol{u}, \tag{83}
$$

which coincides with (47).

## Appendix C Alternative formula 1 of wave energy

In this Appendix, we derive the formula (34), starting from the standard one (37). Moreover, we prove that, for a steady basic flow, the formula of wave energy, of second order in amplitude, for the isomagnetovortical disturbances is expressible by the first-order Lagrangian displacement only.

Let $\boldsymbol{\xi}_2$, $\boldsymbol{\eta}_2$, $\boldsymbol{\zeta}_2$ be the variation of $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}_2 = \nabla\times\boldsymbol{\zeta}_2$ as expressed by (36). Then, the first- and second-order isomagnetovortical perturbations of the velocity and the magnetic field are found, for an incompressible isentropic flow, to be Vladimirov et al. (1999); Isichenko (1998); Kaltsas et al. (2021):

$$
\delta \boldsymbol{u} = \mathcal{P}[\boldsymbol{\xi}\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\boldsymbol{B}], \tag{84}
$$

$$
\delta^2 \boldsymbol{u} = \frac{1}{2}\mathcal{P}\big[\boldsymbol{\xi}\times\delta\boldsymbol{\omega} + \boldsymbol{\xi}_2\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\delta\boldsymbol{b} + \boldsymbol{\eta}_2\times\boldsymbol{B}\big], \tag{85}
$$

$$
\delta \boldsymbol{b} = \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}), \tag{86}
$$

$$\delta^2 \boldsymbol{b} = \frac{1}{2}\left\{\nabla\times(\boldsymbol{\xi}\times\delta\boldsymbol{b}) + \nabla\times\left(\boldsymbol{\xi}_2\times\boldsymbol{B}\right)\right\}. \tag{87}$$

Here we retain the Lagrangian displacements $\boldsymbol{\xi}_2$ and $\boldsymbol{\eta}_2$, of second order, and constructively show below that, when the basic state is steady, these are excluded from the final formula of wave energy (see Fukumoto and Hirota (2008) for a neutral fluid).

We start with the second-order energy (37):

$$\delta^2 H = \frac{1}{2}\int\left\{\delta\boldsymbol{u}\cdot\delta\boldsymbol{u} + 2\boldsymbol{u}\cdot\delta^2\boldsymbol{u} + \delta\boldsymbol{b}\cdot\delta\boldsymbol{b} + 2\boldsymbol{B}\cdot\delta^2\boldsymbol{b}\right\}\mathrm{d}^3x. \tag{88}$$

Upon substitution from (84)–(87), (88) is manipulated as

$$\begin{aligned}
\delta^2 H =& \frac{1}{2}\int\Big\{\delta\boldsymbol{u}\cdot\delta\boldsymbol{u} + \boldsymbol{u}\cdot\left[\boldsymbol{\xi}\times(\nabla\times\delta\boldsymbol{u}) + \boldsymbol{\xi}_2\times\boldsymbol{\omega} + \boldsymbol{\eta}\times\delta\boldsymbol{b} + \boldsymbol{\eta}_2\times\boldsymbol{B}\right] \\
&+ \delta\boldsymbol{b}\cdot\delta\boldsymbol{b} + \boldsymbol{B}\cdot\left[\nabla\times(\boldsymbol{\xi}\times\delta\boldsymbol{b}) + \nabla\times\left(\boldsymbol{\xi}_2\times\boldsymbol{B}\right)\right]\Big\}\mathrm{d}^3x \\
=& \frac{1}{2}\int\Big\{\delta\boldsymbol{u}\cdot\delta\boldsymbol{u} + \boldsymbol{u}\cdot\left[\boldsymbol{\xi}\times(\nabla\times\delta\boldsymbol{u}) + \boldsymbol{\eta}\times\delta\boldsymbol{b}\right] \\
&+ \delta\boldsymbol{b}\cdot\delta\boldsymbol{b} + \boldsymbol{B}\cdot\left[\nabla\times(\boldsymbol{\xi}\times\delta\boldsymbol{b})\right] \\
&- \boldsymbol{\zeta}_2\cdot[\nabla\times(\boldsymbol{u}\times\boldsymbol{B})] - \boldsymbol{\xi}_2\cdot(\boldsymbol{u}\times\boldsymbol{\omega} + \boldsymbol{J}\times\boldsymbol{B})\Big\}\mathrm{d}^3x.
\end{aligned} \tag{89}$$

In (89), the last terms including $\boldsymbol{\zeta}_2$ and $\boldsymbol{\xi}_2$ vanish when the basic flow $\boldsymbol{u}$ and $\boldsymbol{B}$ is a steady solution.

By partial integration, the rest of $\delta^2 H$ is reduced to

$$\delta^2 H = \frac{1}{2}\int\left\{(\delta\boldsymbol{u} + \boldsymbol{\xi}\cdot\nabla\boldsymbol{u} - \boldsymbol{u}\cdot\nabla\boldsymbol{\xi})\cdot\delta\boldsymbol{u} + (\delta\boldsymbol{b} + \boldsymbol{J}\times\boldsymbol{\xi} - \boldsymbol{\eta}\times\boldsymbol{u})\cdot\delta\boldsymbol{b}\right\}\mathrm{d}^3x, \tag{90}$$

which recovers Eq. (34) supplemented by $\rho = 1$.

## Appendix D Alternative formula 2 of wave energy

In this appendix, we show the equivalence of (44), a direct consequence of (43), to (51), a consequence of (34) under the restriction of an incompressible isentropic flow. We take $\rho = 1$.

Substitution of (39), with $\delta\boldsymbol{u}$ provided by (48), into (44) yields, by use of $\nabla\cdot\boldsymbol{u} = 0$ and (49):

$$
\begin{aligned}
\delta^2 H =& \int \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot (\delta \boldsymbol{u} + \boldsymbol{\xi} \cdot \nabla \boldsymbol{u}) - \frac{1}{2} \frac{\partial}{\partial t} \left[ (\boldsymbol{\eta} \times \boldsymbol{B} + (\nabla \boldsymbol{u}) \cdot \boldsymbol{\xi} - \boldsymbol{u} \cdot \nabla \boldsymbol{\xi}) \cdot \boldsymbol{\xi} \right] \right\} \mathrm{d}^3 x \\
=& \int \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot (\delta \boldsymbol{u} + \boldsymbol{\xi} \cdot \nabla \boldsymbol{u}) - \frac{1}{2} \left[ \left( \frac{\partial \boldsymbol{\eta}}{\partial t} \times \boldsymbol{B} + (\nabla \boldsymbol{u}) \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \right) \cdot \boldsymbol{\xi} \right.\right. \\
& \left.\left. + [\boldsymbol{\eta} \times \boldsymbol{B} + (\nabla \boldsymbol{u}) \cdot \boldsymbol{\xi}] \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \right] \right\} \mathrm{d}^3 x \\
=& \int \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \delta \boldsymbol{u} + \frac{1}{2} \frac{\partial \boldsymbol{\zeta}}{\partial t} \cdot \delta \boldsymbol{b} - \frac{1}{2} \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \left[ \boldsymbol{\eta} \times \boldsymbol{B} - \boldsymbol{\xi} \cdot \nabla \boldsymbol{u} + (\nabla \boldsymbol{u}) \cdot \boldsymbol{\xi}) \right] \right\} \mathrm{d}^3 x \\
=& \frac{1}{2} \int \left\{ \frac{\partial \boldsymbol{\xi}}{\partial t} \cdot \delta \boldsymbol{u} + \frac{\partial \boldsymbol{\zeta}}{\partial t} \cdot \delta \boldsymbol{b} \right\} \mathrm{d}^3 x,
\end{aligned}
\tag{91}
$$

completing the proof.

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## Declarations

**Conflicts of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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