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ORIGINAL RESEARCH PAPER

Riesz–Zygmund means and trigonometric approximation in Morrey spaces

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Abstract
We estimate best approximations and moduli of smoothness of functions in Morrey spaces variable exponent spaces by norms of derivatives of Riesz–Zygmund means and partial Fourier sums in these spaces. As a consequence, we obtain a description of Hölder spaces based on the Morrey spaces. The direct results on approximation in these Hölder spaces are also obtained.

Keywords Morrey spaces · Best approximation · Modulus of smoothness · Riesz–Zygmund means of Fourier series · Inverse approximation theorems

Mathematics Subject Classification Primary 42A10 · Secondary 41A17 · 46E30

1 Introduction

Let $L^p_{2\pi}$, $1 \le p < \infty$, be the Lebesgue space of measurable 2π-periodic functions f on $\mathbb{R}$ such that $\|f\|_p^p := \int_0^{2\pi} |f(x)|^p\,dx < \infty$. If $1 \le p < \infty$, $0 < \lambda \le 1$ and f is a measurable 2π-periodic function for which

$$\|f\|_{p,\lambda} = \sup_I \left(|I|^{\lambda-1} \int_I |f(x)|^p\,dx \right)^{1/p} < \infty,$$

where the supremum is taken over all $I = [a, b]$ with $0 \le b - a \le 2\pi$ and $|I|$ is the Lebesgue measure of I, then f belongs to the Morrey space $L^{p,\lambda}_{2\pi}$. We note that the norm $\|\cdot\|_{p,\lambda}$ is invariant with respect to usual translation and that $L^{p,\lambda}_{2\pi} \subset L^p_{2\pi} \subset L^1_{2\pi}$

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for all $0<\lambda\le 1$ and $1\le p<\infty$ (for $\lambda=1$ the space $L^{p,\lambda}_{2\pi}$ coincides with $L^p_{2\pi}$). More about these spaces see in [1, ch. 1].

It is known that in general case $L^{p,\lambda}_{2\pi}$ is not a separable space (see [27, Prop. 2.16]). Therefore we consider a proper subspace $L^{p,\lambda}_{2\pi,0}$ of $L^{p,\lambda}_{2\pi}$, which is the closure of the space of trigonometric polynomials in $L^{p,\lambda}_{2\pi}$ with the same norm $\|\cdot\|_{p,\lambda}$. Then $\lim_{h\to 0}\|f(\cdot+h)-f(\cdot)\|_{p,\lambda}=0$ for $f\in L^{p,\lambda}_{2\pi,0}$ (see Lemma 1). Let

$$a_0(f)/2+\sum_{k=1}^{\infty}(a_k(f)\cos kx+b_k(f)\sin kx)=\sum_{k=0}^{\infty}A_k(f)(x) \tag{1}$$

be the trigonometric Fourier series of $f\in L^1_{2\pi}$ and $S_n(f)(x)=\sum_{k=0}^{n}A_k(f)(x)$ be its n-th partial sum. If T_n is the space of trigonometrical polynomials of order at most $n\in\mathbb{Z}_+=\{0,1,\ldots\}$ and $f\in L^{p,\lambda}_{2\pi,0}$, then $E_n(f)_{p,\lambda}=\inf\{\|f-t_n\|_{p,\lambda}:t_n\in T_n\}$.

For a function $f\in L^{p,\lambda}_{2\pi,0}$ and $m\in\mathbb{N}=\{1,2,\ldots\}$ we consider the difference of order $m\in\mathbb{N}$ with step h

$$\Delta_h^m f(x)=\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x+kh)$$

and the modulus of smoothness

$$\omega_m(f,\delta)_{p,\lambda}=\sup_{|h|\le\delta}\|\Delta_h^m f(x)\|_{p,\lambda}.$$

Let $1<p<\infty$, $1/p+1/q=1$. A weight function w (a 2π-periodic, measurable and positive a.e. on $\mathbb{R}$ function) belongs to the Muckenhoupt class $A_p(\mathbb{T})$, if the inequality

$$\sup_I\left(|I|^{-1}\int_I w^p(x)\,dx\right)^{1/p}\left(|I|^{-1}\int_I w^{-q}(x)\,dx\right)^{1/q}=[v]_{A_p[0,1]}<\infty,$$

holds, where I are intervals of length at most 2π (see [19]). A weight function w belongs to the class $A_1(\mathbb{T})$ if $Mw(x)\le Cw(x)$ a.e. on $\mathbb{R}$. Here $Mw(x)=\sup_{I\ni x}|I|^{-1}\int_I w(x)\,dx$ is the maximal function of w. From the Hölder inequality it follows that $A_1(\mathbb{T})\subset A_{p_1}(\mathbb{T})\subset A_{p_2}(\mathbb{T})$ for $1\le p_1\le p_2<\infty$. A measurable 2π-periodic function f belongs to weighted space $L^p_{w,2\pi}$, $1\le p<\infty$, if $fw\in L^p_{2\pi}$.

Further we use the famous Riesz-Zygmund or typical means of order $r\in\mathbb{N}$ for $f\in L^1_{2\pi}$

$$Z_n^r(f)(x)=\sum_{k=0}^{n}\left(1-\frac{k^r}{(n+1)^r}\right)A_k(x)=\sum_{k=0}^{n}\frac{(k+1)^r-k^r}{n^r}S_k(f)(x), \tag{2}$$

where $n\in\mathbb{Z}_+$. The famous Fejér means $\sigma_n(f)=(n+1)^{-1}\sum_{k=0}^{n}S_k(f)$ coincide with $Z_n^1(f)$.

According to [4, Ch. VIII, §§ 7,14] for a function $f(x) \in L^1_{2\pi}$ there exists a.e. the conjugate function

$$\widetilde{f}(x) = \lim_{\varepsilon \to 0+0} (2\pi)^{-1} \int_{\varepsilon}^{\pi} (f(x-t) - f(x+t)) / \tan(t/2)\, dt,$$

the conjugation operator is bounded in $L^p_{2\pi}$, $1 < p < \infty$ and the Fourier series of the function $\widetilde{f}$ (if $\widetilde{f} \in L^1_{2\pi}$) has the form

$$\sum_{k=1}^{\infty} (a_k(f) \sin kx - b_k(f) \cos kx) =: \sum_{k=1}^{\infty} B_k(f)(x).$$

Let Φ be the space of strictly increasing and continuous on $[0, 2\pi]$ functions $\omega(t)$, with property $\omega(0) = 0$.

We will write $\omega \in B$, if $\omega \in \Phi$ and $\sum_{k=n}^{\infty} k^{-1} \omega(k^{-1}) = O(\omega(n^{-1}))$, $n \in \mathbb{N} = \{1, 2, \ldots\}$.

This class and its equivalent definitions was studied by Bary and Stechkin [3]. If $\omega \in \Phi$ and $\omega(2t) \le \omega(t)$, $t \in [0, \pi]$, then $\omega \in \Delta_2$ (or ω satisfies Δ_2-condition).

For $\omega \in \Phi$ and $m \in \mathbb{N}$ let us consider a Hölder type space

$$H^{m,\omega}_{p,\lambda} = \{ f \in L^{p,\lambda}_{2\pi,0} : \omega_m(f, \delta)_{p,\lambda} \le C\omega(\delta), 2\pi \ge \delta \ge 0 \}, \tag{3}$$

where C depends on f and no depends on δ. The last space with the norm

$$\|f\|_{p,\lambda,m,\omega} = \|f\|_{p,\lambda} + \sup_{0 < t \le 2\pi} \frac{\omega_m(f, t)_{p,\lambda}}{\omega(t)} \tag{4}$$

is a Banach one.

Testici and Israfilov [23] proved

Proposition 1 *Let* $1 < p < \infty$, $0 < \lambda \le 1$, $f \in L^{p,\lambda}_{2\pi}$. *Then*

(i) $\|S_n(f)\|_{p,\lambda} \le C_1 \|f\|_{p,\lambda}$, $n \in \mathbb{N}$;

(ii) $\|\widetilde{f}\|_{p,\lambda} \le C_2 \|f\|_{p,\lambda}$;

(iii) $\|f - S_n(f)\|_{p,\lambda} \le (C_1 + 1) E_n(f)_{p,\lambda}$, $n \in \mathbb{N}$;

where C_1, C_2 does not depend on n and f.

Let $f \in L^p_{2\pi}$, $1 \le p < \infty$, and $t^*_n(f) \in T_n$ be such that $\|f - t^*_n(f)\|_p = \inf_{t_n \in T_n} \|f - t_n\|_p$.

Sunouchi [21] established the following result and its analogue for continuous periodic functions.

Proposition 2 *Let* $f \in L^p_{2\pi}$, $1 \le p < \infty$, $r \in \mathbb{N}$, $0 < \alpha < r$. *Then the conditions*

$$\|f - t^*_n(f)\|_p = O(n^{-\alpha}), \quad n \in \mathbb{N},$$

and

$$\|(t_n^*(f))^{(r)}\|_p = O(n^{r-\alpha}), \quad n \in \mathbb{N},$$

are equivalent.

Zhuk and Natanson [30] obtained the estimate of modulus of smoothness in terms of norms of derivatives of polynomials of best approximation.

Proposition 3 *Let* $1 \le p \le \infty$, $m \in \mathbb{N}$, $r \in \mathbb{Z}_+$, $f \in L^p_{2\pi}$. *If the series*

$$\sum_{k=1}^{\infty} k^{-m-1} \|(t_k^*(f))^{(m+r)}\|_p$$

converges, then for $r \ge 1$ a function f is equivalent to f_0 such that $f_0', \ldots, f_0^{(r-1)}$ are absolutely continuous on each period and $f_0^{(r)} \in L^p_{2\pi}$ (for $r = 0$ we set $f_0 = f$) and the inequality

$$\omega_m(f_0^{(r)}, 1/n)_p \le C \sum_{k=n+1}^{\infty} k^{-m-1} \|(t_k^*(f))^{(m+r)}\|_p, \quad n \in \mathbb{N},$$

holds.

The aim of the present paper is to obtain Sunouchi and Zhuk–Natanson type results in the Morrey space using Riesz–Zygmund means (or Fourier partial sums) instead of polynomials of best approximation. Also we establish a two-sided estimate for the degree of approximation by Riesz–Zygmund means in Morrey space and some direct approximation results for Riesz–Zygmund and Bernstein–Rogosinski means. The approximation in Hölder type spaces based on the Morrey spaces is studied.

2 Auxiliary lemmas

Lemma 1 *Let* $1 \le p < \infty$, $0 < \lambda \le 1$, $f \in L^{p,\lambda}_{2\pi,0}$. *Then*

$$\lim_{h \to 0} \|f(\cdot + h) - f(\cdot)\|_{p,\lambda} = 0$$

Proof By definition for $\varepsilon > 0$ we can find $t_n \in T_n$ such that

$$\|f - t_n\|_{p,\lambda} = \|f(\cdot + h) - t_n(\cdot + h)\|_{p,\lambda} < \varepsilon/3.$$

Since t_n is uniformly continuous. there exists $\delta > 0$ such that $|t_n(x+h) - t_n(x)| < \varepsilon/(6\pi)$ for all $x \in \mathbb{R}$ and $|h| < \delta$. Then

$$\|t_n(\cdot+h)-t_n(\cdot)\|_{p,\lambda}=\sup_I\left(|I|^{\lambda-1}\int_I|t_n(x+h)-t_n(x)|^p\,dx\right)^{1/p}$$
$$\le\sup_I\left(|I|^{\lambda-1}|I|(\varepsilon/(6\pi))^p\right)^{1/p}\le(2\pi)^{\lambda/p}\varepsilon/(6\pi)\le\frac{2\pi\varepsilon}{6\pi}=\frac{\varepsilon}{3}$$

and

$$\|f(\cdot+h)-f(\cdot)\|_{p,\lambda}\le\|f(\cdot+h)-t_n(\cdot+h)\|_{p,\lambda}+\|t_n(\cdot+h)-t_n(\cdot)\|_{p,\lambda}$$
$$+\|f(\cdot)-t_n(\cdot)\|_{p,\lambda}<\varepsilon,\quad |h|<\delta.$$

□

Lemma 2 is known, e.g., it is used in [8] without references. But we can not find it in monographs where Morrey spaces are treated (see [1, 17]) and give a proof here.

Lemma 2 *Let* $1\le p<\infty$, $0<\lambda\le 1$, $f(x,y)$ *is measurable on* $\mathbb{R}^2$ *and* 2π*-periodic in each variable. Then*

$$\left\|\int_0^{2\pi}|f(\cdot,y)|\,dy\right\|_{p,\lambda}\le\int_0^{2\pi}\|f(\cdot,y)\|_{p,\lambda}\,dy.$$

Proof Let $I=[a,b]\subset\mathbb{R}$ and $b-a\le 2\pi$, By the generalized Minkowski inequality we have

$$\left(|I|^{\lambda-1}\int_I\left(\int_0^{2\pi}|f(x,y)|\,dy\right)^p dx\right)^{1/p}$$
$$\le|I|^{(\lambda-1)/p}\int_0^{2\pi}\left(\int_I|f(x,y)|^p\,dx\right)^{1/p}dy\le\int_0^{2\pi}\|f(\cdot,y)\|_{p,\lambda}\,dy. \tag{5}$$

Taking the supremum in the left-hand side of (5) over I we obtain the inequality of Lemma. □

Corollary 1 *Let* $1\le p<\infty$, $h\in L_{2\pi}^{p,\lambda}$, $g\in L_{2\pi}^1$. *Then the convolution* $h*g(x)=\int_0^{2\pi}h(x-y)g(y)\,dy$ *belongs to* $L_{2\pi}^{p,\lambda}$ *and* $\|h*g\|_{p,\lambda}\le\|h\|_{p,\lambda}\|g\|_1$.

Proof We take $f(x,y) = h(x-y)g(y)$ in Lemma 2. Then

$$\|h * g\|_{p,\lambda} \le \left\| \int_0^{2\pi} |h(x-y)||g(y)|\,dy \right\|_{p,\lambda}$$
$$\le \int_0^{2\pi} \|h(\cdot - y)g(y)\|_{p,\lambda}\,dy = \|h\|_{p,\lambda} \int_0^{2\pi} |g(y)|\,dy = \|h\|_{p,\lambda}\|g\|_1.$$

□

Lemma 3 is a variant of Marcinkiewicz multiplier theorem and it is stated in [15] without proof (the author claims that the proof is similar to one of corresponding result in weighted Lebesgue space [18]. But in [18] the multipliers in $\mathbb{R}^n$ were studied). We prove Lemma 3 by the method of Israfilov and Testici [23].

Lemma 3 *Let* $\{\mu_k\}_{k=0}^{\infty}$ *satisfy the conditions*

$$|\mu_k| \le C_1, \quad k \in \mathbb{Z}_+; \quad \sum_{k=2^{m-1}}^{2^m-1} |\mu_k - \mu_{k+1}| \le C_2, \quad m \in \mathbb{N}.$$

If $1 < p < \infty$, $0 < \lambda \le 1$ and $f \in L_{2\pi}^{p,\lambda}$ has the Fourier series (1), then there exists a function $F(f) \in L_{2\pi}^{p,\lambda}$ with the Fourier series $\sum_{k=0}^{\infty} \mu_k A_k(f)(x)$ and $\|F(f)\|_{p,\lambda} \le C_3\|f\|_{p,\lambda}$, where C_3 does not depend on f, p and λ.

Proof Since $L_{2\pi}^{p,\lambda} \subset L_{2\pi}^{p}$ and Lemma 3 is well-known for $\lambda = 1$, i.e. in $L_{2\pi}^{p}$, $1 < p < \infty$ (see [31, Ch. XV, Theorem 4.14]), the function $F(f)$ is correctly defined for $f \in L_{2\pi}^{p,\lambda}$. Coifman and Rochberg [9] proved that for any interval I and its indicator X_I the inequality $M(M(X_I))(x) \le C_1 M(X_I)(x)$ holds a.e. on $\mathbb{R}$. In other words, $M(X_I)$ belongs to the Muckenhoupt class $A_1(\mathbb{T})$. Since $A_1(\mathbb{T}) \subset A_p(\mathbb{T})$ and $F(f)$ is bounded in $L_{w,2\pi}^{p}$ for $1 < p < \infty$, $w \in A_p(\mathbb{T})$ (see [5, Theorem 4.4]), one has for $I \subset [0, 2\pi]$

$$\int_I |F(f)(x)|^p\,dx = \int_0^{2\pi} |F(f)(x)|^p X_I(x)\,dx \le$$
$$\le \int_0^{2\pi} |F(f)(x)|^p M(X_I)(x)\,dx \le C_2 \int_0^{2\pi} |f(x)|^p M(X_I)(x)\,dx.$$

It is known that for $x \in [0, 2\pi]$

$$M(X_I)(x) \asymp X_I(x) + \sum_{k=0}^{\infty} 2^{-k} X_{J_k}(x), \quad J_k = (2^{k+1}I \setminus 2^k I) \cap [0, 2\pi], \tag{6}$$

where $A(x) \asymp B(x)$, $x \in Y$, means that $C_1 A(x) \le B(x) \le C_2 A(x)$ for some $C_2 > C_1 > 0$ and $x \in Y$, and mI is the interval of length $m|I|$ such that the centers of I and mI are the same (see [13]). Using (6) we obtain

$$
\begin{aligned}
&\sup_I |I|^{\lambda-1}\int_I |F(f)(x)|^p\,dx\\
&\quad\le C_3\sup_I |I|^{\lambda-1}\int_I |f(x)|^p\,dx\\
&\qquad + C_3\sum_{k=0}^{\infty}2^{-k}\sup_I |I|^{\lambda-1}\int_{J_k}|f(x)|^p\,dx\\
&\quad\le C_3\left(\|f\|_{p,\lambda}^p+\sum_{k=0}^{\infty}2^{-k}(2^{k+1})^{1-\lambda}\sup_I |2^{k+1}I|^{\lambda-1}\int_{2^{k+1}I}|f(x)|^p\,dx\right)\\
&\quad\le C_4\|f\|_{p,\lambda}^p\left(1+\sum_{k=0}^{\infty}2^{-\lambda}\right)=C_5\|f\|_{p,\lambda}^p.
\end{aligned}
$$

□

For a technical purpose we define the following iterated means

$$
Z_{n,*}^r(f)=\sum_{k=0}^{n}\left(1-\frac{k^r}{(n+1)^r}\right)\left(1-\frac{k^r}{(n+2)^r}\right)A_k(f)=Z_n^r(Z_{n+1}^r(f)).
$$

The result of Lemma 4 for even r may be found in [7] and for odd r in [29].

Lemma 4 *For $f\in L^1_{2\pi}$ the following equalities*

$$
|Z_{n,*}^r(f)-Z_{n-1,*}^r(f)|=\frac{(n+2)^r-n^r}{(n+2)^r n^r}|(U_n^r(f))^{(r)}|,
$$

where $U_n^r(f)=Z_n^r(f)$ for even r and $U_n^r(f)=\widetilde{Z_n^r(f)}$ for odd r

Lemma 5 *Let $r\in\mathbb{N}$, $1\le p<\infty$, $0<\lambda\le 1$. Then the operators Z_n^r are uniformly bounded in $L^{p,\lambda}_{2\pi,0}$ and*

$$
\|f-Z_n^r(f)\|_{p,\lambda}\le C(n+1)^{-r}\sum_{k=0}^{n}(k+1)^{r-1}E_k(f)_{p,\lambda}.
$$

Proof It is easy to see that $Z_n^r(f)=f*F_n^r$, where $F_n^r(x)=\pi^{-1}(1/2+\sum_{k=1}^{n}(1-k^r/(n+1)^r))\cos kx$. Timan [26] proved that $\{\|F_n^r\|_1\}_{n=1}^{\infty}$ is bounded. From this result and Corollary 1 the first statement of Lemma 2.5 follows. We note that for $1<p<\infty$ this assertion may be proved using the equality

$$
Z_n^r(f)(x)=\sum_{k=0}^{n}\frac{(k+1)^r-k^r}{(n+1)^r}S_k(f)(x)
$$

and Proposition 1.

For the second statement we use Timan's method of proof of inequality (1.15) in [26]. Since the last paper is in Russian, we recommend the proof of Lemma 3.8 in

[29] for representation of Timan's method. Here also for $1<p<\infty$ the proof is more brief and uses Proposition 1. Namely,

$$\|f - Z_n^r(f)\|_{p,\lambda} \le \sum_{k=0}^{n} \frac{(k+1)^r - k^r}{(n+1)^r} \|f - S_k(f)\|_{p,\lambda} \le$$
$$\le \frac{C_2}{(n+1)^r} \sum_{k=0}^{n} k^{r-1} E_k(f)_{p,\lambda}.$$

□

The result of Lemma 6 (i) is established by Israfilov and Tozman [12] while the result of (ii) may be found in their paper [13]. These inequalities are known as direct and inverse theorems of approximation by trigonometric polynomials and in general form for the first time were obtained by Stechkin [20] for continuous functions.

Lemma 6 *Let* $m \in \mathbb{N}$, $1<p<\infty$, $0<\lambda\le 1$, $f \in L^{p,\lambda}_{2\pi,0}$. *Then*

(i) $E_n(f)_{p,\lambda} \le C\omega_m(f,(n+1)^{-1})_{p,\lambda}$, $n \in \mathbb{Z}_+$.

(ii) $\omega_m(f,n^{-1})_{p,\lambda} \le Cn^{-m}\sum_{k=0}^{n}(k+1)^{m-1}E_k(f)_{p,\lambda}$, $n \in \mathbb{Z}_+$.

Lemma 7 (i) is due to Nikolskii and Stechkin while part (ii) is established by Stechkin in [20] in the case of continuous functions. We give proof of this part for the utility of a reader.

Lemma 7

(i) Let $m \in \mathbb{N}$, $1<p<\infty$, $0<\lambda\le 1$. If $t_n \in T_n$, $n \in \mathbb{N}$, then

$$\|t_n^{(m)}\|_{p,\lambda} \le \left(\frac{n}{2\sin nh}\right)^m \|\Delta_h^m t_n\|_{p,\lambda}, \quad 0<h\le \pi/(2n).$$

(ii) If $m \in \mathbb{N}$, $f \in L^{p,\lambda}_{2\pi,0}$ and $\tau_n(f) \in T_n$, $n \in \mathbb{N}$, satisfy the inequality

$$\|f - \tau_n(f)\|_{p,\lambda} \le K\omega_m(f,n^{-1})_{p,\lambda}, \quad n \in \mathbb{N}.$$

Then $\omega_m(\tau_n(f),\delta)_{p,\lambda} \le C(K)\omega_m(f,\delta)_{p,\lambda}$ for some $C(K)>0$ and all $\delta \in [0,2\pi]$.

Proof (i) The result of Lemma 7 (i) can be proved by the method of Civin (see [25, Ch. 4, sect. 4.8.61]) or by the method of Zamansky (see [10, Ch. VII, Lemma 2.6]).

(ii) For $\tau_n(f)$ satisfying conditions of (ii) we have

$$\begin{aligned}\omega_m(\tau_n(f),\delta)_{p,\lambda} &\le \omega_m(f,\delta)_{p,\lambda} + \omega_m(f-\tau_n(f),\delta)_{p,\lambda}\\ &\le \omega_m(f,\delta)_{p,\lambda} + 2^m\|f-\tau_n(f)\|_{p,\lambda}\\ &\le \omega_m(f,\delta)_{p,\lambda} + C_1\omega_m(f,n^{-1})_{p,\lambda} \le (C_1+1)\omega_m(f,\delta)_{p,\lambda}\end{aligned} \tag{7}$$

for all $\delta \ge 1/n$. From (i) and (7) we also deduce that

$$\|\tau_n^{(m)}(f)\|_{p,\lambda} \le (n/(2\sin 1))^m \omega_m(\tau_n(f), n^{-1})_{p,\lambda} \le C_2 n^m \omega_m(f, n^{-1}). \quad (8)$$

It is known that usual modulus of smoothness in translation-invariant spaces has the properties $\omega_m(f,\eta)_{p,\lambda} \le C_3(\eta/\delta)^m \omega_m(f,\delta)_{p,\lambda}$, $0<\delta\le\eta\le 2\pi$, and $\omega(\tau_n(f),\delta)_{p,\lambda} \le \|\tau_n^{(m)}(f)\|_{p,\lambda}\delta^m$ (see (7.8) and the proof of (7.12) in [10, Ch. II]). Using these facts and (8) we obtain

$$\omega(\tau_n(f),\delta) \le \delta^m \|\tau_n^{(m)}(f)\|_{p,\lambda} \le C_2(n\delta)^m \omega_m(f,n^{-1})_{p,\lambda} \le \\ \le C_4(n\delta)^m (n^{-1}/\delta)^m \omega(f,\delta)_{p,\lambda} = C_4\omega(f,\delta)_{p,\lambda}$$

for $0<\delta\le n^{-1}$ and (ii) is proved. □

Lemma 8 may be found in [8, Lemma 2.3].

Lemma 8 *Let $r\in\mathbb{N}$, $1<p<\infty$, $0<\lambda\le 1$ and $f\in L^{p,\lambda}_{2\pi,0}$ be such that $f, f', \ldots, f^{(r-1)}$ are absolutely continuous on each period and $f^{(r)} \in L^{p,\lambda}_{2\pi,0}$. Then*

$$E_n(f)_{p,\lambda} \le Cn^{-r}\|f^{(r)}\|_{p,\lambda}, \quad n\in\mathbb{N}.$$

Lemma 9 is proved for $r=1$ by Alexits [2] while its general variant for $r>0$ is established by Joó [16].

Lemma 9 *Let $r>0$, $(X,\|\cdot\|_X)$ be a Banach space and $a_k\in X$, $k\in\mathbb{Z}_+$. Let $R_n^{(r)} = \sum_{k=0}^{n-1}(1-k^r/n^r)a_k$, $T_n^{(r)} = \sum_{k=0}^{n-1}(1-k^r/n^r)k^r a_k$, $n\in\mathbb{N}$. Then the condition $\|T_n^{(r)}\|_X \le C_1$, $n\in\mathbb{N}$, holds if and only if there exists $R\in X$ such that $\|R-R_n^{(r)}\|_X \le C_2 n^{-r}$, $n\in\mathbb{N}$. Note that $C_2 = C(r)C_1$ and vice versa.*

Lemma 10 is proved by Timan [24]. By $D_n(t)$ we denote the trigonometric Dirichlet kernel $\sin(n+1/2)t/(2\sin(t/2))$, $n\in\mathbb{N}$.

Lemma 10 *Let $\alpha_n = \pi k(n)/(2n+1) + O((n\ln(n+1)^{-1}))$, $n\in\mathbb{N}$, $k(n)$ be an even natural number, $|\alpha_n|\le\pi$. Then the norms $\|D_n(\cdot+\alpha_n)+D_n(\cdot-\alpha_n)\|_1$ are bounded.*

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In view of Lemma 6 (ii) Theorem 1 sharpens the estimate of Lemma 5. We note that a similar result in $L^p_{2\pi}$, $1<p<\infty$, was proved by Trigub (see [28, sect. 8.2.6]).

Theorem 1 *Let $m\in\mathbb{N}$, $1<p<\infty$, $0<\lambda\le 1$, $f\in L^{p,\lambda}_{2\pi,0}$. Then*

$$C^{-1}\omega_m(f,1/n)_{p,\lambda} \le \|f - Z_n^m(f)\|_{p,\lambda} \le C\omega_m(f,1/n)_{p,\lambda}, \quad n\in\mathbb{N}, \quad (9)$$

for some $C>0$.

Proof Suppose that $\tau_n(f)\in T_n$ satisfies the equality

$$\|f - \tau_n(f)\|_{p,\lambda} = E_n(f)_{p,\lambda}.$$

Then by Lemma 5 we have

$$\|f - Z_n^m(f)\|_{p,\lambda} \le \|f - \tau_n(f)\|_{p,\lambda} + \|Z_n^m(\tau_n(f)) - \tau_n(f))\|_{p,\lambda} + \\ + \|Z_n^m(f - \tau_n(f))\|_{p,\lambda} \le C_1 E_n(f)_{p,\lambda} + \|Z_n^m(\tau_n(f)) - \tau_n(f))\|_{p,\lambda}.$$

It is easy to see that for $t_n \in T_n$ and even m the equality $|t_n - Z_n^m(t_n)| = (n+1)^{-m}|t_n^{(m)}|$ holds, while for odd m we have $|t_n - Z_n^m(t_n)| = (n+1)^{-m}|\widetilde{t_n^{(m)}}|$. Thus, by Lemmas 6 (i) and 7 (i)

$$\|f - Z_n^m(f)\|_{p,\lambda} \le C_2\omega_m(f, n^{-1})_{p,\lambda} + \\ + C_2(n+1)^{-m} n^m \omega_m(\tau_n(f), n^{-1})_{p,\lambda} \le C_3\omega_m(\tau_n(f), n^{-1})_{p,\lambda}.$$

The right-hand side inequality (9) is proved.

Using again the property $\omega_m(t_n, \delta)_{p,\lambda} \le \|t_n^{(m)}\|_{p,\lambda}\delta^m$ for $t_n \in T_n$, we obtain

$$\omega_m(f, n^{-1})_{p,\lambda} \le \omega_m(f - Z_n^m(f), n^{-1})_{p,\lambda} + \omega_m(Z_n^m(f), n^{-1})_{p,\lambda} \le \\ \le C_4(\|f - Z_n^m(f)\|_{p,\lambda} + n^{-m}\|(Z_n^m(f))^{(m)}\|_{p,\lambda}.$$

Using above equalities for $|t_n - Z_n^m(t_n)|$ we find that

$$n^{-m}\|(Z_n^m(f))^{(m)}\|_{p,\lambda} = \left(\frac{n+1}{n}\right)^m \|U_n^m(f) - Z_n^m(U_n^m(f))\|_{p,\lambda}, \tag{10}$$

where $U_n^m(f) = Z_n^m(f)$ for even m and $U_n^m(f) = \widetilde{Z_n^m}(f)$ for odd m. By Proposition 1 and Lemma 5

$$\omega_m(f, n^{-1})_{p,\lambda} \le C_4\|f - Z_n^m(f)\|_{p,\lambda} + 2^m C_4\|U_n^m(f - Z_n^m(f))\|_{p,\lambda} \le \\ \le C_6\|f - Z_n^m(f)\|_{p,\lambda}$$

and the left-hand side inequality (9) is proved. □

We will write $g \in W^r L_{2\pi,0}^{p,\lambda}$, where $r \in \mathbb{N}$, $1 \le p < \infty$, $0 < \lambda \le 1$, if $g, g', \ldots, g^{(r-1)}$ are absolutely continuous on any period and $g^{(r)} \in L_{2\pi,0}^{p,\lambda}$. Theorem 2 is an analogue and extension of the result of Bustamante [6, Theorem 1] obtained in the case $r = 1$ and $f \in L_{2\pi}^p$, $1 < p < \infty$.

Theorem 2 *Let $r \in \mathbb{N}$, $1 < p < \infty$, $0 < \lambda \le 1$, $f, \widetilde{f} \in W^r L_{2\pi,0}^{p,\lambda}$. Then*

$$\|f - Z_n^r(f)\|_{p,\lambda} \le Cn^{-r}\|f^{(r)}\|_{p,\lambda}, \quad n \in \mathbb{N}. \tag{11}$$

For $r \in \mathbb{N}$, $p = 1$, $0 < \lambda \le 1$ and $f, \widetilde{f} \in W^r L_{2\pi,0}^{p,\lambda}$ we have the inequality

$$\|f - Z_n^r(f)\|_{1,\lambda} \le Cn^{-r}\|(\varphi(f))^{(r)}\|_{1,\lambda}, \quad n \in \mathbb{N},$$

where $\varphi(f) = f$ for even r ($r \in 2\mathbb{N}$) and $\varphi(f) = \widetilde{f}$ for odd r ($r \in 2\mathbb{N} - 1$).

Proof Let $r \in \mathbb{N}$ and $f, \widetilde{f} \in W^r L^{p,\lambda}_{2\pi,0}$. Then the Fourier series of $f^{(r)}$ is

$$\sum_{k=1}^{\infty} k^r(a_k(f)\cos(kx + r\pi/2) + b_k(f)\cos(kx + r\pi/2)) =$$
$$= \begin{cases} \pm\sum_{k=1}^{\infty} k^r A_k(f)(x), & r \in 2\mathbb{N}; \\ \pm\sum_{k=1}^{\infty} k^r B_k(f)(x), & r \in 2\mathbb{N} - 1. \end{cases}$$

In a similar manner, the Fourier series of $\widetilde{f}^{(r)}$ for odd r has the form $\pm\sum_{k=1}^{\infty} k^r A_k(f)(x)$. If $\varphi(f) \in W^r L^{p,\lambda}_{2\pi,0}$, then by Lemma 5 the norms of $T^{(r)}_{n+1} = \sum_{k=0}^{n}(1 - k^r/(n+1)^r)k^r A_k(f)$ in $L^{p,\lambda}_{2\pi}$ are bounded by $C_1\|(\varphi(f))^{(r)}\|_{p,\lambda}$. Thus, by Lemma 9 Riesz-Zygmund means $Z_n^r(f)$ converges to $Z(f)$ in $L^{p,\lambda}_{2\pi}$ and

$$\|Z_n^r(f) - Z(f)\|_{p,\lambda} \le C_2\frac{\|(\varphi(f))^{(r)}\|_{p,\lambda}}{(n+1)^r}, \quad n \in \mathbb{N} \tag{12}$$

for all $1 \le p < \infty$ and $0 < \lambda \le 1$. It is easy to see that Fourier coefficients of $Z(f)$ and f are the same and $Z(f) = f$. Finally, for $1 < p < \infty$ by Proposition 1 (ii) we obtain $\|(\varphi(f))^{(r)}\|_{p,\lambda} \le C_3\|f^{(r)}\|_{p,\lambda}$ and (11) is valid. □

Now we obtain an analogue of Theorem 1 in the case $p = 1$. For $f \in L^{1,\lambda}_{2\pi,0}$, $m \in \mathbb{N}$ and $\varphi(g)$ defined as in Theorem 2 we introduce a K-functional

$$\widetilde{K_m}(f,t)_{1,\lambda} = \inf\{\|f - g\|_{1,\lambda} + t^m\|(\varphi(g))^{(r)}\|_{1,\lambda} : g, \widetilde{g} \in W^m L^{1,\lambda}_{2\pi}\}$$

Theorem 3 *Let $p = 1$, $0 < \lambda \le 1$, $m \in \mathbb{N}$, $f \in L^{1,\lambda}_{2\pi,0}$. Then there exist $K_2 > K_1 > 0$ such that*

$$K_1\widetilde{K_m}(f,n^{-1})_{1,\lambda} \le \|f - Z_n^m(f)\|_{1,\lambda} \le K_2\widetilde{K_m}(f,n^{-1})_{1,\lambda}, \quad n \in \mathbb{N}.$$

Proof It is clear that

$$\widetilde{K_m}(f,n^{-1})_{1,\lambda} \le \|f - Z_n^m(f)\|_{1,\lambda} + n^{-m}\|(\varphi(Z_n^m(f)))^{(m)}\|_{1,\lambda}.$$

By the proof of Theorem 1 and Lemma 5 we have $(\varphi(Z_n^m(f)))^{(m)} = \pm(n+1)^m(Z_n^m(f) - Z_n^m(Z_n^m(f)))$ and

$$\widetilde{K_m}(f, n^{-1})_{1,\lambda} \le \|f - Z_n^m(f)\|_{1,\lambda} + \\ + ((n+1)/n)^m \|Z_n^m(f) - Z_n^m(Z_n^m(f))\|_{1,\lambda} \le C_1 \|f - Z_n^m(f)\|_{1,\lambda}.$$

On the other hand, for every g such that $g, \widetilde{g} \in W^m L_{2\pi}^{1,\lambda}$ we obtain by Lemma 5 and (12) from the proof of Theorem 2

$$\|f - Z_n^m(f)\|_{1,\lambda} \le \|f - g - Z_n^m(f-g)\|_{1,\lambda} + \|g - Z_n^m(g)\|_{1,\lambda} \le \\ \le C_2 \|f - g\|_{1,\lambda} + C_3 \frac{\|(\varphi(g))^{(m)}\|_{1,\lambda}}{n^m}.$$

Taking the infimum over all such g we find that

$$\|f - Z_n^m(f)\|_{1,\lambda} \le \max(C_2, C_3) \widetilde{K_m}(f, n^{-1})_{1,\lambda}.$$

□

From Theorems 1 and 3 we deduce for Fejér means

Corollary 2 *Let* $1 < p < \infty$, $0 < \lambda \le 1$, $f \in L_{2\pi,0}^{p,\lambda}$. *Then for some* $C_2 > C_1 > 0$ *we have*

$$C_1 \omega_1(f, n^{-1})_{p,\lambda} \le \|f - \sigma_n(f)\|_{p,\lambda} \le C_2 \omega_1(f, n^{-1})_{p,\lambda}, n \in \mathbb{N}.$$

If $p = 1$, $0 < \lambda \le 1$, $f \in L_{2\pi,0}^{p,\lambda}$, then for some $C_4 > C_3 > 0$

$$C_3 \widetilde{K_1}(f, n^{-1})_{1,\lambda} \le \|f - \sigma_n(f)\|_{1,\lambda} \le K_2 \widetilde{K_1}(f, n^{-1})_{1,\lambda}, \quad n \in \mathbb{N}.$$

Now we consider Bernstein–Rogosinski means

$$R_{n,\alpha}(f) = 2^{-1}(S_n(f)(\cdot - \alpha_n) + S_n(f)(\cdot + \alpha_n)),$$

where $\{\alpha_n\}_{n=1}^\infty$ satisfies the conditions of Lemma 10. Similar to Riesz–Zygmund means approximation by this means under some restrictions can be estimated by the modulus of smoothness of higher order, namely, of order 2. Such estimate was obtained for continuous functions by Stechkin. In the case $\alpha_n = \pi/(2n)$ see Stechkin's result in [11, Ch. 5, Theorem 2.4].

Theorem 4 *Let* $1 \le p < \infty$, $0 < \lambda \le 1$, $f \in L_{2\pi,0}^{p,\lambda}$. *If*

$$\alpha_n = \frac{\pi k(n)}{2n+1} + O\left(\frac{1}{n \ln(n+1)}\right), \quad n \in \mathbb{N},$$

$\{k(n)\}_{n=1}^\infty$ is a bounded sequence of even natural numbers, then

$$\|f - R_{n,\alpha}(f)\|_{p,\lambda} \le C_2 \omega_2(f, n^{-1})_{p,\lambda}, \quad n \in \mathbb{N}.$$

Proof Let $\tau_n(f) \in T_n$ be such that $\|f - \tau_n(f)\|_{p,\lambda} = E_n(f)_{p,\lambda}$. Then $S_n(\tau_n(f)) = \tau_n(f)$ and we have by virtue of Corollary 1, Lemmas 6 and 10

$$\begin{aligned}
&\|f - R_{n,\alpha}(f)\|_{p,\lambda} \\
&\quad = \|f - (S_n(f - \tau_n(f))(\cdot - \alpha_n) + S_n(f - \tau_n(f))(\cdot + \alpha_n))/2 - \\
&\qquad - (\tau_n(f)(\cdot - \alpha_n) + \tau_n(f)(\cdot + \alpha_n))/2\|_{p,\lambda} \le \|f - (f(\cdot - \alpha_n) + f(\cdot + \alpha_n))/2\|_{p,\lambda} + \\
&\qquad + 2^{-1}\|R_{n,\alpha}(f - \tau_n(f))\|_{p,\lambda} + 2^{-1}\|f(\cdot - \alpha_n) - \tau_n(f)(\cdot - \alpha_n)\|_{p,\lambda} + \\
&\qquad + 2^{-1}\|(f - \tau_n(f))(\cdot + \alpha_n)\|_{p,\lambda} \le \frac{1}{2}\omega_2(f, \alpha_n)_{p,\lambda} + C_1\|f - \tau_n(f)\|_{p,\lambda} + \\
&\qquad + \|f - \tau_n(f)\|_{p,\lambda} \le C_2(\omega_2(f, \alpha_n)_{p,\lambda} + \omega_2(f, n^{-1})_{p,\lambda}).
\end{aligned}$$

Finally, it is known that for $t_2 > t_1 > 0$ one has $\omega_2(f, t_2)_{p,\lambda} \le (2t_2/t_1)^2 \omega_2(f, t_1)_{p,\lambda}$ (see, e.g., [11, Ch. 3, (4.18)] for continuous functions). Thus, $\omega_2(f, \alpha_n)_{p,\lambda} \le C_3 \omega_2(f, n^{-1})_{p,\lambda}$ and the inequality of Theorem 4 holds. □

Corollary 3 *Let* $\alpha_n = \pi/(2n)$, $n \in \mathbb{N}$, $1 \le p < \infty$, $0 < \lambda \le 1$, $f \in L^{p,\lambda}_{2\pi,0}$. *Then*

$$\|f - R_{n,\alpha}(f)\|_{p,\lambda} \le C\omega_2(f, n^{-1})_{p,\lambda}, \quad n \in \mathbb{N}.$$

Proof Let us consider $\beta_n = \pi/n$, $n \in \mathbb{N}$. Then

$$\frac{\pi}{n} - \frac{2\pi}{2n+1} = \frac{2\pi}{2n(2n+1)} = O\left(\frac{1}{n\ln(n+1)}\right)$$

and $\{\beta_n\}_{n=1}^{\infty}$ satisfies the conditions of Theorem 4. But $\alpha_n = \beta_{2n}$ and

$$\begin{aligned}
\|f - R_{n,\alpha}(f)\|_{p,\lambda} &= \|f - R_{2n,\beta}(f)\|_{p,\lambda} \le C_1\omega(f, (2n)^{-1})_{p,\lambda} \\
&\le C_1\omega(f, n^{-1})_{p,\lambda}.
\end{aligned}$$

□

4 Inverse and equivalence theorems of Sunouchi and Zhuk–Natanson type

The counterpart of Theorem 6 was proved in [29] for variable exponent Lebesgue spaces $L^{p(\cdot)}_{2\pi}$. In [29] a general class of exponents $p(\cdot)$ was considered and an analogue of Proposition 1 is not valid in all such spaces $L^{p(\cdot)}_{2\pi}$. Here we give a more simple proof of such result than in [29] in the case $1 < p < \infty$.

The content of Theorem 5 is close to one of Proposition 3. Since S_n are linear operators and $S_n(S_m) = S_{\min(m,n)}$, the proof is simpler than for polynomials of best approximation.

Theorem 5 *Let* $m \in \mathbb{N}$, $1 < p < \infty$, $0 < \lambda \le 1$, $f \in L^{p,\lambda}_{2\pi,0}$. *Then*

$$\omega_m(f,1/n)_{p,\lambda} \le C \sum_{k=n+1}^{\infty} k^{-m-1} \|S_n^{(m)}(f)\|_{p,\lambda}, \quad n \in \mathbb{N}.$$

Proof Using again the property $\omega_m(S_n(f),\delta)_{p,\lambda} \le \|S_n^{(m)}(f)\|_{p,\lambda}\delta^m$, $\delta \in [0,2\pi]$, we have

$$\begin{aligned}\omega_m(f,1/n)_{p,\lambda} &\le \omega_m(f - S_n(f),1/n)_{p,\lambda} + \omega_m(S_n(f),1/n)_{p,\lambda} \le \\ &\le 2^m\|f - S_n(f)\|_{p,\lambda} + n^{-m}\|S_n^{(m)}(f)\|_{p,\lambda}.\end{aligned} \tag{13}$$

Due to Lemma 8 and Proposition 1 (iii) we write for $k \in \mathbb{Z}_+$

$$\begin{aligned}&\|f - S_{2^k n}(f)\|_{p,\lambda} - \|f - S_{2^{k+1}n}(f)\|_{p,\lambda} \\ &\quad \le \|S_{2^{k+1}n}(f) - S_{2^k n}(f)\|_{p,\lambda} = \\ &\quad = \|S_{2^{k+1}n}(f) - S_{2^k n}(S_{2^{k+1}n}(f))\|_{p,\lambda} \le C_1 E_{2^k n}(S_{2^{k+1}n}(f))_{p,\lambda} \le \\ &\quad \le C_2(2^k n)^{-m}\|S^{(m)}_{2^{k+1}n}(f)\|_{p,\lambda}.\end{aligned}$$

We note that for $f \in L_{2\pi}^{p,\lambda}$ and $m,n \in \mathbb{N}$, $m<n$, by Proposition 1 (i)

$$\|S_m(f)\|_{p,\lambda} = \|S_m(S_n(f))\|_{p,\lambda} \le C_3\|S_n(f)\|_{p,\lambda}.$$

Therefore,

$$\|f - S_{2^k n}(f)\|_{p,\lambda} - \|f - S_{2^{k+1}n}(f)\|_{p,\lambda} \le C_4 \sum_{j=2^{k+1}n+1}^{2^{k+2}n} \frac{\|S_j^{(m)}(f)\|_{p,\lambda}}{j^{m+1}}.$$

Summing up these inequalities over $k = 0, 1, \ldots$, we obtain

$$\|f - S_n(f)\|_{p,\lambda} \le C_5 \sum_{j=2n+1}^{\infty} j^{-m-1}\|S_j^{(m)}(f)\|_{p,\lambda}. \tag{14}$$

If we substitute (14) into (13), then we find that

$$\begin{aligned}\omega_m(f,1/n)_{p,\lambda} &\le C_4 \sum_{j=2n+1}^{\infty} j^{-m-1}\|S_j^{(m)}(f)\|_{p,\lambda} + \\ &\quad + 2^{m+1}C_3 \sum_{j=n+1}^{2n} j^{-m-1}\|S_j^{(m)}(f)\|_{p,\lambda} \le C_6 \sum_{j=n+1}^{\infty} j^{-m-1}\|S_j^{(m)}(f)\|_{p,\lambda}.\end{aligned}$$

□

By virtue of Lemma 6 (i) Theorem 6 sharpens the result of Theorem ??.

Theorem 6 *Let* $r, m \in \mathbb{N}$, $1<p<\infty$, $0<\lambda\le 1$, $f \in L_{2\pi,0}^{p,\lambda}$. *Then*

$$\omega_m(f,1/n)_{p,\lambda} \le C \sum_{j=n+1}^{\infty} j^{-m-1} \|(Z_j^r(f))^{(m)}\|_{p,\lambda}.$$

Proof Let us put $\mu_k^{(n)} = (2n+1)^r/((2n+1)^r - k^r)$, $k = 0,1,\ldots,n$, and $\mu_k^{(n)} = 0$ for $k > n$. Then $\mu_k^{(n)}$ increases for $k = 0,1,\ldots,n$ and

$$|\mu_k^{(n)}| = \frac{(2n+1)^r}{(2n+1)^r - n^r} \le \frac{(3n)^r}{(2n)^r - n^r} = \frac{3^r}{2^r - 1}.$$

On the other hand,

$$\sum_{k=0}^{\infty} |\mu_k^{(n)} - \mu_{k+1}^{(n)}| = \sum_{k=0}^{n-1} (\mu_{k+1}^{(n)} - \mu_k^{(n)}) + \mu_n^{(n)} - 0 \le 2\mu_n^{(n)} = 2\frac{3^r}{2^r - 1}.$$

For the operator $F_n(f) = \sum_{k=0}^{\infty} \mu_k^{(n)} A_k(f)$ we have the equality $F_n(Z_{2n}^r(f)) = S_n(f)$. We obtain $\|S_n(f)\|_{p,\lambda} \le C_1 \|Z_{2n}^r(f)\|_{p,\lambda}$, $n \in \mathbb{N}$, where C_1 does not depend on n by Lemma 3. By Theorem 5 we have for $n \in \mathbb{N}$

$$\omega_m(f,1/n)_{p,\lambda} \le C_2 \sum_{j=n+1}^{\infty} j^{-m-1} \|S_j^{(m)}(f)\|_{p,\lambda} \le$$
$$\le C_2 C_1 \sum_{j=n+1}^{\infty} \frac{\|(Z_{2j}^r(f))^{(m)}\|_{p,\lambda}}{j^{m+1}} \le 2^{m+1} C_1 C_2 \sum_{j=n+1}^{\infty} \frac{\|(Z_j^r(f))^{(m)}\|_{p,\lambda}}{j^{m+1}}.$$

□

From Theorem 6 and Lemma 5 (i) we deduce

Corollary 4 *Under conditions of Theorem 6 we have the inequality*

$$E_n(f)_{p,\lambda} \le C \sum_{j=n+1}^{\infty} j^{-m-1} \|(Z_j^r(f))^{(m)}\|_{p,\lambda}, \quad n \in \mathbb{N}.$$

Theorem 7 is a counterpart of Proposition 2.

Theorem 7 *Suppose that $m \in \mathbb{N}$, $1 < p < \infty$, $0 < \lambda \le 1$, $f \in L_{2\pi,0}^{p,\lambda}$ and $\omega \in B \cap \Delta_2$. Then the conditions $f \in H_{p,\lambda}^{m,\omega}$ and*

$$\|(Z_n^m(f))^{(m)}\|_{p,\lambda} = O(n^m \omega(n^{-1})), \quad n \in \mathbb{N}, \tag{15}$$

are equivalent.

Proof If (15) holds, then by Theorem 5 we obtain

$$\omega_m(f,n^{-1})_{p,\lambda} \le C_1 \sum_{k=n+1}^{\infty} k^{-m-1} \|(Z_k^m(f))^{(m)}\|_{p,\lambda} \le$$
$$\le C_2 \sum_{k=n+1}^{\infty} k^{-1}\omega(k^{-1}) \le C_3\omega(n^{-1}), \quad n \in \mathbb{N},$$

due to the condition $\omega \in B$. Since ω satisfies the Δ_2-condition $\omega(2t) \le C_4\omega(t)$, $t \in [0,\pi]$, we have for $n \in \mathbb{N}$ and $\delta \in ((n+1)^{-1}, n^{-1}]$

$$\omega_m(f,\delta)_{p,\lambda} \le \omega_m(f,n^{-1})_{p,\lambda} \le C_3\omega(n^{-1}) \le$$
$$\le C_3\omega(2(n+1)^{-1}) \le C_3C_4\omega((n+1)^{-1}) \le C_3C_4\omega(\delta).$$

For $\delta \in (1, 2\pi]$ a similar inequality follows from monotonicity and boundedness of $\omega_m(f,\delta)_{p,\lambda}$ on $[0,2\pi]$. Thus, $f \in H_{p,\lambda}^{m,\omega}$.

Conversely, let $f \in H_{p,\lambda}^{m,\omega}$. By Lemma 7 (ii) and Theorem 1

$$\|(Z_n^m(f))^{(m)}\|_{p,\lambda} \le C_5 n^m \omega_m(Z_n^m(f), 1/n)_{p,\lambda} \le$$
$$\le C_6 n^m \omega_m(f, 1/n)_{p,\lambda} \le C_7 n^m \omega(1/n)$$

and (15) is proved. □

The statement below sharpens the Proposition 2 in Morrey setting (the case $\alpha = m$ is included).

Corollary 5 *Let $m \in \mathbb{N}$, $1 < p < \infty$, $0 < \lambda \le 1$, $f \in L_{2\pi,0}^{p,\lambda}$ and $0 < \alpha \le m$. Then the conditions $\omega_m(f,\delta)_{p,\lambda} = O(\delta^\alpha)$, $\delta \in [0,2\pi]$, and $\|(Z_n^m(f))^{(m)}\|_{p,\lambda} = O(n^{m-\alpha})$, $n \in \mathbb{N}$, are equivalent.*

In particular, the conditions $\omega_1(f,\delta)_{p,\lambda} = O(\delta^\alpha)$, $\delta \in [0,2\pi]$, and $\|\sigma_n'(f)\|_{p,\lambda} = O(n^{1-\alpha})$, $n \in \mathbb{N}$, are equivalent for $0 < \alpha \le 1$.

5 Approximation in Hölder type spaces

We give an application of Lemma 7 to problems of approximation in Hölder type spaces. The following Theorem 8 is an analogue of the result by Telyakovskii [22] concerning uniform Hölder spaces. Let us remind that Hölder type spaces $H_{p,\lambda}^{m,\omega}$ and its norms $\|\cdot\|_{p,\lambda,m,\omega}$ are defined in (3) and (4).

Theorem 8 *Let $1 \le p < \infty$, $0 < \lambda \le 1$, $m \in \mathbb{N}$, $\omega, \varphi \in \Phi$ and $\eta(t) = \omega(t)/\varphi(t)$ be increasing on $(0, 2\pi]$. If $f \in H_{p,\lambda}^{m,\omega}$ and $t_n \in T_n$, $n \in \mathbb{N}$, are such that the inequality*

$$\|f - t_n\|_{p,\lambda} \le C\omega_m(f, 1/n)_{p,\lambda}, \quad n \in \mathbb{N}. \tag{16}$$

holds, then

$$\|f - t_n\|_{p,\lambda,m,\varphi} \le C\eta(1/n), \quad n \in \mathbb{N}.$$

Proof By the condition of Theorem 8 we have

$$\|f - t_n\|_{p,\lambda} \le C_1\omega(n^{-1}) = C_1\eta(n^{-1})\varphi(n^{-1}) \le C_1\varphi(2\pi)\eta(n^{-1}). \tag{17}$$

Let us estimate $\omega_m(f - t_n, \delta)_{p,\lambda}/\varphi(\delta)$. For $\delta \ge 1/n$ we have

$$\frac{\omega_m(f - t_n, \delta)_{p,\lambda}}{\varphi(\delta)} \le \frac{C_2\|f - t_n\|_{p,\lambda}}{\varphi(n^{-1})} \le C_3\frac{\omega(n^{-1})}{\varphi(n^{-1})} = C_3\eta(n^{-1}). \tag{18}$$

For $0 < \delta < 1/n$ by virtue of Lemma 7 (ii) we find that

$$\frac{\omega_m(f - t_n, \delta)_{p,\lambda}}{\varphi(\delta)} \le \frac{\omega_m(f, \delta)_{p,\lambda} + \omega_m(t_n, \delta)_{p,\lambda}}{\varphi(\delta)} \le C_4\frac{\omega(\delta)}{\varphi(\delta)} \le C_4\eta(n^{-1}). \tag{19}$$

From (17), (18) and (19) we deduce the statement of Theorem 8. □

Now we give some applications of Theorem 8 to approximation by famous means of Fourier series.

Corollary 6 *Let* $1 < p < \infty$, $0 < \lambda \le 1$, $m \in \mathbb{N}$, $\omega, \varphi \in \Phi$ *and* $\eta(t) = \omega(t)/\varphi(t)$ *be increasing on* $(0, 2\pi]$. *If* $f \in H^{m,\omega}_{p,\lambda}$, *then*

$$\|f - S_n(f)\|_{p,\lambda,m,\varphi} \le C\eta(1/n), \quad n \in \mathbb{N}. \tag{20}$$

Proof By Proposition 1 (iii) and Lemma 6 (i) we have

$$\|f - S_n(f)\|_{p,\lambda} \le C_1\omega_m(f, 1/n)_{p,\lambda}, \quad n \in \mathbb{N}.$$

Using Theorem 8 we obtain (20). □

Now we consider the Vallée-Poussin means

$$v_n(f)(x) = 2Z^1_{2n-1}(f)(x) - Z^1_{n-1}(f)(x) = 2\sigma_{2n-1}(f) - \sigma_{n-1}(f).$$

Corollary 7 *Let* $1 \le p < \infty$, $0 < \lambda \le 1$, $m \in \mathbb{N}$, $\omega, \varphi \in \Phi$ *and* $\eta(t) = \omega(t)/\varphi(t)$ *be increasing on* $(0, 2\pi]$. *If* $f \in H^{m,\omega}_{p,\lambda}$, *then*

$$\|f - v_n(f)\|_{p,\lambda,m,\varphi} \le C\eta(1/n), \quad n \in \mathbb{N}. \tag{21}$$

Proof Let$\tau_n(f) \in T_n$ be such that $\|f - \tau_n(f)\|_{p,\lambda} = E_n(f)_{p,\lambda}$. It is known that $v_n(t_n) = t_n$ for $t_n \in T_n$, $n \in \mathbb{N}$, and that $Z^1_n(f) = F^1_n * f$, where $\|F^1_n\|_1 = 1$ (see [4, Ch. 1, § 47,(47.10)]). By Corollary 1 we have $\|v_n(f)\|_{p,\lambda} \le 3\|f\|_{p,\lambda}$, while by the previous inequality and Lemma 6 (i) we find that

$$\|f - v_n(f)\|_{p,\lambda} \le \|f - \tau_n(f)\|_{p,\lambda} + \|\tau_n(f) - v_n(\tau_n(f))\|_{p,\lambda} + \\ + \|v_n(f - \tau_n(f))\|_{p,\lambda} \le 4\|f - \tau_n(f)\|_{p,\lambda} \\ \le C_1\omega_m(f, 1/n)_{p,\lambda}, \quad n \in \mathbb{N}.$$

Using Theorem 8 we obtain (21). □

Corollary 8 *Let* $1 < p < \infty$, $0 < \lambda \le 1$, $m \in \mathbb{N}$, $\omega, \varphi \in \Phi$ *and* $\eta(t) = \omega(t)/\varphi(t)$ *be increasing on* $(0, 2\pi]$. *If* $f \in H^{m,\omega}_{p,\lambda}$, *then*

$$\|f - Z^m_n(f)\|_{p,\lambda,m,\varphi} \le C\lambda(1/n), \quad n \in \mathbb{N}. \tag{22}$$

If $p = 1$, m is even and other conditions above are valid, then (22) also holds. Finally, if $p = 1$, m is odd and $\omega \in B_m$, then (22) is valid.

Proof By Theorem 1 we have for $1 < p < \infty$ and $m \in \mathbb{N}$

$$\|f - Z^m_n(f)\|_{p,\lambda} \le C_1\omega_m(f, n^{-1})_{p,\lambda}, \quad n \in \mathbb{N}. \tag{23}$$

In turn, (26) is valid for $p = 1$ and even m by Theorem 3. Applying Theorem 8 we prove (22) in these cases.

If $p = 1$, $\omega \in B_m$ and $f \in H^{m,\omega}_{p,\lambda}$, the inequality of Lemma 5 together with Lemma 6 (i) gives us

$$\|f - Z^r_n(f)\|_{p,\lambda} \le \frac{C_2}{(n+1)^m}\sum_{k=0}^{n}(k+1)^{r-1}\omega(k^{-1}) \le C_3\omega(n^{-1}).$$

Repeating the proof of Theorem 8 we obtain

$$\frac{\omega_m(f - Z^m_n(f), \delta)_{p,\lambda}}{\varphi(\delta)} \le \frac{C_3\|f - Z^m_n(f)\|_{p,\lambda}}{\varphi(n^{-1})} \le C_3\frac{\omega(n^{-1})}{\varphi(n^{-1})} = C_3\eta(n^{-1}) \tag{24}$$

for $\delta \ge n^{-1}$. Since the translation and the convolution commute, we obtain $\Delta^m_h(f * F^m_n) = \Delta^m_h f * F^m_n$ (the definition of F^m_n see in the proof of Lemma 5) and $\omega_m(Z^m_n(f), \delta)_{p,\lambda} \le C_4\omega_m(f, \delta)_{p,\lambda}$. Now we have

$$\frac{\omega_m(f - Z^m_n(f), \delta)_{p,\lambda}}{\varphi(\delta)} \le \frac{\omega_m(f, \delta)_{p,\lambda} + \omega_m(Z^m_n(f), \delta)_{p,\lambda}}{\varphi(\delta)} \le \\ C_5\frac{\omega(\delta)}{\varphi(\delta)} \le C_5\eta(n^{-1}) \tag{25}$$

for $0 < \delta < n^{-1}$. From (24), (25) and obvious inequality $\|f - Z^r_n(f)\|_{p,\lambda} \le C_3\varphi(1)\eta(n^{-1})$ we deduce (22). □

Corollary 9 *Let* $1 \le p < \infty$, $0 < \lambda \le 1$, $\omega, \varphi \in \Phi$ *and* $\eta(t) = \omega(t)/\varphi(t)$ *be increasing on* $(0, 2\pi]$. *If* $\{\alpha_n\}_{n=1}^{\infty}$ *satisfies the conditions of Theorem* 4, $f \in H^{2,\omega}_{p,\lambda}$, *then*

$$\|f - B_{n,\alpha}(f)\|_{p,\lambda,2,\varphi} \leq C\lambda(n^{-1}), \quad n \in \mathbb{N}. \tag{26}$$

Proof By Theorem 4 we have the inequality $\|f - B_{n,\alpha}(f)\|_{p,\lambda} \leq C_1\omega_2(f, n^{-1})_{p,\lambda}$. Applying Theorem 8, we obtain (26). □

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Declarations

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