



(p, q) -fuzzy aggregation operators and their applications to decision-making

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Abstract

A (p, q) -fuzzy set is the resulting structure when the fuzzy membership and non-membership values are bounded by a general nonlinear relation. This set enhances the range of depicting uncertain information by making the feasible region larger, and thereby widening the purview of decision-making. Data aggregation is crucial in optimal decision-making. This article attempts to formulate aggregation operators for (p, q) -fuzzy sets using additive generators of strict t-norms and strict t-conorms. The utility of these operators is showcased by examining a decision-making problem, where the best decision is obtained by ranking the alternatives based on their score values. A comparative study is also carried out using some existing aggregation operators to test the validity and effectiveness of the introduced operators.

Keywords (p, q) -fuzzy set · Aggregation operators · Strict t-norm · Strict t-conorm · Additive generator

Mathematics Subject Classification 03E72 · 03E75 · 03E99

Abbreviations

AO	Aggregation operator
FFS	Fermatean fuzzy set
IFS	Intuitionistic fuzzy set
MADM	Multi-attribute decision-making
n, m -ROFS	n, m -Rung orthopair fuzzy set
n, m -ROFWPA	n, m -Rung orthopair fuzzy weighted power average
PFS	Pythagorean fuzzy set
(p, q) -FS	(p, q) -Fuzzy set

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(p, q) -FWA	(p, q) -Fuzzy weighted averaging
(p, q) -FWG	(p, q) -Fuzzy weighted geometric
(p, q) -FEWA	(p, q) -Fuzzy Einstein weighted averaging
(p, q) -FEWG	(p, q) -Fuzzy Einstein weighted geometric
(p, q) -FHWA	(p, q) -Fuzzy Hamacher weighted averaging
(p, q) -FHWG	(p, q) -Fuzzy Hamacher weighted geometric
q -ROFS	q -Rung orthopair fuzzy set

1 Introduction

Decision-making in real-world situations faces the challenge of addressing uncertainty. Traditional decision-making approaches predominantly rely on crisp sets and models and tend to overlook factors like vagueness, hesitancy, and uncertainty. To address uncertainty in real-world problems, researchers have introduced several set-theoretic models. In 1965, Lotfi A. Zadeh [26] introduced fuzzy set theory as an extension of crisp sets, offering a means to address uncertainty in decision-making. However, fuzzy set theory faced limitations in its ability to accurately capture human judgments regarding dissatisfaction. Hence, Intuitionistic fuzzy set (IFS) [2] was devised, which depicted both membership and non-membership values, with their sum in the unit interval. Yager introduced a more generalized model called Pythagorean fuzzy set (PFS) [22] by relaxing the condition of IFS that only the square sum of membership and non-membership values is in the unit interval. Subsequently, Senapati and Yager [17] introduced a broader model compared to PFS, known as Fermatean fuzzy set (FFS). The key condition for FFS is that the sum of cubes of membership and non-membership values is in the unit interval. As all these models involve a pair of values depicting competing concepts, they are termed orthopair fuzzy sets. To broaden the range of permissible pairs of membership grades, Yager [24] introduced the concept of q -rung orthopair fuzzy set (q -ROFS), with $q \geq 1$. Lately, Ibrahim *et al.* [12] familiarized the notion of (3, 2)-fuzzy set. Recently, Ibrahim and Alshammari [11] presented a novel notion of n, m -rung orthopair fuzzy set (n, m -ROFS). Al-shami and Mhemdi [1] presented a comprehensive framework for orthopair fuzzy sets known as (m, n) -fuzzy set $((m, n)$ -FS), which serves as an effective solution for addressing problems where the relative importance of membership and non-membership degrees varies and cannot be adequately handled using existing orthopair fuzzy sets. This set is termed as (p, q) -fuzzy set $((p, q)$ -FS) throughout this article. The notations p and q in (p, q) -fuzzy set stand for m and n of (m, n) -FS. The fundamental benefit of using (p, q) -fuzzy sets in various decision-making situations is their ability to describe uncertainty more precisely when compared to other orthopair fuzzy sets.

Multi-attribute decision-making (MADM) involves a procedure through which finite alternatives are ranked based on their attribute values. The initial challenge in decision-making is to find an efficient and precise way to represent attribute values when dealing with uncertain decision information. Given the intricacies of human cognition and the decision-making context, the orthopair fuzzy sets like IFS, PFS, etc., act as tools for expressing fuzzy information. Numerous approaches have been

put forth up to this point to deal with MADM in a fuzzy environment. One such approach involves estimating definite aggregation values for the alternatives. This involves performing an aggregation of the inputs provided by experts. Typically, this process is carried out using an aggregation operator (AO), which takes numerous input values and maps them to a single output value. Both the inputs and outputs belong to the same domain, with the output representing the input data or, at the very least, some of its key aspects. The theory of aggregation operations is extensively studied in [4, 8, 15, 21].

For any type of fuzzy set, fundamental operations play a pivotal role in AOs. To define operational laws in a generalized way for different orthopair fuzzy sets, the primary tool employed is a triple (T, S, N) , where T represents a t -norm, S signifies a t -conorm, and N denotes a fuzzy complement. In the context of this triple (T, S, N) , when T and S are dual with respect to N , this triple is referred to as a dual triple. Based on dual triple, operational rules are formulated and subsequently used to define AOs. When this t -norm T and t -conorm S are continuous and Archimedean, they can be expressed using single-variable functions known as additive generators. The choice of additive generators for both t -norms and t -conorms allows us to formulate diverse sets of operational rules, generating distinct AOs. Researchers have applied this approach to formulate generalized operational rules for developing AOs for various orthopair fuzzy sets. Beliakov *et al.* [3] introduced this approach for constructing AOs for IFSSs. Xia *et al.* [20] reviewed the operators introduced in [3] to prove some of their properties. They illustrated these operators by choosing a few additive generators and demonstrated that many existing operators can be deduced using this approach. Using different families of t -norms and t -conorms, along with the standard complement function specific to a particular fuzzy set, specific AOs are formulated for that fuzzy set. Huang [10] utilized Hamacher product and Hamacher sum, Wang *et al.* [18] used Einstein product and Einstein sum, to develop AOs for IFSSs. Yang *et al.* [25] proposed operations for Pythagorean fuzzy values using dual triples consisting of the standard negation function for PFS, continuous Archimedean t -norms and continuous Archimedean t -conorms. They displayed many existing Pythagorean fuzzy AOs as specific cases of this new approach. Garg [7] employed the Einstein product and Einstein sum, while Wu and Wie [19] utilized the Hamacher product and sum to model operations for developing Pythagorean fuzzy AOs. To create AOs for FFSs, Hadi *et al.* [9] formulated Hamacher operational rules, while Rani and Mishra [16] established Einstein operational laws by utilizing the standard complement function for FFSs. In the context of q -ROFSSs, Darko and Liang [6] proposed AOs using the Hamacher operations.

This article addresses the need for aggregating information represented as (p, q) -fuzzy numbers in decision-making. The challenge lies in developing AOs that yield consistent results on aggregation. The goal is to ensure that aggregation operations for (p, q) -fuzzy sets align with those of IFSSs, PFSs, and FFSs when p and q are equal. To achieve this, the article presents an approach for creating AOs for (p, q) -fuzzy sets, utilizing additive generators for strict t -norms and t -conorms.

Following are the sections of this article. It provides an overview of (p, q) -fuzzy sets in the Sect. 2. Section 3 provides the definitions fundamental to studying aggregation functions on (p, q) -fuzzy sets and develops a few (p, q) -fuzzy aggregation operators.

Additionally, it enlists some properties satisfied by these AOs. Section 4 analyses the suitability of applying the constructed AOs in MADM problems. The article's conclusion outlines the potential future applications of (p, q) -fuzzy sets.

2 Preliminaries

This section outlines the concept and results related to the newly defined (p, q) -fuzzy set, the most generalized orthopair fuzzy set.

Definition 1 [1] Let \mathfrak{X} be a universal set. For $p, q \geq 1$, a (p, q) -fuzzy set S is defined as :

$$S = \{(x, \alpha_S(x), \beta_S(x)) : x \in \mathfrak{X}\},$$

with the condition:

$$0 \leq (\alpha_S(x))^p + (\beta_S(x))^q \leq 1.$$

where $\alpha_S : \mathfrak{X} \rightarrow [0, 1]$ is the membership function, and $\beta_S : \mathfrak{X} \rightarrow [0, 1]$ is the non-membership function.

For each $x \in \mathfrak{X}$, $(x, \alpha_S(x), \beta_S(x))$ is called a (p, q) -fuzzy number. The uncertainty index of $x \in \mathfrak{X}$ with respect to S , defined as $\pi_S(x) = \frac{(p+q)}{2} \sqrt{1 - [(\alpha_S(x))^p + (\beta_S(x))^q]}$, lies in the interval $[0, 1]$. When $\pi_S(x)$ is high, less information is known about x , and vice versa.

Remark 1 IFSs, PFSs and FFSs are subtypes of (p, q) -fuzzy sets for $p = q = 1, 2$ and 3 , respectively. Additionally, when $p = 3$ and $q = 2$, the (p, q) -fuzzy set becomes a $(3, 2)$ -fuzzy set [12].

To establish an ordering for (p, q) -fuzzy numbers, the authors of [1] introduced the concepts of score value and accuracy value for a (p, q) -fuzzy number.

Definition 2 [1]

1. The score value of a (p, q) -fuzzy number, $F = (\alpha, \beta)$, is defined as $score(F) = \alpha^p - \beta^q$.
2. The accuracy value of a (p, q) -fuzzy number, $F = (\alpha, \beta)$, is defined as $accuracy(F) = \alpha^p + \beta^q$.

Definition 3 [1] For (p, q) -fuzzy numbers $F_k = (\alpha_k, \beta_k)$, where $k = 1, 2$, the ordering is as follows:

1. If $score(F_1) < score(F_2)$, then $F_1 < F_2$.
2. If $score(F_1) > score(F_2)$, then $F_1 > F_2$.
3. If $score(F_1) = score(F_2)$, then
 - (a) If $accuracy(F_1) < accuracy(F_2)$, then $F_1 < F_2$.
 - (b) If $accuracy(F_1) > accuracy(F_2)$, then $F_1 > F_2$.

(c) If $accuracy(F_1) = accuracy(F_2)$, then $F_1 = F_2$.

3 (p, q)-fuzzy aggregation operators

Aggregation constitutes a process in which arbitrarily but countably many inputs map to a single output value. It is natural to require all inputs and outputs from the same domain. Comparing aggregation operators is possible through the conventional comparison of functions with n variables. Aggregation operators are mathematical objects that aim to condense a set of numbers into a meaningful representative number.

Definition 4 [4] A function $f : [0, 1]^m \rightarrow [0, 1]$ is called an aggregation function if

1. $f(0, \dots, 0) = 0$ and $f(1, \dots, 1) = 1$.
2. $f(t_1, \dots, t_m) \geq f(t'_1, \dots, t'_m)$ if $t_k \geq t'_k$ for all $k = 1, 2, \dots, m$.

Proposition 1 [15] Let $\varphi : [0, 1] \rightarrow [0, 1]$ be a strictly monotone bijection. For an aggregation function f , the function

$$f_\varphi(t_1, t_2, \dots, t_m) = \varphi^{-1}(f(\varphi(t_1), \varphi(t_2), \dots, \varphi(t_m)))$$

is also an aggregation function on $[0, 1]$, referred to as the φ -transform of f .

Definition 5 [13]

1. A function $n : [0, 1] \rightarrow [0, 1]$ that is non-increasing is referred to as a negation function if $n(0) = 1$ and $n(1) = 0$.
2. A negation function n is called a strict negation function if it is continuous and strictly decreasing.
3. A strict negation function n is called a strong negation function if $n(n(t)) = t$ for all $t \in [0, 1]$.

The negation function considered in the context of (p, q) -fuzzy sets is a strict negation function, $n(t) = (1 - t^q)^{\frac{1}{p}}$. The n -transform of an aggregation function f , denoted as \hat{f} is obtained by using φ mentioned in Proposition 1 as $n(t)$, resulting in the expression $\hat{f}(t_1, \dots, t_m) = n^{-1}(f(n(t_1), \dots, n(t_m)))$.

Definition 6 Let $F_k = (\alpha_k, \beta_k), k = 1 \dots, m$ be a collection of (p, q) -fuzzy numbers, the aggregate of this collection, denoted as $F = \text{agg}(F_1, \dots, F_m)$, is defined as $F = (F_M, F_{NM})$ with $F_M = f(\alpha_1, \alpha_2, \dots, \alpha_m)$ and $F_{NM} = \hat{f}(\beta_1, \beta_2, \dots, \beta_m)$, where f is an aggregation function.

Theorem 1 Let $F_k = (\alpha_k, \beta_k), k = 1 \dots, m$ be a collection of (p, q) -fuzzy numbers and $F = \text{agg}(F_1, \dots, F_m) = (F_M, F_{NM})$ for an aggregation function f . Then F is a (p, q) -fuzzy number, that is F_M and F_{NM} satisfy $(F_M)^p + (F_{NM})^q \leq 1$.

Proof Each F_k is of the form (α_k, β_k) and satisfies $(\alpha_k)^p + (\beta_k)^q \leq 1$. Hence $\alpha_k \leq (1 - \beta_k^q)^{\frac{1}{p}}$ for each k . Now consider F_M :

$$F_M = f(\alpha_1, \alpha_2, \dots, \alpha_m) \leq f\left((1 - \beta_1^q)^{\frac{1}{p}}, (1 - \beta_2^q)^{\frac{1}{p}}, \dots, (1 - \beta_m^q)^{\frac{1}{p}}\right) \text{ (by monotonicity of } f\text{)}.$$

By Definition 6, F_{NM} can be expressed as:

$$F_{NM} = \widehat{f}(\beta_1, \beta_2, \dots, \beta_m) = \left(1 - [f((1 - \beta_1^q)^{\frac{1}{p}}, (1 - \beta_2^q)^{\frac{1}{p}}, \dots, (1 - \beta_m^q)^{\frac{1}{p}})]^p\right)^{\frac{1}{q}}$$

$$F_M^p + F_{NM}^q \leq \left(f((1 - \beta_1^q)^{\frac{1}{p}}, (1 - \beta_2^q)^{\frac{1}{p}}, \dots, (1 - \beta_m^q)^{\frac{1}{p}})\right)^p + \left(1 - [f((1 - \beta_1^q)^{\frac{1}{p}}, (1 - \beta_2^q)^{\frac{1}{p}}, \dots, (1 - \beta_m^q)^{\frac{1}{p}})]^p\right) = 1.$$

□

Hence, aggregating the membership and non-membership values of (p, q) -fuzzy numbers using an aggregation function and its n -transform ensures that the output is a (p, q) -fuzzy number.

Definition 7 [1] Let $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$ be two (p, q) -fuzzy numbers. Then

1. $F_1 \cup F_2 = (\max\{\alpha_1, \alpha_2\}, \min\{\beta_1, \beta_2\})$.
2. $F_1 \cap F_2 = (\min\{\alpha_1, \alpha_2\}, \max\{\beta_1, \beta_2\})$.
3. $F_1^c = (\beta_1^{\frac{q}{p}}, \alpha_1^{\frac{p}{q}})$.

Remark 2 If $f(t_1, t_2) = \max(t_1, t_2)$, then $\widehat{f}(t_1, t_2) = \min(t_1, t_2)$, and if $f(t_1, t_2) = \min(t_1, t_2)$, then $\widehat{f}(t_1, t_2) = \max(t_1, t_2)$. Thus, union and intersection are aggregation operations for (p, q) -fuzzy numbers.

Definition 8 [13] A t-norm is a bivariate aggregation function $\mathfrak{T} : [0, 1]^2 \rightarrow [0, 1]$ satisfying:

1. $\mathfrak{T}(1, t) = t$, for all t .
2. $\mathfrak{T}(t_1, t_2) = \mathfrak{T}(t_2, t_1)$, for all t_1 and t_2 .
3. $\mathfrak{T}(t_1, \mathfrak{T}(t_2, t_3)) = \mathfrak{T}(\mathfrak{T}(t_1, t_2), t_3)$, for all t_1, t_2 and t_3 .

Definition 9 [13] A t-conorm is a bivariate aggregation function $\mathcal{S} : [0, 1]^2 \rightarrow [0, 1]$ satisfying:

1. $\mathcal{S}(0, t) = t$, for all t .
2. $\mathcal{S}(t_1, t_2) = \mathcal{S}(t_2, t_1)$, for all t_1 and t_2 .
3. $\mathcal{S}(t_1, \mathcal{S}(t_2, t_3)) = \mathcal{S}(\mathcal{S}(t_1, t_2), t_3)$, for all t_1, t_2 and t_3 .

Remark 3 Let $\varphi : [0, 1] \rightarrow [0, 1]$ be a strictly decreasing bijection. For any t-norm \mathfrak{T} on $[0, 1]$, $\mathfrak{T}_\varphi(t_1, t_2) = \varphi^{-1}(\mathfrak{T}(\varphi(t_1), \varphi(t_2)))$ is t-conorm on $[0, 1]$, and vice versa. Hence the n -transform of a t-norm for a strict negation function n is a t-conorm, and vice versa.

Definition 10 [4] A t-norm \mathfrak{T} is referred to as a strict t-norm if it is continuous and strictly increasing on $(0, 1)^2$.

Definition 11 [4] A t-conorm \mathcal{S} is referred to as a strict t-conorm if it is continuous and strictly increasing on $[0, 1)^2$.

Proposition 2 [4] Given any continuous strictly decreasing function $g : [0, 1] \rightarrow [0, \infty]$, with $g(1) = 0$ and $g(0) = \infty$, the function $\mathfrak{T}_g(t_1, t_2) = g^{-1}(g(t_1) + g(t_2))$ is a strict t-norm. Conversely, any strict t-norm \mathfrak{T} can be expressed as $\mathfrak{T}(t_1, t_2) = g^{-1}(g(t_1) + g(t_2))$, where g is a continuous strictly decreasing function $g : [0, 1] \rightarrow [0, \infty]$ with $g(1) = 0$ and $g(0) = \infty$.

Proposition 3 [4] Given any continuous strictly increasing function $h : [0, 1] \rightarrow [0, \infty]$, with $h(0) = 0$ and $h(1) = \infty$, the function $\mathcal{S}_h(t_1, t_2) = h^{-1}(h(t_1) + h(t_2))$ is a strict t-conorm. Conversely, any strict t-conorm \mathcal{S} can be expressed as $\mathcal{S}(t_1, t_2) = h^{-1}(h(t_1) + h(t_2))$, where h is a continuous strictly increasing function $h : [0, 1] \rightarrow [0, \infty]$ with $h(0) = 0$ and $h(1) = \infty$.

This article defines aggregation operations for (p, q) -fuzzy sets using the additive generators of strict t-norms and t-conorms, as well as the negation function n employed in the context of (p, q) -fuzzy sets. The n -transform of any strict t-norm is a strict t-conorm, and vice versa. If we choose a continuous additive generator $g(t)$ that generates a t-norm, then $g(n(t))$ serves as an additive generator for the n -transform of the generated t-norm. Similarly, if we choose a continuous additive generator $h(t)$ that generates a t-conorm, then $h(n(t))$ generates the n -transform of the generated t-conorm.

Definition 12 For (p, q) -fuzzy numbers $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$, along with a continuous additive generator $h(t)$ that generates a strict t-conorm \mathcal{S} , a continuous additive generator $g(t)$ that generates a strict t-norm \mathfrak{T} , and a positive real number $a > 0$, the following operations are defined:

1. $F_1 \boxplus F_2 = (\mathcal{S}(\alpha_1, \alpha_2), \mathcal{S}_n(\beta_1, \beta_2))$
 $= (h^{-1}(h(\alpha_1) + h(\alpha_2)), h_n^{-1}(h_n(\beta_1) + h_n(\beta_2)))$
2. $F_1 \boxtimes F_2 = (\mathfrak{T}(\alpha_1, \alpha_2), \mathfrak{T}_n(\beta_1, \beta_2))$
 $= (g^{-1}(g(\alpha_1) + g(\alpha_2)), g_n^{-1}(g_n(\beta_1) + g_n(\beta_2)))$
3. $a.F_1 = (h^{-1}(ah(\alpha)), h_n^{-1}(ah_n(\beta)))$
4. $F_1^a = (g^{-1}(ag(\alpha)), g_n^{-1}(ag_n(\beta)))$

where $h_n(t) = h(n(t))$ and $g_n(t) = g(n(t))$, which turn out to be the additive generators of \mathcal{S}_n and \mathfrak{T}_n respectively.

Remark 4 When $h(t) = -\log(1 - t^p)$, $g(t) = -\log t^p$, and a is a positive real number, we obtain the following operations on (p, q) -fuzzy numbers as defined in [1].

1. $F_1 \boxplus F_2 = (\sqrt[p]{\alpha_1^p + \alpha_2^p - \alpha_1^p \alpha_2^p}, \beta_1 \beta_2)$.
2. $F_1 \boxtimes F_2 = (\alpha_1 \alpha_2, \sqrt[q]{\beta_1^q + \beta_2^q - \beta_1^q \beta_2^q})$.
3. $a.F_1 = (\sqrt[p]{(1 - (1 - \alpha_1^p)^a)}, \beta_1^a)$.
4. $F_1^a = (\alpha_1^a, \sqrt[q]{(1 - (1 - \beta_1^q)^a)})$.

Definition 13 Consider two (p, q) -fuzzy numbers $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$. If $h(t) = \log(\frac{1+t^p}{1-t^p})$, $g(t) = \log(\frac{2-t^q}{t^q})$, and a is a positive real number then

1. $F_1 \boxplus F_2 = \left(\left[\frac{\alpha_1^p + \alpha_2^p}{1 + \alpha_1^p \alpha_2^p} \right]^{\frac{1}{p}}, \left[\frac{\beta_1^q \beta_2^q}{1 + (1 - \beta_1^q)(1 - \beta_2^q)} \right]^{\frac{1}{q}} \right)$
2. $F_1 \boxtimes F_2 = \left(\left[\frac{\alpha_1^p \alpha_2^p}{1 + (1 - \alpha_1^p)(1 - \alpha_2^p)} \right]^{\frac{1}{p}}, \left[\frac{\beta_1^q + \beta_2^q}{1 + \beta_1^q \beta_2^q} \right]^{\frac{1}{q}} \right)$
3. $a.F_1 = \left(\left[\frac{(1 + \alpha_1^p)^a - (1 - \alpha_1^p)^a}{(1 + \alpha_1^p)^a + (1 - \alpha_1^p)^a} \right]^{\frac{1}{p}}, \left[\frac{2^{\frac{1}{q}} \beta_1^q}{((2 - \beta_1^q)^a + (\beta_1^q)^a)^{\frac{1}{q}}} \right] \right)$
4. $F_1^a = \left(\left[\frac{2^{\frac{1}{p}} \alpha_1^p}{((2 - \alpha_1^p)^a + (\alpha_1^p)^a)^{\frac{1}{p}}} \right], \left[\frac{(1 + \beta_1^q)^a - (1 - \beta_1^q)^a}{(1 + \beta_1^q)^a + (1 - \beta_1^q)^a} \right]^{\frac{1}{q}} \right)$.

Definition 14 Consider two (p, q) -fuzzy numbers $F_1 = (\alpha_1, \beta_1)$ and $F_2 = (\alpha_2, \beta_2)$. If $h(t) = \log(\frac{\tau + (1-\tau)(1-t^p)}{1-t^p})$, $g(t) = \log(\frac{\tau + (1-\tau)t^q}{t^q})$, where $\tau > 0$, and let a be a positive real number then

1. $F_1 \boxplus F_2 = \left(\left[\frac{\prod_{k=1}^2 (1 + (\tau - 1)\alpha_k^p) - \prod_{k=1}^2 (1 - \alpha_k^p)}{\prod_{k=1}^2 (1 + (\tau - 1)\alpha_k^p) + (\tau - 1) \prod_{k=1}^2 (1 - \alpha_k^p)} \right]^{\frac{1}{p}}, \left[\frac{\tau \prod_{k=1}^2 \beta_k^q}{\prod_{k=1}^2 (1 + (\tau - 1)(1 - \beta_k^q)) + (\tau - 1) \prod_{k=1}^2 \beta_k^q} \right]^{\frac{1}{q}} \right)$
2. $F_1 \boxtimes F_2 = \left(\left[\frac{\tau \prod_{k=1}^2 \alpha_k^p}{\prod_{k=1}^2 (1 + (\tau - 1)(1 - \alpha_k^p)) + (\tau - 1) \prod_{k=1}^2 \alpha_k^p} \right]^{\frac{1}{p}}, \left[\frac{\prod_{k=1}^2 (1 + (\tau - 1)\beta_k^q) - \prod_{k=1}^2 (1 - \beta_k^q)}{\prod_{k=1}^2 (1 + (\tau - 1)\beta_k^q) + (\tau - 1) \prod_{k=1}^2 (1 - \beta_k^q)} \right]^{\frac{1}{q}} \right)$

$$\begin{aligned}
3. \quad a.F_1 &= \left(\left[\frac{(1+(\tau-1)\alpha_1^p)^a - (1-\alpha_1^p)^a}{(1+(\tau-1)\alpha_1^p)^a + (\tau-1)(1-\alpha_1^p)^a} \right]^{\frac{1}{p}}, \left[\frac{\tau(\beta_1^q)^a}{(1+(\tau-1)(1-\beta_1^q)^a) + (\tau-1)(\beta_1^q)^a} \right]^{\frac{1}{q}} \right) \\
4. \quad F_1^a &= \left(\left[\frac{\tau(\alpha_1^p)^a}{(1+(\tau-1)(1-\alpha_1^p)^a) + (\tau-1)(\alpha_1^p)^a} \right]^{\frac{1}{p}}, \left[\frac{(1+(\tau-1)\beta_1^q)^a - (1-\beta_1^q)^a}{(1+(\tau-1)\beta_1^q)^a + (\tau-1)(1-\beta_1^q)^a} \right]^{\frac{1}{q}} \right).
\end{aligned}$$

When $\tau = 1$, we obtain the operations mentioned in Remark 4 and for $\tau = 2$, the operations in Definition 13.

Definition 15 Suppose $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$, is a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. Then the (p, q) -fuzzy weighted averaging operator is defined as:

$$(p, q) - \text{FWA}(F_1, F_2, \dots, F_m) = \lambda_1 F_1 \boxplus \lambda_2 F_2 \boxplus \dots \boxplus \lambda_m F_m.$$

Theorem 2 Suppose $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$, is a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. Then the (p, q) -fuzzy weighted averaging operator yields a (p, q) -fuzzy number of the form:

$$(p, q) - \text{FWA}(F_1, F_2, \dots, F_m) = \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right).$$

Proof Proof by mathematical induction on k . □

Definition 16 Suppose $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$, is a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. Then the (p, q) -fuzzy weighted geometric operator is defined as:

$$(p, q) - \text{FWG}(F_1, F_2, \dots, F_m) = F_1^{\lambda_1} \boxtimes F_2^{\lambda_2} \boxtimes \dots \boxtimes F_m^{\lambda_m}.$$

Theorem 3 Suppose $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$, is a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. Then the (p, q) -fuzzy weighted geometric operator yields a (p, q) -fuzzy number of the form:

$$(p, q) - \text{FWG}(F_1, F_2, \dots, F_m) = \left(g^{-1} \left(\sum_{k=1}^m \lambda_k g(\alpha_k) \right), g_n^{-1} \left(\sum_{k=1}^m \lambda_k g_n(\beta_k) \right) \right).$$

Proof Proof by mathematical induction on k . \square

Property 1 If all $F_k, k = 1, 2, \dots, m$, are equal, i.e., $F_k = F = (\alpha, \beta)$, for all k , then $(p, q) - \text{FWA}(F_1, F_2, \dots, F_m) = F$.

Proof Let $F_k = F = (\alpha_k, \beta_k)$,

$$\begin{aligned} (p, q) - \text{FWA}(F_1, F_2, \dots, F_m) &= (p, q) - \text{FWA}(F, F, \dots, F) \\ &= \boxplus_{k=1}^m \lambda_k F \\ &= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta) \right) \right) \\ &= (h^{-1}(h(\alpha)), h_n^{-1}(h_n(\beta))) \\ &= (\alpha, \beta) = F \end{aligned}$$

\square

Property 2 Let $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$, be a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ be the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. If $G = (\alpha, \beta)$ is a (p, q) -fuzzy number, then

$$(p, q) - \text{FWA}(F_1 \boxplus G, F_2 \boxplus G, \dots, F_m \boxplus G) = (p, q) - \text{FWA}(F_1, F_2, \dots, F_m) \boxplus G.$$

Proof Since $F_k \boxplus G = (h^{-1}(h(\alpha_k) + h(\alpha)), h_n^{-1}(h_n(\beta_k) + h_n(\beta)))$

$$\begin{aligned} (p, q) - \text{FWA}(F_1 \boxplus G, F_2 \boxplus G, \dots, F_m \boxplus G) &= \left(\begin{array}{c} h^{-1} \left(\sum_{k=1}^m \lambda_k h(h^{-1}(h(\alpha_k) + h(\alpha))) \right), \\ h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(h_n^{-1}(h_n(\beta_k) + h_n(\beta))) \right) \end{array} \right) \\ &= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k (h(\alpha_k) + h(\alpha)) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k (h_n(\beta_k) + h_n(\beta)) \right) \right) \end{aligned}$$

$$\begin{aligned}
& (p, q)\text{-FWA}(F_1, F_2, \dots, F_m) \boxplus G \\
&= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) \boxplus (\alpha, \beta) \\
&= \left(\begin{aligned} & h^{-1} \left(h \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right) \right) + h(\alpha) \right), \\ & h_n^{-1} \left(h_n \left(h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) + h_n(\beta) \right) \end{aligned} \right) \\
&= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) + h(\alpha) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) + h_n(\beta) \right) \right) \\
&= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k (h(\alpha_k) + h(\alpha)) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k (h_n(\beta_k) + h_n(\beta)) \right) \right)
\end{aligned}$$

□

Property 3 Let $F_k = (\alpha_k, \beta_k), k = 1, 2, \dots, m$ be a collection of (p, q) -fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ be the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. If $c > 0$, then $(p, q) - \text{FWA}(cF_1, cF_2, \dots, cF_m) = c(p, q) - \text{FWA}(F_1, F_2, \dots, F_m)$.

Proof As stated in Definition 12, for a (p, q) -fuzzy set $F = (\alpha, \beta)$, and $c > 0$, we have $cF = (h^{-1}(ch(\alpha)), h_n^{-1}(ch_n(\beta)))$.

$$\begin{aligned}
& (p, q) - \text{FWA}(cF_1, cF_2, \dots, cF_m) \\
&= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(h^{-1}(ch(\alpha_k))) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(h_n^{-1}(ch_n(\beta_k))) \right) \right) \\
&= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k (ch(\alpha_k)) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k (ch_n(\beta_k)) \right) \right) \\
&= \left(h^{-1} \left(c \sum_{k=1}^m \lambda_k h(\alpha_k) \right), h_n^{-1} \left(c \sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) \\
& c.(p, q) - \text{FWA}(F_1, F_2, \dots, F_m) \\
&= \left(h^{-1} \left(ch \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right) \right) \right), h_n^{-1} \left(ch_n \left(h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) \right) \right) \\
&= \left(h^{-1} \left(c \sum_{k=1}^m \lambda_k h(\alpha_k) \right), h_n^{-1} \left(c \sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right)
\end{aligned}$$

□

Property 4 Let $F_k = (\alpha_k, \beta_k)$ and $G_k = (\alpha'_k, \beta'_k), k = 1, 2, \dots, m$, be two collections of (p, q) -fuzzy numbers. Let $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ be the weight vector specifying the weights of both F_k and G_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$. Then

$$\begin{aligned} & (p, q)\text{-FWA}(F_1 \boxplus G_1, F_2 \boxplus G_2, \dots, F_m \boxplus G_m) \\ &= (p, q) - \text{FWA}(F_1, F_2, \dots, F_m) \boxplus (p, q) - \text{FWA}(G_1, G_2, \dots, G_m). \end{aligned}$$

Proof $F_k \boxplus G_k = (h^{-1}(h(\alpha_k) + h(\alpha'_k)), h_n^{-1}(h_n(\beta_k) + h_n(\beta'_k)))$

$$\begin{aligned} & (p, q)\text{-FWA}(F_1 \boxplus G_1, F_2 \boxplus G_2, \dots, F_m \boxplus G_m) \\ &= \left(\begin{array}{l} h^{-1} \left(\sum_{k=1}^m \lambda_k h(h^{-1}(h(\alpha_k) + h(\alpha'_k))) \right), \\ h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(h_n^{-1}(h_n(\beta_k) + h_n(\beta'_k))) \right) \end{array} \right) \\ &= \left(\begin{array}{l} h^{-1} \left(\sum_{k=1}^m \lambda_k (h(\alpha_k) + h(\alpha'_k)) \right), \\ h_n^{-1} \left(\sum_{k=1}^m \lambda_k (h_n(\beta_k) + h_n(\beta'_k)) \right) \end{array} \right) \end{aligned}$$

$$\begin{aligned} & (p, q)\text{-FWA}(F_1, F_2, \dots, F_m) \boxplus (p, q) - \text{FWA}(G_1, G_2, \dots, G_m) \\ &= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) \boxplus \\ & \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha'_k) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta'_k) \right) \right) \\ &= \left(\begin{array}{l} h^{-1} \left(h \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right) \right) + h \left(h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha'_k) \right) \right) \right), \\ h_n^{-1} \left(h_n \left(h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \right) + h_n \left(h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta'_k) \right) \right) \right) \end{array} \right) \\ &= \left(h^{-1} \left(\sum_{k=1}^m \lambda_k (h(\alpha_k) + h(\alpha'_k)) \right), h_n^{-1} \left(\sum_{k=1}^m \lambda_k (h_n(\beta_k) + h_n(\beta'_k)) \right) \right) \end{aligned}$$

□

Property 5 Let $F_k = (\alpha_k, \beta_k)$ and $G_k = (\alpha'_k, \beta'_k), k = 1, 2, \dots, m$, be two collections of (p, q) -fuzzy numbers. If $\alpha_k \leq \alpha'_k$ and $\beta_k \geq \beta'_k$ for all k , then

$$\text{score}((p, q) - \text{FWA}(F_1, F_2, \dots, F_m)) \leq \text{score}((p, q) - \text{FWA}(G_1, G_2, \dots, G_m)).$$

Proof Since $h : [0, 1] \rightarrow [0, \infty]$ is a continuous strictly increasing function, its inverse is also strictly increasing. Thus

$$h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha_k) \right) \leq h^{-1} \left(\sum_{k=1}^m \lambda_k h(\alpha'_k) \right) \tag{1}$$

Since h_n is a continuous strictly decreasing function from $[0, 1] \rightarrow [0, \infty]$, its inverse is also strictly decreasing. Hence

$$h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta_k) \right) \geq h_n^{-1} \left(\sum_{k=1}^m \lambda_k h_n(\beta'_k) \right) \tag{2}$$

From (1) and (2)

$$score((p, q) - FWA(F_1, F_2, \dots, F_m)) \leq score((p, q) - FWA(G_1, G_2, \dots, G_m)). \quad \square$$

All the properties listed above also hold for (p, q)-fuzzy weighted geometric operators. This article suggests different aggregation operators by applying various additive generators. Note that the weighted average and weighted geometric operators for (p, q)-fuzzy set defined in [1] differ entirely from those defined here.

Remark 5 Suppose $F_k = (\alpha_k, \beta_k)$, $k = 1, 2, \dots, m$, is a collection of (p, q)-fuzzy numbers, and $\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)^T$ is the weight vector specifying the weights of F_k , where $\lambda_k > 0$ and $\sum_{k=1}^m \lambda_k = 1$.

1. For $h(t) = -\log(1 - t^p)$ and $g(t) = -\log t^p$, we obtain:

$$(p, q) - FWA(F_1, F_2, \dots, F_m) = \left(\left(1 - \prod_{k=1}^m (1 - \alpha_k^p)^{\lambda_k} \right)^{1/p}, \prod_{k=1}^m \beta_k^{\lambda_k} \right)$$

$$(p, q) - FWG(F_1, F_2, \dots, F_m) = \left(\prod_{k=1}^m \alpha_k^{\lambda_k}, \left(1 - \prod_{k=1}^m (1 - \beta_k^q)^{\lambda_k} \right)^{1/q} \right)$$

2. For $h(t) = \log\left(\frac{1+t^p}{1-t^p}\right)$, and $g(t) = \log\left(\frac{2-t^p}{t^p}\right)$, which generate a t-conorm and a t-norm that are generalizations of the Einstein sum and Einstein product, respectively, the corresponding weighted aggregation operators are termed as (p, q)-fuzzy Einstein weighted averaging and (p, q)-fuzzy Einstein weighted geometric operators.

$$\begin{aligned} &(p, q)\text{-FEWA}(F_1, F_2, \dots, F_m) \\ &= \left(\left[\frac{\prod_{k=1}^m (1 + \alpha_k^p)^{\lambda_k} - \prod_{k=1}^m (1 - \alpha_k^p)^{\lambda_k}}{\prod_{k=1}^m (1 + \alpha_k^p)^{\lambda_k} + \prod_{k=1}^m (1 - \alpha_k^p)^{\lambda_k}} \right]^{\frac{1}{p}}, \left[\frac{2 \prod_{k=1}^m \beta_k^{q\lambda_k}}{\prod_{k=1}^m (2 - \beta_k^q)^{\lambda_k} + \prod_{k=1}^m \beta_k^{q\lambda_k}} \right]^{\frac{1}{q}} \right) \end{aligned}$$

$$(p, q)\text{-FEWG}(F_1, F_2, \dots, F_m) = \left(\left[\frac{2 \prod_{k=1}^m \alpha_k^{p\lambda_k}}{\prod_{k=1}^m (2 - \alpha_k^p)^{\lambda_k} + \prod_{k=1}^m \alpha_k^{p\lambda_k}} \right]^{\frac{1}{p}}, \left[\frac{\prod_{k=1}^m (1 + \beta_k^q)^{\lambda_k} - \prod_{k=1}^m (1 - \beta_k^q)^{\lambda_k}}{\prod_{k=1}^m (1 + \beta_k^q)^{\lambda_k} + \prod_{k=1}^m (1 - \beta_k^q)^{\lambda_k}} \right]^{\frac{1}{q}} \right)$$

3. For $h(t) = \log\left(\frac{\tau+(1-\tau)(1-t^p)}{1-t^p}\right)$, $g(t) = \log\left(\frac{\tau+(1-\tau)t^q}{t^q}\right)$, $\tau > 0$, which generate a t-conorm and a t-norm that are generalizations of the Hamacher sum and Hamacher product, respectively, the corresponding weighted aggregation operators are termed as (p, q) -fuzzy Hamacher weighted averaging and (p, q) -fuzzy Hamacher weighted geometric operators.

$$(p, q)\text{-FHWA}(F_1, F_2, \dots, F_m) = \left(\left[\frac{\prod_{k=1}^m (1 + (\tau - 1)\alpha_k^p)^{\lambda_k} - \prod_{k=1}^m (1 - \alpha_k^p)^{\lambda_k}}{\prod_{k=1}^m (1 + (\tau - 1)\alpha_k^p)^{\lambda_k} + (\tau - 1) \prod_{k=1}^m (1 - \alpha_k^p)^{\lambda_k}} \right]^{\frac{1}{p}}, \left[\frac{\tau \prod_{k=1}^m \beta_k^{q\lambda_k}}{\prod_{k=1}^m (1 + (\tau - 1)(1 - \beta_k^q))^{\lambda_k} + (\tau - 1) \prod_{k=1}^m \beta_k^{q\lambda_k}} \right]^{\frac{1}{q}} \right)$$

$$(p, q)\text{-FHWG}(F_1, F_2, \dots, F_m) = \left(\left[\frac{\tau \prod_{k=1}^m \alpha_k^{p\lambda_k}}{\prod_{k=1}^m (1 + (\tau - 1)(1 - \alpha_k^p))^{\lambda_k} + (\tau - 1) \prod_{k=1}^m \alpha_k^{p\lambda_k}} \right]^{\frac{1}{p}}, \left[\frac{\prod_{k=1}^m (1 + (\tau - 1)\beta_k^q)^{\lambda_k} - \prod_{k=1}^m (1 - \beta_k^q)^{\lambda_k}}{\prod_{k=1}^m (1 + (\tau - 1)\beta_k^q)^{\lambda_k} + (\tau - 1) \prod_{k=1}^m (1 - \beta_k^q)^{\lambda_k}} \right]^{\frac{1}{q}} \right)$$

4 Decision-making under (p, q) -fuzzy environment

In this section, we utilize the AOs defined in this article to address a specific type of decision-making problem. We outline the algorithm used in our proposed approach and provide an illustrative example to demonstrate its implementation.

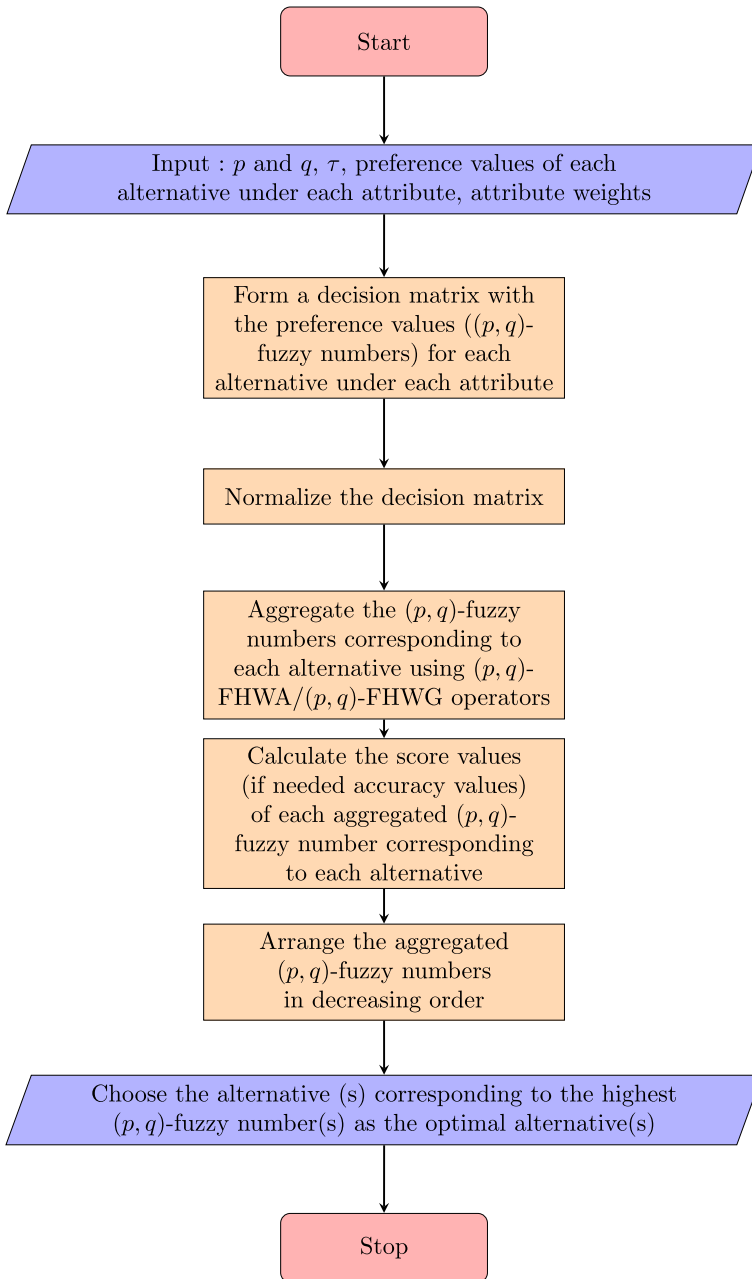


Fig. 1 Flowchart listing the steps in MADM approach based on (p, q) -fuzzy aggregation operators

Multi-attribute decision-making (MADM) offers a swift method for identifying the optimal choice(s) from a range of alternatives based on multiple criteria. To elucidate the approach in (p, q) -fuzzy framework, we begin by defining the alternatives under examination as the set $\mathfrak{X} = \{x_1, x_2, \dots, x_m\}$. The decision-maker assesses them with specific criteria listed as $A = \{a_1, a_2, \dots, a_n\}$. Also, each attribute is assigned with a weight indicating its significance or importance. Since (p, q) -fuzzy sets offer the advantage of encompassing a wide range of membership grades, the preferences of decision-maker are effectively represented using (p, q) -fuzzy numbers: $F_{ij} = (\alpha_{ij}, \beta_{ij})$ where $\alpha_{ij}, \beta_{ij} \in [0, 1]$ and $0 \leq (\alpha_{ij})^p + (\beta_{ij})^q \leq 1$ for $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$ with suitable values assigned to p and q . Here, α_{ij} indicates the degree to which the alternative x_i meets the attribute a_j , while β_{ij} indicates the degree to which x_i does not fulfil a_j . Hence, this MADM problem is briefly depicted as a (p, q) -fuzzy decision matrix with each entry as a (p, q) -fuzzy number. This article presents a method for addressing this MADM problem by adopting an information fusion approach. The AOs designed for (p, q) -fuzzy sets are employed to merge the fuzzy information with attribute weights. This approach constitutes the following sequential steps:

Step 1: Identify the suitable values for p and q to be used in the problem (choose the least p and q which satisfy the condition for membership and non-membership across all the assessments).

Step 2: Represent the decision-maker's assessment of each alternative under each attribute in the form of a (p, q) -fuzzy decision matrix, denoted as $M = (F_{ij})_{m \times n}$.

Step 3: Transform the decision matrix into a normalized decision matrix. There are two types of attributes: benefit type and cost type. Preference values are normalized by checking their attribute type using the formula

$$F_{ij} = \begin{cases} (\alpha_{ij}, \beta_{ij}) & \text{if } a_j \text{ is benefit type} \\ (\alpha_{ij}, \beta_{ij})^c & \text{if } a_j \text{ is cost type} \end{cases}$$

Step 4: Aggregate the (p, q) -fuzzy numbers corresponding to each alternative using the defined AOs.

Step 5: Compute the score value (if needed, accuracy value) to compare the aggregated (p, q) -fuzzy number for each alternative.

Step 6: Arrange the aggregated (p, q) -fuzzy numbers in descending order, following the criteria outlined in Definition 3.

Table 1 (p, q) -fuzzy decision matrix

Winter Coats	R_1	R_2	R_3	R_4	R_5
WC_1	(0.60,0.20)	(0.83,0.50)	(0.20,0.44)	(0.30,0.70)	(0.45,0.22)
WC_2	(0.71,0.40)	(0.60,0.60)	(0.54,0.63)	(0.40,0.76)	(0.15,0.70)
WC_3	(0.25,0.57)	(0.32,0.67)	(0.50,0.90)	(0.84,0.62)	(0.90,0.05)
WC_4	(0.47,0.83)	(0.80,0.20)	(0.85,0.12)	(0.04,0.53)	(0.20,0.78)
WC_5	(0.35,0.83)	(0.55,0.33)	(0.10,0.75)	(0.42,0.62)	(0.38,0.89)

Table 2 Aggregated (p, q)-fuzzy numbers using (p, q)-FHWA operator

Operators	\mathcal{WC}_1	\mathcal{WC}_2	\mathcal{WC}_3	\mathcal{WC}_4	\mathcal{WC}_5
(2,3)-FHWA	(0.5732,0.3815)	(0.4813,0.6442)	(0.7627,0.3084)	(0.5906,0.4141)	(0.4149,0.6348)
(3,2)-FHWA	(0.6030,0.3815)	(0.5048,0.6442)	(0.7797,0.3084)	(0.6354,0.4141)	(0.4264,0.6348)
(4,5)-FHWA	(0.6290,0.3815)	(0.5234,0.6422)	(0.7931,0.3084)	(0.6648,0.4141)	(0.4362,0.6348)
(5,4)-FHWA	(0.6511,0.3815)	(0.5387,0.6442)	(0.8036,0.3084)	(0.6869,0.4141)	(0.4480,0.6348)
(10,100)-FHWA	(0.7193,0.3815)	(0.5879,0.6442)	(0.8336,0.3084)	(0.7442,0.4141)	(0.4780,0.6348)
(100,10)-FHWA	(0.8175,0.3815)	(0.6926,0.6442)	(0.8892,0.3084)	(0.8329,0.4141)	(0.0000,0.6348)

Step 7: Select the alternative(s) corresponding to the highest aggregated (p, q)-fuzzy number(s) as the optimal choice(s).

The flowchart in Fig. 1 illustrates the steps involved in selecting the optimal alternative.

Illustration: *Alternative selection in Online Shopping using (p, q)-fuzzy aggregation operators*

A buyer needs to purchase a thick winter coat to withstand extremely low temperatures. He has the option to explore physical stores or use online retailers for the purchase. When evaluating his options, the buyer takes into consideration several factors, including his prior experiences with products or brands, both positive and negative. To gather information about the product, the buyer relies on sources such as Google and consumer-generated content, including customer reviews, video testimonials from review websites, authentic blog posts, and social media reviews. Based on his research, the buyer selects a specific online store to make the purchase. The customer has specific criteria in mind for the product, such as affordability, additional product benefits, availability, colour, and style. Consequently, he add five winter coats to his cart for final selection. Before making the ultimate purchase decision, the buyer compares these potential selections based on customer reviews and testimonials, focusing on five key factors: *aesthetic appeal*, *value for money*, *material softness*, *material quality*, and *fitting*. The buyer assigns varying degrees of importance to these factors. The goal is to choose the best winter coat from the set of five.

This situation represents a MADM problem, with the five factors serving as attributes and the five potential selections as alternatives. The buyer represents customer reviews for each item as a (p, q)-fuzzy set, with membership values indicating positive comments and non-membership values indicating negative reviews. Let $W = \{\mathcal{WC}_1, \mathcal{WC}_2, \mathcal{WC}_3, \mathcal{WC}_4, \mathcal{WC}_5\}$ represent the set of alternatives (winter coats), and $R = \{R_1, R_2, R_3, R_4, R_5\}$ denote the five attributes for selection: R_1 for *aesthetic appeal*, R_2 for *value for money*, R_3 for *material softness*, R_4 for *material quality*, and R_5 for *fitting*. The data is presented as a decision matrix in Table 1.

The customer gives varying importance to different attributes and provides weights for each attribute, denoted as a vector $\Lambda = [0.10, 0.22, 0.13, 0.25, 0.30]^T$.

Table 3 Ranking using (p, q) -FHHWA operator

(p, q)	$score(WC_1)$	$score(WC_2)$	$score(WC_3)$	$score(WC_4)$	$score(WC_5)$	Ranking
(2, 3)	0.2730	- 0.0357	0.5524	0.2778	- 0.0837	$WC_3 > WC_4 > WC_1 > WC_2 > WC_5$
(3, 2)	0.0737	- 0.2864	0.3789	0.0840	- 0.3255	$WC_3 > WC_4 > WC_1 > WC_2 > WC_5$
(4, 5)	0.1485	- 0.0359	0.3928	0.1832	- 0.0669	$WC_3 > WC_4 > WC_1 > WC_2 > WC_5$
(5, 4)	0.0959	- 0.1269	0.3261	0.1235	- 0.1450	$WC_3 > WC_4 > WC_1 > WC_2 > WC_5$
(10, 100)	0.0371	0.0049	0.1621	0.0521	0.0006	$WC_3 > WC_4 > WC_1 > WC_2 > WC_5$
(100, 10)	-6.5×10^{-5}	- 0.0123	1.83×10^{-7}	-1.48×10^{-4}	- 0.0106	$WC_3 > WC_1 > WC_4 > WC_2 > WC_5$

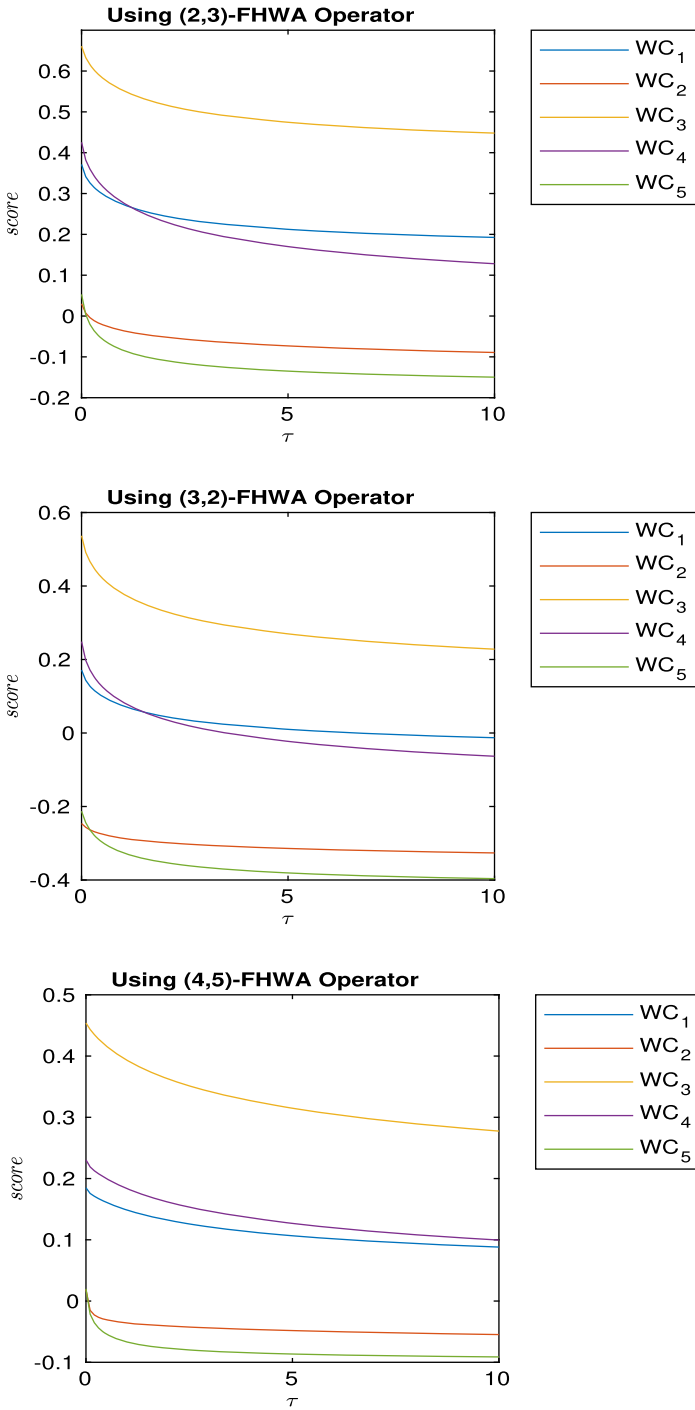


Fig. 2 Score values of alternatives using (p, q)-FHWA operator

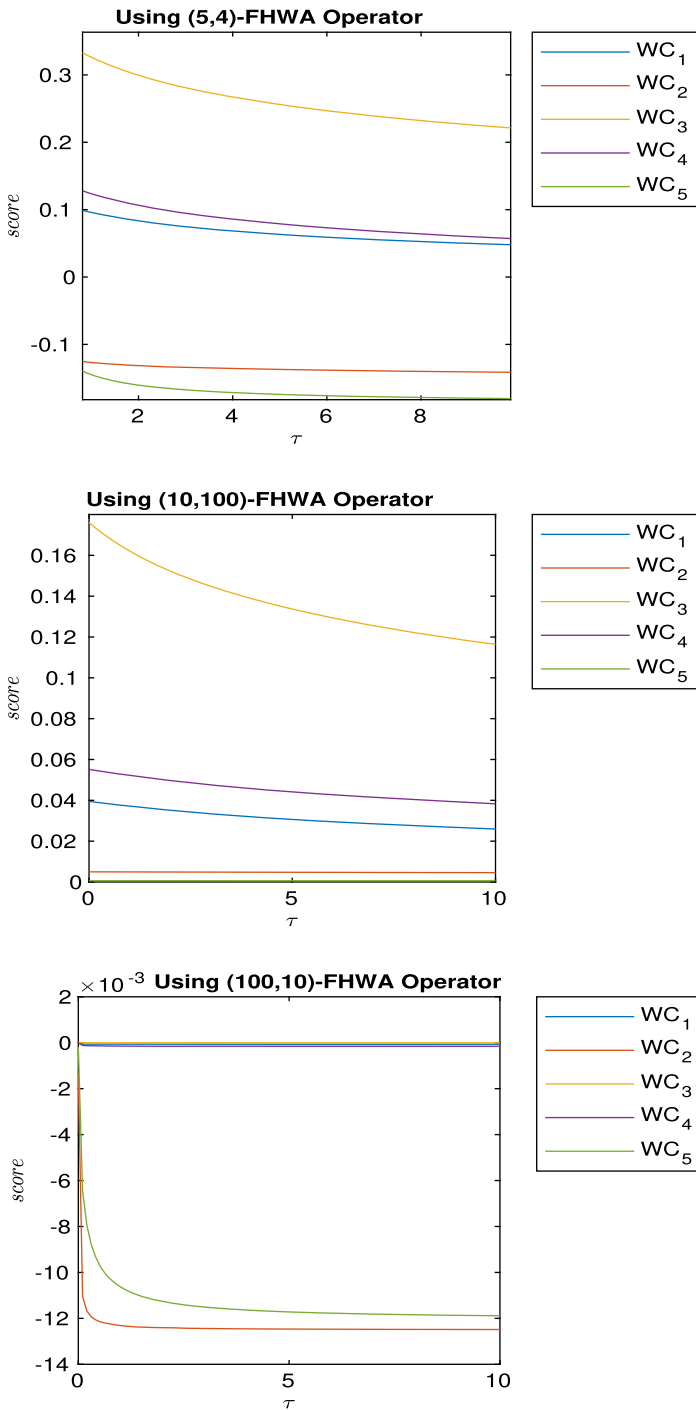


Fig. 3 Score values of alternatives using (p, q) -FHWA operator

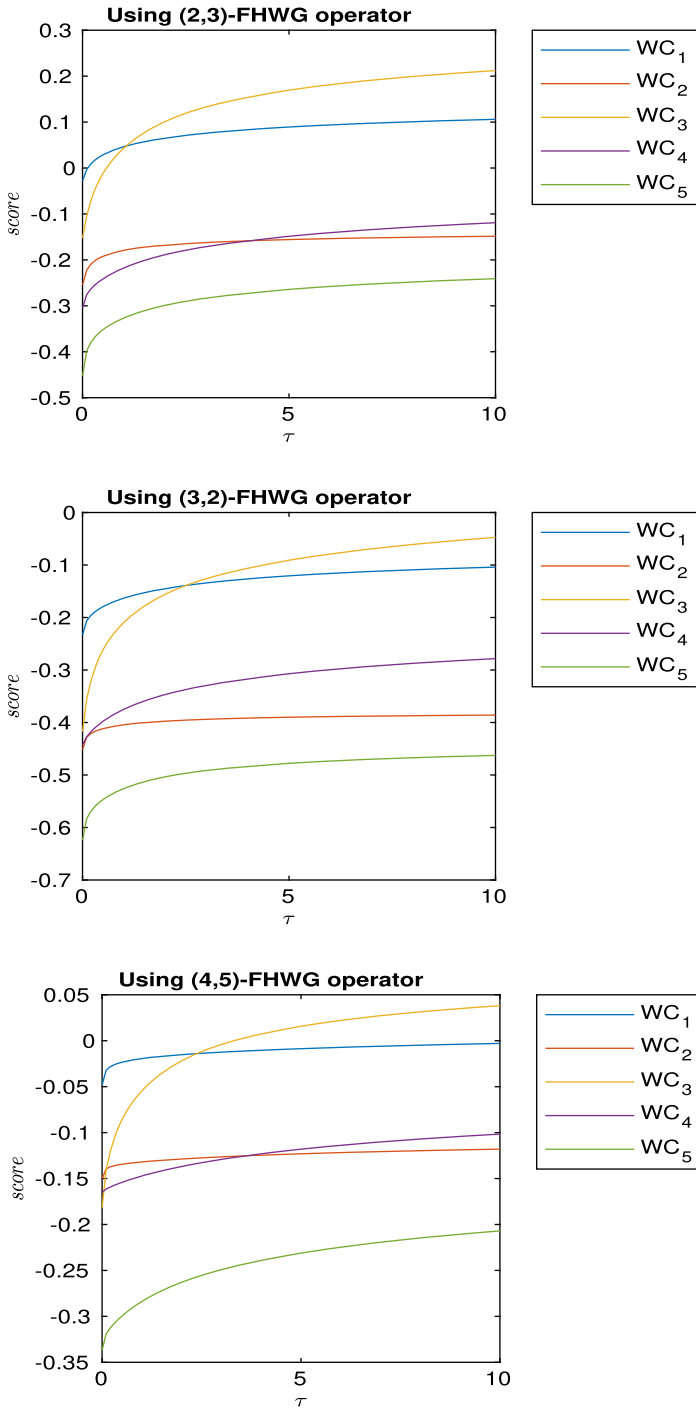


Fig. 4 Score values of alternatives using (p, q)-FHWG operator

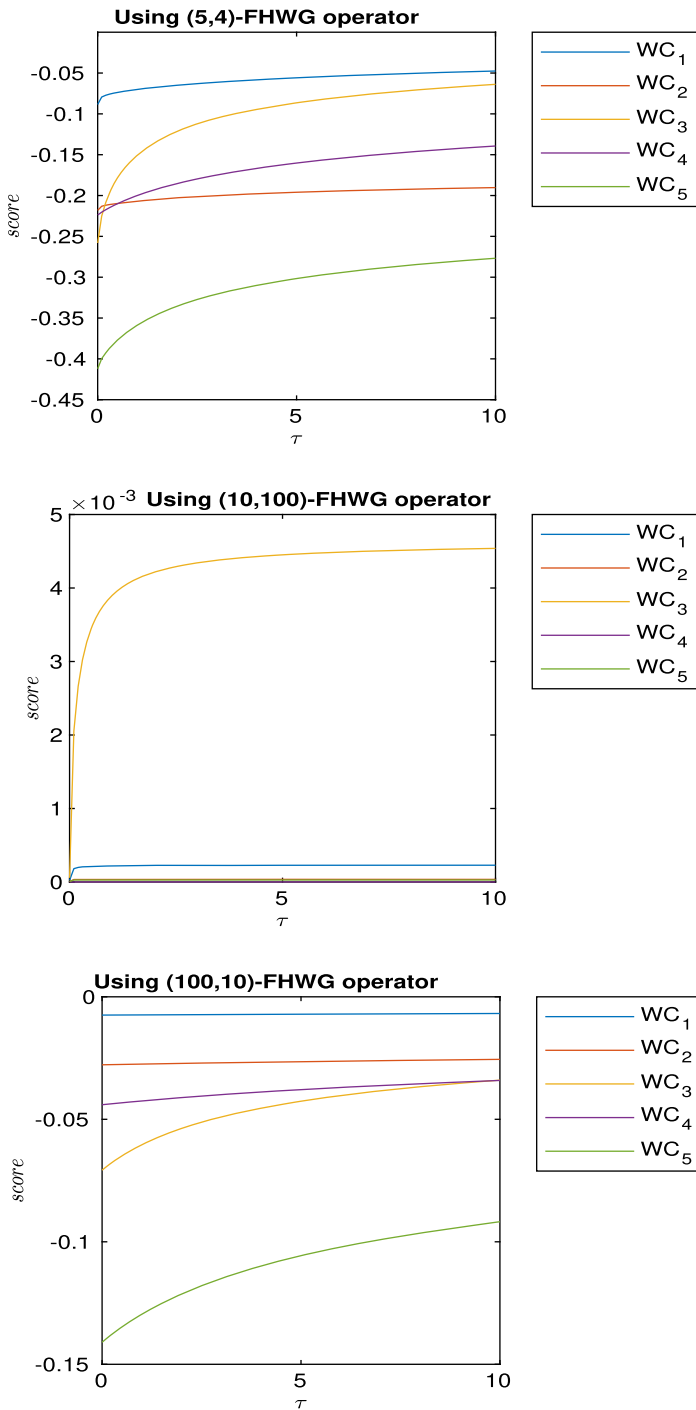


Fig. 5 Score values of alternatives using (p, q) -FHWG operator

Table 4 (p, q)-fuzzy decision matrix

Journals	\mathcal{D}_1	\mathcal{D}_2	\mathcal{D}_3	\mathcal{D}_4	\mathcal{D}_5
Jl_1	(0.61,0.80)	(0.69,0.73)	(0.48,0.79)	(0.59,0.58)	(0.81,0.68)
Jl_2	(0.62,0.81)	(0.81,0.63)	(0.66,0.77)	(0.65,0.76)	(0.75,0.73)
Jl_3	(0.65,0.78)	(0.71,0.79)	(0.75,0.75)	(0.64,0.77)	(0.49,0.91)
Jl_4	(0.81,0.61)	(0.83,0.59)	(0.75,0.74)	(0.85,0.57)	(0.83,0.59)
Jl_5	(0.59,0.86)	(0.81,0.68)	(0.61,0.80)	(0.48,0.79)	(0.47,0.89)

Table 5 Aggregated (p, q)-fuzzy sets using (p, q)-FHWA operator

Operator	Jl_1	Jl_2	Jl_3	Jl_4	Jl_5
(2,3)-FHWA	(0.6787,0.7199)	(0.7091,0.7423)	(0.6506,0.8088)	(0.8140,0.6217)	(0.6066,0.8162)
(3,2)-FHWA	(0.6858,0.7199)	(0.7113,0.7423)	(0.6552,0.8088)	(0.8144,0.6217)	(0.6156,0.8162)
(4,5)-FHWA	(0.6928,0.7199)	(0.7136,0.7423)	(0.6597,0.8088)	(0.8148,0.6217)	(0.6253,0.8162)
(5,4)-FHWA	(0.6998,0.7199)	(0.7160,0.7423)	(0.6642,0.8088)	(0.8152,0.6217)	(0.6355,0.8162)
(10,100)-FHWA	(0.7295,0.7199)	(0.7289,0.7423)	(0.6829,0.8088)	(0.8173,0.6217)	(0.6830,0.8162)
(100,10)-FHWA	(0.8000,0.7199)	(0.7948,0.7423)	(0.7388,0.8088)	(0.8352,0.6217)	(0.7948,0.8162)

Each attribute, R_k , is associated with a (p, q)-fuzzy set, and all of them are benefit-type attributes in this example. Now aggregate these five (p, q)-fuzzy sets using the proposed (p, q)-FHWA operator for $\tau = 1$ (see Table 2). Next, order the (p, q)-fuzzy numbers of the resulting (p, q)-fuzzy set based on score values within the range of $[-1, 1]$. Choose the alternative with the maximum score value as the best option (see Table 3). While this represents the conventional approach to using AOs in MADM problems within a fuzzy environment, presenting the problem’s data in the form of (p, q)-fuzzy sets offers the flexibility to accommodate more orthopairs. Figures 2 and 3 depict the score values obtained for the five different alternatives for various values of p, q, and τ using the (p, q)-FHWA operator.

It is evident from Figures 2, 3, 4 and 5 that the ordering of alternatives changes as the values of τ fluctuate. When different values of τ are assigned to a specific (p, q)-fuzzy aggregation operator, both the score values and the order of alternatives vary. This suggests that the parameter τ represents the decision-maker’s preference. The AOs suggested in this article equip the decision-maker with the flexibility to assign values to the parameter τ . Therefore, when compared to existing AOs for (p, q)-fuzzy sets, the suggested operators are more versatile and adaptable.

Table 6 Ranking using (p, q) -FHWA operator

(p, q)	$score(J_1)$	$score(J_2)$	$score(J_3)$	$score(J_4)$	$score(J_5)$	Ranking	Optimal
(2, 3)	0.0875	0.0939	- 0.1058	0.4223	- 0.1759	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(3, 2)	- 0.1958	- 0.1911	-0.3730	0.1536	- 0.4329	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(4, 5)	0.0370	- 0.0339	- 0.1567	0.3478	- 0.2094	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4
(5, 4)	- 0.1008	- 0.1154	- 0.2987	0.2105	- 0.3403	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4
(10, 100)	0.0427	0.0423	0.0221	0.1330	0.0221	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4
(100, 10)	- 0.0374	- 0.0508	- 0.1198	- 0.0086	- 0.1313	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4

Table 7 Aggregated (p, q)-fuzzy sets using (p, q)-FHWG operator

Operator	J_1	J_2	J_3	J_4	J_5
(2,3)-FHWG	(0.6372,0.7371)	(0.6957,0.7522)	(0.6250,0.8289)	(0.8101,0.6352)	(0.5681,0.8343)
(3,2)-FHWG	(0.6372,0.7351)	(0.6957,0.7511)	(0.6250,0.8272)	(0.8101,0.6328)	(0.5681,0.8329)
(4,5)-FHWG	(0.6372,0.7413)	(0.6957,0.7546)	(0.6250,0.8326)	(0.8101,0.6409)	(0.5681,0.8371)
(5,4)-FHWG	(0.6372,0.7392)	(0.6957,0.7534)	(0.6250,0.8307)	(0.8101,0.6379)	(0.5681,0.8357)
(10,100)-FHWG	(0.6372,0.7897)	(0.6957,0.7975)	(0.6250,0.8988)	(0.8101,0.7289)	(0.5681,0.8793)
(100,10)-FHWG	(0.6372,0.7514)	(0.6957,0.7604)	(0.6250,0.8426)	(0.8101,0.6580)	(0.5681,0.8438)

Comparison with an aggregation operator of n,m-rung orthopair fuzzy sets

An n,m-rung orthopair fuzzy set [11] is a subtype of (p, q)-fuzzy set [1], where n, m ∈ ℕ and n ≠ m. Here, we adapt the numerical example from [11] to demonstrate the validity of the operators defined in this article. The numerical example for MADM depicted in [11] involves selecting the best journal from a set of five journals (alternatives) based on factors, including *Indexing*, *Editorial Board*, *Journal Rank*, *Reviewer Report* and *Impact Factor*. We denote this set of attributes as $D = \{D_1, D_2, D_3, D_4, D_5\}$ where D_1 : denotes *Indexing*, D_2 : denotes *Editorial Board*, D_3 : denotes *Journal Rank*, D_4 : indicates *Reviewer Report* and D_5 : denotes *Impact Factor*. In [11], the decision-maker represented their evaluations using an n, m-rung orthopair fuzzy decision matrix (see Table 4), and authors employed the n, m-rung orthopair fuzzy weighted power average (n, m-ROFWPA) operator with weights 0.21, 0.15, 0.22, 0.13, and 0.29 for each attribute D_j , where $j = 1, 2, \dots, 5$, respectively. They explored various values for p and q to interpret their impact on the results of the MADM problem. This article compares the results obtained in [11] with those obtained by employing (p, q)-FHWA and (p, q)-FHWG operators with $\tau = 1$ for aggregation. Tables 5 and 7 display the aggregated values using these operators for each journal J_i , using values of $p = 2, 3, 4, 5, 10, 100$, and $q = 3, 2, 5, 4, 100, 10$, respectively. Tables 6 and 8 list the score values and subsequent rankings for J_i , $i = 1, 2, \dots, 5$.

The optimal alternative is J_4 irrespective of the values of p and q. Similarly, when utilizing the (p, q)-FHWG operator for $\tau = 1$, J_4 remains the optimal choice. Table 9 presents the rankings of the alternatives, J_i , using the n,m-ROFWPA operator from [11] for n= 2, 3, 4, 5, 10, 100 and m= 3, 2, 5, 4, 100, 10 respectively. The AOs for (p, q)-fuzzy sets defined in this article yield results similar to those obtained using the existing operators in literature. The n,m-ROFWPA operator defined in [11] is expressed as

Table 8 Ranking using (p, q) -FHWG operator

(p, q)	$score(J_1)$	$score(J_2)$	$score(J_3)$	$score(J_4)$	$score(J_5)$	Ranking	Optimal
(2, 3)	0.0055	0.0583	- 0.1790	0.3999	- 0.2579	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(3, 2)	- 0.2816	-0.2275	- 0.4402	0.1311	- 0.5104	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(4, 5)	- 0.0591	- 0.0104	- 0.2475	0.3225	- 0.3068	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(5, 4)	- 0.1936	- 0.1592	- 0.3809	0.1832	- 0.4285	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(10, 100)	0.0110	0.0266	0.0091	0.1217	0.0035	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
(100, 10)	- 0.0574	- 0.0646	- 0.1804	- 0.0152	- 0.1829	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4

Table 9 Ranking of journals using n,m-ROFWPA operator

Operator	$score(J_1)$	$score(J_2)$	$score(J_3)$	$score(J_4)$	$score(J_5)$	Ranking	Optimal
2, 3-ROFWPA	0.0449	0.0736	- 0.1333	0.4073	- 0.2145	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
3,2-ROFWPA	- 0.2274	- 0.2064	- 0.3900	0.1418	- 0.4591	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
4,5-ROFWPA	- 0.0030	0.0116	- 0.1914	0.3309	- 0.2531	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
5,4-ROFWPA	- 0.1338	- 0.1341	- 0.3245	0.1943	- 0.3738	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4
10,100-ROFWPA	0.0412	0.0415	0.0218	0.1318	0.0211	$J_4 > J_2 > J_1 > J_3 > J_5$	J_4
100,10-ROFWPA	- 0.0565	- 0.0639	- 0.1666	- 0.0151	- 0.1760	$J_4 > J_1 > J_2 > J_3 > J_5$	J_4

$$n,m - \text{ROFWPA}(F_1, F_2, \dots, F_m) = \left(\left(\sum_{k=1}^m \lambda_k \alpha_k^p \right)^{\frac{1}{p}}, \left(\sum_{k=1}^m \lambda_k \beta_k^q \right)^{\frac{1}{q}} \right).$$

While this operator pertains to a subtype of (p, q) -fuzzy sets, it differs from the (p, q) -FWA and (p, q) -FHWG operators in that it does not take into account the parameter τ , which reflects the decision maker's preference.

5 Conclusions

A (p, q) -fuzzy set [1] is defined as a structure with its membership and non-membership values satisfying the nonlinear relationship $x^p + y^q \leq 1$. It broadens the scope of representing uncertain information, expanding the feasible region and enhancing the decision-making process. Due to its variable rungs p and q , it retains more fuzzy information compared to IFS, PFS, FFS, and q -ROFS for $q > 3$. Since data aggregation plays a pivotal role in decision-making, this study attempted to develop generalized AOs for (p, q) -fuzzy sets by employing strict t-norms and t-conorms. The AOs discussed in this article, specifically the (p, q) -fuzzy weighted averaging operator and the (p, q) -fuzzy weighted geometric operator defined in this article, generalize similar operators defined for subtypes of (p, q) -fuzzy sets. These generalized definitions of operators allow us to create a variety of operators for (p, q) -fuzzy sets, as well as for any of its subtypes, using unary functions known as additive generators. The article enlists and proves specific beneficial properties of these operators and proposes their application in combining fuzzy information for decision-making. The practicality of this approach is demonstrated through an examination of an MADM problem where the best decision is determined by ranking alternatives based on their score values. A comparative analysis with existing operators is conducted to validate and assess the effectiveness of the generalized operators. Future studies may explore concepts like (p, q) -fuzzy topology and algebraic structures of (p, q) -fuzzy sets.

Appendix A

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Declarations

Conflict of interest All authors declare that they have no conflict of interest.

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