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ORIGINAL RESEARCH PAPER

On Bernstein-type inequalities for polynomials with restricted zeros

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Abstract

For a complex-polynomial $P(z)$ of degree n having no zero in $|z|<1$, it is known that $\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|$. Under same hypothesis, V. K. Jain proved that if $\alpha \in \mathbb{C}$ with $|\alpha| \leq \frac{n}{2}$ then for $|z|=1$,

$$|zP'(z) - \alpha P(z)| \leq \frac{1}{2}\{|n-\alpha| + |\alpha|\}\max_{|z|=1}|P(z)|.$$

In this paper, we obtained an extension of this inequality to mth derivative which also contains a refinement of this inequality. Our result not only generalize some well-known inequalities but also shows that the inequality of Jain holds for wider range of α.

Keywords Polynomials · Bernstein's inequality · Inequalities in the complex domain

Mathematics Subject Classification 30A10 · 30C10 · 41A17

1 Introduction and statement of results

Let $P(z)$ be a polynomial of degree n then according to Bernstein's inequality (for details see [9, p. 508]),

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$$\max_{|z|=1}|P'(z)| \leq n \max_{|z|=1}|P(z)|. \tag{1}$$

The inequality is sharp and equality in (1) holds if $P(z) = az^n$, $a \neq 0$. Bernstein-type inequalities played a fundamental role in the area of Approximation Theory and Polynomial Approximations [4, 8].

Smirnov [10, p. 356] obtained a generalized version of Bernstein's inequality. For $z \in \mathbb{C}$ with $|z| \geq 1$, by $\Omega_{|z|}$ denote the image of the disc $\{t \in \mathbb{C} : |t| < |z|\}$ under the mapping $\psi(t) = t/(1+t)$, then the Smirnov's result can be stated as:

If $P(z)$ be a polynomial of degree at most n and $F(z)$ a polynomial of degree n such that $F(z)$ has all its zeros in $|z| \leq 1$ and $|P(z)| \leq |F(z)|$ for $|z| = 1$, then

$$|zP'(z) - n\alpha P(z)| \leq |zF'(z) - n\alpha F(z)|$$

for all $\alpha \in \Omega_{|z|}$. For $\alpha \in \Omega_{|z|}$, this inequality becomes equality if and only if $P \equiv e^{i\theta}F$, $\theta \in \mathbb{R}$.

The Bernstein's inequality follows from above inequality by taking $\alpha = 0$ and $F(z) = z^n \max_{|z|=1} |P(z)|$.

If the polynomial $P(z)$ has no zero in $|z| < 1$, then the inequality (1) can be improved and the same was conjectured by Erdös [5] and later Lax [7] proved that if $P(z)$ does not vanish in $|z| < 1$, then inequality (1) can take the form:

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2} \max_{|z|=1}|P(z)|.$$

The above result is sharp and equality holds if $P(z) = a + bz^n$, where $|a| = |b| \neq 0$.

Aziz and Dawood [2] refined the above Erdö–Lax theorem by involving minimum of $|P(z)|$ and proved that If the polynomial $P(z)$ has no zero in $|z| < 1$, then

$$\max_{|z|=1}|P'(z)| \leq \frac{n}{2}\left(\max_{|z|=1}|P(z)| - \min_{|z|=1}|P(z)|\right). \tag{2}$$

If T is an operator on the space of polynomials, then the Bernstein's inequality gives us the exact constant C_n in the inequality

$$\max_{|z|=1}|T[P](z)| \leq C_n \max_{|z|=1}|P(z)|$$

for the operator $T \equiv \frac{d}{dz}$. In this case $C_n = n$.

It is interesting to charaterise C_n for different operators defined on the space of complex-polynomials of degree at most n (for some well-known operators, refer to [9, P. 538]). Jain [6], studied the operator $T_\alpha[P](z) := zP'(z) - \alpha P(z)$ and proved that if $P(z)$ is a polynomial of degree n and $\alpha \in \mathbb{C}$ with $|\alpha| \leq n/2$, then

$$\max_{|z|=1}|zP'(z) - \alpha P(z)| \leq |n - \alpha| \max_{|z|=1}|P(z)|. \tag{3}$$

That is, for this operator $C_n = |n - \alpha|$. One can easily observe that Bernstein's inequality is a special of Jain's result and follows by taking $\alpha = 0$.

Jain improved the inequality (3) and proved that if $P(z)$ is an nth-degree polynomial with no zero in the unit disk $|z|<1$, then for any $\alpha \in \mathbb{C}$ with $|\alpha| \leq n/2$,

$$\max_{|z|=1}|zP'(z) - \alpha P(z)| \leq \frac{1}{2}\{|n-\alpha| + |\alpha|\}\max_{|z|=1}|P(z)|. \tag{4}$$

The result is sharp and equality holds if $P(z) = a + bz^n$, where $|a| = |b| \neq 0$.

In this paper, we first present the following sharp estimate for minimum modulus of a polynomial involving mth and $(m-1)$th derivatives of a polynomial $P(z)$ with zeros in closed unit disc.

Theorem 1.1 *Let $P(z)$ be a non-constant polynomial of degree n having all zeros in $|z| \leq 1$. Then for every $\alpha \in \mathbb{C}$ and $m \in \mathbb{N}$ with $\Re(\alpha) \leq \frac{n-m+1}{2}$ and $m \leq n$,*

$$\min_{|z|=1}|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \geq \frac{n!}{(n-m+1)!}|\alpha - (n-m+1)|\min_{|z|=1}|P(z)|. \tag{5}$$

The inequality is sharp and equality holds if $P(z) = ae^{i\gamma}z^n, a > 0$.

By taking $\alpha = 0$ in inequality (5), we obtain the following estimate for minimum modulus of mth derivative of $P(z)$.

Corollary 1.1 *Let $P(z)$ be a polynomial of degree n having all zeros in $|z| \leq 1$, then*

$$\min_{|z|=1}|P^{(m)}(z)| \geq \frac{n!}{(n-m)!}\min_{|z|=1}|P(z)|.$$

The inequality is sharp and equality holds if and only if $P(z) = ae^{i\gamma}z^n, a > 0$

The above Corollary reduces to a result due to Aziz and Dawood [2] for $m = 1$.

The next Corollary is obtained by taking $m = 1$ in Theorem 1.1

Corollary 1.2 *Let $P(z)$ be a polynomial of degree n having all zeros in $|z| \leq 1$. Then for every $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \frac{n}{2}$,*

$$\min_{|z|=1}|zP'(z) - \alpha P(z)| \geq |n-\alpha|\min_{|z|=1}|P(z)|.$$

The inequality is sharp and becomes equality if $P(z) = ae^{i\gamma}z^n, a > 0$

Next, we extend inequality (4) to mth-derivative of $P(z)$ which among other things shows that this inequality of Jain also holds for wider range of α.

Theorem 1.2 *Let $P(z)$ be a non-constant polynomial of degree n and has no zero in $|z|<1$. Then for every $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \frac{n-m+1}{2}$ and $|z| = 1$,*

$$\begin{aligned}&|zP^{(m)}(z)-\alpha P^{(m-1)}(z)|\\&\leq\frac{n!}{2(n-m+1)!}\left[\left\{|\alpha-(n-m+1)|+\delta_{m1}|\alpha|\right\}\max_{|z|=1}|P(z)|\right.\\&\quad\left.-\left\{|\alpha-(n-m+1)|-\delta_{m1}|\alpha|\right\}\min_{|z|=1}|P(z)|\right]\end{aligned}$$

where δ_{m1} denotes Kroneker delta. The inequality is sharp and equality holds if $P(z)=z^n+1$.

For $m=1$, we obtain following result from Theorem 1.2.

Corollary 1.3 *Let $P(z)$ be a polynomial of degree n and has no zero in $|z|<1$. Then for every $\alpha\in\mathbb{C}$ with $\Re(\alpha)\leq\frac{n}{2}$ and $|z|=1$,*

$$|zP'(z)-\alpha P(z)|\leq\frac{1}{2}\left[(|n-\alpha|+|\alpha|)\max_{|z|=1}|P(z)|-(|n-\alpha|-|\alpha|)\min_{|z|=1}|P(z)|\right].$$

The inequality is sharp and equality holds if $P(z)=z^n+1$.

If we take $\alpha=0$ in above inequality, we shall get inequality (2).

Remark 1.1 Since $\Re(\alpha)\leq\frac{n}{2}$ then $|n-\alpha|\geq|\alpha|$. This implies that

$$\begin{aligned}&(|n-\alpha|+|\alpha|)\max_{|z|=1}|P(z)|-(|n-\alpha|-|\alpha|)\min_{|z|=1}|P(z)|\\&\leq(|n-\alpha|+|\alpha|)\max_{|z|=1}|P(z)|.\end{aligned}$$

This shows that Corollary 1.3 not only gives a refinement of inequality (4) but also shows that this inequality holds for all α belonging to the half-plane $|n-\alpha|\geq|\alpha|$.

For $m\geq2$, $\delta_{m1}=0$. By using this fact in Theorem 1.2, we obtain the following Corollary.

Corollary 1.4 *Let $P(z)$ be a polynomial of degree n and has no zero in $|z|<1$. Then for every $\alpha\in\mathbb{C}$ with $\Re(\alpha)\leq\frac{n-m+1}{2}$, $m\geq2$ and $|z|=1$,*

$$|zP^{(m)}(z)-\alpha P^{(m-1)}(z)|\leq\frac{n!|\alpha-(n-m+1)|}{2(n-m+1)!}\left(\max_{|z|=1}|P(z)|-\min_{|z|=1}|P(z)|\right).$$

The inequality is sharp and equality holds if $P(z)=z^n+1$.

2 Lemmas

For the proof of our theorems, we need the following Lemmas.

The first lemma is a generalized version of Walsh's Coincidence theorem, due to Aziz [1], for the case when the circular region is a circle.

Lemma 2.1 *Let $G(z_1, z_2, \ldots, z_n)$ be a symmetric n-linear form of total degree m, $m \le n$, in $z_1, z_2, \ldots, z_n$ and let $\mathcal{C} : |z| \le r$ be a closed circular disc containing the n points $w_1, w_2, \ldots, w_n$. Then in $\mathcal{C}$ there exists atleast one point β such that*

$$G(\beta, \beta, \ldots, \beta) = G(w_1, w_2, \ldots, w_n).$$

Lemma 2.2 *Let $P \in \mathcal{P}_n$ and have all zeros in $|z| \le r$ where $r > 0$, if $\alpha \in \mathbb{C}$ with $\Re(\alpha) \le \frac{n-m+1}{2}$, then all the zeros of $T_{m,\alpha}[P](z) = zP^{(m)}(z) - \alpha P^{(m-1)}(z)$ are also in $|z| \le r$*

Proof Let w be any zero of the polynomial $T_{m,\alpha}[P](z)$, then

$$wP^{(m)}(w) - \alpha P^{(m-1)}(w) = 0. \tag{6}$$

This expression is linear and symmetric in the zeros of $P(z)$. By Lemma 2.1, w will also satisfy the equation obtained by replacing $P(z)$ in (6) by $(z-\beta)^n$, where β is a suitable complex number with $|\beta| \le r$. This implies

$$\begin{aligned} &n(n-1)\ldots(n-m+1)(w-\beta)^{n-m}w \\ &\quad - \alpha\, n(n-1)\ldots.(n-m+2)(w-\beta)^{n-m+1} = 0 \end{aligned}$$

or

$$n(n-1)\ldots(n-m+2)(w-\beta)^{n-m}\{(n-m+1)w - \alpha(w-\beta)\} = 0 \tag{7}$$

Since $\Re(\alpha) \le \frac{n-m+1}{2}$, then $\Re\left(\frac{\alpha}{n-m+1}\right) \le \frac{1}{2}$. This implies that

$$\left|\frac{\alpha}{n-m+1}\right| \le \left|\frac{\alpha}{n-m+1} - 1\right|$$

or

$$|\alpha| \le |\alpha - (n-m+1)| \tag{8}$$

Equation (7) implies that

$$(w-\beta) = 0 \text{ or } (n-m+1)w - \alpha(w-\beta) = 0.$$

Equivalently,

$$w = \beta \text{ or } w = \frac{\alpha\beta}{\alpha - (n-m+1)}.$$

This further implies by using (8) that,

$$|w| = |\beta| \text{ or } |w| = \frac{|\alpha||\beta|}{|\alpha - (n-m+1)|} \leq \frac{|\alpha||\beta|}{|\alpha|}.$$

Thus,

$$\Rightarrow |w| \leq |\beta| \leq r$$

Hence, it follows that all the zeros of $T_{m,\alpha}[P](z)$ also lie in $|z| \leq r$. This completes the proof. □

A proof of Lemma 2.2 also follows from a result due to [3].

A linear operator T on the space of complex-polynomials of degree at most n is called a B_n-operator (see [9, p. 538]) if for every polynomial $P(z)$, of degree at most n, having all its zeros in $|z| \leq 1$, then the polynomial $T[P](z)$ also has all its zeros in $|z| \leq 1$.

The next two lemmas can be found in [9, p. 538, 539].

Lemma 2.3 *Let $h(z)$ be an nth-degree polynomial with all zeros in $|z| \leq 1$ and g a polynomial of degree at most n, such that $|g(z)| \leq |h(z)|$ for $|z| = 1$, then for any B_n operator T, we have*

$$|T[g](z)| \leq |T[h](z)| \text{ for } |z| \geq 1$$

Moreover, $|T[g](z)| = |T[h](z)|$ at some point z outside the closed unit disc if and only if $g(z) = e^{i\theta}h(z)$, $\theta \in \mathbb{R}$.

Lemma 2.4 *Let $P(z)$ be a polynomial of degree n and $Q(z) = z^n\overline{P(1/\bar{z})}$ and $\phi_n(z) = z^n$, then for any B_n-operator T*

$$|T[P](z)| + |T[Q](z)| \leq (|T[1](z)| + |T[\phi_n](z)|) \max_{|z|=1}|P(z)|, \quad |z| \geq 1$$

3 Proof of main results

Proof of Theorem 1.1 If $P(z)$ has a zero on $|z| = 1$, then the Theorem is trivially true. Therefore, suppose all the zeros of $P(z)$ lie in $|z| < 1$. Let $k = \min_{|z|=1}|P(z)|$, then $k > 0$ and $k \leq |P(z)|$ for $|z| = 1$. By Rouche's theorem, the polynomial $g(z) = P(z) - \lambda k z^n$ has all its zeros in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Invoking Lemma 2.2, we conclude that, for any $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \frac{n-m+1}{2}$, the zeros of the polynomial

$$zg^{(m)}(z) - \alpha g^{(m-1)}(z) = \left\{zP^{(m)}(z) - \alpha P^{(m-1)}(z)\right\} - \lambda k \frac{n!}{(n-m+1)!}\{-\alpha + (n-m+1)\}z^{n-m+1}$$

lie in $|z| < 1$. This implies that for any $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \frac{n-m+1}{2}$

$$|zP^{(m)}(z) - \alpha P^{(m-1)}| \geq k\frac{n!}{(n-m+1)!}|\alpha - (n-m+1)||z|^{(n-m+1)} \tag{9}$$

for $|z| \geq 1$. If inequality (9) were not true, then there exists a point $z = z_0$ with $|z_0| \geq 1$ such that

$$|z_0P^{(m)}(z_0) - \alpha P^{(m-1)}(z_0)| < k\frac{n!}{(n-m+1)!}|\alpha - (n-m+1)||z_0|^{(n-m+1)}$$

If we take,

$$\lambda = \frac{z_0P^{(m)}(z_0) - \alpha P^{(m-1)}(z_0)}{\frac{kn!}{(n-m+1)!}\{-\alpha + (n-m+1)\}z_0{}^{(n-m+1)}},$$

then $|\lambda| < 1$ and for this choice of λ, $z_0g^{(m)}(z_0) - \alpha g^{(m-1)}(z_0) = 0$. This contradicts to the fact that all zeros of $zg^{(m)}(z) - \alpha g^{(m-1)}(z)$ lie in $|z| < 1$. Hence, the inequality (9) is valid.

That is for any $\alpha \in \mathbb{C}$ with $\Re(\alpha) \leq \frac{n-m+1}{2}$, we have

$$\min_{|z|=1}|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| \geq \frac{n!}{(n-m+1)!}|\alpha - (n-m+1)| \min_{|z|=1}|P(z)|.$$

This completes the proof. □

Proof of Theorem 1.2 Let $k = \min_{|z|=1}|P(z)|$. If $P(z)$ has no zero on unit circle $|z| = 1$, then by minimum modulus principal, $k < |P(z)|$ for $|z| < 1$. This implies that for any complex number λ with $|\lambda| \leq 1$, the polynomial $g(z) = P(z) - \lambda k$ has no zero in $|z| < 1$. Now, if $P(z)$ has a zero on $|z| = 1$ then $g(z) = P(z)$. Thus, in any case the polynomial $g(z) = P(z) - \lambda k$ does not vanish in the disc $|z| < 1$.

Let $h(z) = z^n\overline{g(1/\overline{z})} = q(z) - k\bar{\lambda}z^n$, where $q(z) = z^n\overline{P(1/\overline{z})}$, then all the zeros of $h(z)$ lie in $|z| \leq 1$. Moreover $|g(z)| = |h(z)|$ for $|z| = 1$, then by Lemmas 2.2 and 2.3, for the B_n-operator $T_{m,\alpha}$, we have

$$|T_{m,\alpha}[g](z)| \leq |T_{m,\alpha}[h](z)| \quad \text{for} \quad |z| \geq 1.$$

This implies,

$$|zg^{(m)}(z) - \alpha g^{(m-1)}(z)| \leq |zh^{(m)}(z) - \alpha h^{(m-1)}(z)| \quad \text{for} \quad |z| \geq 1.$$

Equivalently, for $|z| \geq 1$, we have

$$\begin{aligned}&\left|\left\{zP^{(m)}(z)-\alpha P^{(m-1)}(z)\right\}+\lambda\alpha k\delta_{m1}\right|\\&\quad\leq\left|\left\{zq^{(m)}(z)-\alpha q^{(m-1)}(z)\right\}\right.\\&\qquad\left.-k\overline{\lambda}n(n-1)\ldots(n-m+2)\{(n-m+1)-\alpha\}z^{n-m+1}\right|\end{aligned}\tag{10}$$

Since all the zeros of $q(z)$ lie in $|z|\leq 1$, so by Theorem 1.1, for any $\alpha\in\mathbb{C}$ with $\Re(\alpha)\leq\frac{n-m+1}{2}$ and $|z|=1$,

$$\begin{aligned}|zq^{(m)}(z)-\alpha q^{(m-1)}(z)|&\geq\frac{n!}{(n-m+1)!}|\alpha-(n-m+1)|\min_{|z|=1}|q(z)|\\&=\frac{n!}{(n-m+1)!}|\alpha-(n-m+1)|\min_{|z|=1}|P(z)|\\&=\frac{n!}{(n-m+1)!}|\alpha-(n-m+1)|k\end{aligned}$$

This allows us to choose the argument of λ such that

$$\begin{aligned}&\left|\left\{zq^{(m)}(z)-\alpha q^{(m-1)}(z)\right\}-k\overline{\lambda}\frac{n!}{(n-m+1)!}\{(n-m+1)-\alpha\}z^{n-m+1}\right|\\&\quad=\left|zq^{(m)}(z)-\alpha q^{(m-1)}(z)\right|-k|\overline{\lambda}|\frac{n!}{(n-m+1)!}|(n-m+1)-\alpha||z|^{n-m+1}.\end{aligned}$$

For this argument of λ, the inequality (10) reduces to,

$$\begin{aligned}&|zP^{(m)}(z)-\alpha P^{(m-1)}(z)+k\lambda\alpha\delta_{m1}|\\&\quad\leq|zq^{(m)}(z)-\alpha q^{(m-1)}(z)|-|\lambda|\frac{n!}{(n-m+1)!}|\alpha-(n-m+1)|k\end{aligned}$$

for $|z|=1$. Using triangle inequality in left-hand side of above inequality then letting $|\lambda|\to 1$, for $|z|=1$, we obtain

$$\begin{aligned}&|zP^{(m)}(z)-\alpha P^{(m-1)}(z)|-|zq^{(m)}(z)-\alpha q^{(m-1)}(z)|\\&\quad\leq\left(\delta_{m1}|\alpha|-\frac{n!}{(n-m+1)!}|\alpha-(n-m+1)|\right)k.\end{aligned}\tag{11}$$

Next, applying Lemma 2.4 to $P(z)$ with $T=T_{m,\alpha}$, (as defined in Lemma 2.2), we get for $|z|=1$,

$$\begin{aligned}&|zP^{(m)}(z)-\alpha P^{(m-1)}(z)|+|zq^{(m)}(z)-\alpha q^{(m-1)}(z)|\\&\quad\leq\left\{|T_{m,\alpha}[1](z)|+|T_{m,\alpha}[\phi_n](z)|\right\}\max_{|z|=1}|P(z)|\end{aligned}\tag{12}$$

It is not difficult to see that $T_{m,\alpha}[1](z)=-\delta_{m1}\alpha$ and if $\phi_n(z)=z^n$ then

$T_{m,\alpha}[\phi_n](z) = \frac{n!}{(n-m+1)!}\{(n-m+1)-\alpha\}z^{(n-m+1)}$. Using these values in (12), we have for $|z|=1$,

$$
\begin{aligned}
&|zP^{(m)}(z) - \alpha P^{(m-1)}(z)| + |zq^{(m)}(z) - \alpha q^{(m-1)}(z)| \\
&\leq \left\{ \frac{n!}{(n-m+1)!}|(n-m+1)-\alpha| + \delta_{m1}|\alpha| \right\} \max_{|z|=1}|P(z)|.
\end{aligned} \tag{13}
$$

Note that $\delta_{m1} = \frac{n!}{(n-m+1)!}\delta_{m1}$ as $\delta_{m1} = 0$ for $m > 1$. Finally the conclusion of Theorem 1.2 is obtained by adding inequalities (11) and (13). This completes the proof. □

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Declarations

Conflict of interest Authors declare that they have no have conflict of interest.

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