The Journal of Analysis (2023) 31:1557–1572
https://doi.org/10.1007/s41478-022-00539-2

ORIGINAL RESEARCH PAPER

Some applications of the α-Duhamel product

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Received: 19 July 2022 / Accepted: 3 December 2022 / Published online: 21 December 2022

Abstract

We consider the space $C^{(n)}(\Omega)$, the Banach space of continuous functions with n derivatives and the n th derivative continuous in $\overline{\Omega}$, where $\Omega \subset \mathbb{C}$ is a starlike region with respect to $\alpha \in \Omega$. We use the so-called α-Duhamel product

$$\left(f \underset{\alpha}{\circledast} g\right)(z) := \frac{d}{dz}\int_{\alpha}^{z} f(z+\alpha-t)g(t)dt = \frac{d}{dz}\left(f \underset{\alpha}{*} g\right)(z)$$

to describe usual $\underset{\alpha}{*}$-generators of the Banach algebra $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$, to estimate $\|(I-V_\alpha)^m\|$ and to estimate below the norm $\left\|\delta_A^m\right\|$, where V_α is the Volterra integration operator defined by $V_\alpha f(z) = \int\limits_{\alpha}^{z} f(t)dt$ and δ_A is the inner derivation operator defined by $\delta_A(X) := [X, A]$. We give a new proof of Aleman-Korenblum theorem in one particular case. Namely, we describe V-invariant subspaces in the Hardy space H^p by using Duhamel product.

Keywords α-Duhamel product · Starlike region · Generator · Inner derivation operator · Invariant subspace · Hardy space · Volterra integration operator

Mathematics Subject Classification Primary 46E10; Secondary 46J15

1 Introduction

Let $\alpha \in \mathbb{C}$ be a number. Let $\Omega \subset \mathbb{C}$ be a simply connected bounded region containing the point α, which is a star-like region with respect to the point $z = \alpha$, i.e., $\lambda z + (1-\lambda)\alpha \in \Omega$ for every $z \in \Omega$ and λ, $0 \leq \lambda \leq 1$. We define on Ω the Banach space

Communicated by S Ponnusamy.

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$C^{(n)}(\Omega)$ of all continuous functions on Ω with the n th derivative continuous on $\overline{\Omega}$. The space $C^{(n)} := C^{(n)}(\Omega)$ is a Banach space equipped with the norm

$$\|f\|_n := \max\left\{\max_{z\in\overline{\Omega}}\left|f^{(i)}(z)\right| : i = 1, 2, ..., n\right\}.$$

The α-convolution and α-Duhamel product are defined in $C^{(n)}$, respectively, by

$$\left(f \underset{\alpha}{*} g\right)(z) := \int_{\alpha}^{z} f(z + \alpha - t)g(t)dt \tag{1.1}$$

and

$$\left(f \underset{\alpha}{\circledast} g\right)(z) := \frac{d}{dz}\int_{\alpha}^{z} f(z + \alpha - t)g(t)dt = \int_{\alpha}^{z} f'(z + \alpha - t)g(t)dt + f(\alpha)g(z), \tag{1.2}$$

where the integral is taken over the segment joining the points α and z $(z \in \Omega)$. The α-integration operator V_α is defined on $C^{(n)}$ by $V_\alpha f(z) := \int_\alpha^z f(t)dt$, where the integration is performed as above over straight-line segments connecting the points α and z. Our investigation is motivated by the papers [2, 12, 18, 26, 31], where some properties of Banach algebra $\left(C^{(n)}[0, 1], \underset{\alpha}{\circledast}\right)$ and Volterra integration operator V_α are studied. In the present paper, we describe $\underset{\alpha}{*}$-generators of the Banach algebra $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$ in terms of α-Duhamel product (Sect. 2). In Sect. 3, we characterize V-invariant subspaces of H^p by involving the Duhamel product method, and hence we give a new proof of a result of Aleman and Korenblum in their paper [2]. In Sect. 4, we calculate norm of orbits of operator $I - V_\alpha$ on the Hardy space $H^2 = H^2(\mathbb{D})$ over the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, which is related Esterle-Katznelson-Tzafriri theorem for Cesàro bounded operators with single-point spectrum in Hilbert spaces (more detailly, see [44]). In Sect. 4, we also estimate in terms of α-Duhamel products the norm of orbits of inner-derivation δ_A on $\mathcal{B}\left(C^{(n)}(\Omega)\right)$ defined by $\delta_A(X) := [X, A]$.

2 The $*_\alpha$-generators of algebra $C^{(n)}(\Omega)$

Recall that for a Banach algebra $\mathcal{B}$ the radical $\mathcal{R}$ of $\mathcal{B}$ is the intersection of the kernel of all (strictly) irreducible representations of $\mathcal{B}$. If $\mathcal{R} = \{0\}$, then $\mathcal{B}$ is said to be semi-simple and if $\mathcal{R} = \mathcal{B}$, then $\mathcal{B}$ is called a radical algebra. Equivalently, $\mathcal{B}$ is a radical Banach algebra, if for every element $b \in \mathcal{B}$ the associated multiplication operator $M_b a := ba$ $(a \in \mathcal{B})$, is quasinilpotent on $\mathcal{B}$, i.e., $\sigma(M_b) = \{0\}$.

It is classical that $\lim_{k\to\infty}\left\|f^{\underset{\alpha}{*}k}\right\|^{1/k} = 0$, and so, the space $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$ is a radical Banach algebra with respect to the convolution $\underset{\alpha}{*}$ defined by formula (1); here

$f^{*k}_{\alpha} := \underbrace{f \underset{\alpha}{*} \dots \underset{\alpha}{*} f}_{k}$ is the k^{th} iterated convolution of the function f in $C^{(n)}(\Omega)$. Clearly, $\left(f \underset{\alpha}{*} f\right)(\alpha) = 0$ for any $f \in C^{(n)}(\Omega)$. Also,

$$\left(f \underset{\alpha}{*} f \underset{\alpha}{*} f\right)(\alpha) = f \underset{\alpha}{*} \left(f \underset{\alpha}{*} f\right) = \left(\int_{\alpha}^{z} f(z+\alpha-t) \int_{\alpha}^{t} f(t+\alpha-\tau) f(\tau) d\tau\right)(\alpha) = 0,$$

thus, it is easy to verify that $\left(f^{*k}_{\alpha}\right)(\alpha) = 0$ for all $k = 1, 2, \dots$. Therefore, we see that a necessary condition for $f \in C^{(n)}(\Omega)$ to generate $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$, that is, to yield

$$\operatorname{span}\left\{f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \dots\right\} = C^{(n)}(\Omega)$$

is that $f(\alpha) \neq 0$. However, it is not yet known whether this condition is sufficient, even for $\alpha = 0$ (see, for instance, Ginsberg and Newman [14] and Karaev [25]). For more detail, see [12].

In the present section, we study the above stated question for the Banach algebra $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$ by proving the following theorem, which reduces this question to the case of the subalgebra

$$C^{(n)}_{\alpha}(\Omega) := \left\{f \in C^{(n)}(\Omega) : f(\alpha) = 0\right\}.$$

Before stating our result, let us formulate two auxiliary lemmas, the proofs of which are quite similar to the proofs of Lemmas 2.1 and 2.2 of the paper [12], and therefore we omit it.

Lemma 1 *$\left(C^{(n)}(\Omega), \underset{\alpha}{\circledast}\right)$ is a commutative Banach algebra with the unit element $f = \mathbf{1}$.*

Lemma 2 *The function $f \in C^{(n)}(\Omega)$ is $\underset{\alpha}{\circledast}$-invertible if and only if $f(\alpha) \neq 0$.*

Theorem 1 *Let $f \in C^{(n)}(\Omega)$ be a function such that $f(\alpha) \neq 0$. Let $F(z) = \int_{\alpha}^{z} f(t)dt$. Then f is a $\underset{\alpha}{*}$-generator of the algebra $\left(C^{(n)}(\Omega), \underset{\alpha}{*}\right)$ if and only if F is a $\underset{\alpha}{\circledast}$-generator of the subalgebra $\left(C^{(n)}_{\alpha}(\Omega), \underset{\alpha}{\circledast}\right)$.*

Proof In fact, since $F(z) = \int_{\alpha}^{z} f(t)dt$, we obtain for all $g \in C^{(n)}(\Omega)$ that

$$\left(D_{\alpha,F}g\right)(z) = \frac{d}{dz}\int_{\alpha}^{z} F(z+\alpha-t)g(t)dt = \int_{\alpha}^{z} f(z+\alpha-t)g(t)dt,$$

hence $D_{\alpha,F} = C_{\alpha,f}$. Therefore $F \underset{\alpha}{\circledast} f = f \underset{\alpha}{*} f$, so we have

$$\left(F \underset{\alpha}{\circledast} F\right) \underset{\alpha}{\circledast} f = D^2_{\alpha,F} f = D_{\alpha,F}\left(D_{\alpha,F} f\right) = D_{\alpha,F}\left(C_{\alpha,f} f\right) = C^2_{\alpha,f} f.$$

Thus, by induction we get $C^k_{\alpha,f} f = D^k_{\alpha,F} f$ for all $k \geq 0$, which show that

$$\begin{aligned}
\operatorname{span}\left\{f, f \underset{\alpha}{*} f, f \underset{\alpha}{*} f \underset{\alpha}{*} f, \ldots\right\} &= \operatorname{span}\left\{f, F \underset{\alpha}{\circledast} f, F \underset{\alpha}{\circledast} F \underset{\alpha}{\circledast} f, \ldots\right\} \\
&= \overline{\operatorname{span}\left\{D_{\alpha,f}\left(F^{\underset{\alpha}{\circledast} k}\right) : k \geq 0\right\}} \\
&= \overline{D_{\alpha,f} \operatorname{span}\left\{\left(F^{\underset{\alpha}{\circledast} k}\right) : k \geq 0\right\}} \\
&= D_{\alpha,f} \operatorname{span}\left\{\mathbf{1}, F, F \underset{\alpha}{\circledast} F, F \underset{\alpha}{\circledast} F \underset{\alpha}{\circledast} F, \ldots\right\}.
\end{aligned}$$

Hence, by using the fact that

$$\begin{aligned}
&\operatorname{span}\left\{\mathbf{1}, F, F \underset{\alpha}{\circledast} F, F \underset{\alpha}{\circledast} F \underset{\alpha}{\circledast} F, \ldots\right\} \\
&= \operatorname{span}\{\lambda \mathbf{1} : \lambda \in \mathbb{C}\} \oplus \operatorname{span}\left\{F, F \underset{\alpha}{\circledast} F, F \underset{\alpha}{\circledast} F \underset{\alpha}{\circledast} F, \ldots\right\},
\end{aligned}$$

where $\oplus$ stands for the direct sum of subspaces, we see that

$$\begin{aligned}
&\operatorname{span}\left\{f, f \underset{\alpha}{*} f, \ldots\right\} \\
&= \operatorname{clos}\left\{D_{\alpha,f}\left(\operatorname{span}\{\lambda \mathbf{1} : \lambda \in \mathbb{C}\} \oplus \operatorname{span}\left\{F, F \underset{\alpha}{\circledast} F, \ldots\right\}\right)\right\}.
\end{aligned} \tag{2.1}$$

Since $f(\alpha) \neq 0$, by Lemma 2 the α-Duhamel operator $D_{\alpha,f}$ is invertible on the space $C^{(n)}(\Omega)$. On the other hand, by using that

$$C^{(n)}(\Omega) = \operatorname{span}\{\lambda \mathbf{1} : \lambda \in \mathbb{C}\} \oplus C^{(n)}_{\alpha}(\Omega), \tag{2.2}$$

the assertions of the theorem follow from the invertibility of the operator $D_{\alpha,f}$ and the representations (2.1) and (2.2). The theorem is proved. □

3 On the lattice of *V*-invariant subspaces in H^p

Let $\mathbb{D}$ denote the open unit disc in the complex plane $\mathbb{C}$. The Hardy space $H^p = H^p(\mathbb{D})$, $1 \leq p < \infty$, consist of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H^p}^p := \sup_{0<r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < +\infty.$$

With this norm H^p is a Banach space when $1 \le p < \infty$, while for $0 < p < 1$ it is a topological vector space with the translation invariant metric $d(f,g) = \|f-g\|_{H^p}^p$, $f, g \in H^p$, which is not locally convex. For $p = +\infty$, $H^\infty = H^\infty(\mathbb{D})$ is a Banach algebra with the norm $\|f\|_{H^\infty} := \sup\{|f(z)| : z \in \mathbb{D}\}$.

In this section, we will consider the Volterra integration operator V on the Hardy space H^p $1 \le p < +\infty$ and describe its closed nontrivial invariant subspaces. Also, we calculate the norm of orbits $(I - V_\alpha)^n$, $n = 1, 2, ...,$ on H^2.

Note that the lattice of all V_α-invariant subspaces of H^p was described by Donoghue [7] in the case when $p = 2$ and $\alpha = 0$. Donoghue's method is pure operator theory, and hardly adapted to other values of p and especially if $|\alpha| = 1$. Aleman and Korenblum [2] filled this gap. Their approach is based on classical Borel transforms of complex conjugates of H^p-functions on the unit circle $\mathbb{T} = \partial\mathbb{D}$, which is the entire function defined by

$$\widetilde{h}(\lambda) := \int_{\mathbb{T}} e_\lambda h dm,$$

where $e_\lambda(z) := e^{\lambda z}$ and $dm = \frac{|dz|}{2\pi}$ is the normalized Lebesgue measure on $\mathbb{T}$. Following [2], note that in contrast to the meagerness of results on invariant subspaces of Volterra operators in complex domains, the study of their real-variable analogs has a long history and an extensive literature (see, for instance, the survey paper of Nikolski [33]). The description of the invariant subspaces for the classical Volterra integration operator $V : L^2[0,1] \to L^2[0,1]$,

$$Vf(x) = \int_0^x f(t)dt,$$

is essentially the problem posed in 1938 by Gelfand [13] and first solved by Agmon [1] who showed that all V-invariant subspaces of $L^2[0,1]$ have the form

$$\mathcal{M}_t = \mathcal{X}_{(t,1)} L^2[0,1], \ 0 < t < 1,$$

and hence form a linearly ordered lattice, which means unicellularity of operator V. In the sequel, this result has been extended to a larger class of convolution operators by Kalish [19], Sakhnovich [37], Brodski [5] and Sarason [38] (see also [39, 40] and references therein).

In the following theorem, we give another proof of Aleman-Korenblum theorem [2] in the case when $\alpha = 0$. Our approach is based on the Duhamel product, which was used early by Nagnibida [32], Wigley [46], Tkachenko [42, 43], Dimovski [6], Raychinov [34] and Karaev [20]. The method of Duhamel products is also used in recent works of Ivanova and Melikhov [8–10]. For other applications of Duhamel products method, we refer to the works [4, 6, 11, 15–17, 21–25, 28–30, 35, 36, 41, 47].

Theorem 2 *Let* $V : H^p \to H^p$ *be an integration operator on the Hardy space* H^p $(1 \leq p < \infty)$. *Then*

$$\mathrm{Lat}(V) = \left\{E^{(n)} : n = 0, 1, 2..\right\},$$

where

$$E^{(n)} = \left\{f \in H^p : f(0) = f'(0) = ...f^{(n)}(0) = 0\right\}.$$

Proof It is easy to see that $\mathrm{Lat}(V) \supset \left\{E^{(n)} : n \geq 0\right\}$, i.e., $VE^{(n)} \subset E^{(n)}$ for all $n \geq 0$. Therefore, it remains only to show that $\mathrm{Lat}(V) \subset \left\{E^{(n)} : n \geq 0\right\}$, that is any V-invariant subspace E has the form $E^{(n)}$ for some $n \geq 0$. For this aim, we will need the following lemmas. □

Lemma 3 ([47]). *For any two functions* $f, g \in H^p$ *we have*

$$\|f \circledast g\|_\infty \leq C\|f\|_\infty \|g\|_\infty \text{ (for } p = +\infty),$$

where $C > 0$ is an absolute constant and

$$\|f \circledast g\|_p \leq C_p \|f\|_p \|g\|_p \text{ (for } 1 \leq p < +\infty) \tag{3.1}$$

for some constant $C_p > 0$, i.e., $(H^p, \circledast)$ $(1 \leq p \leq +\infty)$ is a Banach algebra.

The proof of the following lemma is contained, for instance, in Wigley's paper [47].

Lemma 4 *Let* $f \in H^p$ $(1 \leq p \leq \infty)$ *be a nonzero function. Then* f *is* $\circledast$*-invertible if and only if* $f(0) \neq 0$.

The extreme case $p = \infty$ included in Wigley's theorem [47]. So, we prove only the case $1 \leq p < +\infty$. If $f \in H^p$ is $\circledast$-invertible, then there is a function $g \in H^p$ such that

$$f \circledast g = g \circledast f = \mathbf{1}.$$

From this it is easy to see that $(f \circledast g)(0) = f(0)g(0) = 1$, whence $f(0) \neq 0$. Conversely, if $f(0) \neq 0$, then we put $F(z) := f(z) - f(0)$, and consider the Duhamel operator $D_F : H^p \to H^p$ defined by

$$D_F g = (F \circledast g)(z) = \int_0^z F'(z - t)g(t)dt = \int_0^z f'(z - t)g(t)dt \;\; (g \in H^p).$$

According to inequality (3.1), it is a bounded operator on H^p. We will show that D_F is even compact. Indeed, since F is an analytic function on the unit disc $\mathbb{D}$, we have

$$F(z)=\sum_{n=0}^{\infty}\widehat{F}(n)z^n,$$

where $\widehat{F}(n)=\frac{F^{(n)}(0)}{n!}$, $n\geq 0$. We consider the partial sum:

$$F_N:=\sum_{n=0}^{N}\widehat{F}(n)z^n=\sum_{n=1}^{N}\widehat{f}(n)z^n.$$

Then

$$\begin{aligned}D_{F_N}g(z)&=\int_0^z F_N'(z-t)g(t)dt\\&=\sum_{n=1}^{N}n!\widehat{f}(n)\int_0^z\frac{(z-t)^{n-1}}{(n-1)!}g(t)dt\\&=\sum_{n=1}^{N}n!\widehat{f}(n)V^ng(z)\end{aligned}$$

for all $g\in H^p$. Hence

$$D_{F_N}=\sum_{n=1}^{N}n!\widehat{f}(n)V^n.$$

Since, V is compact, we conclude that D_{F_N} is compact on H^p for any $N>0$. So, by (3.1), we have that

$$\|D_F-D_{F_N}\|=\|D_{F-F_N}\|\leq C_p\|F-F_N\|_p \tag{3.2}$$

for $1\leq p<+\infty$. Passing to the limit in (3.2) as $N\to\infty$, we have that D_F is a compact operator.

Now consider operator D_f (with symbol f), and assume that $g\in\ker(D_F)$, that is

$$D_fg(z)=\int_0^z f'(z-t)g(t)dt+f(0)g(z)=0,\ \forall z\in\mathbb{D}.$$

Whence $f(0)g(0)=0$, and hence $g(0)=0$, because $f(\alpha)\neq 0$. Similarly, we get

$$0=\frac{d}{dz}\left(D_fg\right)(z)=\int_0^z f''(z-t)g(t)dt+f'(0)g(z)+f(0)g'(z)$$

for all $z\in\mathbb{D}$, and evaluation at 0 gives $g'(0)=0$. By induction, we obtain that $g^{(n)}(0)=0$, $n\geq 1$, and hence $g\equiv 0$. This shows that $\ker\left(D_f\right)=\{0\}$. Since $D_f=f(0)I+D_F$, thus we deduce by Fredholm alternative that D_f is invertible in H^p. The lemma is proved.

Recall that the function $f\in H^p$ is a cyclic vector for V if

$$\operatorname{span}\{V^nf:n=0,1,2,...\}=H^p.$$

The set of all cyclic vectors of V is denoted by $\operatorname{Cyc}(V)$.

Lemma 5 *Let* $f \in H^p$. *Then* $f \in \mathrm{Cyc}\left(V \mid E^{(n)}\right)_{(n \geq 0)}$ *if and only if* $f \in E^{(n)} \setminus E^{(n+1)}$ $(n \geq 0)$.

Proof The proof of the lemma uses some arguments of the paper [39]. Let us introduce the following Duhamel product in the subspace $E^{(n)}$:

$$\left(g \overset{n}{\circledast} h\right)(z) := \frac{d}{dz} \int_0^z \frac{g(z-t)}{(z-t)^n} h(t)dt \ \left(g, h \in E^{(n)}\right), \ n = 0, 1, \ldots. \tag{3.3}$$

Clearly, for $n = 0$, the product $\overset{0}{\circledast}$ coincides with the usual Duhamel product (1.2).

Let $f \in E^{(n)}$ and $f \notin E^{(n+1)}$. Formula (3.3) implies that

$$V^k g = \frac{z^{n+k}}{k!} \overset{n}{\circledast} g, \ g \in E^{(n)}, \tag{3.4}$$

for each $k \geq 0$ and $n = 0, 1, \ldots$. Expanding function $f \in E^{(n)}$ into the Maclaurin series we have that

$$f(z) = \frac{f^{(n)}(0)}{n!} z^n + \frac{f^{(n+1)}(0)}{(n+1)!} z^{n+1} + \cdots = \frac{f^{(n)}(0)}{n!} z^n + R(z), \tag{3.5}$$

where $f^{(n)}(0) \neq 0$ and

$$R(z) := \sum_{k=n+1}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \in E^{(n)}.$$

Consider the Duhamel operator $D_{f,n}$ acting in the subspace $E^{(n)}$ by the formula $D_{f,n} g := f \overset{n}{\circledast} g$, $g \in E^{(n)}$ (see (3.3)). It follows from (3.4) and (3.5) that

$$D_{f,n} := f^{(n)}(0) I_{E^{(n)}} + D_{R,n}.$$

The same arguments, as in the proof of Lemma 4, allow us to deduce that $D_{f,n}$ is invertible in $E^{(n)}$, which is omitted. On the other hand, since

$$\mathrm{span}\left\{z^{n+k} : k \geq 0\right\} = E^{(n)},$$

according to (3.4), we obtain that

$$\begin{aligned} E_f &:= \text{span}\left\{ \left(V \mid E^{(n)}\right)^k f : k \geq 0 \right\} = \text{span}\{ V^k f : k \geq 0\} \\ &= \text{span}\left\{ \frac{z^{n+k}}{k!} \overset{n}{\circledast} f : k \geq 0 \right\} \\ &= \text{span}\left\{ D_{f,n} \frac{z^{n+k}}{k!} : k \geq 0 \right\} = \overline{D_{f,n}\text{span}\{z^{n+k} : k \geq 0\}} \\ &= \overline{D_{f,n}E^{(n)}} = E^{(n)} \text{ (because } D_{f,n} \text{ is invertible).} \end{aligned}$$

Thus, if $E^{(n)} \setminus E^{(n+1)}$, then $f \in \text{Cyc}\left(V \mid E^{(n)}\right)$.

Conversely, the equality $E_f = E^{(n)}$ implies that $f^{(n+1)}(0) \neq 0$, and hence $f \notin E^{(n+1)}$. Consequently, if $f \in E^{(n)}$ and $f \in \text{Cyc}\left(V \mid E^{(n)}\right)$, then $f \notin E^{(n+1)}$, which proves lemma. Now, we continue the proof of Theorem 2. We will prove that other V-invariant subspaces different from the chain

$$\{0\} \subset \ldots \subset E^{(n+1)} \subset E^{(n)} \subset E^{(n-1)} \subset \ldots \subset E^{(0)} \tag{3.6}$$

do not exist, and hence $\text{Lat}(V) = \left\{E^{(n)} : n = 0, 1, 2, \ldots\right\}$.

In fact, suppose in contrary that there is a nontrivial V-invariant subspace E in H^p which is different from invariant subspaces in (3.6). By virtue of the obvious representation $E = \cup_{g \in E} E_g$ and Lemmas 4 and 5, we see that there exists a function $f \in E$ such that $f(0) \neq 0$. Therefore, by Lemma 4, we deduce that $E = H^p$, which contradicts to our assumption that $\{0\} \neq E \neq H^p$.

So, according to (3.6), $\text{Lat}(V)$ is a linearly ordered set, and hence V is a unicellular operator. The theorem is proven. □

4 On orbits of $I - V_\alpha$ and inner derivation

This section is motivated mostly with the papers [31] by Montes, Sanchez and Zemanek and [26] by Leka.

4.1 Norm of orbits of $I - V_\alpha$ on H^2

In this subsection, we calculate the norm of iterates of operator $I - V_\alpha$, where $V_\alpha f = \int_\alpha^z f(t)dt$ is the Volterra integration operator on H^2. Note that the operator $V = V_0$ is the classical Volterra operator with a long history. Many aspects of the Volterra operator has been widely studied and has a vast literature. In particular, Montes, Sanches and Zemanek [31] studied the asymptotic behavior of the powers $(I - V)^n$ providing sharp estimates on the norms

$$\|(I-V)^n\|_{L^p[0,1]} \asymp n^{\left|\frac{1}{4}-\frac{1}{2p}\right|} \ (n \geq 1).$$

Their result gave a negative answer to the question of whether uniform Kreiss boundedness, in general, implies power boundedness under minimal spectral assumption. They also presented sharp estimates on the norms of the differences of consecutive powers of $I-V$, namely, they obtained that

$$\left\|(I-V)^n-(I-V)^{n+1}\right\|_{L^p[0,1]} \asymp n^{-\frac{1}{2}+\left|\frac{1}{4}-\frac{1}{2p}\right|} \ (n \geq 1).$$

This also showed that Tsedenbayar's [45] earlier result in $L^2[0,1]$ was sharp. The main goal of the paper [26] is to provide a closer look at the orbits $(I-V)^n f$ when f is in the range of Riemann-Liouville fractional integration operator V^α defined on $L^p[0,1]$ by

$$(V^\alpha f)(x)=\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s)ds, \ 0 \leq x \leq 1,$$

where Γ stands for the standard gamma function. More precisely, Leka proved in [26] that

$$\|(I-V^n)V^\alpha\|_{L^p[0,1]} \asymp n^{-\frac{\alpha}{2}+\left|\frac{1}{4}-\frac{1}{2p}\right|} \ (n \geq 1), \ \alpha > 0.$$

The proof of Leka's result in [26] is based on exploiting the earlier method of Montes, Sanches and Zemanek in [31] (see also Leka [27]) and Fejer's asymptotic formula on the Laguerre polynomials. We also note that recently new results and estimates on orbits of operators which are commuting with the Volterra operator have been presented in [3] by Bermudo, Montes and Shkarin. For more details, see [26].

Recall that the Hardy space $H^2 = H^2(\mathbb{D})$ is the Hilbert space of analytic functions

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \text{ such that } \|f\|_2 := \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{1/2} < +\infty.$$

Proposition 1 *Fix an $\alpha \in \mathbb{D}$. Let V_α be the Volterra integration operator on H^2 defined by $V_\alpha f(z) = \int_\alpha^z f(t)dt$. Then*

$$\|(I-V_\alpha)^n\| = \left[\sum_{k=0}^{n}\left|\sum_{j=0}^{k}(-1)^j \alpha^j \frac{n!}{k!(n-k)!j!(k-j)!}\right|^2\right]^{\frac{1}{2}}.$$

Proof The α-Duhamel product is defined by

$$\left(f_1 \underset{\alpha}{\circledast} f_2\right)(z) = \frac{d}{dz}\int_{\alpha}^{z} f_1(z+\alpha-t)f_2(t)dt$$
$$= \int_{\alpha}^{z} f_1'(z+\alpha-t)f_2(t)dt + f_1(\alpha)f_2(z),\ f_1, f_2 \in H^2. \tag{4.1}$$

It is easy to see from (4.1) that $\mathbf{1}\underset{\alpha}{\circledast}f = f\underset{\alpha}{\circledast}\mathbf{1} = f$, for all $f \in H^2$ and

$$V_\alpha^n f = \frac{(z-\alpha)^n}{n!}\underset{\alpha}{\circledast} f,\ \forall f \in H^2. \tag{4.2}$$

The methods of the proofs in [46] and [47] show in particular that $\left(H^2, \underset{\alpha}{\circledast}\right)$ is a Banach algebra (with respect to some equivalent norm of H^2). So, it follows from (4.2) that

$$\|V_\alpha^n f\|_2 \leq \frac{\|(z-\alpha)^n\|_2}{n!}\|f\|_2,\ \forall f \in H^2,\ \forall n \geq 0, \tag{4.3}$$

which implies that $\|V_\alpha^n\| \leq \frac{\|(z-\alpha)^n\|_2}{n!}$, $\forall n \geq 0$. On the other hand, $\|V_\alpha^n \mathbf{1}\|_2 = \left\|\frac{(z-\alpha)^n}{n!} \circledast \mathbf{1}\right\|_2 = \frac{\|(z-\alpha)^n\|_2}{n!}$, and hence,

$$\|V_\alpha^n\| = \frac{1}{n!}\|(z-\alpha)^n\|_2, \forall n \geq 0. \tag{4.4}$$

Similarly, we have

$$\|(I - V_\alpha)^n\| = \left\|(\mathbf{1}-(z-\alpha))^{\underset{\alpha}{\circledast} n}\right\|_2 = \left\|\sum_{k=0}^{n} C_n^k \left(\mathbf{1}^{\underset{\alpha}{\circledast}(n-k)} \underset{\alpha}{\circledast} (z-\alpha)^{\underset{\alpha}{\circledast} k}\right)\right\|_2$$
$$= \left\|\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}(z-\alpha)^{\underset{\alpha}{\circledast} k}\right\|_2 = \left\|\sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\frac{(z-\alpha)^k}{k!}\right\|_2$$
$$= \left\|\sum_{k=0}^{n} \frac{n!}{(k!)^2(n-k)!}\sum_{j=0}^{k} C_k^j z^{k-j}(-1)^j \alpha^j\right\|_2$$
$$= \left\|\sum_{k=0}^{n} \frac{n!}{(k!)^2(n-k)!}\sum_{k=0}^{n} \frac{k!}{(j!)(k-j)!}(-1)^j \alpha^j z^{k-j}\right\|_2$$
$$= \left\|\sum_{k=0}^{n}\sum_{j=0}^{k} \frac{n!(-1)^j \alpha^j}{k!(n-k)!j!(k-j)!} z^{k-j}\right\|_2$$
$$= \left[\sum_{k=0}^{n}\left|\sum_{j=0}^{k} \frac{(-1)^j \alpha^j n!}{k!(n-k)!j!(k-j)!}\right|^2\right]^{\frac{1}{2}},$$

as desired. □

Corollary 1 $\|(I-V)^n\| = \left[\sum_{k=0}^{n}\left(\frac{n!}{(k!)^2(n-k)!}\right)^2\right]^{\frac{1}{2}}.$

The following is immediate from Corollary 1.

Corollary 2 $\|(I-V)^n\| \geq \left(\sum_{k=0}^{n}\frac{1}{(k!)^4}\right)^{\frac{1}{2}}.$

Proposition 2 *We have :*

$$\left\|(I-V_\alpha)^n-(I-V_\alpha)^{n+1}\right\| = \left\|\sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^j\alpha^j\frac{n!}{k!(n-k)!j!(k-j)!}z^{k-j} - \sum_{k=0}^{n+1}\sum_{j=0}^{k}(-1)^j\alpha^j\frac{(n+1)!}{k!(n+1-k)!j!(k-j)!}z^{k-j}\right\|_2.$$

The proof is quite similar to the proof Proposition 1, and therefore it is omitted.

4.2 A lower estimate for the norm of orbits of inner derivation operator

We consider the inner derivation operator on $\mathcal{B}(C^{(m)}(\Omega))$ and estimate the norm of its orbit. Let $A\in\mathcal{B}(C^{(m)}(\Omega))$. The inner derivation operator δ_A is defined on $\mathcal{B}(C^{(m)}(\Omega))$ by the formula

$$\delta_A(X) := [X,A] = XA - AX,\ X\in\mathcal{B}\left(C^{(m)}(\Omega)\right).$$

It is elementary that $\|\delta_A\|\leq 2\|A\|$. Here, we will prove in terms of α-Duhamel product a lower estimate for the orbits δ_A^n, $n=2,3,\ldots$, which improve a result in [18].

Proposition 3 *Let* $A\in\mathcal{B}(C^{(m)}(\Omega))$ *be fixed. Suppose that for every* $n\geq 1$ *and* $X\in\mathcal{B}(C^{(m)}(\Omega))$ *there exists* $f_{n,X}\in C^{(m)}(\Omega)$ *such that*

$$\left(\delta_A^n(X)f_{n,X}\right)(\alpha)\neq 0.$$

Then

$$\sup_{\|X\|\leq 1}\frac{1}{\left\|F_{n,X}\right\|_m}\leq\left\|\delta_A^n\right\|\leq 4\|A\|^n(n\geq 1).$$

Proof Since $\|\delta_A\|\leq 2\|A\|$, the inequality $\|\delta_A\|\leq 4\|A\|$ is trivial. Further, we have

$$\begin{aligned}\left\|\delta_A^2(X)\right\| &= \|[[X,A],A]\| = \|[X,A]A - A[X,A]\| \\ &= \left\|XA^2 - 2AXA + A^2X\right\| \\ &\le 4\|A\|^2\|X\| \text{ for all } X \in \mathcal{B}\left(C^{(m)}(\Omega)\right).\end{aligned}$$

Hence $\left\|\delta_A^2\right\| \le 4\|A\|^2$. Similarly,

$$\begin{aligned}\delta_A^3(X) &= [[[X,A],A],A] \\ &= XA^3 - 3AXA^2 - A^2XA - A^3X,\end{aligned}$$

which implies that $\left\|\delta_A^3\right\| \le 4\|A\|^3$. By induction, we conclude that

$$\left\|\delta_A^n\right\| \le 4\|A\|^n, \text{ for all } n \ge 1,$$

as desired.

Now we prove the lower inequality. According to condition, for every $n \ge 1$ and $X \in \mathcal{B}\left(C^{(m)}(\Omega)\right)$ there exists $f_{n,X} \in C^{(m)}(\Omega)$ such that

$$\left(\delta_A^n(X)f_{n,X}\right)(\alpha) \neq 0. \tag{4.5}$$

Denote $g_{n,X} := \delta_A^n(X)f_{n,X}$. Since $g_{n,X}(\alpha) \neq 0$, by Lemma 2, there exists a unique element $G_{n,X} \in C^{(m)}(\Omega)$ such that

$$G_{n,X} \underset{\alpha}{\circledast} g_{n,X} = g_{n,X} \underset{\alpha}{\circledast} G_{n,X} = \mathbf{1}.$$

Hence, $f_{n,X} \underset{\alpha}{\circledast} G_{n,X} \underset{\alpha}{\circledast} g_{n,X} = f_{n,X}$. We set

$$F_{n,X} := f_{n,X} \underset{\alpha}{\circledast} G_{n,X}.$$

So, it follows from (4.5) that

$$\left(D_{\alpha,F_{n,X}} \delta_A^n(X)\right) f_{n,X} = f_{n,X},$$

which implies that $f_{n,X}$ is an eigenvector of operator $D_{\alpha,F_{n,X}} \delta_A^n(X)$ corresponding to the eigenvalue $1 \in \sigma_p\left(D_{\alpha,F_{n,X}} \delta_A^n(X)\right)$. Then, we obtain that

$$\begin{aligned}1 &\le r\left(D_{\alpha,F_{n,X}} \delta_A^n(X)\right) \le \left\|D_{\alpha,F_{n,X}} \delta_A^n(X)\right\| \\ &\le \left\|D_{\alpha,F_{n,X}}\right\| \left\|\delta_A^n(X)\right\| = \left\|F_{n,X}\right\|_m \left\|\delta_A^n(X)\right\|_{\mathcal{B}(C^{(m)}(\Omega))},\end{aligned}$$

where $r(.)$ denotes the spectral radius of operator. Whence

$$\frac{1}{\left\|F_{n,X}\right\|_m} \leq \left\|\delta_A^n(X)\right\|.$$

By taking supremum over the operators $X \in \mathcal{B}\left(C^{(m)}(\Omega)\right)$ with $\|X\| \leq 1$, we have from this inequality that

$$\sup_{\|X\| \leq 1} \frac{1}{\left\|F_{n,X}\right\|_m} \leq \sup_{\|X\| \leq 1} \left\|\delta_A^n(X)\right\| = \left\|\delta_A^n\right\|.$$

This proves the proposition. □

Data availability Data sharing not applicable to this paper as no data sets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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