The Journal of Analysis (2023) 31:1355–1373
https://doi.org/10.1007/s41478-022-00517-8

ORIGINAL RESEARCH PAPER

Viscosity-type extragradient algorithm for finding common solution of pseudomonotone equilibrium problem and fixed point problem in Hilbert space

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Received: 31 July 2022 / Accepted: 11 October 2022 / Published online: 31 October 2022

Abstract

The primary goal of this research is to find a common solution to the equilibrium problem for pseudomonotone bi-functions satisfying the Lipschitz-type condition as well as the fixed point problem for $\psi-$strongly quasi-nonexpansive mappings in the context of real Hilbert space by combining two different approaches. A viscosity-type extragradient algorithm is presented for solving the problems listed above. Furthermore, with a set of reasonable assumptions, a strong convergence theorem is presented. The fundamental advantage of the suggested approach is that it does not require the use of a linesearch procedure or the knowledge of Lipschitz-type constants in advance, which is a significant advantage. Moreover, we give a numerical example to support and justify our proposed algorithm. In this sense, the findings of this study generalise and extend certain previously published findings.

Keywords Equilibrium problem · $\psi-$Strongly extragradient method · Viscosity approximation method · Pseudomonotone bi-functions · Lipschitz-type condition

Mathematics Subject Classification 47H05 · 47H10 · 47J25

Communicated by S Ponnusamy.

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1 Introduction

Throughout the paper, we let $\mathcal{H}$ be a real Hilbert space and $\mathcal{K}$ be a non-empty subset of $\mathcal{H}$ which is closed and convex.

Recall that in a fixed point problem one needs to find a point $\mathfrak{z} \in \mathcal{K}$ in such a way that

$$S\mathfrak{z} = \mathfrak{z}, \tag{1.1}$$

where $S : \mathcal{K} \to \mathcal{H}$ be a mapping. We indicate the solution set of problem (1.1) by $\Lambda = \{\mathfrak{z} \in \mathcal{K} : S\mathfrak{z} = \mathfrak{z}\}$. Many researchers have studied problem (1.1) and have established various iterative methods to tackle it; see for example [5, 9, 11]. In 2000, Moudafi [25] considered problem (1.1) and proposed well known viscosity approximation method for finding a solution of problem (1.1) as follows: Take $u_0 \in \mathcal{H}$, and formulate an iterative sequence $\{u_n\}$ as follows:

$$u_{n+1} = \psi_n \phi(u_n) + (1 - \psi_n) S u_n, \quad n \geq 0, \tag{1.2}$$

where $\phi : \mathcal{H} \to \mathcal{H}$ is a contraction map and sequence $\{\psi_n\} \in (0, 1)$. He demonstrated that the sequence formulated by (1.2) converges strongly to a unique solution $\mathfrak{z} \in \mathcal{K}$.

On the other hand, a problem in which one needs to find an element $\mathfrak{z} \in \mathcal{K}$ in such a way that

$$g(\mathfrak{z}, v) \geq 0, \quad \forall\, v \in \mathcal{K}, \tag{1.3}$$

where $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a real valued nonlinear bi-function with $g(\mathfrak{z}, \mathfrak{z}) = 0$ for all $\mathfrak{z} \in \mathcal{K}$. The problem (1.3), was first suggested by Fan [15] and further established by Blum and Oettli [2]. Problem (1.3) is now known as equilibrium problem. The solution set of problem (1.3) is represented by $\Gamma = \{\mathfrak{z} \in \mathcal{K} : g(\mathfrak{z}, v) \geq 0, \forall\, v \in \mathcal{K}\}$. Many problems such as medical imaging problems, transportation problems, and financial engineering problems can be converted to find solution of problem (1.3), see, for example [14, 21, 28, 29] and the references therein.

In recent years, many iterative algorithms for solving the problem (1.3) have been developed, including the proximal point algorithm (TPPA) [12, 13], the normal S-iteration algorithm [20] the subgradient algorithm (TSA) [3], the extragradient algorithm (TEA) [17], subgradient extragradient algorithm [10, 19] and the gap function algorithm (TGFA) [24]. The explicit extragradient algorithm (TEEA) for solving problem (1.3) for pseudomonotone bi-functions satisfying Lipschitz-type condition (LTC) in real Hilbert space was introduced by Hieu et al. [30] in 2019 which is defined as following. Choose $u_0 \in \mathcal{K}$ and $\tau_0 > 0$, $\eta \in (0, 1)$, compute the sequences $\{w_n\}$ and $\{u_{n+1}\}$ by

$$w_n = \arg\min_{v\in\mathcal{K}} \{g(u_n, v) + \frac{1}{2\tau_n}\|u_n - v\|^2\},$$
$$u_{n+1} = \arg\min_{v\in\mathcal{K}} \{g(w_n, v) + \frac{1}{2\tau_n}\|u_n - v\|^2\}, \tag{1.4}$$

where the step size τ_n is given as

$$\tau_{n+1} = \min\left\{\tau_n, \frac{\eta(\|u_n - w_n\|^2 + \|u_{n+1} - w_n\|^2)}{2\max\{0, g(u_n, u_{n+1}) - g(u_n, w_n) - g(w_n, u_{n+1})\}}\right\}.$$

They proved the sequence $\{u_n\}$ generated by (1.4) converges weakly to some point $\mathfrak{z} \in \Gamma$.

In this paper, we consider a problem of approximating a common solution of equilibrium problem for pseudomonotone bi-function satisfying Lipschitz-type condition (LTC) and fixed point problem for $\psi-$strongly quasi-nonexpansive mappings in real Hilbert space. i.e., Find $\mathfrak{z} \in \mathcal{K}$ such that

$$\mathfrak{z} \in \Omega := \Gamma \cap \Lambda. \tag{1.5}$$

Inspired and motivated by the work in [25] and Hieu et al. [30], the main goal of this paper is to present a viscosity-type extragradient algorithm which is a combination of extragradient method and viscosity approximation method with a new step size rule for solving problem (1.5) and discuss its convergence analysis. The fundamental advantage of the suggested approach is that it does not require the use of a linesearch procedure or the knowledge of Lipschitz-type constants in advance, which is a significant advantage. In this sense, the findings of this study generalise and extend certain previously published findings.

The following is how this paper is organised: In Sect. 2, we review some of the fundamental definitions and auxiliary results that were used throughout the paper. Our suggested algorithm and its convergence are presented in Sect. 3, and some consequences of our primary findings are discussed in Sect. 4. Moreover, we give a numerical example to support and justify our proposed algorithm in the last section.

2 Preliminaries

Let the inner product and induced norm equipped in Hilbert space $\mathcal{H}$ are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. These convergences are represented by $\rightharpoonup$ and $\rightarrow$ symbols, respectively, when the sequence $\{u_n\} \subset \mathcal{H}$ converges weakly and strongly. We start with some definitions about the monotonicity of bi-function $g : \mathcal{K} \times \mathcal{K} \rightarrow \mathbb{R}$:

Definition 2.1 [2, 16, 26] The bi-function g is said to be

(i) $\gamma-$strongly monotone on $\mathcal{K}$ if there exists $\gamma > 0$ such that

$$g(u, v) + g(v, u) \leq -\gamma\|u - v\|^2, \quad \forall u, v \in \mathcal{K};$$

(ii) monotone if

$$g(u,v)+g(v,u)\leq 0, \quad \forall\, u,v\in\mathcal{K};$$

(iii) $\gamma-$strongly pseudomonotone on $\mathcal{K}$ if there exists $\gamma>0$ such that

$$g(u,v)\geq 0\Rightarrow g(v,u)\leq -\gamma\|u-v\|^2, \quad \forall\, u,v\in\mathcal{K};$$

(iv) pseudomonotone if

$$g(u,v)\geq 0\Rightarrow g(v,u)\leq 0, \quad \forall\, u,v\in\mathcal{K};$$

(v) satisfying the Lipschitz-type condition (LTC) on $\mathcal{K}$ if there exists two positive real numbers λ_1,λ_2 such that

$$g(u,w)\leq g(u,v)+g(v,w)+\lambda_1\|u-v\|^2+\lambda_2\|v-w\|^2, \quad \forall\, u,v,w\in\mathcal{K}.$$

Definition 2.2 [18] The metric projection $P_{\mathcal{K}}(u)$ of u onto a closed, convex subset $\mathcal{K}$ of $\mathcal{H}$ is defined as follows:

$$P_{\mathcal{K}}(u)=\arg\min_{v\in\mathcal{K}}\{\|v-u\|\}.$$

Lemma 2.1 [22] *Let $P_{\mathcal{K}}(u):\mathcal{H}\to\mathcal{K}$ be the metric projection from $\mathcal{H}$ onto $\mathcal{K}$. Then*

(i) $\|u-P_{\mathcal{K}}(v)\|^2+\|P_{\mathcal{K}}(v)-v\|^2\leq\|u-v\|^2, \quad \forall u\in\mathcal{K},\ v\in\mathcal{H},$
(ii) $w=P_{\mathcal{K}}(u)\iff\langle u-w,v-w\rangle\leq 0, \quad \forall\, v\in\mathcal{K}.$

Lemma 2.2 [18] *Suppose that $S:\mathcal{H}\to\mathcal{H}$ is a nonlinear mapping. Then $I-S$ is said to be demiclosed at zero if for any $\{u_n\}\in\mathcal{H}$, the following holds*:

$$u_n\rightharpoonup\mathfrak{z} \text{ and } (I-S)u_n\to 0\Rightarrow\mathfrak{z}\in\Lambda.$$

Definition 2.3 Let $S:\mathcal{H}\to\mathcal{H}$ be a mapping with $\Lambda\neq\emptyset$. Then $S:\mathcal{H}\to\mathcal{H}$ is said to be

(i) firmly nonexpansive if

$$\|Su-Sv\|^2\leq\langle Su-Sv,u-v\rangle, \quad \forall\, u,v\in\mathcal{H},$$

or comparatively

$$\|Su-Sv\|^2\leq\|u-v\|^2-\|(I-S)u-(I-S)v\|^2, \quad \forall\, u,v\in\mathcal{H},$$

(ii) directed if

$$\langle w-Su,u-Su\rangle\leq 0, \quad \forall\, w\in\Lambda,\ u\in\mathcal{H},$$

or comparatively

$$\|Su - w\|^2 \leq \|u - w\|^2 - \|u - Su\|^2, \quad \forall w \in \Lambda,\, u \in \mathcal{H},$$

(iii) $\psi-$strongly quasi-nonexpansive with $\psi > 0$ if

$$\|Su - w\|^2 \leq \|u - w\|^2 - \psi\|u - Su\|^2, \quad \forall w \in \Lambda,\, u \in \mathcal{H},$$

or comparatively

$$\langle Su - u, u - w\rangle \leq \frac{-1-\psi}{2}\|u - Su\|^2, \quad \forall w \in \Lambda,\, u \in \mathcal{H},$$

(iv) quasi-nonexpansive

$$\|Su - w\| \leq \|u - w\|, \quad \forall w \in \Lambda,\, u \in \mathcal{H},$$

(v) $\beta-$demicontractive with $\beta \in [0,\, 1)$

$$\|Su - w\|^2 \leq \|u - w\|^2 + \beta\|u - Su\|^2, \quad \forall w \in \Lambda,\, u \in \mathcal{H},$$

or comparatively

$$\langle u - w, Su - u\rangle \leq \frac{\beta - 1}{2}\|u - Su\|^2, \quad \forall w \in \Lambda,\, u \in \mathcal{H}. \tag{2.1}$$

Recall that the proximal mapping $\text{prox}_{\tau g_1}$ is defined by

$$\text{prox}_{\tau g_1}(u) = \arg\min\{g_1(v) + \frac{1}{2\tau}\|u - v\|^2 : v \in \mathcal{K}\},$$

where $g_1 : \mathcal{K} \to \mathbb{R}$ with a parameter $\tau > 0$ is a proper, convex and lower semicontinuous function .

One can observe the following property of the proximal mapping $\text{prox}_{\tau g_1}$:

Lemma 2.3 [1] *For all $u \in \mathcal{H}$, $v \in \mathcal{K}$ and $\tau > 0$, the following implication holds*:

$$\tau\{g_1(v) - g_1(\text{prox}_{\tau g_1}(u))\} \geq \langle u - \text{prox}_{\tau g_1}(u),\, v - \text{prox}_{\tau g_1}(u)\rangle.$$

Remark 2.1 If $u = \text{prox}_{\tau g_1}(u)$ then

$$u \in \arg\min\{g_1(v) : v \in \mathcal{K}\} := \{u \in \mathcal{K} : g_1(u) = \min_{v \in \mathcal{K}} g_1(v)\}.$$

Lemma 2.4 [23] *Let a sequence $\{b_n\} \subset \mathbb{R}$ such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $b_{n_i} \leq b_{n_{i+1}}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_l\} \subset \mathbb{N}$ such that $m_l \to \infty$ and the following properties are satisfied by all sufficiently large numbers $l \in \mathbb{N}$:*

$$b_{m_l} \leq b_{m_l+1} \text{ and } b_l \leq b_{m_l+1}.$$

In fact, $m_l := \max\{j \leq l : b_j \leq b_{j+1}\}$.

Lemma 2.5 [27, 31] *Let* $\{b_n\}$ *be a sequence of non negative real numbers such that*

$$b_{n+1} \leq (1-\psi_n)b_n + \psi_n\delta_n, \quad \forall n \geq 0,$$

where $\psi_n \in (0, 1)$ and $\delta_n \subset \mathbb{R}$ satisfies the following conditions:

(i) $\sum_{n=0}^{\infty} \psi_n = \infty$;
(ii) $\lim_{n\to\infty} \sup \delta_n \leq 0$. Then $\lim_{n\to\infty} b_n = 0$.

Lemma 2.6 [1] *For every* $u, v \in \mathcal{H}$ *and* $\psi \in \mathbb{R}$, *the following relations are true*:

(i) $\|\psi u + (1-\psi)v\|^2 = \psi\|u\|^2 + (1-\psi)\|v\|^2 - \psi(1-\psi)\|u-v\|^2$,
(ii) $\|u+v\|^2 \leq \|u\|^2 + 2\langle v, u+v\rangle$.

Assumption 2.1 [30] Let a bi-function $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ satisfies the following conditions:

G1: g is pseudomontone on a feasible set $\mathcal{K}$ and for all $u \in \mathcal{K}$, $g(u; u) = 0$;
G2: g satisfy the Lipschitz-type condition (LTC) on $\mathcal{H}$ with positive constants λ_1 and λ_2;
G3: $\lim_{n\to\infty} \sup g(u_n, v) \leq g(\mathfrak{z}, v)$ for every $v \in \mathcal{K}$ and $\{u_n\} \subset \mathcal{K}$ satisfy $u_n \rightharpoonup \mathfrak{z}$;
G4: $g(u, \cdot)$ is convex and subdifferentiable on $\mathcal{K}$ for every $u \in \mathcal{K}$.

3 Main result

In this section, we provide our main algorithm and discuss its convergence analysis under some mild assumptions. Let $S : \mathcal{K} \to \mathcal{H}$ be a $\psi-$strongly quasi-nonexpansive operator such that $I - S$ is demiclosed at zero. Suppose that $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ be a bi-function satisfying Assumptions 2.1 and $\phi : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with constant $\xi \in [0, 1)$. The following is the main algorithm that has been presented:

Algorithm 1 (*A Viscosity-type Extragradient Algorithm*)
Initialization: Choose $u_0 \in \mathcal{K}$ and $\tau_0 > 0$, $\eta \in (0, 1)$. Let sequence $\{\psi_n\} \in (0, 1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \psi_n = 0 \quad \text{and} \quad \sum_{n=0}^{\infty} \psi_n = \infty. \tag{3.1}$$

Iterative steps: Given u_n and τ_n are known for $n \geq 0$.
Step 1: Compute

$$w_n = \arg\min_{v\in\mathcal{K}} \{g(u_n, v) + \frac{1}{2\tau_n}\|u_n - v\|^2\}.$$

If $u_n = w_n$; STOP. Otherwise go to step 2.

Step 2: Compute

$$v_n = \arg\min_{v\in\mathcal{K}} \{g(w_n, v) + \frac{1}{2\tau_n}\|u_n - v\|^2\} \quad \text{and}$$
$$u_{n+1} = \psi_n\phi(u_n) + (1-\psi_n)Sv_n.$$

and set

$$\tau_{n+1} = \min\left\{\tau_n, \frac{\eta(\|u_n - w_n\|^2 + \|v_n - w_n\|^2)}{2\max\{0, g(u_n, v_n) - g(u_n, w_n) - g(w_n, v_n)\}}\right\}. \tag{3.2}$$

Set $n := n + 1$ and return back to **Iterative steps.**

Remark 3.1 Under the Assumption 2.1 (G2), there exist positive constants λ_1 & λ_2 such that

$$\begin{aligned} g(u_n, v_n) - g(u_n, w_n) - g(w_n, v_n) &\le \lambda_1\|u_n - w_n\|^2 + \lambda_2\|v_n - w_n\|^2 \\ &\le \max\{\lambda_1, \lambda_2\}(\|u_n - w_n\|^2 + \|v_n - w_n\|^2). \end{aligned}$$

Thus, from the definition of the sequence $\{\tau_n\}$, this sequence is bounded from below by $\left\{\tau_0, \frac{\eta}{2\max\{\lambda_1, \lambda_2\}}\right\}$. Moreover, the sequence $\{\tau_n\}$ is non-increasing monotone. Thus, there exists $\tau \in \mathbb{R}$ such that $\lim_{n\to\infty} \tau_n = \tau$. In fact, from (3.2), if $g(u_n, v_n) - g(u_n, w_n) - g(w_n, v_n) \le 0$ than $\tau_{n+1} := \tau_n$.

Consequently, we have the following outcomes:

Theorem 3.1 *Let a bi-function $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ satisfying the Assumptions 2.1. Thus, for each $\mathfrak{z} \in \Omega := \Gamma \cap \Lambda \neq \emptyset$, we have*

$$\|v_n - \mathfrak{z}\|^2 \le \|u_n - \mathfrak{z}\|^2 - \left(1 - \frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n - w_n\|^2 - \|v_n - w_n\|^2\right). \tag{3.3}$$

Proof In view of Lemma 2.3 and the definition of sequence $\{v_n\}$ that

$$\langle u_n - v_n, v_n - v\rangle \ge \tau_n g(w_n, v_n) - \tau_n g(w_n, v), \quad \forall v \in \mathcal{K}. \tag{3.4}$$

From the equation (3.2), we obtain

$$g(u_n, v_n) - g(u_n, w_n) - g(w_n, v_n) \le \frac{\eta\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right)}{2\tau_{n+1}},$$

which after multiplying both sides by $\tau_n > 0$, implies that

$$\tau_n g(w_n, v_n) \geq \tau_n \big(g(u_n, v_n) - g(u_n, w_n)\big) - \frac{\eta\tau_n\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right)}{2\tau_{n+1}}, \quad (3.5)$$

combining relations (3.4) and (3.5), we obtain

$$\begin{aligned}\langle u_n - v_n, v_n - v\rangle \geq \tau_n\{g(u_n, v_n) - g(u_n, w_n)\} \\ - \frac{\eta\tau_n}{2\tau_{n+1}}\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right) - \tau_n g(w_n, v).\end{aligned} \quad (3.6)$$

Similarly, from Lemma 2.3 and the definition of the sequence $\{w_n\}$, we also obtain

$$\tau_n\big(g(u_n, v_n) - g(u_n, w_n)\big) \geq \langle w_n - u_n, w_n - v_n\rangle. \quad (3.7)$$

From the relations (3.6) and (3.7), we obtain

$$\begin{aligned}\langle u_n - v_n, v_n - v\rangle \geq \langle w_n - u_n, w_n - v_n\rangle \\ - \frac{\eta\tau_n}{2\tau_{n+1}}\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right) - \tau_n g(w_n, v).\end{aligned} \quad (3.8)$$

Thus, by multiplying both sides of relation (3.8) by 2, we obtain

$$\begin{aligned}2\langle u_n - v_n, v_n - v\rangle \geq 2\langle w_n - u_n, w_n - v_n\rangle \\ - \frac{\eta\tau_n}{\tau_{n+1}}\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right) - 2\tau_n g(w_n, v).\end{aligned} \quad (3.9)$$

We have the following equalities:

$$2\langle u_n - v_n, v_n - v\rangle = \|u_n - v\|^2 - \|v_n - u_n\|^2 - \|v_n - v\|^2; \quad (3.10)$$

$$2\langle w_n - u_n, w_n - v_n\rangle = \|u_n - w_n\|^2 + \|v_n - w_n\|^2 - \|u_n - v_n\|^2. \quad (3.11)$$

Combining the relations (3.9), (3.10) and (3.11), we obtain

$$\begin{aligned}\|v_n - v\|^2 \leq \|u_n - v\|^2 - \left(1 - \frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n - w_n\|^2 + \|v_n - w_n\|^2\right) \\ - 2\tau_n g(w_n, v), \quad \forall\, v \in \mathcal{K}, \forall\, n \geq 0.\end{aligned} \quad (3.12)$$

For each $\mathfrak{z} \in \Gamma$, we have that $g(\mathfrak{z}, w_n) \geq 0$ and by Assumptions 2.1 (G1) that $g(w_n, \mathfrak{z}) \leq 0$. Then using $v = \mathfrak{z} \in \mathcal{K}$ in relation (3.12), we obtain

$$\|v_n - \mathfrak{z}\|^2 \leq \|u_n - \mathfrak{z}\|^2 - \left(1 - \frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n - w_n\|^2 - \|v_n - w_n\|^2\right), \quad \forall\, \mathfrak{z} \in \mathcal{K}, \forall\, n \geq 0.$$

□

Theorem 3.2 *Let a bi-function $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ satisfying Assumptions 2.1. Thus, for each $\mathfrak{z} \in \Omega := \Gamma \cap \Lambda \neq \emptyset$, the sequence $\{u_n\}$ generated by Algorithm 1 is bounded.*

Proof It is given that $\mathfrak{z} \in \Omega$. Since $\lim_{n\to\infty} \tau_n = \tau > 0$,

$$\lim_{n\to\infty}\left(1-\frac{\eta\tau_n}{\tau_{n+1}}\right)=1-\eta>0.$$

Thus, there exists $n_0\geq 1$ such that

$$1-\frac{\eta\tau_n}{\tau_{n+1}}>0, \quad \forall n\geq n_0. \tag{3.13}$$

From the Theorem 3.1 and relation (3.13), we obtain

$$\|v_n-\mathfrak{z}\|^2\leq\|u_n-\mathfrak{z}\|^2. \tag{3.14}$$

From the definition of $\{u_{n+1}\}$ and due to the fact that ϕ is a contraction with $\xi\in[0,1)$, we have

$$\begin{aligned}\|u_{n+1}-\mathfrak{z}\|&=\|\psi_n\phi(u_n)+(1-\psi_n)Sv_n-\mathfrak{z}\|\\&=\|\psi_n(\phi(u_n)-\mathfrak{z})+(1-\psi_n)(Sv_n-\mathfrak{z})\|\\&\leq\psi_n\|\phi(u_n)-\mathfrak{z}\|+(1-\psi_n)\|Sv_n-\mathfrak{z}\|\\&\leq\psi_n\|\phi(u_n)-\phi(\mathfrak{z})\|+\psi_n\|\phi(\mathfrak{z})-\mathfrak{z}\|+(1-\psi_n)\|v_n-\mathfrak{z}\|\\&\leq\psi_n\xi\|u_n-\mathfrak{z}\|+\psi_n\|\phi(\mathfrak{z})-\mathfrak{z}\|+(1-\psi_n)\|v_n-\mathfrak{z}\|.\end{aligned} \tag{3.15}$$

Combining relations (3.1), (3.13) and (3.15), we obtain

$$\begin{aligned}\|u_{n+1}-\mathfrak{z}\|&\leq\psi_n\xi\|u_n-\mathfrak{z}\|+\psi_n\|\phi(\mathfrak{z})-\mathfrak{z}\|+(1-\psi_n)\|u_n-\mathfrak{z}\|\\&=(1-\psi_n+\psi_n\xi)\|u_n-\mathfrak{z}\|+\psi_n(1-\xi)\frac{\|\phi(\mathfrak{z})-\mathfrak{z}\|}{1-\xi}\\&\leq\max\left\{\|u_n-\mathfrak{z}\|,\frac{\|\phi(\mathfrak{z})-\mathfrak{z}\|}{1-\xi}\right\},\end{aligned}$$

continuing in the same way, we obtain

$$\|u_{n+1}-\mathfrak{z}\|\leq\max\left\{\|u_0-\mathfrak{z}\|,\frac{\|\phi(\mathfrak{z})-\mathfrak{z}\|}{1-\xi}\right\}.$$

Thus, we conclude that the sequence $\{u_n\}$ is bounded. □

Theorem 3.3 *Let a bi-function $g:\mathcal{K}\times\mathcal{K}\to\mathbb{R}$ satisfying Assumptions 2.1. Thus, for each $\mathfrak{z}\in\Omega:=\Gamma\cap\Lambda\neq\emptyset$, the sequence $\{u_n\}$ generated by Algorithm 1 converges strongly to $\mathfrak{z}$, where $\mathfrak{z}=P_\Omega\phi(\mathfrak{z})$.*

Proof By using Lemma 2.1 (ii), we have

$$\langle\phi(\mathfrak{z})-\mathfrak{z},v-\mathfrak{z}\rangle\leq 0, \quad \forall v\in\Gamma. \tag{3.16}$$

By Lemma 2.6 (i) and Theorem 3.1, we obtain

$$
\begin{aligned}
\|u_{n+1}-\mathfrak{z}\|^2 &= \|\psi_n\phi(u_n)+(1-\psi_n)Sv_n-\mathfrak{z}\|^2 \\
&= \|\psi_n(\phi(u_n)-\mathfrak{z})+(1-\psi_n)(Sv_n-\mathfrak{z})\|^2 \\
&= \psi_n\|\phi(u_n)-\mathfrak{z}\|^2+(1-\psi_n)\|Sv_n-\mathfrak{z}\|^2-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2 \\
&\leq \psi_n\|\phi(u_n)-\mathfrak{z}\|^2+(1-\psi_n)\|v_n-\mathfrak{z}\|^2-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2 \\
&\leq \psi_n\|\phi(u_n)-\mathfrak{z}\|^2+(1-\psi_n)\left[\|u_n-\mathfrak{z}\|^2-\left(1-\frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n-w_n\|^2\right.\right. \\
&\quad \left.\left. +\|v_n-w_n\|^2\right)\right]-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2 \\
&\leq \psi_n\|\phi(u_n)-\mathfrak{z}\|^2+(1-\psi_n)\|u_n-\mathfrak{z}\|^2-(1-\psi_n)\left(1-\frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n-w_n\|^2\right. \\
&\quad \left. +\|v_n-w_n\|^2\right)-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2.
\end{aligned} \tag{3.17}
$$

The rest of the proof shall be divided into two cases:

Case I: Assume that there is a fixed number $N_1 \in \mathbb{N}$ such that

$$\|u_{n+1}-\mathfrak{z}\| \leq \|u_n-\mathfrak{z}\|, \quad \forall n \geq N_1. \tag{3.18}$$

Thus, above relation implies that $\lim_{n\to\infty}\|u_n-\mathfrak{z}\|$ exists and let $\lim_{n\to\infty}\|u_n-\mathfrak{z}\|=l$. From (3.17), we obtain

$$
\begin{aligned}
&(1-\psi_n)\left(1-\frac{\eta\tau_n}{\tau_{n+1}}\right)\left(\|u_n-w_n\|^2-\|v_n-w_n\|^2\right) \\
&\quad \leq \psi_n\|\phi(u_n)-\mathfrak{z}\|^2+\|u_n-\mathfrak{z}\|^2-\|u_{n+1}-\mathfrak{z}\|^2-\psi_n\|u_n-\mathfrak{z}\|^2-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2.
\end{aligned}
$$

Since $\lim_{n\to\infty}\|u_n-\mathfrak{z}\|=l$ and $\lim_{n\to\infty}\psi_n=0$, then from (3.13) and the above relation we obtain

$$\lim_{n\to\infty}\|u_n-w_n\|=\lim_{n\to\infty}\|v_n-w_n\|=0. \tag{3.19}$$

It follows from the above relation that

$$\lim_{n\to\infty}\|u_n-v_n\| \leq \lim_{n\to\infty}\|u_n-w_n\|+\lim_{n\to\infty}\|w_n-v_n\|=0. \tag{3.20}$$

We can also obtain

$$
\begin{aligned}
\|u_{n+1}-v_n\|^2 = \psi_n\|\phi(u_n)-v_n\|^2+(1-\psi_n)\|Sv_n-v_n\|^2 \\
-\psi_n(1-\psi_n)\|\phi(u_n)-Sv_n\|^2.
\end{aligned} \tag{3.21}
$$

and

$$\begin{aligned} &\|u_{n+1} - \mathfrak{z}\|^2 - \|v_n - \mathfrak{z}\|^2 - \|u_{n+1} - v_n\|^2 \\ &= 2\langle u_{n+1} - v_n, v_n - \mathfrak{z}\rangle \\ &= 2\psi_n\langle \phi(u_n) - v_n, v_n - \mathfrak{z}\rangle + 2(1-\psi_n)\langle Sv_n - v_n, v_n - \mathfrak{z}\rangle \\ &\leq 2\psi_n\langle \phi(u_n) - v_n, v_n - \mathfrak{z}\rangle - (1+\psi)(1-\psi_n)\|v_n - Sv_n\|^2. \end{aligned} \tag{3.22}$$

From relations (3.21) and (3.22), we obtain

$$\begin{aligned} \|u_{n+1} - \mathfrak{z}\|^2 - \|v_n - \mathfrak{z}\|^2 \leq \psi_n\|\phi(u_n) - v_n\|^2 - \psi_n(1-\psi_n)\|\phi(u_n) - Sv_n\|^2 \\ + 2\psi_n\langle \phi(u_n) - v_n, v_n - \mathfrak{z}\rangle - \psi(1-\psi_n)\|v_n - Sv_n\|^2. \end{aligned}$$

Therefore

$$\begin{aligned} \psi(1-\psi_n)\|v_n - Sv_n\|^2 \leq \psi_n\|\phi(u_n) - v_n\|^2 - \psi_n(1-\psi_n)\|\phi(u_n) - Sv_n\|^2 \\ + 2\psi_n\langle \phi(u_n) - v_n, v_n - \mathfrak{z}\rangle - \|u_{n+1} - \mathfrak{z}\|^2 + \|v_n - \mathfrak{z}\|^2. \end{aligned}$$

Using relation (3.18) and the fact that $\lim_{n\to\infty}\|u_n - \mathfrak{z}\|$ exists, we obtain

$$\lim_{n\to\infty}\|Sv_n - v_n\| = 0. \tag{3.23}$$

Next, we show that $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. Consider

$$\begin{aligned} \|u_{n+1} - u_n\| &= \|u_{n+1} - Sv_n + Sv_n - v_n + v_n - u_n\| \\ &\leq \|u_{n+1} - Sv_n\| + \|Sv_n - v_n\| + \|v_n - u_n\| \\ &\leq \psi_n\|\phi(u_n) - Sv_n\| + \|Sv_n - v_n\| + \|v_n - u_n\|. \end{aligned}$$

By using relations (3.1), (3.20) and (3.23), we obtain

$$\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0. \tag{3.24}$$

Since, the sequences $\{u_n\}, \{w_n\}$ and $\{v_n\}$ are bounded. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\} \rightharpoonup \hat{\mathfrak{z}} \in \mathcal{H}$. Thus, by relation (3.23) and Lemma 2.2, we can conclude that $\hat{\mathfrak{z}} \in Fix(T)$. Next, we need to show that $\hat{\mathfrak{z}} \in \Gamma$. Since $\|u_n - w_n\| \to 0$, we also have that $\{w_{n_k}\} \rightharpoonup \hat{\mathfrak{z}}$. Passing to the limit in relation (3.3) as $k \to \infty$ and using Assumptions 2.1 (G3), the relation (3.19) and the fact that $\lim_{n\to\infty}\tau_n = \tau > 0$, we obtain

$$\begin{aligned} g(\hat{\mathfrak{z}}, v) &\geq \lim_{k\to\infty}\sup g(w_{n_k}, v) \\ &\geq \frac{1}{2\tau_n}\lim_{k\to\infty}\sup\left(\|v_n - v\|^2 - \|u_n - v\|^2\right), \quad \forall v \in \mathcal{K}. \end{aligned} \tag{3.25}$$

On the other hand, by the triangle inequality, we have

$$\left|\|v_n - v\|^2 - \|u_n - v\|^2\right| \le \|v_n - u_n\|\left(\|v_n - u_n\| + \|u_n - v\|\right).$$

Thus, from the boundedness of the sequence $\{u_n\}$ and the relation (3.20), we get for each $v \in \mathcal{K}$

$$\lim_{n\to\infty}\left|\|v_n - v\|^2 - \|u_n - v\|^2\right| = 0. \tag{3.26}$$

Combining the relations (3.25) and (3.26), we get $g(\hat{\mathfrak{z}}, v) \ge 0$ for all $v \in \mathcal{K}$ so $\hat{\mathfrak{z}} \in \Gamma$. Therefore $\hat{\mathfrak{z}} \in \Omega := \Gamma \cap \Lambda$. Next, we consider

$$\begin{aligned}\lim_{n\to\infty}\ \sup\,\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_n - \mathfrak{z}\rangle &= \lim_{k\to\infty}\ \sup\,\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n_k} - \mathfrak{z}\rangle\\ &= \langle\phi(\mathfrak{z}) - \mathfrak{z}, \hat{\mathfrak{z}} - \mathfrak{z}\rangle \le 0.\end{aligned} \tag{3.27}$$

We have $\lim_{n\to\infty}\|u_{n+1} - u_n\| = 0$. We can deduce that

$$\begin{aligned}&\lim_{n\to\infty}\ \sup\,\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad\le \lim_{n\to\infty}\ \sup\,\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - u_n\rangle\\ &\qquad + \lim_{n\to\infty}\ \sup\,\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_n - \mathfrak{z}\rangle\\ &\quad\le 0.\end{aligned} \tag{3.28}$$

From Lemma 2.6 (ii) and relation (3.3), we have

$$\begin{aligned}&\|u_{n+1} - \mathfrak{z}\|^2\\ &\quad= \|\psi_n\phi(u_n) + (1-\psi_n)Sv_n - \mathfrak{z}\|^2\\ &\quad= \|\psi_n(\phi(u_n) - \mathfrak{z}) + (1-\psi_n)(Sv_n - \mathfrak{z})\|^2\\ &\quad\le (1-\psi_n)^2\|Sv_n - \mathfrak{z}\|^2 + 2\psi_n\langle\phi(u_n) - \mathfrak{z}, (1-\psi_n)(Sv_n - \mathfrak{z}) + \psi_n(\phi(u_n) - \mathfrak{z})\rangle\\ &\quad= (1-\psi_n)^2\|v_n - \mathfrak{z}\|^2 + 2\psi_n\langle\phi(u_n) - \phi(\mathfrak{z}) + \phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad= (1-\psi_n)^2\|v_n - \mathfrak{z}\|^2 + 2\psi_n\langle\phi(u_n) - \phi(\mathfrak{z}), u_{n+1} - \mathfrak{z}\rangle + 2\psi_n\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad\le (1-\psi_n)^2\|v_n - \mathfrak{z}\|^2 + 2\psi_n\xi\langle u_n - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle + 2\psi_n\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad\le (1+{\psi_n}^2 - 2\psi_n)\|u_n - \mathfrak{z}\|^2 + 2\psi_n\xi\|u_n - \mathfrak{z}\|^2 + 2\psi_n\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad= (1-2\psi_n)\|u_n - \mathfrak{z}\|^2 + {\psi_n}^2\|u_n - \mathfrak{z}\|^2 + 2\psi_n\xi\|u_n - \mathfrak{z}\|^2 + 2\psi_n\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle\\ &\quad= \left[1 - 2\psi_n(1-\xi)\right]\|u_n - \mathfrak{z}\|^2 + 2\psi_n(1-\xi)\left[\frac{\psi_n\|u_n - \mathfrak{z}\|^2}{2(1-\xi)} + \frac{\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle}{1-\xi}\right].\end{aligned} \tag{3.29}$$

It follows from relations (3.28) and (3.29), that

$$\lim_{n\to\infty}\ \sup\left[\frac{\psi_n\|u_n - \mathfrak{z}\|^2}{2(1-\xi)} + \frac{\langle\phi(\mathfrak{z}) - \mathfrak{z}, u_{n+1} - \mathfrak{z}\rangle}{1-\xi}\right] \le 0. \tag{3.30}$$

Choose $n \ge N_2 \in \mathbb{N}\,(N_2 \ge N_1)$ large enough such that $2\psi_n(1-\xi) < 1$. By using relations (3.29) and (3.30) and applying Lemma 2.5, we conclude that $\lim_{n\to\infty} u_n \to \mathfrak{z}$.

Case II: Assume that there is a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\|u_{n_i} - \mathfrak{z}\| \le \|u_{n_{i+1}} - \mathfrak{z}\|, \quad \forall i \in \mathbb{N}.$$

Thus, by Lemma 2.4, there is a sequence $\{m_k\} \subset \mathbb{N}$ as $\lim_{k\to\infty} m_k = \infty$ such that

$$\|u_{m_k} - \mathfrak{z}\| \le \|u_{m_k+1} - \mathfrak{z}\| \quad \text{and} \quad \|u_k - \mathfrak{z}\| \le \|u_{m_k+1} - \mathfrak{z}\|, \quad \forall k \in \mathbb{N}. \tag{3.31}$$

Similar to **Case I**, the relation (3.17) provides that

$$(1-\psi_{m_k})\left(1 - \frac{\eta\tau_{m_k}}{\tau_{m_k+1}}\right)\left(\|u_{m_k} - w_{m_k}\|^2 + \|v_{m_k} - w_{m_k}\|^2\right) \tag{3.32}$$

$$\begin{aligned}\le \psi_{m_k}\|\phi(u_{m_k}) - \mathfrak{z}\|^2 + \|u_{m_k} - \mathfrak{z}\|^2 - \|u_{m_k+1} - \mathfrak{z}\|^2 - \psi_{m_k}\|u_{m_k} - \mathfrak{z}\|^2 \\ - \psi_{m_k}(1-\psi_{m_k})\|\phi(u_{m_k}) - Sv_{m_k}\|^2.\end{aligned} \tag{3.33}$$

By the relations (3.1), (3.13) and (3.31), we obtain

$$\lim_{k\to\infty} \|u_{m_k} - w_{m_k}\| = \lim_{k\to\infty} \|v_{m_k} - w_{m_k}\| = 0. \tag{3.34}$$

Also, we can obtain as similar to **Case I**

$$\lim_{k\to\infty} \|Sv_{m_k} - v_{m_k}\| = 0 \tag{3.35}$$

and

$$\lim_{k\to\infty} \|u_{m_k+1} - u_{m_k}\| = 0. \tag{3.36}$$

We have to use the same justification as in **Case I**, such that

$$\lim_{k\to\infty} \sup \langle \phi(\mathfrak{z}) - \mathfrak{z}, u_{m_k+1} - \mathfrak{z}\rangle \le 0. \tag{3.37}$$

By using relations (3.29) and (3.31), we obtain

$$\begin{aligned}\|u_{m_k+1} - \mathfrak{z}\|^2 &\le [1 - 2\psi_{m_k}(1-\xi)]\|u_{m_k} - \mathfrak{z}\|^2 \\ &\quad + 2\psi_{m_k}(1-\xi)\left[\frac{\psi_{m_k}\|u_{m_k} - \mathfrak{z}\|^2}{2(1-\xi)} + \frac{\langle \phi(\mathfrak{z}) - \mathfrak{z}, u_{m_k+1} - \mathfrak{z}\rangle}{1-\xi}\right] \\ &\le [1 - 2\psi_{m_k}(1-\xi)]\|u_{m_k+1} - \mathfrak{z}\|^2 \\ &\quad + 2\psi_{m_k}(1-\xi)\left[\frac{\psi_{m_k}\|u_{m_k} - \mathfrak{z}\|^2}{2(1-\xi)} + \frac{\langle \phi(\mathfrak{z}) - \mathfrak{z}, u_{m_k+1} - \mathfrak{z}\rangle}{1-\xi}\right].\end{aligned} \tag{3.38}$$

It follows that

$$\|u_{m_k+1} - \mathfrak{z}\|^2 \leq \frac{\psi_{m_k}\|u_{m_k} - \mathfrak{z}\|^2}{2(1-\xi)} + \frac{\langle \phi(\mathfrak{z}) - \mathfrak{z}, u_{m_k+1} - \mathfrak{z}\rangle}{1-\xi}. \tag{3.39}$$

By the relations (3.1) and (3.31), relations (3.37) and (3.39) implies that

$$\lim_{k\to\infty} \|u_{m_k+1} - \mathfrak{z}\|^2 = 0.$$

Thus, the above relation implies that

$$\lim_{k\to\infty} \|u_k - \mathfrak{z}\|^2 \leq \lim_{k\to\infty} \|u_{m_k} - \mathfrak{z}\|^2 \leq 0. \tag{3.40}$$

Consequently, the sequence $\{u_n\}$ converges strongly to $\mathfrak{z} \in \Omega := \Gamma \cap \Lambda$. □

4 Applications

Application to pseudomonotone equilibrium problems:

Set $S = I$ in Algorithm 1, then we have the following strong convergence algorithm for pseudomonotone equilibrium problem:

Corollary 4.1 *Assume that* $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ *is satisfying Assumption* 2.1. *Let the sequence* $\{u_n\}, \{w_n\}$ *and* $\{v_n\}$ *be generated in the following manner: Choose* $u_0 \in \mathcal{K}$, *and* $\tau_0 > 0$, $\eta \in (0, 1)$. *Compute*

$$\begin{aligned} w_n &= \mathrm{prox}_{\tau_n g(u_n)}(u_n), \\ v_n &= \mathrm{prox}_{\tau_n g(w_n)}(u_n), \\ u_{n+1} &= \psi_n \phi(u_n) + (1-\psi_n) v_n, \end{aligned}$$

and set

$$\tau_{n+1} = \min\left\{\tau_n, \frac{\eta(\|u_n - w_n\|^2 + \|v_n - w_n\|^2)}{2\max\{0, g(u_n, v_n) - g(u_n, w_n) - g(w_n, v_n)\}}\right\}.$$

Then the sequences $\{u_n\}, \{w_n\}$ and $\{v_n\}$ converge strongly to the solution $\mathfrak{z}$ of Γ.

Application to pseudomonotone variational inequality problems:

Recall that in the problem of classical variational inequality, one needs to find a point $\mathfrak{z} \in \mathcal{K}$ such that

$$\langle \mathcal{A}(\mathfrak{z}), v - \mathfrak{z}\rangle \geq 0, \quad \forall\, v \in \mathcal{K},$$

where $\mathcal{A} : \mathcal{H} \to \mathcal{H}$ is an operator. We denote the solution set of classical variational inequality by the symbol $VI(\mathcal{A}, \mathcal{K})$. Set the bi-function $g(u, v) := \langle \mathcal{A}(u), v - u\rangle$ for all $u, v \in \mathcal{K}$ in Algorithm 1, we have

$$\begin{aligned} w_n &= \arg\min_{v\in\mathcal{K}} \{g(u_n, v) + \frac{1}{2\tau_n}\|u_n - v\|^2\} \\ &= \arg\min_{v\in\mathcal{K}} \{\langle \mathcal{A}(u_n), v - u_n\rangle + \frac{1}{2\tau_n}\|u_n - v\|^2\} \\ &= \arg\min_{v\in\mathcal{K}} \{\langle \mathcal{A}(u_n), v - u_n\rangle + \frac{1}{2\tau_n}\|u_n - v\|^2\} + \frac{\tau_n^2}{2}\|\mathcal{A}(u_n)\|^2 - \frac{\tau_n^2}{2}\|\mathcal{A}(u_n)\|^2 \\ &= \arg\min_{v\in\mathcal{K}}\{\frac{1}{2\tau_n}\|v - (u_n - \tau_n\mathcal{A}(u_n))\|^2\} - \frac{\tau_n}{2}\|\mathcal{A}(u_n)\|^2 \\ &= P_{\mathcal{K}}(u_n - \tau_n\mathcal{A}(u_n)). \end{aligned}$$

Similarly,

$$v_n = P_{\mathcal{K}}(u_n - \tau_n\mathcal{A}(v_n)).$$

Assumption 4.1 Assume that $\mathcal{A}$ is satisfying the following assumptions:

$\mathcal{A}_1$: $\mathcal{A}$ is pseudomonotone on $\mathcal{K}$, that is, for all $u, v \in \mathcal{K}$,

$$\langle \mathcal{A}(u), v - u\rangle \geq 0 \Rightarrow \langle \mathcal{A}(v), u - v\rangle \leq 0.$$

and $VI(\mathcal{A}, \mathcal{K})$ is non-empty.

$\mathcal{A}_2$: $\mathcal{A}$ is Lipschitz continuous on $\mathcal{K}$ with $L > 0$, that is, for all $u, v \in \mathcal{K}$,

$$\|\mathcal{A}(u) - \mathcal{A}(v)\| \leq L\|u - v\|.$$

$\mathcal{A}_3$: $\lim_{n\to\infty}\sup\langle \mathcal{A}(u_n), v - u_n\rangle \leq \langle \mathcal{A}(\mathfrak{z}), v - \mathfrak{z}\rangle$ for every $v \in \mathcal{K}$ and $\{u_n\} \subset \mathcal{K}$ satisfying $u_n \rightharpoonup \mathfrak{z}$.

Many researchers have studied variational inequality problem [8] and have established various iterative methods to tackle it; see for example [4, 6, 7, 32]. We have the following strong convergence theorem about the pseudomonotone variational inequality problem [8]:

Corollary 4.2 *Assume that* $\mathcal{A} : \mathcal{K} \to \mathcal{H}$ *is satisfying Assumptions* 4.1. *Let the sequences* $\{u_n\}, \{w_n\}$ *and* $\{v_n\}$ *be generated in the following manner: Choose* $u_0 \in \mathcal{H}$ *and* $\tau_0 > 0$, $\eta \in (0, 1)$. *Compute*

$$\begin{aligned} w_n &= P_{\mathcal{K}}\big(u_n - \tau_n\mathcal{A}(u_n)\big), \\ v_n &= P_{\mathcal{K}}\big(u_n - \tau_n\mathcal{A}(w_n)\big), \\ u_{n+1} &= \psi_n\phi(u_n) + (1 - \psi_n)v_n, \end{aligned}$$

and set

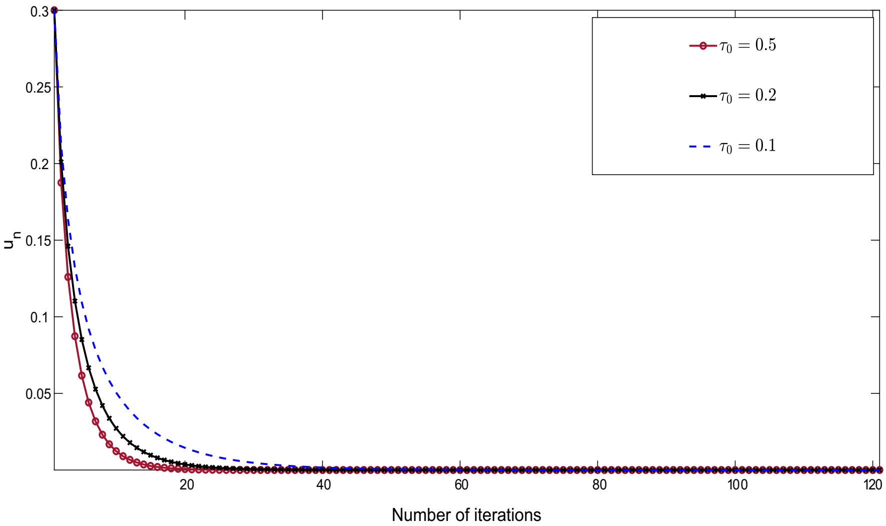

Fig. 1 Graphical representation of the sequence $\{u_n\}$ for initial value $u_0 = 0.3$ and different choices of step size τ_0.

$$\tau_{n+1} = \min\left\{\tau_n, \frac{\eta(\|u_n - w_n\|^2 + \|v_n - w_n\|^2)}{2\max\{0, \langle \mathcal{A}(u_n), v_n - w_n\rangle - \langle \mathcal{A}(w_n), v_n - w_n\rangle\}}\right\}.$$

Then the sequences $\{u_n\}.\{w_n\}$ and $\{v_n\}$ strongly converge to the solution $\mathfrak{z}$ of $VI(\mathcal{A}, \mathcal{K})$.

5 Numerical illustrations

In this section, we provide a numerical example to support and justify our proposed algorithm. All codes are written in Matlab (2021a).

Example 5.1 Suppose that $\mathcal{H} = \mathbb{R}$ with the inner product $\langle u, v\rangle := u \cdot v, \quad \forall\, u, v \in \mathcal{H}$. and the induced norm $\|u\| := |u|, \quad \forall\, u \in \mathcal{H}$. Let $\mathcal{K} := \{u \in \mathcal{H} : |u| \leq 1\}$ be the unit ball and defined an operator $\mathcal{A} : \mathcal{K} \to \mathcal{H}$ by

$$\mathcal{A}(u) := (u + |u|)/2.$$

Clearly, $\mathcal{A}$ is $1-$Lipschitz continuous and pseudomonotone operator on $\mathcal{K}$. We consider a contraction mapping $g(u) = u/2$ for all $u \in \mathcal{H}$ with $\xi = 1/2$. The solution set of variational inequality problem (VI) is given by $VI(\mathcal{A}, \mathcal{K}) = \{0\} \neq \emptyset$. Moreover, with respect to corollary 4.2, we take $\psi_n = \dfrac{1}{1+n}$, $\eta = 0.33$, $u_0 = 0.3$.

Numerical results of the sequence $\{u_n\}$ generated by Corollary 4.2 for initial value $u_0 = 0.3$ and different choices of step size τ_0.

Numerical results for initial value $u_0 = 0.3$.

Number of iterations	$\tau_0 = 0.1$	$\tau_0 = 0.2$	$\tau_0 = 0.5$
1	0.300000	0.300000	0.300000
15	0.026389	0.009698	0.002684
30	0.004712	0.000538	0.000033
45	0.000955	0.000033	0.000000
69	0.000082	0.000000	0.000000
75	0.000045	0.000000	0.000000
90	0.000010	0.000000	0.000000
121	0.000000	0.000000	0.000000

Remark 5.1 In view of the above graphical representation (Fig. 1) of the sequence $\{u_n\}$, we see that the proposed algorithm work better when the value of the step size τ_0 is larger.

Author contributions All authors contributed equally to this manuscript.

Data availibility Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors

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