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ORIGINAL RESEARCH PAPER

An elementary best proximity point theorem in metric spaces

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Abstract

Let us consider two nonempty compact subsets A and B of a metric space X and a continuous mapping $f : A \cup B \to A \cup B$ satisfying $f(A) \subseteq B$, $f(B) \subseteq A$. In this manuscript, we provide sufficient conditions for the existence of a point $x_0 \in A$ holding the condition that $d(x_0, fx_0) = \inf\{d(a, b) : a \in A, b \in B\}$. When $A = B$, our main result reduces to the well-known fixed point theorem for continuous mapping (Lemma 1) in metric spaces.

Keywords Fixed point · $\varepsilon-$close mapping · $P-$property · $UC-$property · Best proximity point

1 Introduction

A topological space X is said to have the *fixed point property* if every continuous function $f : X \to X$ has at least one point $x \in X$ such that $x = f(x)$. Using intermediate value theorem, it is easy to see that a closed and bounded interval $[a, b]$ in $\mathbb{R}$ has fixed point property. More generally, in 1910, Brouwer [2] proved that the closed unit ball $\mathbf{B} = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ in $\mathbb{R}^n$ has fixed point property. Since the fixed point property is a topological property, one can generally say that every nonempty closed bounded convex subset of a finite dimensional normed linear space has fixed point property. For different proofs of Brouwer's fixed point theorem one may refer [3, 4].

Due to the wide applications in Economics, Game Theory, etc., Brouwer's theorem has attracted many researchers to obtain interesting generalizations of it. Schauder's fixed point theorem [5] is one of such novel generalizations of Brouwer's Theorem. In 1930, Schauder extended Brouwer's theorem to infinite dimensional

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Extended author information available on the last page of the article spaces. It states that every nonempty compact convex subset of a normed linear space has fixed point property. The following elementary fixed point lemma is used to prove the Schauder fixed point theorem. (See [6]).

Lemma 1 [1] *Let X be a compact metric space. Suppose that for each* $\varepsilon > 0$, *there exists a continuous function* $g_\varepsilon : X \to X$ *satisfying*:

1. $d(x, g_\varepsilon(x)) < \varepsilon$, for all $x \in X$,
2. $g_\varepsilon(X)$ has the fixed point property.

Then X has the fixed point property.

One may refer [6] for the proof of Lemma 1. In [1], the author used the Lemma 1 and established a Brouwer type fixed point theorem which ensures the existence of a fixed point for a continuous function on a nonempty star-like compact subset of $\mathbb{R}^2$.

Now, let us consider two nonempty compact subsets A and B of a metric space X and a continuous mapping $T : A \cup B \to A \cup B$ satisfying the cyclic condition $T(A) \subseteq B$, $T(B) \subseteq A$. Let $\text{dist}(A, B) := \inf\{d(a, b) : a \in A, b \in B\}$. If $\text{dist}(A, B) > 0$, then there is no $x \in A \cup B$ which satisfies $x = T(x)$, since $0 < \text{dist}(A, B) \leq d(x, T(x))$. In this juncture, any $x \in A \cup B$ satisfying the condition $d(x, T(x)) = \text{dist}(A, B)$ is an optimal solution to the minimization problem

$$\min_{x \in A \cup B} d(x, T(x)).$$

A point $x \in A \cup B$ satisfying $d(x, T(x)) = \text{dist}(A, B)$ is known as a best proximity point of T. If $\text{dist}(A, B) = 0$, then the best proximity points of T are nothing but the fixed points of T. In this context, best proximity point theorems are considered as generalized fixed point theorems. The following example shows that, in general, the cyclic continuous mapping on $A \cup B$ need not have best proximity point.

Example 1 Consider $\mathbb{R}^2$ with usual metric. Let $A := \{(x, y) : x \in \{-2, 1\}, 0 \leq y \leq 1\}$ and $B := \{(x, y) : x \in \{-1, 2\}, 0 \leq y \leq 1\}$. Then A and B are nonempty compact subsets of $\mathbb{R}$ with $\text{dist}(A, B) = 1$. Let $T : A \cup B \to A \cup B$ be a mapping defined by $T(x, y) = (-x, y)$, for all $(x, y) \in A \cup B$. Clearly T is a continuous mapping satisfying $T(A) \subseteq B$, $T(B) \subseteq A$. Since $d((x, y), T(x, y)) \geq 2$, for all $(x, y) \in A \cup B$, T has no best proximity point.

Hence, it is of interest to investigate sufficient conditions to ensure the existence of at least one best proximity point of a cyclic continuous mapping. In this manuscript, first we prove that every pair (A, B) of compact subsets of a metric space having $P-$property (Definition 1) also satisfies the $UC-$property (Definition 2). Using this geometric idea, we provide sufficient conditions for the existence of best proximity points of a cyclic continuous mapping on a pair of compact subsets. Our main result serves as a generalization of Lemma 1, by considering $A = B = X$ in our main result.

2 Preliminaries

This section provides few known results and standard notations which we use in our main results. Let A and B be two nonempty subsets of a metric space X. Let us fix the following notations :

$$A_0 := \{x \in A : d(x, y) = \text{dist}(A, B), \text{for some } y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = \text{dist}(A, B), \text{for some } x \in A\}.$$

In general, the sets A_0 and B_0 may be empty. If A_0 and B_0 is nonempty, then the pair (A_0, B_0) is known as the proximal pair associated with the given pair (A, B) of subsets. It is easy to see that if A, B are compact subsets of X, then A_0, B_0 are nonempty subsets of A, B respectively. For any subset A of X and $x \in X$, the distance between the set A and x is defined as $\text{dist}(x, A) := \inf\{d(x, a) : a \in A\}$.

Let A be a nonempty compact subset of X. We recall the metric projection mapping $P_A : X \to 2^A$ such that $P_A(x) := \{a \in A : d(x, a) = \text{dist}(x, A)\}$, where 2^A denotes the set of all nonempty subsets of A. i.e., P_A is a multivalued mapping. Suppose that A, B are nonempty compact subsets of X. Then, let us define a mapping $P : A \cup B \to 2^B \cup 2^A$ called *projection operator* as follows:

$$P(x) := \begin{cases} P_B(x), & \text{if } x \in A, \\ P_A(x), & \text{if } x \in B. \end{cases} \tag{1}$$

Thus, $P(A_0) \subseteq B_0$ and $P(B_0) \subseteq A_0$. We use the following $P-$property to restrict the above operator P to be a single valued mapping on $A_0 \cup B_0$.

Definition 1 [7] A pair (A,B) of nonempty subsets of a metric space X is said to have $P-$property if and only if for any $x_1, x_2 \in A$ and $y_1, y_2 \in B$ with

$$\left.\begin{array}{l} d(x_1, y_1) = \text{dist}(A, B) \\ d(x_2, y_2) = \text{dist}(A, B) \end{array}\right\} \Rightarrow d(x_1, x_2) = d(y_1, y_2).$$

It is well-known fact, see [8], that every pair (A, B) of nonempty closed convex subsets of a strictly convex Banach space has $P-$property. Also, for any nonempty subset A of a metric space X, the pair (A, A) has $P-$property. Suppose that A and B are nonempty compact subsets of a metric space X such that the pair (A, B) has $P-$property. Then it is easy to verify the following facts.

1. A_0, B_0 are nonempty compact subsets of A, B respectively (the $P-$property is not necessary to prove this statement).
2. the projection operator $P : A_0 \cup B_0 \to A_0 \cup B_0$ is single valued and $P(A_0) \subseteq B_0$ and $P(B_0) \subset A_0$.
3. Let $x_0 \in A_0$ and $y_0 \in B_0$. Then $d(x_0, y_0) = \text{dist}(A, B)$ if and only if $x_0 = P(y_0)$.

We use the following $UC-$property in our main result.

Definition 2 [9] Let A,B be nonempty subsets of a metric space X. The pair (A,B) is said to satisfy $UC-$property if the following holds:

If $\{x_n\}$ and $\{z_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$\lim_{n\to\infty} d(x_n, y_n) = \text{dist}(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = \text{dist}(A, B)$, then $\lim_{n\to\infty} d(x_n, z_n) = 0$ holds.

In [9], the authors proved that if $\text{dist}(A, B) = 0$, then the pair (*A*, *B*) has $UC-$property. Also, every pair (*A*, *B*) of nonempty closed convex subsets of a uniformly convex Banach space has $UC-$property.

Let *A* and *B* be two nonempty subsets of a metric space *X*. A mapping $T : A \cup B \to A \cup B$ is said to be a relatively nonexpansive mapping if it satisfies the following conditions:

1. $T(A) \subseteq B$ and $T(B) \subseteq A$,
2. $d(T(x), T(y)) \leq d(x, y)$, for all $x \in A$ and $y \in B$.

It is worth mentioning that a relatively nonexpansive need not be continuous. Also, it is easy to see that if $d(x_0, y_0) = \text{dist}(A, B)$, then $d(T(x), T(y)) = \text{dist}(A, B)$. In [10], the authors introduced a geometric notion called proximal normal structure and provided sufficient conditions for the existence of best proximity points for relatively nonexpansive mappings.

3 Main results

We begin our main results with the following proposition.

Proposition 1 *Let A and B be two nonempty compact subsets of a metric space X such that the pair (A, B) has P−property. Then (A, B) has UC−property.*

Proof Let $\{x_n\}$ and $\{z_n\}$ be sequences in *A* and $\{y_n\}$ be a sequence in *B* such that $\lim_{n\to\infty} d(x_n, y_n) = \text{dist}(A, B)$ and $\lim_{n\to\infty} d(z_n, y_n) = \text{dist}(A, B)$. Suppose $d(x_n, z_n) \not\to 0$. Then there is an $\varepsilon > 0$ and subsequences $\{x_{n_k}\}$, $\{z_{n_k}\}$ such that $d(x_{n_k}, z_{n_k}) \geq \varepsilon$, for all $k \in \mathbb{N}$. Since *A* and *B* are compact, without loss of generality, let us assume that $x_{n_k} \to x$, $z_{n_k} \to z$ and the corresponding $y_{n_k} \to y$. Then $d(x, z) \geq \varepsilon$ and $d(x, y) = \text{dist}(A, B)$, $d(z, y) = \text{dist}(A, B)$, which contradict the $P-$property of (*A*, *B*). Hence (*A*, *B*) has $UC-$property. □

Proposition 2 *Let A and B be two nonempty compact subsets of a metric space X. Suppose that the pair (A, B) has P−property. Then, the projection operator* $P : A_0 \cup B_0 \to A_0 \cup B_0$ *is a continuous single valued mapping and satisfy* $P(A_0) \subseteq B_0$ *and* $P(B_0) \subseteq A_0$.

Proof Let us show that *P* is continuous.

Let $x_0 \in A_0$ and $\{x_n\}$ be a sequence in A_0 such that $x_n \to x_0$ as $n \to \infty$. Then, for each $n \in \mathbb{N}$, there exist $y_0, y_n \in B_0$ such that $d(x_0, y_0) = \text{dist}(A, B)$ and $d(x_n, y_n) = \text{dist}(A, B)$. Thus, $P(x_0) = y_0$ and $P(x_n) = y_n$, for all $n \in \mathbb{N}$. Then, by $P-$property, we have

$$d(P(x_n), P(x_0)) = d(y_n, y_0) = d(x_n, x_0) \to 0.$$

Hence $P(x_n) \to P(x_0)$. i.e., P is continuous on A_0. Similarly, P is continuous on B_0 also. □

Let us define a new notion called $\varepsilon-$close mapping.

Definition 3 Let A and B be nonempty subsets of a metric space X such that $A_0 \neq \emptyset$. Let $\varepsilon > 0$. A mapping $h : A \cup B \to A \cup B$ is said to be $\varepsilon-$close mapping if

1. $h(A_0) \subseteq B_0$ and $h(B_0) \subseteq A_0$,
2. $d(x, h(x)) < \text{dist}(A, B) + \varepsilon$, for all $x \in A_0 \cup B_0$.

Let us give some examples of $\varepsilon-$close mappings.

Example 2 Let A and B be nonempty compact subsets of a metric space X such that the pair (A, B) has $P-$property. Then the mapping defined in (1) is $\varepsilon-$close mapping, for any $\varepsilon > 0$.

Example 3 Let A and B be nonempty weakly compact convex subsets of a strictly convex Banach space. Then the mapping defined in (1) is $\varepsilon-$close mapping, for any $\varepsilon > 0$.

Example 4 Let $A := \{(0, 0)\}$ and $B := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Clearly, $\text{dist}(A, B) = 1$ and $A_0 = A, B_0 = B$. Then any mapping $h : A \cup B \to A \cup B$ satisfying $h(A) \subseteq B, h(B) \subseteq A$ is an $\varepsilon-$close mapping.

Now, we define a new class of $\varepsilon-$perturbed cyclic mapping.

Definition 4 Let A and B be two nonempty subsets of a metric space X with $A_0 \neq \emptyset$. Let $\varepsilon > 0$. A relatively nonexpansive mapping $T : A \cup B \to A \cup B$ is said to be $\varepsilon-$perturbed mapping if $T(x_0) \in B(y_0, \varepsilon)$ and $T(y_0) \in B(x_0, \varepsilon)$, whenever $d(x_0, y_0) = \text{dist}(A, B)$.

Example 5 Let $A := \{(0, x) : x \in \mathbb{R}\}$ and $B := \{(1, y) : y \in \mathbb{R}\}$. Let $\varepsilon > 0$ be given. Let $T : A \cup B \to A \cup B$ be mapping defined by

$$T(x, y) := \begin{cases} \left(1, y + \frac{\varepsilon}{2}\right), & \text{if } x = 0, \\ \left(0, y + \frac{\varepsilon}{2}\right), & \text{if } x = 1. \end{cases}$$

Then T is a $\varepsilon-$perturbed mapping.

Proposition 3 *Let A and B be nonempty subsets of a metric space X such that the pair (A, B) has P−property. Let $\varepsilon > 0$. Then every $\varepsilon-$perturbed mapping is $\varepsilon-$close.*

Proof Let $T : A \cup B \to A \cup B$ be an $\varepsilon-$perturbed mapping. Being relatively nonexpansive, it satisfies the condition $T(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. Now, let $x_0 \in A_0$. By $P-$property, there is unique $y_0 \in B_0$ such that $d(x_0, y_0) = \text{dist}(A, B) =$

$d(T(x_0),\ T(y_0))$ and $T(y_0) \in B(x_0, \varepsilon)$. Then, $d(x_0, T(x_0)) \leq d(x_0, T(y_0)) + d(T(y_0),\ T(x_0)) < \varepsilon + \text{dist}(A, B)$. i.e., T is $\varepsilon-$close. □

The following example shows that a cyclic continuous mapping on a pair (A, B) of nonempty compact subsets need not be $\varepsilon-$close.

Example 6 Let $A := \{(0, x) : 0 \leq x \leq 1\}$ and $B := \{(1, x) : 0 \leq x \leq 1\}$. Clearly, A and B are nonempty compact convex subsets of $\mathbb{R}^2$ such that $A_0 = A$, $B_0 = B$. Let $h : A \cup B \to A \cup B$ be a mapping defined as

$$h(u, v) = \begin{cases} (1, 1), & \text{if } u = 0, \\ (0, 0), & \text{if } u = 1. \end{cases}$$

Then $h(A) \subseteq B$, $h(B) \subseteq A$ and h is continuous. But, for any $0 < \varepsilon < (\sqrt{2} - 1)$, h is not an $\varepsilon-$close mapping.

Definition 5 A pair (A,B) of nonempty subsets of a metric space is said to have the best proximity point property if every cyclic continuous mapping $h : A \cup B \to A \cup B$ has at least one best proximity point on $A \cup B$.

Now, let us state our main result.

Theorem 1 *Let A and B be nonempty compact subsets of a metric space X such that the pair (A,B) has the $P-$property. Assume that, for each $\varepsilon > 0$, there is a continuous, $\varepsilon-$close function $h_\varepsilon : A \cup B \to A \cup B$ such that the pair $(h_\varepsilon(B_0), h_\varepsilon(A_0))$ has best proximity point property. Then, for any cyclic continuous function $h : A \cup B \to A \cup B$ satisfying $h(A_0) \subseteq B_0$, $h(B_0) \subseteq A_0$ has at least one best proximity point in $A \cup B$.*

Proof Let $h : A \cup B \to A \cup B$ be a continuous mapping satisfying $h(A_0) \subseteq B_0$ and $h(B_0) \subseteq A_0$. Let $\varepsilon > 0$ be fixed. Then there is a $\varepsilon-$close mapping $h_\varepsilon : A \cup B \to A \cup B$ satisfying $h_\varepsilon(A_0) \subseteq B_0$, $h_\varepsilon(B_0) \subseteq A_0$ and $d(x, h_\varepsilon(x)) < \text{dist}(A, B) + \varepsilon$, for all $x \in A_0 \cup B_0$.

Note that $h_\varepsilon(P(h(A_0))) \subseteq h_\varepsilon(P(B_0)) \subseteq h_\varepsilon(A_0)$, where P is the mapping given in (1). Also, in similar manner we have $h_\varepsilon(P(h(B_0))) \subseteq h_\varepsilon(B_0)$.

Consider the mapping $f := h_\varepsilon \circ P \circ h : A_0 \cup B_0 \to h_\varepsilon(A_0) \cup h_\varepsilon(B_0)$. By Proposition 2, the mapping f is continuous. Restricting f to $h_\varepsilon(A_0) \cup h_\varepsilon(B_0)$, we get $f : h_\varepsilon(A_0) \cup h_\varepsilon(B_0) \to h_\varepsilon(A_0) \cup h_\varepsilon(B_0)$.

Now, $f(h_\varepsilon(A_0)) \subseteq f(B_0) = h_\varepsilon(P(h(B_0))) \subseteq h_\varepsilon(B_0)$. Similarly, $f(h_\varepsilon(B_0)) \subseteq f(A_0) = h_\varepsilon(P(h(A_0))) \subseteq h_\varepsilon(A_0)$. Thus, $f : h_\varepsilon(A_0) \cup h_\varepsilon(B_0) \to h_\varepsilon(A_0) \cup h_\varepsilon(B_0)$ is a cyclic continuous function satisfying $f(h_\varepsilon(A_0)) \subseteq h_\varepsilon(B_0)$ and $f(h_\varepsilon(B_0)) \subseteq h_\varepsilon(A_0)$. By assumption, there is $x_\varepsilon \in h_\varepsilon(B_0)$ such that $d(x_\varepsilon, f(x_\varepsilon)) = \text{dist}(A, B)$. i.e., for each $\varepsilon > 0$, there is $x_\varepsilon \in h_\varepsilon(B_0)$ such that

$$d(x_\varepsilon, h_\varepsilon(P(h(x_\varepsilon)))) = \text{dist}(A, B). \tag{2}$$

Since h_ε is $\varepsilon-$close, we have

$$d(P(h(x_\varepsilon)), h_\varepsilon(P(h(x_\varepsilon)))) < \text{dist}(A,B) + \varepsilon. \tag{3}$$

Hence, from (2) and (3), we have $d(x_\varepsilon, h_\varepsilon(P(h(x_\varepsilon)))) \to \text{dist}(A,B)$ and $d(P(h(x_\varepsilon)), h_\varepsilon(P(h(x_\varepsilon)))) \to \text{dist}(A,B)$ as $\varepsilon \to 0$. By applying $UC-$property, we get

$$d(x_\varepsilon, P(h(x_\varepsilon))) \to 0 \text{ as } \varepsilon \to 0. \tag{4}$$

By the compactness of A_0, without loss of generality, there is $x_0 \in A_0$ such that $x_\varepsilon \to x_0$ as $\varepsilon \to 0$. From the continuity of $P \circ h$, we have $P(h(x_\varepsilon)) \to P(h(x_0))$. By (4), we have $d(x_0, P(h(x_0))) = 0$. i.e., $x_0 = P(h(x_0))$ and hence $d(x_0, h(x_0)) = \text{dist}(A,B)$. □

Let us illustrate the above theorem with the following example.

Example 7 Consider $\mathbb{R}^2$ with usual metric. Let $A = \{(0,x) : 0 \leq x \leq 1\}$ and $B = \{(1,x) : 0 \leq x \leq 1\}$ be nonempty compact subsets $\mathbb{R}^2$ with $A_0 = A, B_0 = B$. For each $\varepsilon > 0$, define

$$P_\varepsilon(x) := \begin{cases} P_B(x), & \text{if } x \in A, \\ P_A(x), & \text{if } x \in B. \end{cases} \tag{5}$$

It is easy to see that P_ε is a single valued continuous and $\varepsilon-$close mapping satisfies Theorem 1. Hence, any cyclic continuous function on $A \cup B$ has best proximity point.

Now, let A be a nonempty compact subset of a metric space X. Then the pair (A, A) has both $P-$property and $UC-$property. Also, $A_0 = A$ and $\text{dist}(A,A) = 0$. Using these fact in Theorem 1, we obtain the following fixed point theorem.

Corollary 1 *Let A be a nonempty compact subset of a metric space X. Suppose that for each $\varepsilon > 0$, there exists a continuous function $g_\varepsilon : A \to A$ satisfying*:

1. $d(x, g_\varepsilon(x)) < \varepsilon$, for all $x \in A$,
2. $g_\varepsilon(A)$ has the fixed point property.

Then every continuous function $f : A \to A$ has at least one fixed point in A.

Declarations

Conflict of interest The first author declares that he has no conflict of interest. The second author declares that she has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors

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