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ORIGINAL RESEARCH PAPER

On the bounds of eigenvalues of matrix polynomials

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Abstract

Let $M(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$ be a matrix polynomial, whose coefficients $A_k \in \mathbb{C}^{n\times n}$, $\forall\, k = 0, 1, \ldots, m$, satisfying the following dominant property

$$\|A_m\| > \|A_k\|,\ \forall\, k = 0, 1, \ldots, m-1,$$

then it is known that all eigenvalues λ of $M(z)$ locate in the open disk

$$|\lambda| < 1 + \|A_m\| \|{A_m}^{-1}\|.$$

In this paper, among other things, we prove some refinements of this result, which in particular provide refinements of some results concerning the distribution of zeros of polynomials in the complex plane.

Keywords Matrix polynomial · Polynomial eigenvalue problem · Bounds

Mathematics Subject Classification 15A18 · 15A42 · 65F15

1 Introduction

Let $\mathbb{C}^{n\times n}$ be the set of all $n \times n$ matrices whose entries are in $\mathbb{C}$. By *matrix polynomial*, we mean the matrix-valued function of a complex variable of the form

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$$M(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0, \tag{1}$$

where $A_i \in \mathbb{C}^{n\times n}$ for all $i = 0, 1, 2, \ldots, m$. If $A_m \neq 0$, $M(z)$ is called a matrix polynomial of degree m. A number λ is called an *eigenvalue* of the matrix polynomial $M(z)$, if there exits a nonzero vector $X \in \mathbb{C}^n$, such that $M(\lambda)X = 0$. The vector X is called an *Eigenvector* of $M(z)$ associated to the eigenvalue λ. It should be noted that each finite eigenvalue of $M(z)$ is a root of the characteristic polynomial $\det(M(z))$. The polynomial eigenvalue problem is to find an eigenvalue λ and a non-zero vector $X \in \mathbb{C}^n$ such that $M(\lambda)X = 0$. For $m = 1$, it is actually the generalized eigenvalue problem

$$AX = \lambda BX,$$

and in addition, if $B = I$, we have the standard eigenvalue problem

$$AX = \lambda X.$$

Computing eigenvalues of a matrix polynomial is a hard problem. There are iterative methods to compute these eigenvalues (for reference see [5]). Moreover, when computing pseudospectra of matrix polynomials, which provide information about the global sensitivity of the eigenvalues, a particular region of the (possibly extended) complex plane must be identified that contains the eigenvalues of interest, and bounds clearly help to determine such region (for details see [6]). Therefore, it is useful to find the location of these eigenvalues. Note that, if A_0 is singular, then 0 is an eigenvalue of $M(z)$, and if A_m is singular, then 0 is an eigenvalue of the matrix polynomial $z^m M(1/z)$. Therefore, to locate the eigenvalues of these matrix polynomials, we always assume that A_0 and A_m are non-singular.

Notations:

For a matrix $A \in \mathbb{C}^{n\times n}$, the notation $A \geq 0$ means "*A is positive semidefnite*", that is for every vector $X \in \mathbb{C}^n$ we have $X^*AX \geq 0$. By $A > 0$, we mean "*A is positive definite*", that is $X^*AX > 0$ for every $X \in \mathbb{C}^n$. Also in this paper for any two matrices $A, B \in \mathbb{C}^{n\times n}$, the notation $A \geq B$ means $A - B \geq 0$. *Throughout this paper,* $\|.\|$ *denotes a subordinate matrix norm.*

In the theory of distribution of zeros of polynomials with complex coefficients, we have the following result due to Cauchy [4, p. 123]

Theorem A *Let* $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$ *be a polynomial of degree m and*

$$\mathcal{M} = \max\left\{ \left|\frac{a_{m-1}}{a_m}\right|, \left|\frac{a_{m-2}}{a_m}\right|, \left|\frac{a_{m-3}}{a_m}\right|, \ldots, \left|\frac{a_1}{a_m}\right|, \left|\frac{a_0}{a_m}\right| \right\},$$

then all the zeros of $P(z)$ *lie in the circle* $|z| < 1 + \mathcal{M}$.

As an application of Theorem A, Dehmer [3] proved the following (also see [4, Theorem 27.2])

Theorem B *Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$, $a_m \neq 0$, $m \geq 1$ be a complex polynomial of degree m, such that $|a_m| > |a_i|$ for all $i = 0, 1, \ldots, m-1$, then all the zeros of $P(z)$ lie in the disk $|z| < 2$.*

Trı́nh et al. [2] extended this result to the matrix polynomials and proved the following:

Theorem C *Let $M(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial, whose coefficients $A_i \in \mathbb{C}^{n\times n}$ satisfying the following dominant property*

$$\|A_m\| > \|A_i\| \ \forall\, i = 0, 1, 2, \ldots, m-1.$$

Then each eigenvalue λ of *$M(z)$ locate in the open disk*

$$|\lambda| < 1 + \|A_m\| \|A_m^{-1}\|. \tag{2}$$

In the same paper, they proved the direct extension of Theorem A to matrix polynomial in the form of the following result:

Theorem D *Let $M(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial of degree m, whose coefficients $A_k \in \mathbb{C}^{n\times n}$.*

Then each eigenvalue λ of $M(z)$ satisfies

$$|\lambda| < 1 + \mathcal{M}, \tag{3}$$

where

$$\mathcal{M} = \max_{0 \leq k \leq m-1} \|A_k\| \|A_m^{-1}\|.$$

In this paper, we obtain an annulus containing all the eigenvalues of the matrix polynomial $M(z)$ and use it to obtain a refinement of Theorem D. In this direction, we have the following:

Theorem 1.1 *Let $M(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$ be a matrix polynomial of degree m, where $A_k \in \mathbb{C}^{n\times n}$ and A_0, A_m are invertible. If $\alpha_1, \alpha_2, \ldots, \alpha_m$ are m non-zero real or complex numbers, such that $\sum_{k=1}^{m} |\alpha_k| \leq 1$, then each eigenvalue λ of $M(z)$ satisfies $r_1 \leq |\lambda| \leq r_2$, where*

$$r_1 = \min_{1 \leq k \leq m} \left| \frac{\alpha_k}{\|A_k\| \|A_0^{-1}\|} \right|^{1/k}$$

and

$$r_2 = \max_{1 \leq k \leq m} \left| \frac{1}{\alpha_k} \|A_{m-k}\| \|A_m^{-1}\| \right|^{1/k}.$$

Remark 1 *If we take $n = 1$ and let $A_i = [a_i], i = 0, 1, \ldots, m$, we get a result due to Aziz and Qayoom* [1] *for the zeros of a polynomial with real or complex coefficients.*

Now as a refinement of Theorem D, we have the following:

Theorem 1.2 *Let*

$$M(z) = A_m z^m + A_p z^p + A_{p-1} z^{p-1} + \cdots + A_1 z + A_0$$
$$= A_m z^m + \sum_{k=0}^{p} A_k z^k, \quad 0 \le p \le m - 1$$

be a matrix polynomial of degree m, whose coefficients $A_k \in \mathbb{C}^{n \times n}$ $(k = 0, 1, 2, \ldots, p, m)$ and let

$$\mathcal{M} = \max_{0 \le k \le p} \|A_k\| \|A_m^{-1}\|,$$

then every eigenvalue λ of $M(z)$ satisfies

$$|\lambda| \le \{(1 + \mathcal{M})^{p+1} - 1\}^{\frac{1}{m}}. \tag{4}$$

For $p = m - 1$, we have the following:

Corollary 1 *Let*

$$M(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$$

be a matrix polynomial of degree m, whose coefficients $A_k \in \mathbb{C}^{n \times n}$ $(k = 0, 1, 2, \ldots, m)$ and let

$$\mathcal{M} = \max_{0 \le k \le m-1} \|A_k\| \|A_m^{-1}\|,$$

then every eigenvalue λ of $M(z)$ satisfies

$$|\lambda| \le \{(1 + \mathcal{M})^{m} - 1\}^{\frac{1}{m}}. \tag{5}$$

This result is a refinement of Theorem D.

Remark 2 *A result of Aziz and Qayoom* [1] *for the distribution of zeros of polynomials with real or complex coefficients is a special case of Theorem1.2, when we take $n = 1$ and $A_i = [a_i]$, forall, $i = 0, 1, 2, \ldots, m$.*

In particular, if $\|A_m\| > \|A_i\|$, $\forall\, i = 0, 1, 2, \ldots, p$, then from Theorem 1.2, we get the following:

Corollary 2 *Let*

$$M(z) = A_m z^m + A_p z^p + A_{p-1} z^{p-1} + \cdots + A_1 z + A_0$$
$$= A_m z^m + \sum_{k=0}^{p} A_k z^k, \quad 0 \le p \le m-1$$

be a matrix polynomial of degree m, *whose coefficients* $A_k \in \mathbb{C}^{n \times n}$ $(k = 0, 1, 2, \ldots, p, m)$ *and let* $\|A_m\| > \|A_i\|, \forall\, i = 0, 1, 2, \ldots, p$, *then each eigenvalue* λ *of* $M(z)$ *satisfies*

$$|\lambda| \le \{(1 + \|A_m\| \|A_m^{-1}\|)^{p+1} - 1\}^{\frac{1}{m}}. \tag{6}$$

If in Corollary 2, we choose $p = m-1$, we get the following:

Corollary 3 *Let*

$$M(z) = A_m z^m + A_{m-1} z^{m-1} + \cdots + A_1 z + A_0$$

be a matrix polynomial of degree m, *whose coefficients* $A_k \in \mathbb{C}^{n \times n}$ $(k = 0, 1, 2, \ldots, m)$ *and let* $\|A_m\| > \|A_i\|, \forall\, i = 0, 1, 2, \ldots, m-1$, *then each eigenvalue* λ of $M(z)$ *satisfies*

$$|\lambda| \le \{(1 + \|A_m\| \|A_m^{-1}\|)^{m} - 1\}^{\frac{1}{m}}. \tag{7}$$

This result is a refinement of Theorem C.

2 Lemmas

Before proving the main results, let us first prove the following lemma:

Lemma 1 *Let* $M(z) = A_m z^m + A_p z^p + \cdots + A_1 z + A_0$, $0 \le p \le m-1$ *be a matrix polynomial of degree* m, *where* $A_k \in \mathbb{C}^{n \times n}$ *and* A_m *is invertible. If* $\alpha_1, \alpha_2, \ldots, \alpha_{p+1}$ *are* $p+1$ *non-zero real or complex numbers, such that* $\sum_{k=1}^{p+1} |\alpha_k| \le 1$, *then each eigenvalue* λ *of* $M(z)$ *lie in the disk*

$$|\lambda| \le \max_{1 \le k \le p+1} \left| \frac{1}{\alpha_k} \|A_{p-k+1}\| \|A_m^{-1}\| \right|^{\frac{1}{m-p+k-1}}.$$

Proof Let

$$r = \max_{1 \le k \le p+1} \left| \frac{1}{\alpha_k} \|A_{p-k+1}\| \|A_m^{-1}\| \right|^{\frac{1}{m-p+k-1}}.$$

This implies

$$r^{m-p+k-1} \geq \frac{1}{|\alpha_k|}\|A_{p-k+1}\|\|A_m^{-1}\|,$$

or

$$|\alpha_k| \geq \|A_{p-k+1}\|\|A_m^{-1}\|\frac{1}{r^{m-p+k-1}},$$

therefore, we have

$$\begin{aligned}\sum_{k=1}^{p+1}|\alpha_k| &\geq \sum_{k=1}^{p+1}\|A_{p-k+1}\|\|A_m^{-1}\|\frac{1}{r^{m-p+k-1}}\\ &= \|A_0\|\|A_m^{-1}\|\frac{1}{r^m} + \cdots + \|A_p\|\|A_m^{-1}\|\frac{1}{r^{m-p}}.\end{aligned}$$

This implies

$$\sum_{k=1}^{p+1}|\alpha_k| \geq \sum_{j=0}^{p}\|A_j\|\|A_m^{-1}\|\frac{1}{r^{m-j}}. \tag{8}$$

Let λ be any eigenvalue of $M(z)$ and X be corresponding unit eigenvector. If possible, suppose that $|\lambda| > r$, then we have by using (8)

$$\begin{aligned}\|M(\lambda)X\| &= \|(A_m\lambda^m + A_p\lambda^p + \cdots + A_1\lambda + A_0)X\|\\ &\geq |\lambda|^m\|A_m^{-1}\|^{-1}\left\{1 - \sum_{j=0}^{p}\|A_j\|\|A_m^{-1}\|\frac{1}{|\lambda|^{m-j}}\right\}\\ &> |\lambda|^m\|A_m^{-1}\|^{-1}\left\{1 - \sum_{j=0}^{p}\|A_j\|\|A_m^{-1}\|\frac{1}{r^{m-j}}\right\}\\ &\geq |\lambda|^m\|A_m^{-1}\|^{-1}\left\{1 - \sum_{k=1}^{p+1}|\alpha_k|\right\}\\ &\geq 0.\end{aligned}$$

Therefore $\|M(\lambda)X\| > 0$ for $|\lambda| > r$. Hence our supposition is wrong and thus each eigenvalue λ of $M(z)$ lies in

$$|\lambda| \leq r = \max_{1\leq k\leq p+1}\left|\frac{1}{\alpha_k}\|A_{p-k+1}\|\|A_m^{-1}\|\right|^{m-p+k+1}.$$

□

3 Proofs of theorems

Proof of 1.1 We have $M(z) = A_0 + A_1z + \cdots + A_mz^m$. Let λ be an eigenvalue of $M(z)$ and $X \in \mathbb{C}^n$ be the corresponding eigenvector of $M(z)$. We first show that each eigenvalue λ of $M(z)$ lie in $|\lambda| \leq r_2$.

We have

$$r_2 = \max_{1 \leq k \leq m} \left| \frac{1}{\alpha_k} \|A_{m-k}\| \|A_m^{-1}\| \right|^{\frac{1}{k}}. \tag{9}$$

Applying the case when $p = m - 1$, we have from Lemma 1 that each eigenvalue λ of $M(z)$ lies in

$$|\lambda| \leq r = \max_{1 \leq k \leq m} \left| \frac{1}{\alpha_k} \|A_{m-k}\| \|A_m^{-1}\| \right|^{k} = r_2.$$

Hence we conclude that $|\lambda| \leq r_2$.

Now, consider the matrix polynomial

$$M^*(z) = z^m M\left(\frac{1}{z}\right) = A_0 z^m + A_1 z^{m-1} + \cdots + A_m.$$

Proceeding similarly as above, we have each eigenvalue λ of $M^*(z)$ satisfies

$$\begin{aligned} |\lambda| &\leq \max_{1 \leq k \leq m} \left| \frac{1}{\alpha_k} \|A_k\| \|A_0^{-1}\| \right|^{\frac{1}{k}} \\ &= \frac{1}{\min_{1 \leq k \leq m} \left| \frac{\alpha_k}{\|A_0^{-1}\| \|A_k\|} \right|^{\frac{1}{k}}} \\ &= \frac{1}{r_1}. \end{aligned}$$

Replacing z by $1/z$ and noting that $M(z) = z^m M^*\left(\frac{1}{z}\right)$, we conclude that if λ is an eigenvalue of $M(z)$, then

$$|\lambda| \geq r_1 = \min_{1 \leq k \leq m} \left| \frac{\alpha_k}{\|A_0^{-1}\| \|A_k\|} \right|^{\frac{1}{k}}.$$

This completes the proof of theorem. □

Proof of 1.2 We have

$$\mathcal{M} = \max_{1 \leq k \leq p+1} \|A_{p-k+1}\| \|A_m^{-1}\|,$$

therefore

$$\|A_{p-k+1}\| \|A_m^{-1}\| \leq \mathcal{M} \text{ for } k = 1, 2, 3, \ldots, p+1. \tag{10}$$

Now let us choose

$$\alpha_k = \left\{\frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\right\}\left\{\frac{\|A_{p-k+1}\|\|A_m^{-1}\|}{(1+\mathcal{M})^{m-p+k-1}}\right\} \text{ for } k=1,2,3,\ldots,m. \tag{11}$$

This implies

$$\frac{1}{\alpha_k}\|A_{p-k+1}\|\|A_m^{-1}\| = \frac{(1+\mathcal{M})^{m-p+k-1}\left[(1+\mathcal{M})^{p+1}-1\right]}{(1+\mathcal{M})^m} \text{ for } k=1,2,3,\ldots,m. \tag{12}$$

Since $A_{p-k+1}=0$ for $k=p+2,p+3,\ldots,m$, therefore $\alpha_k=0,$ for $k=p+2,p+3,\ldots,m$. Hence from (11), we have by using (10)

$$\begin{aligned}
\sum_{k=1}^{m}|\alpha_k| &= \sum_{k=1}^{p+1}|\alpha_k| \\
&= \sum_{k=1}^{p+1}\left|\left\{\frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\right\}\left\{\frac{\|A_{p-k+1}\|\|A_m^{-1}\|}{(1+\mathcal{M})^{m-p+k-1}}\right\}\right| \\
&= \frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\sum_{k=1}^{p+1}\|A_{p-k+1}\|\|A_m^{-1}\|\frac{1}{(1+\mathcal{M})^{m-p}(1+\mathcal{M})^{k-1}} \\
&\le \frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\sum_{k=1}^{p+1}\frac{\mathcal{M}}{(1+\mathcal{M})^{m-p}}\frac{1}{(1+\mathcal{M})^{k-1}} \\
&= \left(\frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\right)\left(\frac{\mathcal{M}}{(1+\mathcal{M})^{m-p}}\right)\sum_{k=1}^{p+1}\frac{1}{(1+\mathcal{M})^{k-1}} \\
&= \left(\frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\right)\left(\frac{\mathcal{M}}{(1+\mathcal{M})^{m-p}}\right)\left(\frac{1-\frac{1}{(\mathcal{M}+1)^{p+1}}}{1-\frac{1}{(\mathcal{M}+1)}}\right) \\
&= \left(\frac{(1+\mathcal{M})^m}{(1+\mathcal{M})^{p+1}-1}\right)\left(\frac{\mathcal{M}}{(1+\mathcal{M})^{m-p}}\right)\frac{(1+\mathcal{M})^{p+1}-1}{(1+\mathcal{M})^p\mathcal{M}} \\
&= 1.
\end{aligned}$$

That is

$$\sum_{k=1}^{m}|\alpha_k| \le 1.$$

Now by Lemma 1 and with the help of (12), we have that each eigenvalue λ of $M(z)$ satisfies

$$
\begin{aligned}
|\lambda| \leq r_2 &= \max_{1 \leq k \leq p+1} \left\{ \frac{1}{\alpha_k} \|A_{p-k+1}\| \|A_m^{-1}\| \right\}^{\frac{1}{m-p+k-1}} \\
&= \max_{1 \leq k \leq p+1} \left\{ \frac{(1+\mathcal{M})^{m-p+k-1}\left[(1+\mathcal{M})^{p+1}-1\right]}{(1+\mathcal{M})^m} \right\}^{\frac{1}{m-p+k-1}} \\
&= (1+\mathcal{M}) \max_{1 \leq k \leq p+1} \left\{ \frac{(1+\mathcal{M})^{p+1}-1}{(1+\mathcal{M})^m} \right\}^{\frac{1}{m-p+k-1}} \\
&= (1+\mathcal{M}) \left\{ \frac{(1+\mathcal{M})^{p+1}-1}{(1+\mathcal{M})^m} \right\}^{\frac{1}{m}} \\
&= \left((1+\mathcal{M})^{p+1}-1\right)^{\frac{1}{m}}.
\end{aligned}
$$

This completes the proof of theorem. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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