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ORIGINAL RESEARCH PAPER

Refinements of some Berezin number inequalities and related questions

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Abstract
In this paper, we obtain some inequalities for the Berezin number of operators on reproducing kernel Hilbert spaces. These inequalities improve on some earlier related inequalities. In addition, related Berezin number inequalities are also given.

Keywords Berezin number · Reproducing kernel Hilbert space · Inequality

Mathematics Subject Classification Primary 47A63

1 Introduction and terminologies

Throughout this paper, $\mathcal{B}(\mathcal{H})$ denotes the C^*- algebra of all linear bounded operators acting on a non trivial complex Hilbert space $\mathcal{H}$ with inner product $\langle .,. \rangle$ and associated norm $\|.\|$. Let Ω be a nonempty set. A functional Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$ is a Hilbert space of complex valued functions, which has the property that point evaluations are continuous, i.e., for each $\lambda \in \Omega$ the map $f \longmapsto f(\lambda)$ is a continuous linear functional on $\mathcal{H}$. The Riesz representation theorem ensues that for each $\lambda \in \Omega$ there exists a unique element $k_\lambda \in \mathcal{H}$ suchthat $f(\lambda) = \langle f, k_\lambda \rangle$ for all $f \in \mathcal{H}$. The set $\{k_\lambda : \lambda \in \Omega\}$ is called the reproducing kernel of the space $\mathcal{H}$. If $\{e_n\}_{n \geq 0}$ is an orthonormal basis for a functional Hilbert space $\mathcal{H}$, then the reproducing kernel of $\mathcal{H}$

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is given by $k_\lambda(z) = \sum_{n=0}^{+\infty} \overline{e_n(\lambda)} e_n(z)$ (see [8]). For $\lambda \in \Omega$, let $\hat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of $\mathcal{H}$. Let A a bounded linear operator on $\mathcal{H}$, the Berezin symbol of A, which firstly have been introduced by Berezin [3, 4] is the function $\tilde{A}$ on Ω defined by

$$\tilde{A}(\lambda) := \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.$$

The Berezin set and the Berezin number of the operator A are defined respectively by:

$$\mathbf{Ber}(A) := \left\{ \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle : \lambda \in \Omega \right\},$$

and

$$\mathbf{Ber}(A) := \sup\left\{ \left| \left\langle A\hat{k}_\lambda, \hat{k}_\lambda \right\rangle \right| : \lambda \in \Omega \right\}.$$

It is clear that the Berezin symbol $\tilde{A}$ is the bounded function on Ω whose value lies in the numerical range of the operator A and hence for any $A \in \mathcal{B}(\mathcal{H})$,

$$\mathbf{Ber}(A) \subset W(A) \text{ and } \mathbf{ber}(A) \leq \omega(A),$$

where

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \ \|x\| = 1\},$$

is the numerical range of the operator A and

$$\omega(A) = \{|\langle Ax, x\rangle| : x \in \mathcal{H}, \ \|x\| = 1\},$$

is the numerical radius of A.

Moreover, the Berezin number of an operator A satisfies the following properties:

(i) $\mathbf{ber}(A) \leq \|A\|$.
(ii) $\mathbf{ber}(\alpha A) = |\alpha| \mathbf{ber}(A)$ for all $\alpha \in \mathbb{C}$.
(iii) $\mathbf{ber}(A + B) \leq \mathbf{ber}(A) + \mathbf{ber}(B)$ for all $A, B \in \mathcal{B}(\mathcal{H})$.

Notice that, in general, the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions, (for instance on the unit disc $D = \{z \in \mathbb{C} : |z < 1|\}$), then $\mathbf{ber}(.)$ defines a norm on $\mathcal{B}(\mathcal{H}(D))$ (see [9, 10]).

The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. A nice property of the Berezin symbol is mentioned next. If $\tilde{A}(\lambda) = \tilde{B}(\lambda)$ for all $\lambda \in \Omega$, then $A = B$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol and Berezin number have been studied by many mathematicians over the years, a few of them are [2, 5–7, 14–16].

In this paper we introduce some refnements of Berezin number inequalities on reproducing kernel Hilbert spaces. Some of the obtained results refinement the following paper [1, 5, 12].

2 Main results

In this section, we present our results. First, we start with the following lemmas that will be used to develop new results in this paper.

Lemma 1 [13] *Let a_i $(i = 1, 2, ..., n)$ be apositive real number and $r \geq 1$. Then*

$$\left(\sum_{i=1}^{n} a_i\right)^r \leq n^{r-1} \sum_{i=1}^{n} a_i^r.$$

Lemma 2 [11] *Let $T \in B(\mathcal{H})$ be positive operator and let $x \in \mathcal{H}$ such that $\|x\| = 1$. Then*

$$\langle Tx, x\rangle^r \leq \langle T^r x, x\rangle \text{for } r \geq 1.$$

Lemma 3 [11] *Let $T \in B(\mathcal{H})$ and let f and g be non-negative continuous functions on $[0, +\infty)$ such that $f(t)g(t) = t$ for all $t \in [0, +\infty)$. Then*

$$|\langle Tx, y\rangle|^2 \leq \left\langle f^2(|T|)x, x\right\rangle \left\langle g^2(|T^*|)y, y\right\rangle,$$

for all $x, y \in \mathcal{H}$.

The following Theorem is proved in [12].

Theorem 1 *Let $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$, $r \geq 1$ and let f and g be non-negative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2} \mathbf{ber}\left(\sum_{i=1}^{n} \left(\left[A_i^* g^2\left(\left|X_i^*\right|\right) A_i\right]^r + \left[B_i^* f^2(|X_i|) B_i\right]^r\right)\right).$$

The first main result in this section is refinement of the above Theorem.

Theorem 2 *Let $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$ and let f and g be non-negative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\le\frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right]^r+\left[B_i^*f^2(|X_i|)B_i\right]^r\right)\right)-\frac{n^{r-1}}{2}\inf_{\left\|\hat{k}_\lambda\right\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right)=\sum_{i=1}^{n}\left(\left\langle\left[A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right]^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

for $r\ge 1$.

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$, then

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&=\left|\sum_{i=1}^{n}\left\langle\left(A_i^*X_iB_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&\le\sum_{i=1}^{n}\left|\left\langle A_i^*X_iB_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&=\sum_{i=1}^{n}\left|\left\langle X_iB_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle\right|^r\\
&\le\left(\sum_{i=1}^{n}\left\langle f^2(|X_i|)B_i\hat{k}_\lambda,B_i\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle g^2\left(\left|X_i^*\right|\right)A_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^r\\
&\quad\text{(by using Lemma 2)}\\
&\le n^{r-1}\sum_{i=1}^{n}\left\langle f^2(|X_i|)B_i\hat{k}_\lambda,B_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle g^2\left(\left|X_i^*\right|\right)A_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\\
&\quad\text{(by using Lemma 2)}\\
&\le n^{r-1}\sum_{i=1}^{n}\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\\
&\quad\text{(by using Lemma 1)}\\
&=\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r k_\lambda,k_\lambda\right\rangle+\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r k_\lambda,k_\lambda\right\rangle\right)\right]\\
&\quad-\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r k_\lambda,k_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r k_\lambda,k_\lambda\right\rangle^{\frac{1}{2}}\right)^2\right]\\
&\text{(since }\sqrt{a}\sqrt{b}=\frac{a+b}{2}-\frac{\left(\sqrt{a}-\sqrt{b}\right)^2}{2}\text{ where }a,b\ge 0)\\
&=\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left\langle\left(\left(B_i^*f^2(|X_i|)B_i\right)^r+\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right]\\
&\quad-\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2\right]\\
&\le\frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right]^r+\left[B_i^*f^2(|X_i|)B_i\right]^r\right)\right)\\
&\quad-\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2\right].
\end{aligned}$$

Now, by taking supremum over $\lambda\in\Omega$, we get the desired inequality. □

Corollary 1 *Let* $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$, $r \geq 1$ *and* $0 \leq p \leq 1$. *Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^*\left|X_i^*\right|^{2(1-p)}A_i\right]^r + \left[B_i^*|X_i|^{2p}B_i\right]^r\right)\right) - \frac{n^{r-1}}{2}\inf_{\left\|\hat{k}_\lambda\right\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle\left[A_i^*\left|X_i^*\right|^{2(1-p)}A_i\right]^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle\left[B_i^*|X_i|^{2p}B_i\right]^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

Proof The result follows immediately from Theorem 2 for $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \leq p \leq 1)$. □

For $A_i = B_i = I$ $(i = 1, 2, ..., n)$ in Theorem 2 we get the following result.

Corollary 2 *Let* $X_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n\ )$ *and let* f *and* g *as in Theorem 2. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n} g^{2r}\left(\left|X_i^*\right|\right) + f^{2r}(|X_i|)\right) - \frac{n^{r-1}}{2}\inf_{\left\|\hat{k}_\lambda\right\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle g^{2r}\left(\left|X_i^*\right|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle f^{2r}(|X_i|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

for $r \geq 1$.

Remark 1 (1) If we take $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \leq p \leq 1)$ in Corollary 2, then

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}|X_i|^{2rp} + \left|X_i^*\right|^{2r(p-1)}\right) - \frac{n^{r-1}}{2}\inf_{\left\|\hat{k}_\lambda\right\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle\left|X_i^*\right|^{2r(p-1)}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle|X_i|^{2rp}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

for $r \geq 1$.

(2) Taking $f(t) = g(t) = t^{\frac{1}{2}}$ $(t \in [0, +\infty))$ and $r = 1$in Corollary 2, we get

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{1}{2}\mathbf{ber}\left(\sum_{i=1}^{n}|X_i| + |X_i^*|\right) - \frac{1}{2}\inf_{\|\hat{k}_\lambda\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle |X_i^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle |X_i|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

For $X_i = I$ $(i = 1, 2, ..., n)$, $r \geq 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ $(t \in [0, +\infty))$ in Theorem 2 we get the following inequality.

Corollary 3 *Let $A_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$ and $r \geq 1$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{n^{r-1}}{2}\mathbf{ber}\left(\sum_{i=1}^{n}\left(|A_i|^{2r} + |B_i|^{2r}\right)\right) - \frac{n^{r-1}}{2}\inf_{\|\hat{k}_\lambda\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle |A_i|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle |B_i|^{2r}\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

Remark 2 For $n = 1$ in Corollary 2, we get

$$\mathbf{ber}^r(A^*B) \leq \frac{1}{2}\mathbf{ber}((A^*A)^r + (B^*B)^r) - \frac{1}{2}\inf_{\|\hat{k}_\lambda\|=1}\eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \left(\left\langle (A^*A)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle (B^*B)^r\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

In [5] the authors proved the following theorem.

Theorem 3 *Let $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$, $r \geq 1$ and let f and g be non-negative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then for all $r \geq 1$, we have.*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{\sqrt{2}}\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^* g^2\left(|X_i^*|\right)A_i\right]^r + i\left[B_i^* f^2(|X_i|)B_i\right]^r\right)\right).$$

In the following theorem we introduce a refinement of the above Theorem.

Theorem 4 *Let $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$ and let f and g be non-negative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then for all $r \geq 1$, we have.*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{\sqrt{2}} \mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^* g^2\left(\left|X_i^*\right|\right)A_i\right]^r + i\left[B_i^* f^2(|X_i|)B_i\right]^r\right)\right) - \frac{n^{r-1}}{2} \inf_{\|\hat{k}_\lambda\|=1} \eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle\left[A_i^* g^2\left(\left|X_i^*\right|\right)A_i\right]^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle\left[B_i^* f^2(|X_i|)B_i\right]^r \hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$, then

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n}A_i^*X_iB_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&=\left|\sum_{i=1}^{n}\left\langle\left(A_i^*X_iB_i\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&\le\sum_{i=1}^{n}\left|\left\langle A_i^*X_iB_i\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right|^r\\
&=\sum_{i=1}^{n}\left|\left\langle X_iB_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle\right|^r\\
&\le\left(\sum_{i=1}^{n}\left\langle f^2(|X_i|)B_i\hat{k}_\lambda,B_i\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle g^2\left(\left|X_i^*\right|\right)A_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^r\\
&\quad\text{(by using Lemma 3)}\\
&\le n^{r-1}\sum_{i=1}^{n}\left\langle f^2(|X_i|)B_i\hat{k}_\lambda,B_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\left\langle g^2\left(\left|X_i^*\right|\right)A_i\hat{k}_\lambda,A_i\hat{k}_\lambda\right\rangle^{\frac{r}{2}}\\
&\quad\text{(by using Lemma 1)}\\
&\le n^{r-1}\sum_{i=1}^{n}\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\\
&\quad\text{(by using Lemma 2)}\\
&=\frac{n^{r-1}}{2}\left[\sum_{i=1}^{n}\left(\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)\right]\\
&\quad-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2\\
&\le\frac{n^{r-1}}{\sqrt{2}}\left[\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle+i\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right)\right]\\
&\quad-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2\\
&=\frac{n^{r-1}}{\sqrt{2}}\left[\sum_{i=1}^{n}\left\langle\left(\left(A_i^*f^2(|X_i|)A_i\right)^r+i\left(B_i^*g^2\left(\left|X_i^*\right|\right)B_i\right)^r\right)\hat{k}_\lambda,\hat{k}_\lambda\right\rangle\right]\\
&\quad-\frac{n^{r-1}}{2}\sum_{i=1}^{n}\left(\left\langle\left(B_i^*f^2(|X_i|)B_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left(A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right)^r\hat{k}_\lambda,\hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2
\end{aligned}$$

(as $|a+b|\le\sqrt{2}|a+ib|$, $\forall a,b\in\mathbb{R}$)

$$\le\frac{n^{r-1}}{\sqrt{2}}\,\mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right]^r+i\left[B_i^*f^2(|X_i|)B_i\right]^r\right)\right)-\frac{n^{r-1}}{2}\inf_{\|\hat{k}_\lambda\|=1}\eta\left(\hat{k}_\lambda\right),$$

.

where

$$\eta(k_\lambda)=\sum_{i=1}^{n}\left(\left\langle\left[A_i^*g^2\left(\left|X_i^*\right|\right)A_i\right]^rk_\lambda,k_\lambda\right\rangle^{\frac{1}{2}}-\left\langle\left[B_i^*f^2(|X_i|)B_i\right]^rk_\lambda,k_\lambda\right\rangle^{\frac{1}{2}}\right)^2.$$

By taking supremum over $\lambda\in\Omega$, we get

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \le \frac{n^{r-1}}{\sqrt{2}} \mathbf{ber}\left(\sum_{i=1}^{n}\left(\left[A_i^* g^2\left(\left|X_i^*\right|\right)A_i\right]^r + i\left[B_i^* f^2(|X_i|)B_i\right]^r\right)\right) - \frac{n^{r-1}}{2} \inf_{\|\hat{k}_\lambda\|=1} \eta\left(\hat{k}_\lambda\right).$$

This completes the proof. □

Putting $f(t) = t^p$ and $g(t) = t^{1-p}$ $(0 \le p \le 1)$ in Theorem 4 we get the following corollary.

Corollary 4 *Let $A_i, X_i, B_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$ and let f and g be non-negative continuous functions on $[0, +\infty)$ satisfying the relation $f(t)g(t) = t$ $(t \in [0, +\infty))$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} X_i\right) \le \frac{n^{r-1}}{\sqrt{2}} \mathbf{ber}\left(\sum_{i=1}^{n}\left(g^{2r}\left(\left|X_i^*\right|\right) + i f^{2r}(|X_i|)\right)\right) - \frac{n^{r-1}}{2} \inf_{\|\hat{k}_\lambda\|=1} \eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle g^{2r}\left(\left|X_i^*\right|\right)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle f^{2r}(|X_i|)\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

for all $r \ge 1$.

Proof Taking $A_i = B_i = I$ $(i = 1, 2, ..., n)$ in Theorem 4 we deduce the desired result. □

Next, taking $n = 1$, $r = 1$ and $f(t) = g(t) = t^{\frac{1}{2}}$ in Corollary 4 we get the following inequality

Corollary 5 *Let $T \in B(\mathcal{H})$. Then*

$$\mathbf{ber}(T) \le \frac{1}{\sqrt{2}} \mathbf{ber}(|T^*| + i|T|) - \frac{1}{2} \inf_{\|\hat{k}_\lambda\|=1} \eta\left(\hat{k}_\lambda\right),$$

where

$$\eta\left(\hat{k}_\lambda\right) = \sum_{i=1}^{n}\left(\left\langle |T^*|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}} - \left\langle |T|\hat{k}_\lambda, \hat{k}_\lambda\right\rangle^{\frac{1}{2}}\right)^2,$$

Now, we are in a position to prove the following result.

Theorem 5 *Let $T_i \in B(\mathcal{H})$ $(i = 1, 2, ..., n)$ with the Cartesian decomposition $T_i = A_i + iB_i$ for $i = 1, 2, ..., n$. Then*

$$\mathbf{ber}^r\left(\sum_{i=1}^{n} T_i\right) \leq 2^{\frac{r}{2}-1} n^{r-1} \sum_{i=1}^{n} \mathbf{ber}((|A_i|^r + |B_i|^r)).$$

Proof Let $\hat{k}_\lambda$ be the normalized reproducing kernel of $\mathcal{H}$, then we have

$$\begin{aligned}
&\left|\left\langle \left(\sum_{i=1}^{n} T_i\right)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^r \\
&= \left|\sum_{i=1}^{n} \left\langle T_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^r \\
&\leq \left(\sum_{i=1}^{n} \left|\left\langle T_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|\right)^r \\
&= \left(\sum_{i=1}^{n} \left|\left\langle A_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle + i\left\langle B_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|\right)^r \\
&= \left(\sum_{i=1}^{n} \left(\left|\left\langle A_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^2 + \left|\left\langle B_i\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right|^2\right)^{\frac{1}{2}}\right)^r \\
&\leq \left(\sum_{i=1}^{n} \left(\left\langle |A_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^2 + \left\langle |B_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^2\right)^{\frac{1}{2}}\right)^r \\
&\quad \text{(by using Lemma 3)} \\
&\leq n^{r-1} \sum_{i=1}^{n} \left(\left\langle |A_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^2 + \left\langle |B_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^2\right)^{\frac{r}{2}} \\
&\quad \text{(by using Lemma 1)} \\
&\leq 2^{\frac{r}{2}-1} n^{r-1} \sum_{i=1}^{n} \left(\left\langle |A_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^r + \left\langle |B_i|\hat{k}_\lambda, \hat{k}_\lambda \right\rangle^r\right) \\
&\quad \text{(by using Lemma 1)} \\
&\leq 2^{\frac{r}{2}-1} n^{r-1} \sum_{i=1}^{n} \left(\left\langle |A_i|^r\hat{k}_\lambda, \hat{k}_\lambda \right\rangle + \left\langle |B_i|^r\hat{k}_\lambda, \hat{k}_\lambda \right\rangle\right) \\
&= 2^{\frac{r}{2}-1} n^{r-1} \sum_{i=1}^{n} \left\langle (|A_i|^r + |B_i|^r)\hat{k}_\lambda, \hat{k}_\lambda \right\rangle.
\end{aligned}$$

Now, taking the supremum over all $\lambda \in \Omega$ we get the desired result. □

For $n = 1$ we get the following inequality.

Corollary 6 *Let $T \in B(\mathcal{H})$ with the Cartesian decomposition $T = A + iB$ and $r \geq 1$. Then*

$$\mathbf{ber}^r(T) \leq 2^{\frac{r}{2}-1} \mathbf{ber}((|A|^r + |B|^r)).$$

Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors

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