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ORIGINAL RESEARCH PAPER

Fixed point results for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space with an applications

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Abstract

This paper aims to prove fixed point results for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric spaces. We also provide an illustrative example to support our results. The results proved here will be utilized to show the existence of the solution of a nonlinear matrix equation as an application.

Keywords Fixed point · Interpolation · ψ-Hardy-Rogers type · Quasi-partial b-metric space · Non-linear matrix equation

Mathematics Subject Classification 47H10 · 54H25

1 Introduction and preliminaries

In 1989, Bakhtin [5] introduced the concept of b-metric space. Later, Czerwik [9] proved the results on b-metric spaces by weakening the triangle inequality coefficient and generalized Banach's contraction principle [6] to these spaces. Since then, several researchers have published their work in fixed point theory for various classes of the single and multivalued maps in b-metric space. For more literature on it, one can see [8, 22, 31, 43] and the references therein. In 2004, Ran and Reurings [40] followed by Nieto and Rodríguez-López [37] in 2006 introduced the study of fixed

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point theorems for partially ordered sets along with the application in the matrix and ordinary differential equations. For more applications, we refer the reader to [7, 19, 32, 42] and the references therein.

Matthew [34] introduced non-zero self-distance, which is applied in computer networking, data structure, and computer programming languages. The non-self distance generalizes the metric to partial metric axioms, accommodating both metric and topological properties of abstract spaces. Some of these properties are complete spaces, Cauchy sequences and contraction fixed point theorem, which generalizes the Banach contraction principle. Motivated by the work of Matthew [34], Karapinar et al. [22] initiated the concept of quasi-partial metric space and proved results for the existence of fixed points of self-mapping for this space. Later, Gupta and Gautam [16, 17] further generalized the quasi-partial metric space concept to the class of quasi-partial b-metric spaces. Furthermore, Gautam and Verma [12] discussed fixed point results via implicit mapping in quasi-partial b-metric space.

Recently, Karapinar [24] modified the classical Kannan [20, 21] contraction phenomena to an interpolative Kannan contraction one to maximize the rate of convergence of an operator to a unique fixed point. However, Karapinar and Agarwal [25] found a little gap in their work [24] about the assumption of the fixed point being unique. They provided a counter example to verify that the fixed point need not be unique and invalidate the assumption of a unique fixed point. Since then, several results for variants of interpolative mapping proved for single and multivalued in various abstract spaces. Aydi et al. [3] proved an interpolative Ćirić-Reich-Rus type contractions via the Branciari distance. Karapinal et al. [26] proved an interpolative Ćirić-Reich-Rus type contractions fixed point result on partial metric space. In 2019, Karapinar [23] proved the results for interpolative Hardy-Rogers type contractions in metric space. More information can be found in [2, 11] and the references therein. Thereafter, Muhammad et al. [36] proved the results on the fixed point theorem of the rational interpolative-type operator by using Dass-Gupta contraction mapping in metric space. In 2021, Gautam et al. [15] proved an Interpolative Chatterjea and cyclic Chatterjea contraction on quasi-partial b-metric space.

Furthermore, Errai et al. [10] gave some new results of interpolative Hardy-Rogers and Ćirić-Reich-Rus type contraction. Mishra et al. [35] proved the common fixed point theorems for interpolative Hardy-Rogers and Ćirić-Reich-Rus and Hardy-Rogers type contraction on quasi partial b-metric space. Aydi et al. [4] proved ω-interpolative Ćirić-Reich-Rus-type contractions. For more literature, we refer the reader to [1, 38, 46] and the references therein. Finally, Karapinar et al. [27] proved the fixed point theory in the setting of $(\alpha, \beta, \psi, \phi)$-interpolative contractions. Karapınar [28] gave the results for interpolative Kannan-Meir-Keeler type contraction. Khan et al. [30] proved the result on the interpolative (ϕ, ψ)-type Z-contraction. Karapinar et al. [29] proved new results on Perov-interpolative contractions of Suzuki type mappings, Pant and Shukla [39] proved new fixed point results for Proinov-Suzuki type contractions in metric spaces. Recently, Gautam et al. [13, 14] extended some results to interpolative type contractions.

We describe some definitions and theorems, which will help to develop our main results.

The property of quasi-partial b-metric space introduced in [16] is as follows:

Definition 1 [16] A quasi-partial b-metric space on a non empty set $\mathscr{M}$ is a mapping $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ such that for some real number $s \geq 1$ and all $\sigma, \varsigma, \zeta \in \mathscr{M}$:

(QPB1) $qp_b(\sigma, \varsigma) = qp_b(\sigma, \varsigma) = qp_b(\varsigma, \varsigma) \Rightarrow \sigma = \varsigma$;
(QPB2) $qp_b(\sigma, \sigma) \leq qp_b(\sigma, \varsigma)$;
(QPB3) $qp_b(\sigma, \sigma) \leq qp_b(\varsigma, \sigma)$; and
(QPB4) $qp_b(\sigma, \varsigma) \leq s[qp_b(\sigma, \zeta) + qp_b(\zeta, \varsigma)] - qp_b(\zeta, \zeta)$.

A quasi-partial b-metric space is a pair $(\mathscr{M}, qp_b)$ such that $\mathscr{M}$ is a non-empty set and $(\mathscr{M}, qp_b)$ is a quasi partial b-metric on $\mathscr{M}$. The number s is called the coefficient of $(\mathscr{M}, qp_b)$.

For a quasi-partial b-metric space $(\mathscr{M}, qp_b)$, the function $d_{qpb} : \mathscr{M} \times \mathscr{M} \to R^+$ defined by $d_{qp_b}(\sigma, \varsigma) = qp_b(\sigma, \varsigma) + qp_b(\varsigma, \sigma) - qp_b(\sigma, \sigma) - qp_b(\varsigma, \varsigma)$ is a b-metric on $\mathscr{M}$.

We defined $\mathbb{R}^+$ a set of all positive real number.

Lemma 1 [16] *Every quasi-partial metric space is a quasi-partial b-metric space, but the converse need not be true.*

The following are fundamental convergence properties of quasi- partial b- metric spaces.

Definition 2 [16] Let $(\mathscr{M}, qp_b)$ be a quasi partial b- metric space, then:

(i) a sequence $\{\sigma_i\} \subset \mathscr{M}$ converges to a point $\sigma \in \mathscr{M}$ if and only if

$$qp_b(\sigma, \sigma) = \lim_{i \to \infty} qp_b(\sigma_i, \sigma) = \lim_{i \to \infty} qp_b(\sigma, \sigma_i),$$

(ii) a sequence $\{\sigma_i\}$ of elements of $\mathscr{M}$ is called a Cauchy sequence if and only if

$$\lim_{i,j \to \infty} qp_b(\sigma_i, \sigma_j) \text{ and } \lim_{i,j \to \infty} qp_b(\sigma_j, \sigma_i)$$

exists and is finite,

(iii) the quasi-partial b-metric space $(\mathscr{M}, qp_b)$ is said to be complete if every Cauchy sequence $\{\sigma_i\} \subset \mathscr{M}$ converges with respect τ_{qp_b}, to a point $\sigma \in \mathscr{M}$ such that

$$\lim_{i,j \to \infty} qp_b(\sigma_i, \sigma_j) = qp_b(\sigma_j, \sigma_i).$$

Lemma 2 [16] *Let $(\mathscr{M}, qp_b)$ be a quasi-partial b-metric space and $(\mathscr{M}, d_{qp_b})$ be the corresponding b-metric space. Then $(\mathscr{M}, d_{qp_b})$ is complete if $(\mathscr{M}, qp_b)$ is complete.*

Lemma 3 [16] *Let $(\mathscr{M}, qp_b)$ be a quasi-partial b-metric space. Then the following hold*:

(i) if $qp_b(\sigma, \varsigma) = 0$, then $\sigma = \varsigma$.
(ii) if $\sigma \neq \varsigma$, then $qp_b(\sigma, \varsigma) > 0$ and $qp_b(\varsigma, \sigma) > 0$

Examples of quasi-partial b-metric space are given in [16] and [12] as follows:

Example 1 [12, 16] Following are the examples on quasi-partial b-metric space:

(i) Let $\mathscr{M} = [0, 1]$. Define $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ as $qp_b(\sigma, \varsigma) =| \sigma - \varsigma | + \sigma$. It is easy to show that $(\mathscr{M}, qp_b)$ is a quasi-partial b-metric space.
(ii) Let $\mathscr{M} = [0, \infty)$. Define $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ as $qp_b(\sigma, \varsigma) = \ln(\sigma, \varsigma)$. Then $(\mathscr{M}, qp_b)$ is a quasi-partial b-metric space.
(iii) Let $\mathscr{M} = [0, \frac{\pi}{4}]$. Define $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ as $qp_b(\sigma, \varsigma) = \sin\sigma + \sin\varsigma$. Then $(\mathscr{M}, qp_b)$ is a quasi-partial b-metric space.
(iv) Let $\mathscr{M} = [0, 1]$. Define $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ as $qp_b(\sigma, \varsigma) = (\sigma - \varsigma)^2 + \sigma$. $(\mathscr{M}, qp_b)$ is a quasi-partial b-metric space.
(v) Let $\mathscr{M} = [0, \infty)$. Define $qp_b : \mathscr{M} \times \mathscr{M} \to \mathbb{R}^+$ as $qp_b(\sigma, \varsigma) = e^\sigma + e^\varsigma$. Then $(\mathscr{M}, qp_b)$ is a quasi-partial b-metric space.

The notion of an almost altering distance function introduced by Popa [44].

Definition 3 [44] A function $\psi : [0, \infty) \to [0, \infty)$ is almost altering distance if

(Ψ1) ψ is continuous;
(Ψ2) $\psi(t) = 0$ if and only if $t = 0$.

Example 2 Following function satisfies the conditions imposed in Definition 3 on the function ψ:

$$\psi(t) = \begin{cases} t, & \text{for } t = [0, 1]; \\ \dfrac{1}{2t + 1} & \text{for } t \in (1, \infty). \end{cases}$$

Now, we introduce some preliminary results:

The following results for interpolative Kannan contraction has been proved in [24] as follows:

Definition 4 [24] Let $(\mathscr{M}, \varrho)$ be a metric space, the mapping $\kappa : \mathscr{M} \to \mathscr{M}$ is said to be interpolative Kannan contraction mappings if

$$\varrho(\kappa\sigma, \kappa\varsigma) \leq \lambda[\varrho(\sigma, \kappa\sigma)]^\alpha . [\varrho(\varsigma, \kappa\varsigma)]^{1-\alpha}, \tag{1}$$

for all $\sigma, \varsigma \in \mathscr{M}$ with $\sigma \neq \kappa\sigma$, where $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$.

Theorem 1 [24] *Let $(\mathscr{M}, \varrho)$ be a complete metric space and κ be an interpolative Kannan type contraction. Then κ has a unique fixed point in $\mathscr{M}$.*

Karapinar et al. [25] gave a counterexample to Theorem 1, showing that the fixed point may not be unique. Then prove the following Theorem.

Theorem 2 [24] *Let $(\mathscr{M}, \varrho)$ be a complete metric space. $\kappa : \mathscr{M} \to \mathscr{M}$ be a mapping such that*

$$\rho(\kappa\sigma, \kappa\varsigma) \leq \lambda[\rho(\sigma, \kappa\sigma)]^{\alpha}.[\rho(\varsigma, \kappa\varsigma)]^{1-\alpha}, \tag{2}$$

for all $\sigma, \varsigma \in \mathscr{M} \backslash Fix(\kappa)$ where $Fix(\kappa) = \{\sigma \in \mathscr{M}, \kappa\sigma = \sigma\}$. Then κ has a fixed point in $\mathscr{M}$.

Further, Karapinar et al. [23] introduced the concept of interpolative Hardy-Rogers type contractions as follows:

Definition 5 [24] Let $(\mathscr{M}, \varrho)$ be a metric space. We say that the self mapping $\kappa : \mathscr{M} \to \mathscr{M}$ is an interpolative Hardy-Rogers type contraction if there exists $\lambda \in [0, 1)$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that

$$\rho(\kappa\sigma, \kappa\varsigma) \leq \lambda[\rho(\sigma, \varsigma)]^{\beta}.[\rho(\sigma, \kappa\sigma)]^{\alpha}.[\rho(\varsigma, \kappa\varsigma)]^{\gamma}.\left[\frac{1}{2}(\rho(\sigma, \kappa\varsigma) + \rho(\varsigma, \kappa\sigma))\right]^{1-\alpha-\beta-\gamma}, \tag{3}$$

for all $\sigma, \varsigma \in \mathscr{M} \backslash Fix(\kappa)$.

Using Definition 5, Karapinar et al. [23] proved the following theorem:

Theorem 3 *Let $(\mathscr{M}, \rho)$ be a complete metric space and κ be an interpolative Hardy-Rogers type contraction. Then κ has a fixed point in $\mathscr{M}$.*

In this paper, motivated by the results of Gupta and Gautam [16, 17], Karapinar et al. [23], Muhammad et al. [36], Gautam et al. [15] and Mishra et al. [35], we prove a fixed point theorem for interpolative ψ-Hardy-Rogers type contraction mappings in quasi-partial b-metric space and obtain a fixed point results of such mappings. Some examples are provided to demonstrate our results. Finally, some applications to non-linear matrix equations are given to validate the usefulness of the result.

2 Main results

We commence this section by introducing the concept of interpolative Hardy-Rogers type contractions in quasi partial b-metric space.

The following definition is the extended version of Definition 5 from metric space to quasi partial b-metric space setting.

Definition 6 Let $(\mathscr{M}, qp_b, s)$ be a quasi partial b- metric space. We say that the self mapping $\boldsymbol{\kappa} : \mathscr{M} \to \mathscr{M}$ is an interpolative ψ-Hardy-Rogers type contraction if there exists $s \geq 1$, $\delta \in [0, \frac{1}{s})$ and $\alpha, \beta, \gamma \in (0, 1)$ with $\alpha + \beta + \gamma < 1$, such that

$$\begin{aligned}\psi(qp_b(\kappa\sigma, \kappa\varsigma)) \leq \delta\Big[qp_b(\sigma, \varsigma)\Big]^{\beta}.\Big[qp_b(\sigma, \kappa\sigma)\Big]^{\alpha}.\Big[qp_b(\varsigma, \kappa\varsigma\Big]^{\gamma}.\\ \left[\frac{1}{s^2}(qp_b(\sigma, \kappa\varsigma) + qp_b(\varsigma, \kappa\sigma))\right]^{1-\alpha-\beta-\gamma}\end{aligned} \tag{4}$$

for all $\sigma, \varsigma \in \mathscr{M} \backslash Fix(\kappa)$.

We prove the following theorem.

Theorem 4 *Let $(\mathscr{M}, qp_b, s)$ be a complete quasi partial b-metric space and κ be an interpolative Hardy-Rogers type contraction. Then κ posses a fixed point in $\mathscr{M}$.*

Proof Let σ_0 be an arbitrary point in $\mathscr{M}$. Define $\{\sigma_i\} \subset \mathscr{M}$ by $\sigma_i = \sigma_{i-1}$ for all $i \geq 1$, if there exists $i_0 \in \mathbb{N}$ with $\sigma_i = \sigma_{i+1}$, then σ_i is a fixed point of $\mathscr{M}$. This complete the proof. Suppose now that $\sigma_i \neq \sigma_{i+1}$, for all $i \in \mathbb{N}$. Then $qp_b(\sigma_i, \sigma_{i+1}) > 0$ for $i \in \mathbb{N}$. By taking $\sigma = \sigma_{i-1}$, $\varsigma = \sigma_i$ and , using (4) we get

$$\begin{aligned} qp_b(\sigma_i, \sigma_{i+1}) &= \psi(qp_b(\kappa\sigma_{i-1}, \kappa\sigma_i) < qp_b(\kappa\sigma_{i-1}, \kappa\sigma_i), \\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\beta}.\Big[qp_b(\sigma_{i-1}, \kappa\sigma_{i-1})\Big]^{\alpha}.\Big[qp_b(\sigma_i, \kappa\sigma_i)\Big]^{\gamma} \\ &\quad \left[\frac{1}{s^2}\Big[qp_b(\sigma_{i-1}, \kappa\sigma_i) + qp_b(\sigma_i, \kappa\sigma_{i-1})\Big]\right]^{1-\alpha-\beta-\gamma}, \\ &\leq \boldsymbol{\delta}\Big[qp_b(\mu_{n-1}, \mu_i)\Big]^{\beta}.\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\alpha}.\Big[qp_b(\sigma_i, \sigma_{i+1})\Big]^{\gamma}, \\ &\quad \left[\frac{1}{s^2}\Big[qp_b(\sigma_{i-1}, \sigma_{i+1}) + qp_b(\sigma_i, \sigma_i)\Big]\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{5}$$

By ($QPSb4$) and (5) we get

$$\begin{aligned} &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\beta}.\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\alpha}.\Big[qp_b(\sigma_i, \sigma_{i+1})\Big]^{\gamma} \\ &\quad \bigg[\frac{1}{s^2}\Big[s\Big(qp_b(\sigma_{i-1}, \sigma_i) + qp_b(\sigma_i, \sigma_{i+1})\Big) \\ &\quad - qp_b(\sigma_i, \sigma_i) + qp_b(\sigma_i, \sigma_i)\Big]\bigg]^{1-\alpha-\beta-\gamma} \\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\beta}.\Big[qp_b(\sigma_{i-1}, \sigma_i)\Big]^{\alpha}.\Big[qp_b(\sigma_i, \sigma_{i+1})\Big]^{\gamma} \\ &\quad \left[\frac{1}{s}\Big[\Big(qp_b(\sigma_{i-1}, \sigma_i) + qp_b(\sigma_i, \sigma_{i+1})\Big)\Big)\Big]\right]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{6}$$

Suppose that

$$qp_b(\sigma_{i-1}, \sigma_i) < qp_b(\sigma_i, \sigma_{i+1}),$$

for some $i \geq 1$. Thus

$$\frac{1}{s}\Big(qp_b(\sigma_{i-1}, \sigma_i) + qp_b(\sigma_i, \sigma_{i+1})\Big) \leq qp_b(\sigma_i, \sigma_{i+1}). \tag{7}$$

Consequently, using (7) in (6), we obtain

$$\begin{aligned}
qp_b(\sigma_i,\sigma_{i+1}) &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\beta}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\alpha}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\gamma} \\
&\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{1-\alpha-\beta-\gamma}, \\
&\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\beta+\alpha}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{1-\alpha-\beta}, \\
\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha+\beta} &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\alpha+\beta}.
\end{aligned} \tag{8}$$

Since $qp_b(\sigma_{i-1},\sigma_i) < qp_b(\sigma_i,\sigma_{i+1})$ is a contradiction. Equation 8 is equivalently to

$$qp_b(\sigma_i,\sigma_{i+1}) \leq \boldsymbol{\delta} qp_b(\sigma_{i-1},\sigma_i). \tag{9}$$

By induction in (9), we get

$$\begin{aligned}
qp_b(\sigma_i,\sigma_{i+1}) &\leq \boldsymbol{\delta} qp_b(\sigma_{i-1},\sigma_i), \\
&\leq \boldsymbol{\delta}^2 qp_b(\sigma_{i-1},\sigma_i), \\
&\leq \ldots\ldots \\
&\leq \boldsymbol{\delta}^i qp_b(\sigma_0,\sigma_1).
\end{aligned}$$

Therefore, $\lim_{i\to\infty} qp_b(\sigma_i,\sigma_{i+1}) = 0$.

Now, we prove that $qp_b(\sigma_i,\sigma_{i+1})$ is a Cauchy sequence. Let $i,j \in \mathbb{N}$, for any positive integers such that $j > i$, using ($QPSb4$) we have

$$\begin{aligned}
& qp_b(\sigma_i,\sigma_j) \leq s[qp_b(\sigma_i,\sigma_{i+1}) + qp_b(\sigma_{i+1},\sigma_j)] - qp_b(\sigma_{i+1},\sigma_{i+1}), \\
& = sqp_b(\sigma_i,\sigma_{i+1}) + sqp_b(\sigma_{i+1},\sigma_j) - qp_b(\sigma_{i+1},\sigma_{i+1}), \\
& \leq sqp_b(\sigma_i,\sigma_{i+1}) + s^2 qp_b(\sigma_{i+1},\sigma_{i+2}) + \cdots + s^{i-j-1} qp_b(\sigma_{j-1},\sigma_j) \\
& \leq [s\boldsymbol{\delta}^i + s^2\boldsymbol{\delta}^{i+1} + s^3\boldsymbol{\delta}^{i+2} + \cdots + s^{j-i-1}\boldsymbol{\delta}^{j-1}] qp_b(\sigma_0,\sigma_1), \\
& \leq s\boldsymbol{\delta}^i[1 + s\boldsymbol{\delta} + s^2\boldsymbol{\delta}^2 + \cdots + s^{j-i}\boldsymbol{\delta}^{j-i-1}] qp_b(\sigma_0,\sigma_1), \\
& \leq \frac{s\boldsymbol{\delta}^i}{1 - s\boldsymbol{\delta}} qp_b(\sigma_0,\sigma_1).
\end{aligned} \tag{10}$$

Since $\boldsymbol{\delta} \in [0,\frac{1}{s})$, we conclude that $\frac{s\boldsymbol{\delta}^i}{1-s\boldsymbol{\delta}} qp_b(\sigma_0,\sigma_1) \to 0$ as $i \to \infty$. Therefore, $\{\sigma_i\}$ is a Cauchy sequence in $\mathscr{M}$. Thus $qp_b(\sigma_i,\sigma_j) \to 0$ as $i,j \to \infty$.

Similarly, suppose that $\sigma = \sigma_i$ and $\varsigma = \sigma_{i-1}$ in (4), we have

$$\begin{aligned} qp_b(\sigma_{i+1},\sigma_i) &= \psi(qp_b(\kappa\sigma_i,\kappa\sigma_{i-1})) < qp_b(\kappa\sigma_i,\kappa\sigma_{i-1}),\\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\sigma_{i-1})\Big]^{\beta}.\Big[qp_b(\sigma_i,\kappa\sigma_i)\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\kappa\sigma_{i-1})\Big]^{\gamma}\\ &\quad \left[\frac{1}{s^2}\Big[qp_b(\sigma_i,\kappa\sigma_{i-1})+qp_b(\sigma_{i-1},\kappa\sigma_i)\Big]\right]^{1-\alpha-\beta-\gamma},\\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\sigma_{i-1})\Big]^{\beta}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\gamma},\\ &\quad \left[\frac{1}{s^2}\Big[qp_b(\sigma_i,\sigma_i)+qp_b(\sigma_{i-1},\sigma_{i+1})\Big]\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{11}$$

By ($QPSb4$), Lemma 3, Definition 2 and (11) we get

$$\begin{aligned} &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\sigma_{i-1})\Big]^{\beta}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\gamma}\\ &\quad \left[\frac{1}{s^2}\Big[s\Big(qp_b(\sigma_{i-1},\sigma_i)+qp_b(\sigma_i,\sigma_{i+1})\Big)\right.\\ &\quad \left.-\,qp_b(\sigma_i,\sigma_i)+qp_b(\sigma_i,\sigma_i)\Big]\right]^{1-\alpha-\beta-\gamma}\\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\beta}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\gamma}\\ &\quad \left[\frac{1}{s}\Big[\Big(qp_b(\sigma_{i-1},\sigma_i)+qp_b(\sigma_i,\sigma_{i+1})\Big)\Big]\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{12}$$

Suppose that,

$$qp_b(\sigma_{i-1},\sigma_i) < qp_b(\sigma_i,\sigma_{i+1}),$$

for some $i \geq 1$. Therefore,

$$\frac{1}{s}\Big(qp_b(\sigma_{i-1},\sigma_i)+qp_b(\sigma_i,\sigma_{i+1})\Big) \leq qp_b(\sigma_{i-1},\sigma_i). \tag{13}$$

Consequently, using (13) in (12), we obtain

$$\begin{aligned} qp_b(\sigma_{i+1},\sigma_i) &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_n)\Big]^{\beta}.\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\gamma}\\ &\quad \Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{1-\alpha-\beta-\gamma},\\ &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\sigma_{i+1})\Big]^{\alpha}.\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{1-\alpha},\\ \Big[qp_b(\sigma_{i+1},\sigma_i)\Big]^{1-\alpha} &\leq \boldsymbol{\delta}\Big[qp_b(\sigma_{i-1},\sigma_i)\Big]^{\alpha+\beta}. \end{aligned} \tag{14}$$

Since $qp_b(\sigma_{i-1},\sigma_i) < qp_b(\sigma_i,\sigma_{i+1})$ is a contradiction. Equation 14 implies that

$$qp_b(\sigma_{i+1},\sigma_i) \leq \boldsymbol{\delta} qp_b(\sigma_i,\sigma_{i-1}). \tag{15}$$

There exists $\boldsymbol{\delta} \in [0,\frac{1}{s})$ such that

$$qp_b(\sigma_{i+1}, \sigma_i) \leq \boldsymbol{\delta} qp_b(\sigma_i, \sigma_{i-1}).$$

Through repeating the above procedure and (15) i-times, we conclude that

$$\begin{aligned} qp_b(\sigma_{i+1}, \sigma_i) &\leq \boldsymbol{\delta} qp_b(\sigma_i, \sigma_{i-1}), \\ &\leq \boldsymbol{\delta}^2 qp_b(\sigma_i, \sigma_{i-1}), \\ &\leq \ldots\ldots \\ &\leq \boldsymbol{\delta}^i qp_b(\sigma_1, \sigma_0). \end{aligned} \tag{16}$$

Therefore, $\lim_{i\to\infty} qp_b(\sigma_{i+1}, \sigma_i) = 0$, for $i \geq 1$.

Now, we show that $qp_b(\sigma_{i+1}, \sigma_i)$ is a Cauchy sequence. Let $i,j \in \mathbb{N}$, for any positive integers such that $i > j$, using ($QPSb4$) we have

$$\begin{aligned} qp_b(\sigma_i, \sigma_i) &\leq s[qp_b(\sigma_i, \sigma_{i-1}) + qp_b(\sigma_{i-1}, \sigma_j)] - qp_b(\sigma_{i-1}, \sigma_{i-1}), \\ &= sqp_b(\sigma_i, \sigma_{i-1}) + sqp_b(\sigma_{i-1}, \sigma_j) - qp_b(\sigma_{i-1}, \sigma_{i-1}), \\ &\leq sqp_b(\sigma_i, \sigma_{i-1}) + s^2 qp_b(\sigma_{i-1}, \sigma_{i-2}) + \cdots + s^{j-i-1} qp_b(\sigma_{i-1}, \sigma_i) \\ &\leq [s\boldsymbol{\delta}^{i-1} + s^2\boldsymbol{\delta}^{i-2} + s^3\boldsymbol{\delta}^{i-3} + \cdots + s^{j-i-1}\boldsymbol{\delta}^j] qp_b(\sigma_1, \sigma_0), \\ &\leq s\boldsymbol{\delta}^{i-1}[1 + s\boldsymbol{\delta}^{-1} + s^2\boldsymbol{\delta}^{-2} + \cdots + s^{j-i-2}\boldsymbol{\delta}^{j-i+1}] qp_b(\sigma_1, \sigma_0), \\ &\leq \frac{s\boldsymbol{\delta}^{i-1}}{1 - s\boldsymbol{\delta}^{-1}} qp_b(\sigma_1, \sigma_0). \end{aligned} \tag{17}$$

RSince $\boldsymbol{\delta} \in [0, \frac{1}{s})$, we conclude that $\frac{s\boldsymbol{\delta}^{i-1}}{1-s\boldsymbol{\delta}^{-1}} qp_b(\sigma_1, \sigma_0) \to 0$ as $i \to \infty$. Therefore, $\{\sigma_i\}$ is a Cauchy sequence in $\mathscr{M}$. Thus, $qp_b(\sigma_i, \sigma_j) \to 0$ as $i,j \to \infty$. □

Since $(\mathscr{M}, qp_b)$ is complete, $\{\sigma_i\}$ converges to some point $\zeta \in \mathscr{M}$ with $qp_b(\zeta, \zeta) = 0$. By Lemma 3 and Definition 2, we have

$$qp_b(\zeta, \zeta) = \lim_{i\to\infty} qp_b(\sigma_i, \zeta) = \lim_{i,j\to\infty} qp_b(\sigma_i, \sigma_j).$$

Suppose that $\sigma_i \neq \kappa\sigma_i$ for each $i \geq 1$ and the increasing property of ψ we have

$$qp_b(\sigma_{i+1}, \kappa\zeta) = \psi(qp_b(\kappa\sigma_i, \kappa\zeta)) < qp_b(\kappa\sigma_i, \kappa\zeta).$$

By taking $\sigma = \sigma_i$ and $\varsigma = \zeta$ in inequality (4), one gets

$$\begin{aligned} qp_b(\kappa\sigma_i,\kappa\zeta) \leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\zeta)\Big]^{\beta}.\Big[qp_b(\sigma_i,\kappa\sigma_i)\Big]^{\alpha}.\Big[qp_b(\zeta,\kappa\zeta)\Big]^{\gamma} \\ \left[\frac{1}{s^2}\Big[s\Big(qp_b(\sigma_{i-1},\sigma_i)+qp_b(\sigma_i,\sigma_{i+1})\Big) \right. \\ \left. -\, qp_b(\sigma_i,\sigma_i)+qp_b(\sigma_i,\sigma_i)\Big]\right]^{1-\alpha-\beta-\gamma} \\ \leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\zeta)\Big]^{\beta}.\Big[qp_b(\sigma_i,\kappa\sigma_i)\Big]^{\alpha}.\Big[qp_b(\zeta,\kappa\zeta)\Big]^{\gamma} \\ \left[\frac{1}{s}\Big[\Big(qp_b(\sigma_i,\kappa\zeta)+qp_b(\zeta,\kappa\sigma_i)\Big)\Big]\right]^{1-\alpha-\beta-\gamma} \end{aligned} \tag{18}$$

Suppose that,

$$qp_b(\sigma_i,\kappa\zeta) < qp_b(\zeta,\kappa\sigma_i),$$

for some $i \geq 1$. Therefore,

$$\frac{1}{s}\Big(qp_b(qp_b(\sigma_i,\kappa\zeta)+qp_b(\zeta,\kappa\sigma_i)\Big) \leq qp_b(\zeta,\kappa\sigma_i). \tag{19}$$

Consequently, using (19) in (18) as $i \to \infty$ we get

$$\begin{aligned} & qp_b(\kappa\sigma_i,\kappa\zeta) \leq \boldsymbol{\delta}\Big[qp_b(\sigma_i,\zeta)\Big]^{\beta}.\Big[qp_b(\sigma_i,\kappa\sigma_i)\Big]^{\alpha}.\Big[qp_b(\zeta,\kappa\zeta)\Big]^{\gamma} \\ & \Big[qp_b(\zeta,\kappa\sigma_i)\Big]^{1-\alpha-\beta-\gamma}, \\ & qp_b(\zeta,\kappa\zeta) \leq \boldsymbol{\delta}\Big[qp_b(\zeta,\zeta)\Big]^{\beta}.\Big[qp_b(\zeta,\kappa\zeta)\Big]^{\alpha}.\Big[qp_b(\zeta,\kappa\zeta)\Big]^{\gamma} \\ & \Big[qp_b(\zeta,\kappa\zeta)\Big]^{1-\alpha-\beta-\gamma}, \\ & \leq \boldsymbol{\delta}\Big[qp_b(\zeta,\kappa\zeta)\Big]^{1-\beta}, \\ & \Big[qp_b(\zeta,\kappa\zeta)\Big] \leq \boldsymbol{\delta}\Big[qp_b(\zeta,\kappa\zeta)\Big]^{1-\beta}, \end{aligned} \tag{20}$$

we find that $qp_b(\zeta,\kappa\zeta)=0$, which is a contradiction. Hence, $\kappa\zeta=\zeta$.

By using the condition imposed in Definition 6, we formulate the following corollaries:

Corollary 1 *Let $(\mathscr{M}, qp_b)$ be a complete quasi partial b-metric space. Assume that $\kappa : \mathscr{M} \to \mathscr{M}$ satisfy the conditions of Theorem 4 if any of the following contractions is applied:*

(i) The following contraction is called Reich-Rus-Ćirić type contraction mapping

$$\psi(qp_b(\kappa\sigma,\kappa\varsigma)) \leq \mathscr{K}\Big[qp_b(\sigma,\varsigma)\Big]^{\beta}.\Big[qp_b(\sigma,\kappa\sigma)\Big]^{\alpha}.\Big[qp_b(\varsigma,\kappa\varsigma\Big]^{1-\alpha-\beta}, \tag{21}$$

for all $\sigma,\varsigma \in \mathscr{M}\backslash Fix(\kappa)$, where $\mathscr{K} \in [0,\frac{1}{3})$ and $\alpha+\beta<1$.

(ii) The following results for Chatterjea type contraction mappings

$$\psi(qp_b(\kappa\sigma,\kappa\varsigma)) \leq \mathscr{P}\Big[qp_b(\sigma,\kappa\varsigma)\Big]^{\alpha}.\Big[\frac{1}{s^2}qp_b(\varsigma,\kappa\sigma\Big]^{1-\alpha}, \tag{22}$$

for all $\sigma,\varsigma \in \mathscr{M}\backslash Fix(\kappa)$, where $\mathscr{P} \in [0,\frac{1}{s})$ and $\alpha \in (0,1)$.

We demonstrate our results with an example below.

Example 3 Let $\mathscr{M} = \{\epsilon, \varepsilon, \theta, \zeta\}$. Define complete quasi partial b-metric as $qp_b(\sigma,\varsigma) = (\sigma-\varsigma)^2 + \sigma$ with $\psi(t) = \frac{1}{2t+1}$ and $s = 2$, that is

The geometrical representation of $\mathscr{M}$ defined by $qp_b : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ given in Table 2 below.

We define self mapping $\mathscr{T}$ on $\mathscr{M}$ as

$$\kappa : \begin{pmatrix} \zeta & \epsilon & \varepsilon & \theta \\ \epsilon & \varepsilon & \epsilon & \varepsilon \end{pmatrix},$$

shown in Tables 1 and 2. Thus, ϵ and ε are two fixed points of κ. To see this, consider the following cases

Case 1

For $(\sigma,\varsigma) = (\epsilon,\epsilon)$ we have

$$\begin{aligned}
qp_b(\sigma,\varsigma) &= qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = 0 + \epsilon = \epsilon,\\
qp_b(\kappa\sigma,\kappa\varsigma) &= qp_b(\kappa\epsilon,\kappa\epsilon) = qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = \epsilon,\\
qp_b(\sigma,\kappa\sigma) &= qp_b(\epsilon,\kappa\epsilon) = qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = \epsilon,\\
qp_b(\varsigma,\kappa\varsigma) &= qp_b(\epsilon,\kappa\epsilon) = qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = \epsilon,\\
qp_b(\sigma,\kappa\varsigma) &= qp_b(\epsilon,\kappa\epsilon) = qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = \epsilon,\\
qp_b(\varsigma,\kappa\sigma) &= qp_b(\epsilon,\kappa\epsilon) = qp_b(\epsilon,\epsilon) = (\epsilon-\epsilon)^2 + \epsilon = \epsilon.
\end{aligned}$$

Using all of the above inequalities in (4) and the property of ψ, we obtain

Table 1 The partial quasi partial b-metric for an element in $\mathscr{M}$

$qp_b(\sigma,\varsigma)$	ζ	ϵ	ε	θ
ζ	ζ	$(\zeta-\epsilon)^2+\zeta$	$(\zeta-\varepsilon)^2+\zeta$	$(\zeta-\theta)^2+\zeta$
ϵ	$(\epsilon-\zeta)^2+\epsilon$	ϵ	$(\epsilon-\varepsilon)^2+\epsilon$	$(\epsilon-\theta)^2+\epsilon$
ε	$(\varepsilon-\zeta)^2+\varepsilon$	$(\varepsilon-\epsilon)^2+\varepsilon$	ε	$(\varepsilon-\theta)^2+\varepsilon$
θ	$(\theta-\zeta)^2+\theta$	$(\theta-\epsilon)^2+\theta$	$(\theta-\varepsilon)^2+z$	θ

Table 2 The geometrical/cartesian product form for an element in $\mathscr{M}$

$qp_b : \mathscr{M} \times \mathscr{M}$	ζ	ϵ	ε	θ
ζ	(ζ,ζ)	(ζ,ϵ)	(ζ,ε)	(ζ,θ)
ϵ	(ϵ,ζ)	(ϵ,ϵ)	(ϵ,ε)	(ε,θ)
ε	(ε,ζ)	(ε,ϵ)	$(\varepsilon,\varepsilon)$	(ε,θ)
θ	(θ,ζ)	(θ,ϵ)	(θ,ε)	(θ,θ)

$$\begin{aligned}\psi(\epsilon) &\leq \delta\Big[\epsilon\Big]^{\beta}.\Big[\epsilon\Big]^{\alpha}.\Big[\epsilon\Big]^{\gamma}.\Big[\frac{1}{s^2}(\epsilon+\epsilon)\Big]^{1-\alpha-\beta-\gamma},\\ &\leq \delta\Big[x\Big]^{\beta+\alpha+\gamma}.\Big[\frac{2\epsilon}{s^2}\Big]^{1-\alpha-\beta-\gamma},\\ \frac{1}{2\epsilon+1} &\leq \delta\Big[\epsilon\Big]^{\beta+\alpha+\gamma}.\Big[\frac{2\epsilon}{s^2}\Big]^{1-\alpha-\beta-\gamma}.\end{aligned} \tag{23}$$

Case 2

For $(\sigma,\varsigma)=(\varepsilon,\varepsilon)$ we have

$$\begin{aligned} qp_b(\sigma,\varsigma) = qp_b(\varepsilon,\varepsilon) &= (\varepsilon-\varepsilon)^2+\varepsilon = 0+\varepsilon = \varepsilon,\\ qp_b(\kappa\sigma,\kappa\varsigma) = qp_b(\kappa\varepsilon,\kappa\varepsilon) &= qp_b(\varepsilon,\varepsilon) = (\varepsilon-\varepsilon)^2+\varepsilon=\varepsilon,\\ qp_b(\sigma,\kappa\sigma) = qp_b(\varepsilon,\kappa\varepsilon) &= qp_b(\varepsilon,\varepsilon) = (\varepsilon-\varepsilon)^2+\varepsilon=\varepsilon,\\ qp_b(\varsigma,\kappa\varsigma) = qp_b(\varepsilon,\kappa\varepsilon) &= qp_b(\varepsilon,\varepsilon) = (\varepsilon-\varepsilon)^2+\varepsilon=\varepsilon,\\ qp_b(\sigma,\kappa\varsigma) = qp_b(\varepsilon,\kappa\varepsilon) &= qp_b(\varepsilon,\varepsilon) = (\varepsilon-\varepsilon)^2+\varepsilon=\varepsilon,\\ qp_b(\varsigma,\kappa\sigma) = qp_b(\varepsilon,\kappa\varepsilon) &= qp_b(\varepsilon,\varepsilon) = (\varepsilon-\varepsilon)^2+\varepsilon=\varepsilon.\end{aligned}$$

Using all of the above inequalities in (4) and the property of ψ, we obtain

$$\begin{aligned}\psi(y) &\leq \delta\Big[y\Big]^{\beta}.\Big[y\Big]^{\alpha}.\Big[y\Big]^{\gamma}.\Big[\frac{1}{s^2}(\varepsilon+\varepsilon)\Big]^{1-\alpha-\beta-\gamma},\\ &\leq \delta\Big[\varepsilon\Big]^{\beta+\alpha+\gamma}.\Big[\frac{2\varepsilon}{s^2}\Big]^{1-\alpha-\beta-\gamma},\\ \frac{1}{2\varepsilon+1} &\leq \delta\Big[\varepsilon\Big]^{\beta+\alpha+\gamma}.\Big[\frac{2\varepsilon}{s^2}\Big]^{1-\alpha-\beta-\gamma}.\end{aligned} \tag{24}$$

By choosing the appropriate values of $\epsilon, \varepsilon, \boldsymbol{\delta}, \alpha, \beta$ and γ in $\mathscr{M}$, using (23) and (24), we conclude that all assumptions of Theorem 4 are satisfied. It is observed that a self-mapping κ has more than one fixed point in $\mathscr{M}$. Thus, κ posses a fixed point.

3 An application to non linear matrix equations

In this section, we prove the existence of the solution for the non-linear matrix equation. We use one application to utilize the results obtained in Theorem 4 where a fixed point solution is applied in quasi partial b-metric space setting. In literature, the non-linear matrix equation initiated by Ran and Reurings [40, 41] proved a fixed point theorem in partially ordered sets and some applications to matrix equations.

The Hermitian solution of the equation $X = Q + \mathcal{N} X^{-1} \mathcal{N}^*$ is the matrix equation arising from the Gaussian process. The equation admits both positive definite solution and negative definite solution if and only if $\mathcal{N}$ is non-singular. If $\mathcal{N}$ is singular, no negative definite solution exists. Non-linear matrix equations play an important role in several problems that arise in the analysis of control theory [45, 47] and system theory [48–50].

The main concern of this section is to apply Theorem 4 to study the following non-linear matrix equations which are motivated by Jain et al. [18], Lim et al. [32, 33], Ran and Reurings [40, 41] and several others.

$$\sigma = Q + \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i, \tag{25}$$

$$Q = \sigma - \mathcal{N}_1^* \kappa(\sigma) \mathcal{N}_1 - \ldots - \mathcal{N}_n^* \kappa(\sigma) \mathcal{N}_n, \tag{26}$$

where $\mathscr{H}(n)$ is a set of $n \times n$ Hermitian matrices, $\mathfrak{p}(n)$ is a set of $n \times n$ positive definite matrices and $\mathfrak{p}(n) \subseteq \mathscr{H}(n)$, $\mathrm{Q} \in \mathfrak{p}(n)$, $\mathcal{N}_i$ is $n \times n$ matrices and $\kappa; \mathfrak{p}(n) \to \mathfrak{p}(n)$ is a continuous order preserving map such that $\kappa(0) = 0$.

The set $\mathscr{H}(n)$ equipped with the trace norm $\|.\|_{tr}$ is a complete partial b- metric space [40] and partially ordered with partial ordering $\preceq$, where $\sigma \preceq \varsigma \Rightarrow \varsigma \succeq \sigma$.

The following lemmas are inspired by Ran and Reurings [40, 41] will be useful for development of our results.

Lemma 4 [40] *If* $\sigma, \varsigma \succeq 0$ *are* $n \times n$ *matrices, then*

$$0 \leq tr(\sigma, \varsigma) \leq \|\varsigma\| |tr(\sigma)|. \tag{27}$$

Lemma 5 [40] *If* $\sigma, \varsigma \preceq I_n$, *then*

$$\|\sigma\| < 1. \tag{28}$$

We prove our results by establishing a fixed point for a self mappings for the following non-linear matrix equation in quasi-partial b metric space.

$$\sigma = Q + \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i, \tag{29}$$

where $\mathrm{Q} \in \mathfrak{p}(n)$, $\mathcal{N}_i$ is $n \times n$ matrices, $\mathcal{N}_i^*$ stands for conjugate transpose of $\mathcal{N}_i \in \mathscr{H}(n)$ and $\kappa; \mathfrak{p}(n) \to \mathfrak{p}(n)$ is a continuous order preserving map such that $\kappa(0) = 0$.

Theorem 5 *Consider the class of non-linear matrix Equation 29 and suppose the following condition holds.*

(i) there exists $Q \in \mathfrak{p}(n)$ with

$$Q = Q + \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(Q) \mathcal{N}_i,$$

(ii) for all $\sigma, \varsigma \in \mathfrak{p}(n)$,

$$\sigma \leq \varsigma \Rightarrow \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i \leq \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\varsigma) \mathcal{N}_i,$$

(iii) there exists $\delta \in (0, \frac{1}{2})$ for which $\sum_{i=1}^{n} \mathcal{N}_i^* \mathcal{N}_i < \delta \mathcal{I}_n$ and $\sum_{i=1}^{n} \mathcal{N}_i^* \kappa(Q) \mathcal{N}_i > 0$ such that for all $\mu \leq v$ we have

$$\sigma \leq \varsigma \Rightarrow$$

(iv) there exists $\sigma, \varsigma \in \mathscr{P}(n)$ and $\delta \in (0, \frac{1}{2})$ such that

$$\begin{aligned} \|\kappa\sigma - \kappa\varsigma\|_{tr} \leq \varsigma \leq \delta \Big[\|\sigma - \varsigma\|_{tr}\Big]^{\beta} . \Big[\|\sigma - \kappa\sigma\|_{tr}\Big]^{\alpha} . \Big[\|\varsigma - \kappa\varsigma\|_{tr}\Big]^{\gamma} . \\ \left[\frac{1}{s^2}\left(\|\sigma - \kappa\varsigma\|_{tr} + \|\varsigma - \kappa\sigma\|_{tr}\right)\right]^{1-\alpha-\beta-\gamma} . \end{aligned} \tag{30}$$

Then, the non linear matrix equation (29) has a solution in $\mathfrak{p}(n) \subseteq \mathscr{H}(n)$.

Proof Define $\kappa : \mathfrak{p}(n) \to \mathfrak{p}(n)$ by

$$\kappa(\sigma) = Q + \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i, \tag{31}$$

for all $\sigma \in \mathfrak{p}(n)$. Then the fixed point of the mapping κ is a solution of the matrix equation (29).

We define a quasi partial b-metric $qp_b : \mathfrak{p}(n) \to \mathfrak{p}(n) \to \mathbb{R}_+$ by

$$qp_b(\sigma, \varsigma) = \|\sigma - \varsigma\|_{tr}^2 + \|\sigma\|_{tr}.$$

Let $\sigma, \varsigma \in \mathfrak{p}(n)$ with $\sigma \preceq \varsigma$, then $\kappa(\sigma) \preceq \kappa(\varsigma)$. So for $qp_b(\sigma, \varsigma) > 0$ and by using $(i) - (iv)$ we have

$$\begin{aligned} qp_b(\kappa\sigma, \kappa\varsigma) &= \|\kappa\sigma - \kappa\varsigma\|_{tr} \Rightarrow \|\kappa\sigma - \kappa\varsigma\|_{1,} \\ &= \|\kappa\sigma - \kappa\varsigma\|^2 + \|\kappa\sigma\|, \\ &= \|\sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i - \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\varsigma) \mathcal{N}_i\| \\ &+ \|Q + \sum_{i=1}^{n} \mathcal{N}_i^* \kappa(\sigma) \mathcal{N}_i\|. \end{aligned} \tag{32}$$

$$\begin{aligned} qp_b(\sigma,\varsigma) &= \|\sigma-\varsigma\|_{tr} \Rightarrow \|\sigma-\varsigma\|_{1,} \\ &= \|\sigma-\varsigma\|^2+\|\sigma\|. \end{aligned} \tag{33}$$

$$\begin{aligned} qp_b(\sigma,\kappa\sigma) &= \|\sigma-\kappa\sigma\|_{tr} \Rightarrow \|\sigma-\kappa\sigma\|_{1,} \\ &= \|\sigma-\kappa\sigma\|^2+\|\sigma\|, \\ &= \|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\mathcal{T}(\sigma)\mathcal{N}_i\|^2+\|\sigma\|. \end{aligned} \tag{34}$$

$$\begin{aligned} qp_b(\varsigma,\kappa\varsigma) &= \|\varsigma-\kappa\varsigma\|_{tr} \Rightarrow \|\varsigma-\kappa\varsigma\|_{1,} \\ &= \|\varsigma-\kappa\varsigma\|^2+\|\varsigma\|, \\ &= \|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|^2+\|\sigma\|. \end{aligned} \tag{35}$$

$$\begin{aligned} qp_b(\sigma,\kappa\varsigma) &= \|\sigma-\kappa\varsigma\|_{tr} \Rightarrow \|\sigma-\kappa\varsigma\|_{1,} \\ &= \|\sigma-\kappa\varsigma\|^2+\|\sigma\|, \\ &= \|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\varsigma)\mathcal{N}_i\|^2+\|\sigma\|. \end{aligned} \tag{36}$$

$$\begin{aligned} qp_b(\varsigma,\kappa\sigma) &= \|\varsigma-\kappa\sigma\|_{tr} \Rightarrow \|\varsigma-\kappa\sigma\|_{1,} \\ &= \|\varsigma-\kappa\sigma\|^2+\|\varsigma\|, \\ &= \|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|^2+\|\varsigma\|. \end{aligned} \tag{37}$$

Using (32, 33, 34, 35, 36, 37) in (30) we obtains

$$\begin{aligned} &\|\sum_{i=1}^{n}\mathcal{N}_i^*\mathcal{N}_i(\kappa(\sigma)-\kappa(\varsigma))\|+ \\ &\|Q+\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|\leq\delta\Big[\|\sigma-\varsigma\|^2+\|\sigma\|\Big]^{\beta}. \\ &\Big[\|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|^2+\|\sigma\|\Big]^{\alpha}. \\ &\Big[\|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|^2+\|\sigma\|\Big]^{\gamma}. \\ &\Big[\frac{1}{s^2}(\|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\varsigma)\mathcal{N}_i\|^2+\|\sigma\|+ \\ &\|\sigma-Q-\sum_{i=1}^{n}\mathcal{N}_i^*\kappa(\sigma)\mathcal{N}_i\|^2+\|\varsigma\|)\Big]^{1-\alpha-\beta-\gamma}. \end{aligned} \tag{38}$$

This implies that

$$\|\kappa\sigma - \kappa\varsigma\|_1 \leq \varsigma \leq \delta\Big[\|\sigma - \varsigma\|_1\Big]^{\beta} \cdot \Big[\|\sigma - \kappa\sigma\|_1\Big]^{\alpha} \cdot \Big[\|\varsigma - \kappa\varsigma\|_1\Big]^{\gamma} \cdot \Big[\frac{1}{s^2}\left(\|\sigma - \kappa\varsigma\|_1 + \|\varsigma - \kappa\sigma\|_1\right)\Big]^{1-\alpha-\beta-\gamma}. \tag{39}$$

Consequently by the property of ψ we get

$$\psi(qp_b(\kappa\sigma, \kappa\varsigma)) \leq \delta\Big[qp_b(\sigma, \varsigma)\Big]^{\beta} \cdot \Big[qp_b(\sigma, \kappa\sigma)\Big]^{\alpha} \cdot \Big[qp_b(\varsigma, \kappa\varsigma\Big]^{\gamma} \cdot \Big[\frac{1}{s^2}\left(qp_b(\sigma, \kappa\varsigma) + qp_b(\varsigma, \kappa\sigma)\right)\Big]^{1-\alpha-\beta-\gamma}, \tag{40}$$

Therefore, from $\sum_{i=1}^{n} \mathscr{N}_i^* \kappa(Q)\mathscr{N}_i > 0$, we have $Q \leq \kappa(Q)$. Thus, by using Theorem 4 we conclude that κ posses a fixed point in $\mathfrak{p}(n)$ and $\mathfrak{p}(n) \in \mathscr{M}$. □

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Declarations

Conflict of interest The authors declare that they have no conflicts of interest.

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