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ORIGINAL RESEARCH PAPER

The 2-D Hyper-complex Gabor quadratic-phase Fourier transform and uncertainty principles

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Abstract

In this paper, we present a novel integral transform known as the 2-D Hyper-complex (quaternion) Gabor quadratic-phase Fourier transform (Q-GQPFT), which is embodiment of several well known signal processing tools. We first define the 2-D Hyper-complex(quaternion) quadratic-phase Fourier transform (Q-QPFT) and then we propose the definition of novel Q-GQPFT, which is a modified version of the classical windowed quadratic-phase Fourier transform to quaternion-valued signals and we study various properties of the proposed Q-GQPFT, including Moyal's formula, reconstruction formula, isometry and reproducing kernel formula. We also establish the Heisenberg and logarithmic uncertainty inequalities for the Q-GQPFT.

Keywords 2-D Hyper-complex quadratic-phase Fourier transform · 2-D Hyper-complex Gabor quadratic-phase Fourier transform · Isometry · Uncertainty principle

Mathematics Subject Classification Primary 42C40 · Secondary 42C15 · 47G10 · 42A38 · 42B10

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1 Introduction

In time-frequency analysis transformations like Fourier transform, fractional Fourier transform and linear canonical transform have been studied as various types of the well known integral transforms. Recently Castro et al. [10] have defined the quadratic-phase Fourier transform (QPFT) as a generalization of the classical integral transform by taking the kernel in the conventional exponential form. With a slight modification in [10], the authors in [7, 22] have defined QPFT with five parameters A, B, C, D, E of a function which are indeed real parameters as

$$\mathcal{Q}_\mu[f](w) = \int_{\mathbb{R}} f(x)\Lambda_\mu(x, w)dx, \tag{1.1}$$

where $\Lambda_\mu(x, w)$ is a quadratic-phase kernel and is given by

$$\Lambda_\mu(x, w) = \sqrt{\frac{B}{2\pi\imath}}e^{-i(Ax^2+Bxw+Cw^2+Dx+Ew)} \tag{1.2}$$

and the corresponding inversion formula is given by

$$f(x) = \int_{\mathbb{R}} \mathcal{Q}_\mu[f](w)\overline{\Lambda_\mu(x, w)}dw, \tag{1.3}$$

where $A, B, C, D, E \in \mathbb{R}$, $B \neq 0$. I It is here worth to mention that if we take all parameters equal to zero except $B = -1$, the QPFT boils down to the classical Fourier transform. If we take $D = E = 0$, we will end up with linear canonical transform. On taking $A = C = \cot\theta$, $B = -\csc\theta$ and $D = E = 0$ the conventional fractional Fourier transform is obtained. Moreover we can obtain Fresnel transform with $A = -B = C = \frac{1}{\beta}$ and $D = E = 0$.

Talking again about these real arbitrary parameters, an appropriate selection of them is important. Due to this a sense of rotation as well as shift can be inculcated in both the axes of time and frequency domain. Hence can be used in the better analysis of chirp-like signals which are employed in radar and other communications systems. Due to the freedom of degrees, the QPFT has arrived an efficient tool in solving the problems harmonic analysis, image processing, sampling and so on. The generalization of integral transforms from real and complex numbers to the quaternion setting is popular nowadays for the study of higher dimension viz: the quaternion Fourier transform (QFT) [14], the quaternion linear canonical transform (QLCT) [19], the fractional quaternion Fourier transform (Fr-QFT) [28], the quaternion offset linear canonical transform (QOLCT) [5, 6]. In past decades, quaternion algebra has become a leading area of research with its applications in color image processing, image filtering, watermarking, edge detection and pattern recognition (see [23, 26]). The Fourier transform (FT) in quaternion setting i.e.the quaternion Fourier transform (QFT) [8] plays a significant role in the representation of hyper-complex signals in signal processing which is believed to be the substitute of the commonly used two-dimensional Complex Fourier Transform (CFT). The QFT has wide range of applications see ([2]). On the other hand the uncertainty principle (UP) plays a vital role in various scientific fields such as mathematics, quantum physics, signal processing and information theory [11]. The uncertainty principles associated with QFT are given in [17, 18] and the extension of UPs in the domains of QLCT,QOLCT are given in [1, 20]. These UPs have many applications in the analysis of optical systems, signal recovery and so on see ([27]). Therefore modern era of information processing is in dire need of quaternionic valued signals and therefore is a very hot area of research. Since the QPFT is a five parameter class of linear integral transform and has more degrees of freedom and is more flexible than the FT, the FRFT, the LCT but with similar computation cost as the conventional FT. Due to the mentioned advantages, it is natural to generalize the classical QPFT to the quaternionic algebra.

So motivated and inspired by this, we shall propose the novel quaternion quadratic phase Fourier transform. Furthermore, keeping in mind that the Q-QPFT takes quaternion signals from time domain to the frequency domain but is unable to perform time frequency localization simultaneously due to its global kernel. So to overcome this drawback we used Q-QPFT to generate a new transform coned as the 2-D Hyper-complex (quaternion) Gabor quadratic-phase Fourier transform (Q-GQPFT). It is embodiment of several well known signal processing tools. Keeping in view the contemporary trends in the time-frequency analysis, it is both theoretically interesting and practically useful to propose a generalized quaternion Gabor quadratic phase Fourier transform that can efficiently localize the quadratic-phase spectrum of a non-transient quaternion signal in the time-frequency plane.The main purpose of this paper is to rigorously study the 2-D Hyper-complex (quaternion) Gabor quadratic phase Fourier transform.

The highlights of this study are itemized below:

- To propose the definition of the novel 2-D Hyper-complex (Quaternion) Qudratic-phase Fourier Transform (Q-QPFT).
- To propose the definition of novel integral transform coined as the 2-D Hyper-complex (quaternion) quadratic-phase Fourier transform (Q-QPFT).
- To study various important properties of the proposed transform, including the Moyal's formula, reconstruction formula, isometry and reproducing kernel formula.
- To establish the Heisenberg and logarithmic uncertainty inequalities associated with the 2-D Hyper-complex(quaternion) quadratic-phase Fourier transform (Q-QPFT).

The rest of the paper is organised as. In Sect. 2, we introduced the definition of the novel 2-D Hyper-complex(Quaternion) Qudratic-phase Fourier Transform (Q-QPFT) and some basic results which are used in subsequent sections. In Sect. 3, we formally introduced the notion of novel 2-D Hyper-complex (Quaternion) Gabor Quadratic-Phase Fourier transforms (Q-GQPFTs).

The relationship between Quaternion Gabor Quadratic-Phase Fourier Transforms and Quaternion Quadratic-Phase Fourier Transforms is established here and several basic properties are investigated. In Sect. 4, we established an analogue of the well-known Heisenberg's uncertainty inequality and the corresponding logarithmic uncertainty principle for the Q-GQPFT. Finally, a conclusion is extracted in Sect. 4.

2 2-D Hyper-complex Fourier transform and 2-D Hyper-complex Quadratic-phase Fourier transform

There are many possible definitions of hyper-complex Fourier transforms dictated by the choice of the algebra of imaginary units $\{e_1, e_2, ..., e_n\}$, . In this paper, the dominant role plays the hyper-complex FT with imaginary units satisfying the multiplication rules of the Cayley-Dickson algebra.

2.1 2-D Hyper-complex Qudratic-phase Fourier Transform

In this subsection we will introduce the definition of the novel 2-D Hyper-complex (Quaternion) Qudratic-phase Fourier Transform (Q-QPFT) which is a generalization of the classical Qudratic-phase Fourier transform [9]. Because of non-commutative property of quaternion multiplication, there are three different types of the Q-QPFT: the left-sided Q-QPFT, the right-sided Q-QPFT, and the two-sided Q-QPFT. In this paper, we mainly focus on the two-sided Q-QPFT.

Definition 2.1 (Q-QPFT). Let $\mu_s = (A_s, B_s, C_s, D_s, E_s)$ for $s = 1, 2$ then the two-sided Q-QPFT of any signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w}) = \frac{\mathbf{1}}{\mathbf{2}\pi} \int_{\mathbb{R}^2} \Lambda^{\mathbf{i}}_{\mu_1}(\mathbf{t_1}, \mathbf{w_1})\mathbf{f}(\mathbf{t})\Lambda^{\mathbf{j}}_{\mu_2}(\mathbf{t_2}, \mathbf{w_2})\mathbf{dt} \tag{2.1}$$

where $\mathbf{w} = (w_1, w_2) \in \mathbb{R}^2$,$\mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $\Lambda^{\mathbf{i}}_{\mu_1}(t_1, w_1)$ and $\Lambda^{\mathbf{j}}_{\mu_2}(t_2, w_2)$ are kernel signals given by

$$\Lambda^{\mathbf{i}}_{\mu_1}(t_1, w_1) = \exp\{i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1 + D_1 t_1 + E_1 w_1)\}. \tag{2.2}$$

$$\Lambda^{\mathbf{j}}_{\mu_2}(t_2, w_2) = \exp\{j(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2 + D_2 t_2 + E_2 w_2)\}. \tag{2.3}$$

Where $A_s, B_s, C_s, D_s, E_s \in \mathbb{R}, B_s \neq 0$ and $s = 1, 2$.

Proposition 2.2 *(Moyal's formula for Q-QPFT). Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ be two quaternion signals, then we have*

$$\left\langle \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[g] \right\rangle = \frac{1}{|B_1 B_2|} \langle f, g \rangle. \tag{2.4}$$

For $f = g$, we have

$$\|f\|^2 = |B_1B_2|\|\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]\|^2. \tag{2.5}$$

Lemma 2.3 (*Reconstruction formula for Q-QPFT*). *Every signal* $f \in L^2(\mathbb{R}^2, \mathbb{H})$, *can be reconstructed back by the formula*:

$$\begin{aligned} f(\mathbf{t}) &= \mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]\}(\mathbf{t}) \\ &= \frac{|B_1B_2|}{2\pi}\int_{\mathbb{R}^2} \overline{\Lambda^{\mathbf{i}}_{\mu_1}(t_1,w_1)}\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]\}(\mathbf{w})\overline{\Lambda^{\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})}\mathbf{dw}. \end{aligned} \tag{2.6}$$

3 2-D Hyper-complex Gabor Qudratic-phase Fourier transform

In this section, we formally introduce the notion of novel 2-D Hyper-complex (Quaternion) Gabor Quadratic-Phase Fourier transforms (Q-GQPFTs). We shall establish the relation between Quaternion Gabor Quadratic-Phase Fourier Transforms and Quaternion Quadratic-Phase Fourier Transforms and then investigate several basic properties of (Two-sided)Q-GQPFT which are important for signal representation in signal processing. Let us start with definition of Q-GQPFT.

Definition 3.1 (Q-GQPFT). For $s = 1, 2$ let $\mu_s = (A_s, B_s, C_s, D_s, E_s), B_s \neq 0$ be a given set of real parameters then the two-sided Q-GQPFT of signal $f \in L^2(\mathbb{R}^2, \mathbb{H})$ is denoted and defined by

$$\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\pi}\int_{\mathbb{R}^2} \Lambda^{\mathbf{i}}_{\mu_1}(\mathbf{t_1},\mathbf{w_1})\mathbf{f}(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}\Lambda^{\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})\mathbf{dt} \tag{3.1}$$

where $\mathbf{w} = (w_1, w_2), \mathbf{t} = (t_1, t_2) \in \mathbb{R}^2$ and $\Lambda^{\mathbf{i}}_{\mu_1}(t_1, w_1)$ and $\Lambda^{\mathbf{j}}_{\mu_2}(t_2, w_2)$ are kernel signals given by (2.2) and (2.3), respectively.

By appropriately choosing parameters in $\mu_s = (A_s, B_s, C_s, D_s, E_s), s = 1, 2$ the Q-GQPFT (3.1) gives birth to the following existing time-frequency transforms:

- For $\mu_s = (0, 1, 0, 0, 0), s = 1, 2$, the Q-GQPFT (3.1) boils down to the Quaternion-Gabor Fourier Transform:

$$\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x}) = \frac{\mathbf{1}}{\mathbf{2}\pi}\int_{\mathbb{R}^2} \mathbf{e}^{\mathbf{i}\mathbf{w_1}\mathbf{t_1}}\mathbf{f}(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}\mathbf{e}^{\mathbf{j}\mathbf{w_2}\mathbf{t_2}}\mathbf{dt}. \tag{3.2}$$

- For $\mu_s = (A_s/2B_s, -1/B_s, C_s/2B_s, 0, 0), s = 1, 2$ and multiplying the right side of (2.2) by $1/\sqrt{iB_1}$ and right side of (2.3) by $1/\sqrt{jB_2}$, the Q-GQPFT (3.1) reduces to the Quaternion-Gabor Linear Canonical Transform [12].
- For $\mu_s = (A_s/2B_s, -1/B_s, C_s/2B_s, p_s/B_s, -(p_sq_s + B_sC_s)/B_s^2p_s), s = 1, 2$ and multiplying the right side of (2.2) by $1/\sqrt{iB_1}$ and right side of (2.3)by $1/\sqrt{jB_2}$, the Q-GQPFT (3.1) reduces to the Quaternion-Gabor Offset Linear Canonical Transform [3, 32].

For lucid illustration of the proposed quaternion Gabor- quadratic phase Fourier transform (Q-GQPFT) we present an example:

Example Given the the rectangular window function

$$\phi(t) = \begin{cases} 1, & \text{if } |t_1| \quad \frac{1}{2}, |t_2| \quad \frac{1}{2} \\ 0, & \text{elsewhere} \end{cases}$$

Consider a 2D Gaussian quaternionic function of the form $f(t_1, t_2) = e^{-(\beta_1 t_1^2 + \beta_2 t_2^2)}$, for $\beta_1, \beta_2 \in \mathbb{R}$ are positive real constants.

The Q-GQPFT of f is given by

$$\begin{aligned}
&\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w}, \mathbf{x}) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}^2} \Lambda^{\mathbf{i}}_{\mu_1}(t_1, w_1) f(\mathbf{t}) \overline{\phi(\mathbf{t} - \mathbf{x})} \Lambda^{\mathbf{j}}_{\mu_2}(t_2, w_2) d\mathbf{t} \\
&= \frac{1}{2\pi} \int_{x_1-1/2}^{x_1+1/2} \int_{x_2-1/2}^{x_2+1/2} e^{i(A_1 t_1^2 + B_1 t_1 w_1 + C_1 w_1 + D_1 t_1 + E_1 w_1) - \beta_1 t_1^2} \\
&\quad \times e^{i(A_2 t_2^2 + B_2 t_2 w_2 + C_2 w_2 + D_2 t_2 + E_2 w_2) - \beta_2 t_2^2} d\mathbf{t} \\
&= \frac{1}{2\pi} e^{i(C_1 w_1^2 + E_1 w_1)} \int_{x_1-1/2}^{x_1+1/2} e^{i((A_1 + i\beta_1) t_1^2 + B_1 t_1 w_1 + C_1 w_1 + D_1 t_1} dt_1 \\
&\quad \times \int_{x_2-1/2}^{x_2+1/2} e^{j((A_2 + j\beta_2) t_2^2 + B_2 t_2 w_2 + C_2 w_2 + D_2 t_2} dt_2 \times e^{j(C_2 w_2^2 + E_2 w_2)}.
\end{aligned}$$

For simplicity, we choose $\beta_1 = iA_1$ and $\beta_2 = jA_2$, we obtain from (3.3)

$$\begin{aligned}
&\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w}, \mathbf{x}) \\
&= \frac{1}{2\pi} e^{i(C_1 w_1^2 + E_1 w_1)} \int_{x_1-1/2}^{x_1+1/2} e^{i(B_1 w_1 + D_1) t_1} dt_1 \\
&\quad \times \int_{x_2-1/2}^{x_2+1/2} e^{j(B_2 w_2 + D_2 t_2) t_2} dt_2 \times e^{j(C_2 w_2^2 + E_2 w_2)} \\
&= \frac{e^{i(B_1 w_1 x_1 + C_1 w_1^2 + D x_1 + E_1 w_1)}}{2\pi (B_1 w_1 + D_1)} \left(e^{i\frac{B_1 w_1 + D_1}{2}} - e^{-i\frac{B_1 w_1 + D_1}{2}} \right) \\
&\quad \times \left(e^{j\frac{B_2 w_2 + D_2}{2}} - e^{-j\frac{B_2 w_2 + D_2}{2}} \right) \frac{e^{j(B_2 w_2 x_2 + C_2 w_2^2 + D x_2 + E_2 w_2)}}{(B_2 w_2 + D_2)} \\
&= \frac{i}{\sqrt{2\pi}} e^{i(B_1 w_1 x_1 + C_1 w_1^2 + D x_1 + E_1 w_1)} sinc\left(\frac{B_1 w_1 + D_1}{2} \right) \\
&\quad \times sinc\left(\frac{B_2 w_2 + D_2}{2} \right) \frac{j}{\sqrt{2\pi}} e^{j(B_2 w_2 x_2 + C_2 w_2^2 + D x_2 + E_2 w_2)}
\end{aligned}$$

Before studying some vital properties of Q-GQPFT, we first establish the relation between Q-GQPFT and Q-QPFT. Let us begin by revisiting the Definition (3.1):

$$\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x}) = \mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[\mathbf{f}(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}](\mathbf{w}). \tag{3.3}$$

Applying inverse Q-QPFT on both sides of (3.3), we have

$$\begin{aligned} f(t)\overline{\phi(t-x)} &= \mathbb{Q}^{-1}_{\mu_1,\mu_2}[\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})](\mathbf{t}) \\ &= \mathbb{Q}^{\mathbb{H}}_{-\mu_1,-\mu_2}[\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})](\mathbf{t}) \\ &= \frac{|B_1B_2|}{2\pi}\int_{\mathbb{R}^2}\overline{\Lambda^{\mathbf{i}}_{\mu_1}(t_1,w_1)}\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\}(\mathbf{w})\overline{\Lambda^{\mathbf{j}}_{\mu_2}(t_2,w_2)}d\mathbf{w} \\ &= \frac{|B_1B_2|}{2\pi}\int_{\mathbb{R}^2}\Lambda{-i}_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\}(\mathbf{w})\Lambda{-j}_{\mu_2}(t_2,w_2)d\mathbf{w}. \end{aligned} \tag{3.4}$$

Now we will study some fundamental properties of Q-GQPFT (3.1)

3.1 Some properties of Q-GQPFT

In this subsection, we establish some fundamental properties of the Q-GQPFT. These properties are vital in signal processing.

Theorem 3.2 *Let $\phi,\psi \in L^2(\mathbb{R}^2,\mathbb{H})$ be a non zero window function and $f,g \in L^2(\mathbb{R}^2,\mathbb{H})$, then Q-GQPFT defined by (3.1) satisfies the following properties :*

1. **(Linearity)**

$$\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\alpha f+\beta g](\mathbf{w},\mathbf{x}) = \alpha\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x}) + \beta\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{g}](\mathbf{w},\mathbf{x}). \tag{3.5}$$

2. **(Translation)**

$$\begin{aligned} &\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f(\mathbf{t}-\mathbf{k})](\mathbf{w},\mathbf{x}) \\ &\quad = e^{i(A_1k_1^2+B_1k_1w_1+D_1k_1)}\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[e^{2iA_1k_1t_1}f(\mathbf{t}) \\ &\quad\quad \times e^{2iA_2k_1t_2}](\mathbf{w},\mathbf{x}-\mathbf{k})\mathbf{e}^{\mathbf{i}(\mathbf{A_2k_2^2}+\mathbf{B_2k_2w_2}+\mathbf{D_2k_2})}. \end{aligned} \tag{3.6}$$

3. **(Modulation)**

$$\begin{aligned} &\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[e^{i\gamma_1t_1}f(\mathbf{t})\mathbf{e}^{\mathbf{j}\gamma_2\mathbf{t_2}}](\mathbf{w},\mathbf{x}) \\ &= e^{\frac{-i}{B_1^2}(\gamma_1^2+2\gamma_1B_1w_1+\gamma_1E_1B_1)} \\ &\quad \times \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f(\mathbf{t})]\left(\mathbf{w}+\frac{\gamma}{\mathbf{B}},\mathbf{x}\right)e^{\frac{-j}{B_2^2}(\gamma_2^2+2\gamma_2B_2w_2+\gamma_2E_2B_2)}. \end{aligned} \tag{3.7}$$

4. **(Conjugation)**

$$\mathbb{Q}^{\mathbb{H}}_{\overline{\phi},\mu_1,\mu_2}[\overline{f(\mathbf{t})}](\mathbf{w},\mathbf{x}) = \overline{\mathbb{Q}^{\mathbb{H}}_{\phi,-\mu_1,-\mu_2}[\mathbf{f}(\mathbf{t})](\mathbf{w},\mathbf{x})}. \tag{3.8}$$

5. **(Anti-Linearity)**

$$\mathbb{Q}^{\mathbb{H}}_{\alpha\phi+\beta\psi,\mu_1,\mu_2}[f(\mathbf{t})](\mathbf{w},\mathbf{x}) = \overline{\alpha}\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}(\mathbf{t})](\mathbf{w},\mathbf{x}) + \overline{\beta}\mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[\mathbf{f}(\mathbf{t})](\mathbf{w},\mathbf{x}). \tag{3.9}$$

6. **(Parity)**

$$\mathbb{Q}^{\mathbb{H}}_{P\phi,\mu_1,\mu_2}[Pf(\mathbf{t})](\mathbf{w},\mathbf{x}) = \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1',\mu_2'}[\mathbf{f}(\mathbf{t})](-\mathbf{w},-\mathbf{x}), \tag{3.10}$$

where $P\phi(t) = \phi(-t)$, and $\mu_s' = (A_s, B_s, C_s, -D_s, -E_s)$,$s = 1,2$.

Proof We avoid the proof of Theorem 3.3 as it is straightforward. □

Lemma 3.3 *Let $f \in L^p(\mathbb{R}^2,\mathbb{H})$, $\phi \in L^q(\mathbb{R}^2,\mathbb{H})$, and $p,q \in [1,\infty)$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f(\mathbf{t})](\mathbf{w},\mathbf{x})| \leq \frac{\mathbf{1}}{\mathbf{2}\pi}\|\mathbf{f}\|_{\mathbf{L^p}(\mathbb{R}^2,\mathbb{H})}\|\phi\|_{\mathbf{L^q}(\mathbb{R}^2,\mathbb{H})}. \tag{3.11}$$

Proof We have from Definition (3.1) and Hölder inequality

$$\begin{aligned}
&\left|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f(\mathbf{t})](\mathbf{w},\mathbf{x})\right| \\
&= \left|\frac{1}{2\pi}\int_{\mathbb{R}^2}\Lambda^{\mathbf{i}}_{\mu_1}(t_1,w_1)f(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}\Lambda^{\mathbf{j}}_{\mu_2}(t_2,w_2)d\mathbf{t}\right| \\
&\leq \left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\left|\Lambda^{\mathbf{i}}_{\mu_1}(t_1,w_1)\right|\left|f(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}\right|\left|\Lambda^{\mathbf{j}}_{\mu_2}(t_2,w_2)\right|d\mathbf{t}\right) \\
&\leq \frac{1}{2\pi}\left(\int_{\mathbb{R}^2}|f(\mathbf{t})|^p d\mathbf{t}\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^2}\left|\overline{\phi(\mathbf{t}-\mathbf{x})}\right|^q d\mathbf{t}\right)^{\frac{1}{q}} \\
&= \frac{1}{2\pi}\|f\|_{L^p(\mathbb{R}^2,\mathbb{H})}\|\phi\|_{L^q(\mathbb{R}^2,\mathbb{H})}.
\end{aligned} \tag{3.12}$$

This completes the proof. □

Note for $p=q=2$ (3.11) gives the following important result.

Theorem 3.4 *Let $f,\phi \in L^2(\mathbb{R}^2,\mathbb{H})$, where ϕ is a non zero window function then Q-GQPFT defined by (3.1) is bounded and uniformly continuous on the time–frequency plane $\mathbb{R}^2 \times \mathbb{R}^2$ and satisfies*

$$|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f(\mathbf{t})](\mathbf{w},\mathbf{x})| \leq \frac{\mathbf{1}}{\mathbf{2}\pi}\|\mathbf{f}\|_{\mathbf{L^2}(\mathbb{R}^2,\mathbb{H})}\|\phi\|_{\mathbf{L^2}(\mathbb{R}^2,\mathbb{H})}. \tag{3.13}$$

In the remaining part of this section we are going to prove main properties of Q-GQPFT viz: Moyal's formula, reconstruction formula , isometry and reproducing kernel. Before investigating them we shall introduce two quaternion valued constants as:

For nonzero pair $\{\phi,\psi\}$ of window functions in $L^2(\mathbb{R}^2,\mathbb{H})$, we define quaternion valued constants $\mathcal{C}_{\phi,\psi}$ and $\mathcal{C}_\phi$ with $0<\mathcal{C}_{\phi,\psi},\mathcal{C}_\phi<\infty$ by

$$\mathcal{C}_{\phi,\psi} = \langle \phi, \psi \rangle_{L^2(\mathbb{R}^2,\mathbb{H})} \quad and \quad \mathcal{C}_{\phi} = \|\phi\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{3.14}$$

Theorem 3.5 *(Moyal's formula). Let $f, g \in L^2(\mathbb{R}^2, \mathbb{H})$ and ϕ, ψ are non-zero quaternion valued window functions. Then we have*

$$\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[g] \right\rangle\right]_0 = \frac{1}{|B_1 B_2|}\left[\mathcal{C}_{\phi,\psi}\langle f, g\rangle\right]_0. \tag{3.15}$$

Proof We have

$$\begin{aligned}
&\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[g] \right\rangle\right]_0 \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\overline{\mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[\mathbf{g}](\mathbf{w},\mathbf{x})}\right]_0 d\mathbf{w}d\mathbf{x} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\left(\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\overline{\int_{\mathbb{R}^2}\Lambda^{\mathbf{i}}_{\mu_1}(\mathbf{t_1},\mathbf{w_1})\mathbf{g}(\mathbf{t})\overline{\psi(\mathbf{t}-\mathbf{x})}\Lambda^{\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})d\mathbf{t}}\right]_0\right) d\mathbf{w}d\mathbf{x} \\
&= \frac{1}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\int_{\mathbb{R}^2}\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda^{-\mathbf{i}}_{\mu_1}(\mathbf{t_1},\mathbf{w_1})\overline{\mathbf{g}(\mathbf{t})}\psi(\mathbf{t}-\mathbf{x})\Lambda^{-\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})\right]_0 d\mathbf{t}d\mathbf{w}d\mathbf{x} \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\left(\frac{1}{2\pi}\int_{\mathbb{R}^2}\Lambda^{-i}_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda^{-\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})d\mathbf{w}\right)\psi(\mathbf{t}-\mathbf{x})\overline{\mathbf{g}(\mathbf{t})}\right]_0 d\mathbf{t}d\mathbf{x}.
\end{aligned} \tag{3.16}$$

Applying (3.4) in (3.16), we obtain

$$\begin{aligned}
&\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[g] \right\rangle\right]_0 \\
&= \frac{1}{|B_1 B_2|}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[f(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}\psi(\mathbf{t}-\mathbf{x})\overline{\mathbf{g}(\mathbf{t})}\right]_0 d\mathbf{t}d\mathbf{x} \\
&= \frac{1}{|B_1 B_2|}\left[\int_{\mathbb{R}^2}\psi(\mathbf{t}-\mathbf{x})\overline{\phi(\mathbf{t}-\mathbf{x})}d\mathbf{x}\int_{\mathbb{R}^2}\mathbf{f}(\mathbf{t})\overline{\mathbf{g}(\mathbf{t})}d\mathbf{t}\right]_0 \\
&= \frac{1}{|B_1 B_2|}\left[\mathcal{C}_{\phi,\psi}\langle f, g\rangle\right]_0.
\end{aligned} \tag{3.17}$$

Which completes the proof. □

As an easy consequences of the previous theorem, we immediately obtain the following important results:

1. If $f = g$, then

$$\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[f] \right\rangle\right]_0 = \frac{1}{|B_1 B_2|}\left[\mathcal{C}_{\phi,\psi}\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\right]_0. \tag{3.18}$$

2. If $\phi = \psi$,then

$$\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[g]\right\rangle\right]_0 = \frac{1}{|B_1B_2|}\left[\mathcal{C}_\phi\langle f,g\rangle\right]_0. \tag{3.19}$$

3. If $f = g$ and $\phi = \psi$,then

$$\left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\right\rangle\right]_0 = \frac{1}{|B_1B_2|}\left[\mathcal{C}_\phi\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}\right]_0. \tag{3.20}$$

Note (3.20) is known as the **energy preserving relation** for the proposed Q-GQPFT. And for $\mathcal{C}_\phi = 1$ the proposed Q-GQPFT (3.1) becomes an **isometry** from $L^2(\mathbb{R}^2,\mathbb{H})$ into $L^2(\mathbb{R}^2,\mathbb{H})$. In other words, up to the factor $|B_1B_2|^{-1}$, the total energy of a quaternion-valued signal computed in the quaternion Gabor quadratic-phase Fourier domain is equal to the total energy computed in the spatial domain.

The up comming theorem guarantees the reconstruction of the input signal from the corresponding Q-GQPFT(3.1).

Theorem 3.6 (*Reconstruction formula*). *Every 2-D quaternion signal* $f \in L^2(\mathbb{R}^2,\mathbb{H})$ *can be fully reconstructed by the formula*

$$f(\mathbf{t}) = \frac{|B_1B_2|}{2\pi\mathcal{C}_{\phi,\psi}}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\Lambda^{i}_{-\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda^{\mathbf{j}}_{-\mu_2}(\mathbf{t_2},\mathbf{w_2})\psi(\mathbf{t}-\mathbf{x})\right]_0 d\mathbf{w}d\mathbf{x} \tag{3.21}$$

where $\mathcal{C}_{\phi,\psi}$ is given by (3.14)

Proof By employing Moyal's formula(3.15), we have for any arbitrary $g \in L^2(\mathbb{R}^2,\mathbb{H})$

$$\begin{aligned}
&\frac{1}{|B_1B_2|}\left[\mathcal{C}_{\phi,\psi}\langle f,g\rangle\right]_0\\
&= \left[\left\langle \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f], \mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[g]\right\rangle\right]_0\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left[\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\overline{\mathbb{Q}^{\mathbb{H}}_{\psi,\mu_1,\mu_2}[\mathbf{g}](\mathbf{w},\mathbf{x})}\right]_0 d\mathbf{w}d\mathbf{x}\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^4}\left[\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\left(\overline{\int_{\mathbb{R}^2}\Lambda^{\mathbf{i}}_{\mu_1}(\mathbf{t_1},\mathbf{w_1})\mathbf{g}(\mathbf{t})\overline{\psi(\mathbf{t}-\mathbf{x})}\Lambda^{\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})d\mathbf{t}}\right)\right]_0 d\mathbf{w}d\mathbf{x}\\
&= \frac{1}{2\pi}\int_{\mathbb{R}^6}\left[\Lambda^{-i}_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda^{-\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})\psi(\mathbf{t}-\mathbf{x})\overline{\mathbf{g}(\mathbf{t})}\right]_0 d\mathbf{w}d\mathbf{x}d\mathbf{t}\\
&= \frac{1}{2\pi}\left[\left\langle\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Lambda^{-i}_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda^{-\mathbf{j}}_{\mu_2}(\mathbf{t_2},\mathbf{w_2})\psi(\mathbf{t}-\mathbf{x})d\mathbf{w}d\mathbf{x}, \mathbf{g}(\mathbf{t})\right\rangle\right]_0.
\end{aligned}$$

Since g was chosen arbitrary element of $L^2(\mathbb{R}^2,\mathbb{H})$, it follows that

$$f(\mathbf{t}) = \frac{|B_1B_2|}{2\pi\mathcal{C}_{\phi,\psi}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \left[\Lambda_{\mu_1}^{-i}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\Lambda_{\mu_2}^{-\mathbf{j}}(\mathbf{t_2},\mathbf{w_2})\psi(\mathbf{t}-\mathbf{x})\right]_0 d\mathbf{wdx}.$$

Which completes the proof. □

Remark 3.7 By using the relation between Q-GQPFT and Q-QPFT defined in (3.3) and (3.4), we can derive the alternative form of reconstruction formula as:

On multiplying both sides of (3.4) from the right by $\phi(\mathbf{t}-\mathbf{x})$ and integrating with respect to $d\mathbf{x}$, we get

$$\begin{aligned} f(\mathbf{t}) &\int_{\mathbb{R}^2} |\phi(\mathbf{t}-\mathbf{x})|^{\mathbf{2}}\mathbf{dx} \\ &= \frac{|B_1B_2|}{2\pi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Lambda - i_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\}(\mathbf{w})\Lambda - j_{\mu_2}(t_2,w_2)\phi(\mathbf{t}-\mathbf{x})\mathbf{dwdx}. \end{aligned} \tag{3.22}$$

Now using (3.14) in (3.22), we obtain

$$f(\mathbf{t}) = \frac{|B_1B_2|}{2\pi\mathcal{C}_\phi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Lambda - i_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\}(\mathbf{w})\Lambda - j_{\mu_2}(t_2,w_2)\phi(\mathbf{t}-\mathbf{x})\mathbf{dwdx}. \tag{3.23}$$

Which is required alternative form.

In the next theorem, we shall investigate the reproducing kernel property for the Q-GQPFT(3.1). Firstly we define a family of analyzing quaternion functions (daughter window functions) depending on parameter $\mu_s = (A_s, B_s, C_s, D_s, E_s)$, $s = 1, 2$ as

$$\phi^{\mu_1,\mu_2}_{w,x} = \frac{1}{2\pi}\Lambda i_{\mu_1}(t_1,w_1)\phi(\mathbf{t}-\mathbf{x})\Lambda \mathbf{j}_{\mu_2}(\mathbf{t_2},\mathbf{w_2}) \tag{3.24}$$

where $\Lambda i_{\mu_1}(t_1,w_1)$ and $\Lambda j_{\mu_2}(t_2,w_2)$ are given by (2.2) and (2.3) respectively.

We are now ready to investigate the reproducing kernel property for the Q-GQPFT (3.1).

Theorem 3.8 (*Reproducing kernel*). *Let $\mathcal{C}_\phi \in \mathbb{H}$ defined in* (3.14). *If*

$$\mathbb{K}_\phi(\mathbf{w},\mathbf{x};\mathbf{w}',\mathbf{x}') = \frac{|\mathbf{B_1B_2}|}{(\mathbf{2}\pi)^{\mathbf{2}}\mathcal{C}_\phi}\left\langle \phi^{\mu_1,\mu_2}_{\mathbf{w},\mathbf{x}}, \phi^{\mu_1,\mu_2}_{\mathbf{w}',\mathbf{x}'}\right\rangle, \tag{3.25}$$

then $\mathbb{K}_\phi(\mathbf{w},\mathbf{x};\mathbf{w}',\mathbf{x}')$ is a reproducing kernel, i.e.

$$\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w}',\mathbf{x}') = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x})\mathbb{K}_\phi(\mathbf{w},\mathbf{x};\mathbf{w}',\mathbf{x}')\mathbf{dwdx}. \tag{3.26}$$

Proof By virtue of (3.23), the definition of Q-GQPFT 3.1) becomes

$$\begin{aligned}
&\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w}',\mathbf{x}')\\
&\quad=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Lambda^{i}_{\mu_1}(t_1,w_1')f(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x}')}\Lambda^{j}_{\mu_2}(t_2,w_2')d\mathbf{t}\\
&\quad=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Bigg\{\Lambda^{i}_{\mu_1}(t_1,w_1')\left(\frac{|B_1B_2|}{2\pi\mathcal{C}_\phi}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\Lambda^{-i}_{\mu_1}(t_1,w_1)\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\right.\\
&\quad\times\left.\Lambda{-j}_{\mu_2}(t_2,w_2)\phi(\mathbf{t}-\mathbf{x})\mathbf{dwdx}\right)\overline{\phi(\mathbf{t}-\mathbf{x}')}\Lambda^{j}_{\mu_2}(t_2,w_2')\Bigg\}d\mathbf{t}.
\end{aligned}\tag{3.27}$$

By using (3.24) in (3.27), we have

$$\begin{aligned}
&\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w}',\mathbf{x}')\\
&\quad=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\left(\frac{|\mathbf{B_1B_2}|}{(\mathbf{2}\pi)^{\mathbf{2}}\mathcal{C}_\phi}\int_{\mathbb{R}^2}\phi^{\mu_1,\mu_2}_{\mathbf{w},\mathbf{x}}\overline{\phi^{\mu_1,\mu_2}_{\mathbf{w}',\mathbf{x}'}}\mathbf{dt}\right)\right\}d\mathbf{x}d\mathbf{w}\\
&\quad=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\mathbb{K}_\phi(\mathbf{w},\mathbf{x};\mathbf{w}',\mathbf{x}')\mathbf{dwdx}.
\end{aligned}\tag{3.28}$$

Which completes the proof. □

4 Uncertainty principles for quaternion Q-GQPFT

Heisenberg's uncertainty principle lies at the heart of any time-frequency transform, as it enables us to detect the optimal simultaneous resolution in time and frequency domains.In signal processing an uncertainty principle states that the product of the variances of the signal in the time and frequency domains has a lower bound.In Refs. [13, 21] uncertainty principles for the two sided quaternion Fourier transform and the two sided quaternion Gabor Fourier transform had been studied. The uncertainty principles for linear canonical transform,windowed linear canonical transform and their counter parts in quaternion domain had been discussed in Refs. [3, 4, 15, 16, 29–31]. Recently, the authors established the uncertainty principles associated quadratic-phase Fourier transform and short time quadratic-phase Fourier transform in Refs. [24, 25]. In this Section, we shall establish an analogue of the well-known Heisenberg's uncertainty inequality and the corresponding logarithmic uncertainty principle for the Q-GQPFT as defined by (3.1). Prior to establishing the Heisenberg's uncertainty principle for the proposed transform, we have the following lemma.

Lemma 4.1 *Let $\phi\in L^2(\mathbb{R}^2,\mathbb{H})$ be a nonzero quaternion window function and let $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]\in L^2(\mathbb{R}^2,\mathbb{H})$ be the Q-GQPFT of any signal f, then we have*

$$\mathcal{C}_\phi\int_{\mathbb{R}^2}t_s^2|f(\mathbf{t})|^{\mathbf{2}}\mathbf{dt}=\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\mathbf{t}_\mathbf{s}^\mathbf{2}|\mathbb{Q}^{-\mathbf{1}}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x})\}(\mathbf{t})|^{\mathbf{2}}\mathbf{dtdx},\tag{4.1}$$

where $s=1,2$

Proof Using the relation between Q-GQPFT and Q-QPFT (3.4), (3.14) and elementary properties of quaternions, we have

$$\begin{aligned}\mathcal{C}_\phi \int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})|^2 dt &= \int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})|^2 d\mathbf{t} \int_{\mathbb{R}^2} |\phi(\mathbf{t}-\mathbf{x})|^{\mathbf{2}} \mathbf{dx} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})|^{\mathbf{2}} |\phi(\mathbf{t}-\mathbf{x})|^{\mathbf{2}} \mathbf{dtdx} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})|^{\mathbf{2}} |\overline{\phi(\mathbf{t}-\mathbf{x})}|^{\mathbf{2}} \mathbf{dtdx} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})\overline{\phi(\mathbf{t}-\mathbf{x})}|^{\mathbf{2}} \mathbf{dtdx} \\ &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} t_s^2 |\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}(\mathbf{t})|^{\mathbf{2}} \mathbf{dtdx}.\end{aligned}$$

□

We are now in a position to prove the Heisenberg uncertainty principle for the Q-GQPFT

Theorem 4.2 (*Heisenberg for Q-GQPFT*). *Let $\phi \in L^2(\mathbb{R}^2, \mathbb{H})$ be a nonzero quaternion window function and let $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f] \in L^2(\mathbb{R}^2, \mathbb{H})$ be the Q-GQPFT of any signal f, then the following inequality holds*:

$$\begin{aligned}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_s^2 |\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}} \mathbf{dwdx}\right\}^{1/2} \\ \times \left\{\int_{\mathbb{R}^2} t_s^2 |f(\mathbf{t})|^2 d\mathbf{t}\right\}^{1/2} \geq \frac{\sqrt{\mathcal{C}_\phi}}{2|B_1B_2|}\|f\|^2,\end{aligned} \tag{4.2}$$

where $s = 1, 2$.

Proof Let $f \in L^2(\mathbb{R}^2, \mathbb{H})$ be a quaternion valued signal and $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w}) \in L^2(\mathbb{R}^2, \mathbb{H})$, then Heisenberg-Pauli-Weyl inequality in the quaternion QPFT domain is given by

$$\int_{\mathbb{R}^2} w_s^2 |\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w})|^2 d\mathbf{w} \int_{\mathbb{R}^2} t_s^2 |f(t)|^2 d\mathbf{t} \geq \frac{\mathbf{1}}{\mathbf{4|B_1B_2|}}\left\{\int_{\mathbb{R}^2} |\mathbf{f(t)}|^{\mathbf{2}} \mathbf{dt}\right\}^{\mathbf{2}}, \tag{4.3}$$

where $s = 1, 2$.

By virtue of the Moyal's formula (2.5) and reconstruction formula (2.6), (4.3) can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}^2} t_s^2 |\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]\}|^2 d\mathbf{t} \int_{\mathbb{R}^2} \mathbf{w_s^2} |\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[\mathbf{f}](\mathbf{w})|^{\mathbf{2}} \mathbf{dw} \\ \geq \frac{1}{4}\left(\int_{\mathbb{R}^2} |\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w})|^2 d\mathbf{w}\right)^2.\end{aligned} \tag{4.4}$$

Since $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f] \in L^2(\mathbb{R}^2, \mathbb{H})$, therefore we can replace $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]$ by $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]$ on the both sides of (4.4) to obtain

$$\int_{\mathbb{R}^2} t_s^2 |\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}|^{\mathbf{2}}\mathbf{dt}\int_{\mathbb{R}^2}\mathbf{w_s^2}|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dw} \geq \frac{1}{4}\left(\int_{\mathbb{R}^2}|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dw}\right)^2. \tag{4.5}$$

Taking the square root on both sides of (4.5) and integrating both sides with respect to $d\mathbf{x}$, we obtain

$$\int_{\mathbb{R}^2}\left\{\int_{\mathbb{R}^2} t_s^2 |\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}|^{\mathbf{2}}\mathbf{dt}\right\}^{1/2} \times\left\{\int_{\mathbb{R}^2} w_s^2|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dw}\right\}^{1/2} d\mathbf{x} \geq \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\right|^2 d\mathbf{w}d\mathbf{x}. \tag{4.6}$$

Furthermore, an implication of the well known Cauchy-Schwartz inequality on 4.6 yields

$$\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} t_s^2 |\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}|^{\mathbf{2}}\mathbf{dtdx}\right\}^{1/2} \times\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_s^2|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}\right\}^{1/2} \geq \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\left|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\right|^2 d\mathbf{w}d\mathbf{x}. \tag{4.7}$$

Now by applying Lemma4.1 on L.H.S and (3.20) on R.H.S of the above inequality, we have

$$\left\{\mathcal{C}_\phi\int_{\mathbb{R}^2}|f(\mathbf{t})|^2 d\mathbf{t}\right\}^{1/2}\left\{\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_s^2|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}\right\}^{1/2} \geq \frac{\mathcal{C}_\phi}{2|B_1B_2|}\|f\|^2. \tag{4.8}$$

On further simplifying (4.8),we get

$$\left(\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w_s|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}\right)^{1/2}\left(\int_{\mathbb{R}^2} t_s^2|f(t)|^2 d\mathbf{t}\right)^{1/2} \geq \frac{\sqrt{\mathcal{C}_\phi}}{2|B_1B_2|}\|f\|^2. \tag{4.9}$$

Which completes the proof. □

Remark 4.3 By changing the parameter $\mu_s = (A_s, B_s, C_s, D_s, E_s), \quad s = 1, 2$ as in definition 3.1, the Heisenberg's uncertainty inequality for Q-GQPFT 4.2 boils down to Heisenberg inequality pertaining to quaternion-Gabor Fourier transform, quaternion-Gabor linear canonical transform, quaternion-Gabor offset linear canonical transform and etc.

Next, we derive the logarithmic uncertainty inequality for the Q-GQPFT (3.1). In order to prove logarithmic uncertainty inequality, we need to introduce the following definition.

Definition 4.4 For a multi-index $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^+ \times \mathbb{R}^+$, the Schwartz space in è$^2(\mathbb{R}^2, \mathbb{H})$ is defined by

$$\mathcal{S}(\mathbb{R}^2, \mathbb{H}) = \left\{ f \in \mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H}) : \sup_{t \in \mathbb{R}^2} (1 + |t|^s) \left| \frac{\partial^{\alpha_1 + \alpha_2 |f(t)|}}{\partial_{t_1}^{\alpha_1} \partial_{t_2}^{\alpha_2}} \right| < \infty \right\}, \tag{4.10}$$

where $\mathbb{C}^\infty(\mathbb{R}^2, \mathbb{H})$ is the class of smooth functions from $\mathbb{R}^2$ to $\mathbb{H}$.

Lemma 4.5 *For $f, \phi \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, where ϕ is a nonzero window function. Then we have*

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \ln |\mathbf{t}| |\mathbb{Q}^{-1}_{\mu_1, \mu_2} \{ \mathbb{Q}^{\mathbb{H}}_{\phi, \mu_1, \mu_2} [f](\mathbf{w}, \mathbf{x}) \} (\mathbf{t})|^{\mathbf{2}} \mathbf{dtdx} = \mathcal{C}_\phi \int_{\mathbb{R}^\mathbf{2}} \ln |\mathbf{t}| |\mathbf{f}(\mathbf{t})|^\mathbf{2} \mathbf{dt}. \tag{4.11}$$

Proof By just changing t_s to $\ln |\mathbf{t}|$ in (4.1), we get the desired result. □

Theorem 4.6 (*Q-GQPFT Logarithmic Uncertainty Principle*). *Let $\phi \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$ be a nonzero quaternion window function and let $\mathbb{Q}^{\mathbb{H}}_{\phi, \mu_1, \mu_2}[f]$ be the Q-GQPFT of any signal $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, then the following logarithmic inequality holds*:

$$\begin{aligned} \mathcal{C}_\phi \int_{\mathbb{R}^2} \ln |\mathbf{t}| |f(\mathbf{t})|^\mathbf{2} \mathbf{dt} + |\mathbf{B_1 B_2}| \int_{\mathbb{R}^\mathbf{2}} \int_{\mathbb{R}^\mathbf{2}} \ln |\mathbf{w}| |\mathbb{Q}^{\mathbb{H}}_{\phi, \mu_\mathbf{1}, \mu_\mathbf{2}} [\mathbf{f}](\mathbf{w}, \mathbf{x})|^\mathbf{2} \mathbf{dwdx} \\ \geq \left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi - \ln |B_1 B_2| \right) \mathcal{C}_\phi \|f\|^2_{L^2(\mathbb{R}^2, \mathbb{H})}. \end{aligned} \tag{4.12}$$

Proof For any signal $f \in \mathcal{S}(\mathbb{R}^2, \mathbb{H})$, the logarithmic uncertainty principle for the Q-QPFT reads

$$\begin{aligned} \int_{\mathbb{R}^2} \ln |\mathbf{t}| |f(\mathbf{t})|^2 d\mathbf{t} + |B_1 B_2| \int_{\mathbb{R}^2} \ln |\mathbf{w}| |\mathbb{Q}^{\mathbb{H}}_{\mu_1, \mu_2} [f](\mathbf{w})|^2 d\mathbf{w} \\ \geq \left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln \pi - \ln |B_1 B_2| \right) \int_{\mathbb{R}^2} |f(\mathbf{t})|^2 d\mathbf{t}, \end{aligned} \tag{4.13}$$

where Γ is a gamma function and $\Gamma'(t) = \frac{d}{dt}$.

By invoking of reconstruction formula (2.6) on L.H.S and Moyal's formula (2.5) on R.H.S, Lemma (4.5) can be rewritten as

$$\int_{\mathbb{R}^2} \ln|\mathbf{t}||\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w})\}(\mathbf{t})|^{\mathbf{2}}\mathbf{dt} + |\mathbf{B_1B_2}| \int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[\mathbf{f}](\mathbf{w})|^{\mathbf{2}}\mathbf{dw}$$
$$\geq |B_1B_2|\left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|B_1B_2|\right)\int_{\mathbb{R}^2}|\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f](\mathbf{w})|^2 d\mathbf{w}. \tag{4.14}$$

As both $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]$ and $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]$ are in $\mathcal{S}(\mathbb{R}^2,\mathbb{H})$, therefore by replacing $\mathbb{Q}^{\mathbb{H}}_{\mu_1,\mu_2}[f]$ with $\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f]$ in (4.14), we get

$$\int_{\mathbb{R}^2} \ln|\mathbf{t}||\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}(\mathbf{t})|^{\mathbf{2}}\mathbf{dt}$$
$$+ |B_1B_2| \int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dw}$$
$$\geq |B_1B_2|\left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|B_1B_2|\right)\int_{\mathbb{R}^2}|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dw}. \tag{4.15}$$

Integrating both sides of (4.15) with respect to x, we obtain

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{t}||\mathbb{Q}^{-1}_{\mu_1,\mu_2}\{\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})\}(\mathbf{t})|^{\mathbf{2}}\mathbf{dtdx} + |\mathbf{B_1B_2}|$$
$$\times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}$$
$$\geq |B_1B_2|\left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|B_1B_2|\right)$$
$$\times \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}. \tag{4.16}$$

Applying Lemma (4.5) into the first term on the left-hand side of (4.16), yields

$$\mathcal{C}_\phi \int_{\mathbb{R}^2} \ln|\mathbf{t}||f(\mathbf{t})|^{\mathbf{2}}\mathbf{dt} + |\mathbf{B_1B_2}| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}$$
$$\geq |B_1B_2|\left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|B_1B_2|\right)\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[f](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}. \tag{4.17}$$

Now applying the (3.20) in the right hand side of (4.17), we obtain our desired result,

$$\mathcal{C}_\phi \int_{\mathbb{R}^2} \ln|\mathbf{t}||f(\mathbf{t})|^{\mathbf{2}}\mathbf{dt} + |\mathbf{B_1B_2}| \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \ln|\mathbf{w}||\mathbb{Q}^{\mathbb{H}}_{\phi,\mu_1,\mu_2}[\mathbf{f}](\mathbf{w},\mathbf{x})|^{\mathbf{2}}\mathbf{dwdx}$$
$$\geq \left(\frac{\Gamma'(1/4)}{\Gamma(1/4)} - \ln\pi - \ln|B_1B_2|\right)\mathcal{C}_\phi\|f\|^2_{L^2(\mathbb{R}^2,\mathbb{H})}. \tag{4.18}$$

Which completes the proof. □

5 Conclusion

In this paper we presented a novel integral transform designated as the 2-D Hyper-complex(quaternion) Gabor quadratic- phase Fourier transform (Q-GQPFT), and is embodiment of several well known signal processing tools. We defined the 2-D Hyper-complex(quaternion) quadratic-phase Fourier transform (Q-QPFT) and then proposed the definition of novel Q-GQPFT, which is a modified version of the classical windowed quadratic-phase Fourier transform to quaternion-valued signals . we studied various properties of the proposed Q-GQPFT, including Moyal's formula, reconstruction formula, isometry and reproducing kernel formula. Further We also established the Heisenberg and logarithmic uncertainty inequalities for the Q-GQPFT.

This work can play a vital role in sampling theory, signal synthesis and optics. Further it can be extended to the domains of complex Clifford valued signals.

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Declarations

Conflict of interest The authors have no competing interests.

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