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ORIGINAL RESEARCH PAPER

Local generalizations of Banach's contraction mapping principle

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Abstract

In this paper we prove certain generalized local versions of the Banach's contraction mapping principle. We give a comparison amongst these results and discuss their relationships with certain existing results. We present two illustrative examples. Through these examples we establish a hierarchy amongst some of these results.

Keywords η-chainable metric space · Meir–Keeler contraction · Boyd–Wong contraction · Local contraction · Banach contraction · Fixed point

Mathematics Subject Classification Primary 47H10 · 54H25 · Secondary 54E50

1 Introduction and preliminaries

There have been several generalizations of the celebrated Banach's contraction mapping principle [4, 13] in different directions. Even after a century of its introduction, its generalizations and extensions are being actively pursued. A few instances of these works are [2, 5, 9–11, 14–17, 19]. We consider two of these generalizations, one due to Boyd and Wong [5] and the other due to Meir and Keeler [14]. Eventually, the former is contained in the latter.

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Definition 1.1 (*Boyd–Wong contraction*) [5] Let (X, d) be a metric space and $T : X \to X$ be a function. The map T is called a Boyd–Wong contraction if for every $x, y \in X$,

$$d(Tx, Ty) \leq \phi(d(x, y)) \tag{1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is upper semicontinuous from right and satisfies $\phi(t) < t$ for $t > 0$.

Theorem 1.1 (*Boyd–Wong*) [5] *If (X, d) is a complete metric space and T satisfies relation (1), then T has a unique fixed point $\bar{x}$. Moreover, for any $x \in X$, the sequence of Picard iterates $\{T^n x\}$ of x, converges to $\bar{x}$.*

There are several works, reference [3] for instance, where the ϕ function mentioned above has been used.

Definition 1.2 (*Meir–Keeler contraction*) [14] Let (X, d) be a metric space and $T : X \to X$ be a self-map. T is called a Meir–Keeler contraction if for a given $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x, y \in X$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \tag{2}$$

Theorem 1.2 (*Meir–Keeler*) [14] *If (X, d) is a complete metric space and T satisfies relation (2), then T has a unique fixed point $\bar{x}$. Moreover, for any $x \in X$, the sequence of Picard iterates $\{T^n x\}$ of x, converges to $\bar{x}$.*

In 1961, a local contraction was introduced by Edelstein [10]. In this work (η, λ)-contraction condition was introduced in the following way.

Definition 1.3 (*Local contraction*) [6, 10] Let $(X, d))$ be a metric space. A self map $T : X \to X$ is called locally (Banach) contractive if for each $x \in X$, there exist two real numbers η, λ with $\eta > 0$, and $0 \leq \lambda < 1$, such that

$$\forall a, b \in B(x, \eta), d(Ta, Tb) < \lambda d(a, b).$$

T is called an (η, λ)-uniformly locally contractive map if T is locally contractive and both λ and η do not depend on the point x.

Remark 1.1 As it can be seen from the above that a Banach's contraction is trivially an (η, λ)-contraction for all $\eta > 0$.

In [10], Edelstein established a fixed point theorem for the above local contraction which is derivable as a corollary to our result. We will discuss it in the next section.

In this paper, our primary aim is to introduce two local contractions of the types of Boyd–Wong and Meir–Keeler and obtain fixed point results for them. Further we will show that the local Boyd–Wong type result is effectively contained in local Meir–Keeler result, while both of them effectively contain the theorem of Edelstein, creating a hierarchy of local contractions.

In the following, we define a local notion of Boyd–Wong contraction .

Definition 1.4 (*Uniform local Boyd–Wong contraction*) Let (*X*, *d*) be a metric space, $T : X \to X$ be a function and $\eta > 0$ be a real number. *T* is called an η-uniform local Boyd–Wong contraction if for all $x, y \in X$ with $d(x, y) < \eta$, we have

$$d(Tx, Ty) \leq \phi(d(x, y)), \tag{3}$$

where $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from right and satisfies, $\phi(t) < t$ for $t > 0$.

Remark 1.2 If *T* is an η-uniform local Boyd–Wong contraction then for each $x, y \in X$ with $0 < d(x, y) < \eta$, we have

$$d(Tx, Ty) < d(x, y). \tag{4}$$

Therefore, *T* is a strict local contraction and so continuous.

Remark 1.3 It is clear that every Boyd–Wong contraction map is an η-uniform local Boyd–Wong contraction for all $\eta > 0$.

Next we define the notion of local Meir–Keeler contraction.

Definition 1.5 (*Uniform local Meir–Keeler contraction*) Let (*X*, *d*) be a metric space, $T : X \to X$ be a function and $\eta > 0$ be a real number. *T* is an η-uniform local Meir–Keeler contraction if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in X$ with $d(x, y) < \eta$,

$$\varepsilon \leq d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \tag{5}$$

Remark 1.4 From the above definition it can be seen that if ε is chosen to be greater than or equal to η, then relation-(5) is vacuously satisfied as there exist no *x*, *y* in *X* with $d(x, y) < \eta$ such that $\varepsilon \leq d(x, y)$. In case, $\varepsilon \in (0, \eta)$, the corresponding choice of δ can be made to fall within $(0, \eta - \varepsilon]$, because if $\delta + \varepsilon > \eta$ then selecting $\delta' = \min\{\delta, \eta - \varepsilon\}$ we make δ' to be less than or equal to $\eta - \varepsilon$.

Remark 1.5 If *T* is an η-uniform local Meir–Keeler contraction then for each $x, y \in X$ with $0 < d(x, y) < \eta$, it trivially follows that

$$d(Tx, Ty) < d(x, y). \tag{6}$$

So, *T* is a strict local contraction and thus, continuous.

Remark 1.6 It is clear that every Meir–Keeler contraction map is an η-uniform local Meir–Keeler contraction for all $\eta > 0$.

Definition 1.6 (*η-chainable metric space*) [6, 10] Let $\eta > 0$ be a real number. A metric space (*X*, *d*) is called η-chainable if for every pair of points $x, y \in X$ there exists η-chain from the point *x* to the point *y*, that is, there are finite number of points $a_0, a_1, \ldots, a_n$ in *X* such that $x = a_0,\ y = a_n$ and $d(a_i, a_{i+1}) < \eta$ for $0 \leq i \leq n - 1$.

2 Main results

Theorem 2.1 *Let η be a positive real number and (X, d) be a complete η-chainable metric space. If the map $T : X \rightarrow X$ is η-uniform local Meir–Keeler contraction then T has a unique fixed point in X.*

Proof First we shall assure the existence of a fixed point and then show its uniqueness.

Let us select an element $x \in X$ arbitrarily and construct the sequence $\{x_n\}$ as $x_0 = x,\ x_n = T^n x,\ \forall n \in \mathbb{N}$. If $Tx = x$, then x is a fixed point and this part of the theorem is proved. Now assume $Tx \neq x$. Since X is an η-chainable metric space, let $x = a_0, a_1, ..., a_n = Tx$ be an η-chain from x to Tx. Thus,

$$d(a_i, a_{i+1}) < \eta, \quad \text{for } 0 \leq i \leq n-1. \tag{7}$$

Therefore, each pair of the consecutive elements from the chain satisfies the relation-(6). Therefore,

$$d(Ta_i, Ta_{i+1}) < d(a_i, a_{i+1}) < \eta.$$

By repeated application of the above result, we have $d(T^m a_i, T^m a_{i+1}) < \eta$, for all $m \in \mathbb{N}$. Let us take $R^i_m = d(T^m a_i, T^m a_{i+1})$. By using (6), we have

$$R^i_{m+1} = d(T^{m+1} a_i, T^{m+1} a_{i+1}) < d(T^m a_i, T^m a_{i+1}) = R^i_m.$$

Thus, $\{R^i_m\}$ is a decreasing sequence of non-negative real numbers. Hence, it is convergent. Let $\lim_{m \rightarrow \infty} R^i_m = R^i$. We note that $0 \leq R^i \leq R^i_m < \eta, \forall m \in \mathbb{N}$.

We claim that $R^i = 0$, otherwise, if $R^i > 0$ then, since $R^i < \eta$, from the remark 1.4, there exists $\delta(R^i) \in (0, \eta - R^i)$ such that for all $x, y \in X$ with $d(x, y) < \eta$,

$$R^i \leq d(x, y) < R^i + \delta \Rightarrow d(Tx, Ty) < R^i. \tag{8}$$

Now, as $\{R^i_m\}$ decreases to R^i, there exists $N \in \mathbb{N}$ such that $R^i \leq R^i_N = d(T^N a_i, T^N a_{i+1}) < R^i + \delta$, which, by virtue of the relation (8), implies that $R^i_{N+1} = d(T^{N+1} a_i, T^{N+1} a_{i+1}) < R^i$. This leads to a contradiction to the fact that $R^i \leq R^i_m$, for all $m \in \mathbb{N}$. Thus, $R^i = 0$.

Also,

$$\begin{aligned} \lim_{m\to\infty} d(x_m, x_{m+1}) &= \lim_{m\to\infty} d(T^m x, T^m(Tx)) \\ &\leq \lim_{m\to\infty} \sum_{i=0}^{n-1} d(T^m a_i, T^m a_{i+1}) \\ &= \lim_{m\to\infty} \sum_{i=0}^{n-1} R^i_m \\ &\leq \sum_{i=0}^{n-1} \lim_{m\to\infty} R^i_m \\ &= \sum_{i=0}^{n-1} R^i \\ &= 0. \end{aligned} \tag{9}$$

Hence, $d(x_m, x_{m+1}) \to 0$ as $m \to \infty$.

We now show that $\{x_n\}$ is a Cauchy sequence. Let ε be any positive real number such that $\varepsilon < \eta$ and set $\varepsilon' = \min\{\varepsilon, \eta - \varepsilon\}$. Since T is η-uniform local Meir–Keeler contraction, for the $\varepsilon > 0$ there exists $\delta(\varepsilon) \in (0, \varepsilon')$ such that the implication (5) holds for all $x, y \in X$ with $d(x, y) < \eta$.

Again, as $\lim_{m\to\infty} d(x_m, x_{m+1}) = 0$, $\exists N_0 \in \mathbb{N}$ such that

$$d(x_m, x_{m+1}) < \delta \text{ for all } m \geq N_0. \tag{10}$$

Let $k \geq N_0$ be an integer. We now prove, by induction, that for all $p \in \mathbb{N}$,

$$d(x_k, x_{k+p}) < \varepsilon + \delta. \tag{11}$$

Putting $m = k$ in (10) we get $d(x_k, x_{k+1}) < \delta < \varepsilon + \delta$. Thus, relation (11) is true for $p = 1$.

Let as assume that the relation (11) is true for some $p = n \in \mathbb{N}$. Then $d(x_k, x_{k+n}) < \varepsilon + \delta$. If $\varepsilon \leq d(x_k, x_{k+n})$, then from relation (5) we have $d(x_{k+1}, x_{k+n+1}) = d(Tx_k, Tx_{k+n}) < \varepsilon$. On the other hand, if $d(x_k, x_{k+n}) < \varepsilon$ then from relation (6) we have $d(x_{k+1}, x_{k+n+1}) = d(Tx_k, Tx_{k+n}) < d(x_k, x_{k+n}) < \varepsilon$. Thus, in any case $d(x_{k+1}, x_{k+n+1}) < \varepsilon$.

Now,

$$\begin{aligned} d(x_k, x_{k+n+1}) &\leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+n+1}) \\ &< \varepsilon + \delta. \end{aligned} \tag{12}$$

Therefore, (11) is true for $p = n + 1$ whenever it is true for $p = n$, and thus, relation (11) is true for all $p \in \mathbb{N}$.

Hence, $d(x_k, x_{k+p}) < \varepsilon + \delta < 2\varepsilon$, $\forall k \geq N_0$, and $\forall p \in \mathbb{N}$. Therefore, the sequence $\{x_m\}$ is Cauchy. The space X being complete, there exists $\bar{x} \in X$ such that $\lim_{n\to\infty} x_n = \bar{x}$. The continuity of T (see remark 1.5) then implies $\bar{x} = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T\bar{x}$, showing $\bar{x}$ is a fixed point of T.

We claim that $\bar{x}$ is the unique fixed point for T. If not, and if possible suppose that there exists another fixed point $\bar{y} \in X$. Thus, $d(\bar{x}, \bar{y}) > 0$. For the η-chain $\bar{x} = b_0, b_1, ..., b_p = \bar{y}$ from $\bar{x}$ to $\bar{y}$, we have $d(b_i, b_{i+1}) < \eta$, for $0 \leq i \leq p-1$. Now, following the similar argument, as we have obtained relation (9), we conclude that

$$d(\bar{x}, \bar{y}) = d(T^m \bar{x}, T^m \bar{y}) \leq \sum_{i=0}^{p-1} d(T^m b_i, T^m b_{i+1}). \tag{13}$$

Passing to the limit as $m \longrightarrow \infty$ in the above relation we get the contradiction that $d(\bar{x}, \bar{y}) = 0$.

Thus, the fixed point of T is unique. □

Lemma 2.1 *Let* (X, d) *be a metric space. If* $T: X \to X$ *is an* η*-uniform local Boyd–Wong contraction then it is an* η*-uniform local Meir–Keeler contraction.*

Proof Since T is an η-uniform local Boyd–Wong contraction, for all $x, y \in X$ with $d(x, y) < \eta$ condition (3) holds. Without loss of any generality first fix $\varepsilon > 0$ such that $\varepsilon < \eta$ and then let $\varepsilon' = \varepsilon - \phi(\varepsilon)$. Clearly, $\varepsilon' > 0$ as $\phi(\varepsilon) < \varepsilon$. The function ϕ being upper semicontinuous from right at ε, for the $\varepsilon' > 0$ there exists $0 < \delta < \eta - \varepsilon$ such that

$$t \in [\varepsilon, \varepsilon + \delta) \Rightarrow \phi(t) < \phi(\varepsilon) + \varepsilon' = \varepsilon. \tag{14}$$

Thus, from (14) we have for all $x, y \in X$ with $d(x, y) < \eta$

$$d(x, y) \in [\varepsilon, \varepsilon + \delta) \Rightarrow \phi(d(x, y)) < \varepsilon. \tag{15}$$

Therefore, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(Tx, Ty) \leq \phi(d(x, y)) < \varepsilon$, whenever $\varepsilon \leq d(x, y) < \varepsilon + \delta$ with $d(x, y) < \eta$. Hence, T is an η-uniform local Meir–Keeler contraction. □

Theorem 2.2 *Let* (X, d) *be a complete*, η*-chainable metric space for some real number* $\eta > 0$. *If* $T: X \to X$ *is* η*-uniform local Boyd–Wong contraction*, *then* T *has a unique fixed point in* X.

Proof Let T satisfies the relation (3) for all $x, y \in X$ with $d(x, y) < \eta$. Then by lemma 2.1, the map T satisfies all the conditions of Theorem 2.1. Thus, T has a unique fixed point in X. □

In the following we describe two corollaries which are derived from the above theorems.

Corollary 2.1 (*Edelstein*) [10] *Let* (X, d) *be a complete* η*-chainable metric space for some real number* $\eta > 0$ *and* $T: X \to X$ *be* (η, λ)*-uniformly locally contractive for some real number* $\lambda \in [0, 1)$, *i.e. for each* $x, y \in X$ *if* $d(x, y) < \eta$ *then*

$$d(Tx, Ty) \leq \lambda(d(x, y)). \tag{16}$$

Then T has a unique fixed point in X.

Proof Let for all $x, y \in X$ with $d(x, y) < \eta$, the map T satisfy relation (16). By taking $\phi(t) = \lambda t$ in (3) we see that the function T is an η-uniform local Boyd–Wong contraction. The result then follows from Theorem 2.2. □

Corollary 2.2 [6] *Let* (X, d) *be a complete*, η*-chainable metric space for some real number* $\eta > 0$ *and* $\psi : [0, \infty) \to [0, \infty)$ *be a prior given nondecreasing continuous function such that for* $t > 0$, $\psi(t) > 0$ *and* $\psi(0) = 0$. *If* $T : X \to X$ *is such that for all* $x, y \in X$, *with* $d(x, y) < \eta$, $d(Tx, Ty) \le d(x, y) - \psi(d(x, y))$ *then* T *has a unique fixed point in* X.

Proof Choosing $\phi(t) = t - \psi(t)$ we observe that all the conditions of Theorem 2.2 are satisfied. The corollary then follows as an application of Theorem 2.2. □

3 Illustrative examples

Example 3.1 Let $A = \{(\cos\theta, \sin\theta) | 0 \le \theta \le \frac{3\pi}{2}\}$, $B = \mathbb{N} \times \{0\}$ and consider the metric space (X, d) where $X = A \cup B$ and d is the usual distance on $\mathbb{R}^2$. Clearly X is complete and η-chainable for $\eta = 1.1$. Define $T : X \to X$ by

$$Tx = \begin{cases} (\cos\frac{\theta}{2}, \sin\frac{\theta}{2}) & \text{if } x = (\cos\theta, \sin\theta) \in A, \\ (1, 0) & \text{if } x = (2n, 0), n \in \mathbb{N}, \\ (\cos(\frac{\pi}{3} - \frac{1}{2n+1}), \sin(\frac{\pi}{3} - \frac{1}{2n+1})) & \text{if } x = (2n+1, 0), n \in \mathbb{N}. \end{cases}$$

We note that, for the above η and for $\lambda = \frac{1}{2\cos\frac{5\pi}{48}}$, the map T is an (η, λ) uniform local Banach contraction if applied on the pair of points x, y where both $x, y \in A$. Thus, T is an η-uniform local Meir–Keeler contraction, when restricted to A.

Now, without loss of generality fix $\varepsilon > 0$ such that $\varepsilon < \eta$ and consider the following two cases.

Case I: If $0 < \varepsilon < 1$, then we choose $\delta(\varepsilon) = 1 - \varepsilon$. In this case the inequality $\varepsilon \le d(x, y) < \varepsilon + \delta$ is satisfied only if $x, y \in A$. Hence, T being an (η, λ)-uniform local Banach contraction on A, and thus, being η-uniform local Meir–Keeler contraction on A, T satisfies the relation (5) in this case for all $x, y \in X$ with $d(x, y) < \eta$.

Case II: If $1 \le \varepsilon < \eta$, then we choose $\delta(\varepsilon) = \eta - \varepsilon$, and consider the points $x, y \in X$ such that $\varepsilon \le d(x, y) < \varepsilon + \delta$. Here consider the following three subcases.

Subcase II(a): If both $x, y \in A$ and $\varepsilon \le d(x, y) < \varepsilon + \delta$, then relation (5) is satisfied.

Subcase II(b): If $x = (2n, 0)$, $y = (2n+1, 0)$ for some $n \in \mathbb{N}$, then

$$d(Tx, Ty) = \sqrt{\left(\cos\left(\frac{\pi}{3} - \frac{1}{2n+1}\right) - 1\right)^2 + \left(\sin\left(\frac{\pi}{3} - \frac{1}{2n+1}\right) - 0\right)^2} < 1 \le \varepsilon.$$

Thus, (5) is satisfied also in this case.

Subcase II(c): If $x \in A$ and $y \in B$ then $d(x, y) < \eta$ only when $y = (2, 0)$. Then for $x = (\cos\theta, \sin\theta) \in A$ and $y = (2, 0)$, with $\varepsilon \le d(x, y) < \varepsilon + \delta$ we have

$$d(Tx, Ty) = \sqrt{\left(\cos\frac{\theta}{2} - 1\right)^2 + \left(\sin\frac{\theta}{2} - 0\right)^2} < 1 \leq \varepsilon.$$

Thus, (5) is satisfied also in this case.

Hence, in any case, for the above ε and the corersponding δ, the map T satisfy the relation (5) for all $x, y \in X$ with $d(x, y) < \eta$.

Thus, all the criteria of Theorem 2.1 are met, and thus, by application of Theorem 2.1 we have a unique fixed point for T.

It is observed that (1, 0) is the unique fixed point of T in X.

Remark 3.1 In the Example 3.1, let $x = (1, 0)$ and $y = (0, -1)$. Then, $d(x, y) = \sqrt{2}$ and $d(Tx, Ty) = \sqrt{2 + \sqrt{2}}$. Therefore, $d(Tx, Ty) > d(x, y)$. So, the map T is not even a contraction and consequently not a Meir–Keeler map.

Thus, the Meir–Keeler theorem is not applicable in Example 3.1. On the other hand from remark 1.6 it is seen that every Meir–Keeler contraction map is an η-uniform local Meir–Keeler contraction for all $\eta > 0$. This shows that Theorem 2.1 effectively generalizes the result due to Meir and Keeler[14] (Theorem 1.2) in the context of η-chainable metric spaces.

Remark 3.2 In the Example 3.1, the map T is not an η-uniform local Boyd–Wong map for any $\eta > 1$. If not, let there exist a function ϕ such that T satisfies the relation (3) for all $x, y \in X$ with $d(x, y) < \eta$ for $\eta > 1$; where $\phi : [0, \infty) \to [0, \infty)$ is an upper semicontinuous function from right and satisfies, $\phi(t) < t$ for $t > 0$. Let $x = (2n, 0)$ and $y = (2n + 1, 0)$. Then, from (3) we have

$$\sqrt{\left(\cos\left(\frac{\pi}{3} - \frac{1}{2n+1}\right) - 1\right)^2 + \left(\sin\left(\frac{\pi}{3} - \frac{1}{2n+1}\right) - 0\right)^2} = d(Tx, Ty)$$
$$\leq \phi(d(x, y)) = \phi(1).$$

Taking limit $n \to \infty$ on both sides, we get $1 \leq \phi(1)$, which contradicts the properties of ϕ. So, the map T is not a local Boyd–Wong contraction.

This shows that an η-uniform local Meir–Keeler contraction may not be an η-uniform local Boyd–Wong contraction. But in lemma 2.1 we have proved that every η-uniform local Boyd–Wong contraction is an η-uniform local Meir–Keeler contraction. Thus, Theorem 2.2 is effectively contained in Theorem 2.1.

Example 3.2 Suppose $X = A \cup B$ where $A = \left\{\left(x(t), y(t)\right) : x(t) = t, y(t) = 0, 0 \leq t \leq \frac{1}{2}\right\}$, and $B = \left\{\left(x(s), y(s)\right) : x(s) = \frac{1}{2}, y(s) = s - \frac{1}{2}, \frac{3}{4} \leq s \leq 1\right\}$. Then (X, d) is a complete metric space with respect to the metric d induced by usual Euclidean distance on $\mathbb{R}^2$. Let the map $T : X \to X$ be defined by

$$T\left(x(t), y(t)\right) = \left(t - \frac{1}{2}t^2, 0\right), 0 \leq t \leq 1.$$

Let us choose $\eta = 0.36$. With this value of η, (X, d) is an η-chainable metric space.

(In fact this is true for all $\eta > \frac{1}{4}$). We next prove that the map T is an η-uniform local Boyd–Wong contraction.

Let $\phi : [0, \infty) \to [0, \infty)$ be the function, defined as

$$\phi(x) = \begin{cases} x - \frac{1}{2}x^2, & \text{when } 0 \le x < 1, \\ \frac{x}{2}, & \text{when } x > 1. \end{cases}$$

Clearly ϕ is continuous and for $x > 0$, $0 \le \phi(x) < x$.

Now we consider the following cases.

Case I: Let us consider the pair of points $\tilde{p}\big(x(t), y(t)\big) \in A$, $0 \le t \le \frac{1}{2}$ and $\tilde{q}\big((x(s), y(s)\big) \in B$, $\frac{3}{4} \le s \le 1$.

Then $d(\tilde{p}, \tilde{q}) = \sqrt{\left(\frac{1}{2} - t\right)^2 + \left(s - \frac{1}{2}\right)^2}$.

So, $\phi(d(\tilde{p}, \tilde{q})) = \sqrt{\left(\frac{1}{2} - t\right)^2 + \left(s - \frac{1}{2}\right)^2} - \frac{1}{2}\left(\left(\frac{1}{2} - t\right)^2 + \left(s - \frac{1}{2}\right)^2\right)$.

Now, $d(T\tilde{p}, T\tilde{q}) = \left(s - \frac{1}{2}s^2\right) - \left(t - \frac{1}{2}t^2\right)$.

Consider the function $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by

$$F(t, s) = \left\{\sqrt{\left(\frac{1}{2} - t\right)^2 + \left(s - \frac{1}{2}\right)^2} - \frac{1}{2}\left(\left(\frac{1}{2} - t\right)^2 + \left(s - \frac{1}{2}\right)^2\right)\right\} - \left\{\left(s - \frac{1}{2}s^2\right) - \left(t - \frac{1}{2}t^2\right)\right\}.$$

The function F is continuous at $\left(\frac{1}{2}, \frac{3}{4}\right)$ and also $F\left(\frac{1}{2}, \frac{3}{4}\right) > 0$. Thus, there exists $\delta > 0$ such that F takes only positive values in the δ-neighborhood of $\left(\frac{1}{2}, \frac{3}{4}\right)$.

Now,for the parameters $t = \frac{1}{2}$, $s = \frac{3}{4}$ the corresponding points $\tilde{p_0}(\frac{1}{2}, 0) \in A$, and $\tilde{q_0}(\frac{1}{2}, \frac{1}{4}) \in B$ satisfies $\phi(d(\tilde{p_0}, \tilde{q_0})) - d(T\tilde{p_0}, T\tilde{q_0}) > 0$, as $F\left(\frac{1}{2}, \frac{3}{4}\right) > 0$. Thus, $\tilde{p_0}, \tilde{q_0}$ satisfy relation (3). Therefore, there exists $\delta > 0$ with $\delta < \frac{1}{4}$, such that for all pair of points $\tilde{u}\big(x(t), y(t)\big) \in A$, $\tilde{v}\big((x(s), y(s)\big) \in B)$, with $t \in \left(\frac{1}{2} - \delta, \frac{1}{2}\right]$ and $s \in \left[\frac{3}{4}, \frac{3}{4} + \delta\right)$ the relation (3) is satisfied.

Now let us consider a real number α such that $0 < \alpha \le \delta$. Then, the points corresponding to $t = \frac{1}{2} - \alpha$ and $s = \frac{3}{4} + \alpha$ are respectively $\tilde{p}\left(\frac{1}{2} - \alpha, 0\right)$ and $\tilde{q}(\frac{1}{2}, \frac{1}{4} + \alpha)$. Thus,

$$d(\tilde{p}, \tilde{q}) = d\left(\left(\frac{1}{2} - \alpha, 0\right), \left(\frac{1}{2}, \frac{1}{4} + \alpha\right)\right) = \sqrt{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2}.$$

$$\text{So, } \phi(d(\tilde{p}, \tilde{q})) = \sqrt{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2} - \frac{1}{2}\left\{\alpha^2 + \left(\frac{1}{4} + \alpha\right)^2\right\} = \Delta(\alpha)(say).$$

Now, for $d(\tilde{p}, \tilde{q}) < 1$, $\Delta(\alpha) = d(\tilde{p}, \tilde{q}) - \frac{1}{2}(d(\tilde{p}, \tilde{q}))^2$ is a strictly increasing function of $d(\tilde{p}, \tilde{q})$. Thus, $\Delta(\alpha)$ is strictly increasing with respect to α, for $0 \le \alpha < \frac{1}{4}$.

Therefore, the minimum value of $\Delta(\alpha)$ for $\alpha \in [0, \frac{1}{4})$ is $\Delta(0) = \frac{7}{32}$.

Also, $d(T\tilde{p}, T\tilde{q}) = \left\{\left(\frac{3}{4} + \alpha\right) - \frac{1}{2}\left(\frac{3}{4} + \alpha\right)^2\right\} - \left\{\left(\frac{1}{2} - \alpha\right) - \frac{1}{2}\left(\frac{1}{2} - \alpha\right)^2\right\} = \frac{3}{32} + \frac{3}{4}\alpha = f(\alpha)$ (say), which is clearly a continuous and increasing function of α.

For $\alpha = 0.1$, we have $d(T\tilde{p}, T\tilde{q}) = \frac{27}{160} < \frac{7}{32}$, the minimum value of the R.H.S of (3) in this case.

Hence, we have, $f(\alpha) < f(0.1) = \frac{27}{160} < \frac{7}{32} = \Delta(0) < \Delta(\alpha)$, that is, $d(T\tilde{p}, T\tilde{q}) < \phi(d(\tilde{p}, \tilde{q}))$. Therefore, for $\alpha \leq 0.1$, relation (3) is satisfied. Here we take $\delta = 0.1$.

Hence, any pair of points $\tilde{p} \in A$ and $\tilde{q} \in B$, whose distance is lesser than $\eta = 0.36$, satisfies the inequality (3).

Case II: Let both the points $\tilde{p}\big(x(s_1), y(s_1)\big), \tilde{q}\big(x(s_2), y(s_2)\big) \in B$, where, $\frac{3}{4} \leq s_1 \leq s_2 \leq 1$. Thus, $d(\tilde{p}, \tilde{q}) = s_2 - s_1$ and

$$
\begin{aligned}
d(T\tilde{p}, T\tilde{q}) &= \left\{s_2 - \frac{1}{2}{s_2}^2\right\} - \left\{s_1 - \frac{1}{2}{s_1}^2\right\} \\
&= (s_2 - s_1) - \frac{1}{2}\left({s_2}^2 - {s_1}^2\right).
\end{aligned}
$$

Here, $\phi(d(\tilde{p}, \tilde{q})) = d(\tilde{p}, \tilde{q}) - \frac{1}{2}(d(\tilde{p}, \tilde{q}))^2 = (s_2 - s_1) - \frac{1}{2}(s_2 - s_1)^2$. Since $\frac{3}{4} \leq s_1 \leq s_2$, we note that ${s_2}^2 - {s_1}^2 = (s_2 + s_1)(s_2 - s_1) \geq (s_2 - s_1)(s_2 - s_1) = (s_2 - s_1)^2$. Thus, we have $(s_2 - s_1) - \frac{1}{2}\left\{{s_2}^2 - {s_1}^2\right\} \leq (s_2 - s_1) - \frac{1}{2}(s_2 - s_1)^2$.

Thus, relation (3) is satisfied in this case. So, in particular, for $d(\tilde{p}, \tilde{q}) < \eta$, relation (3) remains satisfied.

Case III: Let both the points $\tilde{p}\big(x(t_1), y(t_1)\big), \tilde{q}\big(x(t_2), y(t_2)\big) \in A$, where, $0 \leq t_1 \leq t_2 \leq \frac{1}{2}$. Then $d(\tilde{p}, \tilde{q}) = t_2 - t_1$ and $d(T\tilde{p}, T\tilde{q}) = \left\{t_2 - \frac{1}{2}{t_2}^2\right\} - \left\{t_1 - \frac{1}{2}{t_1}^2\right\} = (t_2 - t_1) - \frac{1}{2}\left\{{t_2}^2 - {t_1}^2\right\}$.

Now, $\phi(d(\tilde{p}, \tilde{q})) = d(\tilde{p}, \tilde{q}) - \frac{1}{2}(d(\tilde{p}, \tilde{q}))^2 = (t_2 - t_1) - \frac{1}{2}(t_2 - t_1)^2$.

Since $0 \leq t_1 \leq t_2$, we note that ${t_2}^2 - {t_1}^2 = (t_2 + t_1)(t_2 - t_1) \geq (t_2 - t_1)(t_2 - t_1) = (t_2 - t_1)^2$.

Thus, we have $(t_2 - t_1) - \frac{1}{2}\left\{{t_2}^2 - {t_1}^2\right\} \leq (t_2 - t_1) - \frac{1}{2}(t_2 - t_1)^2$.

Therefore, relation (3) is satisfied and thus, in particular, for $d(\tilde{p}, \tilde{q}) < \eta$, relation (3) is satisfied.

Thus, by the above three cases, T is η-uniform local Boyd–Wong contraction, for $\eta = 0.36$.

Thus, all the conditions of the Theorem 2.2 are satisfied.

We observe that $(0, 0)$ is the unique fixed point of T.

Remark 3.3 The map T in the Example 3.2, fails to be uniform local (Banach) contraction. If not, then there exist constants η, λ,such that $\eta > 0$, and $0 \leq \lambda < 1$, satisfying

$$\forall \tilde{p}, \tilde{q} \in X \text{ if } d(\tilde{p}, \tilde{q}) < \eta \text{ then } d(T\tilde{p}, T\tilde{q}) < \lambda\, d(\tilde{p}, \tilde{q}). \tag{17}$$

Let $t = \min\{\eta, \frac{1-\lambda}{2}\}$ and consider the two points $\tilde{p}(0,0)$ and $\tilde{q}(t,0)$. Then, we have, $d(\tilde{p}, \tilde{q}) = t$, $d(T\tilde{p}, T\tilde{q}) = t - \frac{1}{2}t^2$. Since $d(\tilde{p}, \tilde{q}) = t < \eta$, form relation (17) we get,

$$\begin{aligned} & t - \frac{t^2}{2} < \lambda t \\ or,\ & t - \frac{t^2}{2} < (1-2t)t \qquad \left[\text{Since } t \le \frac{1-\lambda}{2} \text{ or, } \lambda \le 1 - 2t.\right] \\ or,\ & -\frac{t^2}{2} < -2t^2. \end{aligned}$$

The above is impossible to hold and hence, the map T fails to be uniform local (Banach) contraction.

This shows that an η-uniform local Boyd–Wong contraction may not be an uniform local (Banach) contraction. In view of Corollary 2.1 we conclude that Theorem 2.2 effectively includes the result of Edelstein [10].

4 Discussion and conclusion

In view of Remarks 3.2 and 3.3 we observe that the local Meir–Keeler result (Theorem 2.1), the local Boyd–Wong result (Theorem 2.2), and the theorem of Edelstein (Corollary 2.1) form a hierarchy in that the former effectively contains the successor. In this context we mention the weak contraction mapping principle due to Rhoades[18]. Although this result is contained in the theorem of Boyd and Wong, and is published subsequently in 2001, it opened a new avenue of research in which not only the Banach's result was generalized but also a new chapter with weak contraction inequalities was opened in fixed point theory. See [1, 7, 8, 12] and references therein for instances. In a recent paper [6] the present authors have established a local version of the weak contraction principle which is derived here as Corollary 2.2. It is shown in [6] that for a prior given ψ, the Banach's contraction mapping theorem may not be included in that result. Therefore, the local contraction theorem of Edelstein is also not included in the above result. For this reason Corollary 2.2 is not a part of the hierarchy of the local fixed point results mentioned above.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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