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ORIGINAL RESEARCH PAPER

Characterization of *n*-Jordan multipliers through zero products

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Abstract

Let A be a unital C^*-algebra and X be a unital Banach A-bimodule. In this paper, we characterize n-Jordan multipliers $T: A \longrightarrow X$ through the action on zero product. We prove that each continuous linear mapping T from group algebra $L^1(G)$ into unital Banach A-bimodule X which satisfies a related condition is an n-Jordan multiplier.

Keywords n-Jordan multiplier · Bilinear maps · C^*-algebra · Unital A-bimodule

Mathematics Subject Classification Primary 47B47 · 47B49 · Secondary 15A86

1 Introduction and preliminaries

Let A be a Banach algebra and X be an A-bimodule. A linear map $T: A \longrightarrow X$ is called *left multiplier* [*right multiplier*] if for all $a, b \in A$,

$$T(ab) = T(a)b, \quad [T(ab) = aT(b)],$$

and T is called a *multiplier* if it is both left and right multiplier. Also, T is called *left Jordan multiplier* [*right Jordan multiplier*] if for all $a \in A$,

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$$T(a^2) = T(a)a, \quad [T(a^2) = aT(a)],$$

and T is called a *Jordan multiplier* if T is a left and a right Jordan multiplier.

It is clear that every left (right) multiplier is a left (right) Jordan multiplier, but the converse is not true in general, as was demonstrated in [9, Example 2.6]. Concerning the characterization of Jordan multiplier, see [12] and [13].

A linear map D from Banach algebra A into an A-bimodule X is called *derivation* [*Jordan derivation*] if

$$D(ab) = D(a)b + aD(b), \quad [D(a^2) = D(a)a + aD(a)], \quad a, b \in A.$$

Note that every derivation is a Jordan derivation, but the converse is fails in general [4, 10]. It is proved by Johnson in [10, Theorem 6.3] that every Jordan derivation from C^*-algebra A into any A-bimodule X is a derivation.

Characterizing homomorphisms, derivations and multipliers on Banach algebras through the action on zero products have been studied by many authors. We refer the reader to [2, 3, 8, 9, 14] for a full account of the topic and a list of references.

Let A be a unital Banach algebra with unit e_A. An A-bimodule X is called *unital* if $e_A x = x e_A = x$, for all $x \in X$, and it is called *symmetric* if $ax = xa$ for every $a \in A$ and $x \in X$. For example, A^* is an unital A-bimodule with the following actions.

$$(a \cdot f)(b) = f(ba), \quad (f \cdot a)(b) = f(ab), \quad a, b \in A, \quad f \in A^*.$$

Moreover, if A is commutative, then A^* is symmetric.

Recall that a bounded approximate identity for Banach algebra A is a bounded net $(e_\alpha)_{\alpha \in I}$ in A such that $ae_\alpha \longrightarrow a$ and $e_\alpha a \longrightarrow a$, for all $a \in A$. For example, it is known that the group algebra $L^1(G)$, for a locally compact group G, and C^*-algebras have a bounded approximate identity bounded by one [6]. If A has a bounded approximate identity $(e_\alpha)_{\alpha \in I}$, then A-bimodule X is unitary if $e_\alpha x \longrightarrow x$ and $xe_\alpha \longrightarrow x$, for all $x \in X$.

Definition 1.1 [7] Let A be a Banach algebra, X be a right A-module and let $T : A \longrightarrow X$ be a linear map. Then T is called *left n-Jordan multiplier* if for all $a \in A$,

$$T(a^n) = T(a^{n-1})a.$$

The *right n-Jordan multiplier* and *n-Jordan multiplier* can be defined analogously.

The two following results concerning characterization of n-Jordan multiplier presented by the first author in [15].

Theorem 1.2 [15, *Theorem* 2.3] *Let A be a unital Banach algebra and X be a unital Banach left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map such that*

$$a, b \in A, \quad ab = e_A \quad \Longrightarrow \quad T(ab) = aT(b). \tag{1}$$

Then T is a right n-Jordan multiplier.

Lemma 1.3 [15, *Lemma* 2.1] *Let A be a Banach algebra, X be a left A-module and let $T : A \longrightarrow X$ be a right Jordan multiplier. Then T is a right n-Jordan multiplier for each $n \geq 2$.*

Let A be a Banach algebra and X be a arbitrary Banach space. Then the continuous bilinear mapping $\phi : A \times A \longrightarrow X$ preserves zero products if

$$ab = 0 \quad \Longrightarrow \quad \phi(a, b) = 0, \quad a, b \in A. \tag{2}$$

The study of zero products preserving bilinear maps was initiated in [1] for a very special setting and then it was studied in [2] for the general case. Motivated by (2) the following concept was introduced in [2].

Definition 1.4 A Banach algebra A *has the property* $(\mathbb{B})$ if for every continuous bilinear mapping $\phi : A \times A \longrightarrow X$, where X is an arbitrary Banach space, the condition (2) implies that $\phi(ab, c) = \phi(a, bc)$, for all $a, b, c \in A$.

Let $\mathfrak{J}(A)$ denote the subalgebra of A generated by all idempotents in A. If $A = \overline{\mathfrak{J}(A)}$, then we say that the Banach algebra A is generated by idempotents. Some examples of Banach algebras with the property that $A = \overline{\mathfrak{J}(A)}$, are the following:

(a) The Banach algebra $C(X)$, where X is a compact Hausdorff space.
(b) Topologically simple Banach algebras containing a non-trivial idempotent.
(c) W^*-algebras. Indeed, the linear span of projections is norm dense in a W^*-algebra. Following [11], we recall that a C^*-algebra A is called a W^*-algebra if it is a dual space as a Banach space, see also [6]. Note that every W^*-algebra is unital.

Another classes of Banach algebras A with the property that $A = \overline{\mathfrak{J}(A)}$ are given in [2]. By Examples 1.3 and Theorem 2.11 of [2] we have the following.

Remark 1.5

(i) *Every C^*-algebra A has the property* $(\mathbb{B})$.
(ii) *Let $A = L^1(G)$ for a locally compact group G. Then A has the property* $(\mathbb{B})$.
(iii) *Let A be a unital Banach algebra that is generated by idempotents, then A has the property* $(\mathbb{B})$.

Consider the following condition on a linear map T from Banach algebra A into a Banach A-bimodule X which is closely related to the condition (1).

$$a, b \in A, \quad ab = 0 \Longrightarrow \quad aT(b) = 0. \qquad (\mathbb{M})$$

A rather natural weakening of condition $(\mathbb{M})$ is the following:

$$a, b \in A, \quad ab = ba = 0 \Longrightarrow \quad aT(b) + bT(a) = 0. \qquad (\mathbb{JM})$$

In this paper we investigate whether those conditions characterizes n-Jordan multipliers. We prove that when A is a unital C^*-algebra and X is a symmetric unital A-

bimodule, condition $(\mathbb{JM})$ implies that T is of the form $D+\psi$, where $D : A \longrightarrow X$ is a (Jordan) derivation and $\psi : A \longrightarrow X$ is a (Jordan) multiplier.

2 Characterization of *n*-Jordan multipliers

Since all results which are true for left versions have obvious analogue statements for right versions, we will focus in the sequel just the right versions.

Theorem 2.1 *Let A be a unital C^*-algebra and X be a unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{M})$. Then T is a right n-Jordan multiplier.*

Proof Define a continuous bilinear mapping $\phi : A \times A \longrightarrow X$ by $\phi(a,b) = aT(b)$. Then $\phi(a,b) = 0$ whenever $ab = 0$. Hence the property $(\mathbb{B})$ gives

$$abT(c) = \phi(ab,c) = \phi(a,bc) = aT(bc), \qquad a,b,c \in A.$$

Taking $b = c$ and $a = e_A$, we conclude that $T(b^2) = bT(b)$ for all $b \in A$. Therefore, T is a right Jordan multiplier and hence it is a right n-Jordan multiplier by Lemma 1.3. □

We mention that Theorem 2.1 is also true for non-unital case, because every C^*-algebra A has a bounded approximate identity.

Moreover, by Remark 1.5 it is true for group algebra and for each unital Banach algebra that is generated by idempotents.

In view of Theorem 2.1, the following question can be raised. Dose Theorem 2.1 remain valid with condition $(\mathbb{M})$ replaced by $(\mathbb{JM})$?

Theorem 2.2 [3, *Theorem* 2.2] *Let A be a C^*-algebra and X be a Banach space and let $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$ab = ba = 0 \implies \phi(a,b) = 0, \qquad a,b \in A.$$

Then

$$\phi(ax,by) + \phi(ya,xb) = \phi(a,xby) + \phi(yax,b),$$

for all $a,b,x,y \in A$.

Our first main result is the following.

Theorem 2.3 *Let A be a unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying $(\mathbb{JM})$. Then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Proof Define a bilinear mapping $\phi : A \times A \longrightarrow X$ by

$$\phi(a,b) = aT(b) + bT(a),$$

for all $a,b \in A$. Then $ab = ba = 0$ implies that $\phi(a,b) = 0$. Hence by Theorem 2.2,

$$\phi(ax, by) + \phi(ya, xb) = \phi(a, xby) + \phi(yax, b), \tag{3}$$

for all $a, b, x, y \in A$. Replacing a, b by e_A in (3), we get

$$\phi(x, y) + \phi(y, x) = \phi(e_A, xy) + \phi(yx, e_A),$$

for all $x, y \in A$. This meanes that

$$xT(y) + yT(x) + yT(x) + xT(y) = T(xy + yx) + xyT(e_A) + yxT(e_A). \tag{4}$$

Replacing y by x in (4) and since X is symmetric, we arrive at

$$T(x^2) = \big(xT(x) + T(x)x\big) - x^2T(e_A). \tag{5}$$

Define $\psi : A \longrightarrow X$ via $\psi(a) = aT(e_A)$. Then clearly ψ is a Jordan multiplier. Now let $D = T - \psi$. Then D is a Jordan derivation. Indeed, from (5), we get

$$\begin{aligned} D(a^2) &= T(a^2) - \psi(a^2) \\ &= (aT(a) + T(a)a) - 2a^2T(e_A) \\ &= a\big(T(a) - aT(e_A)\big) + \big(T(a) - aT(e_A)\big)a \\ &= aD(a) + D(a)a. \end{aligned}$$

Consequently, $T = D + \psi$. This finishes the proof. □

Recall that the Banach algebra A is called *weakly amenable* if $\mathcal{H}^1(A, A^*) = \{0\}$, i.e., each derivation from A into the dual module A^* is inner.

It is known-that each C^*-algebra and every commutative Banach algebra with the property that $A = \overline{\mathfrak{J}(A)}$ are weakly amenable [6].

Corollary 2.4 *Let A be a commutative unital C^*-algebra and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear mapping such that the condition* $(\mathbb{JM})$ *holds. Then T is an n-Jordan multiplier.*

Proof By Theorem 2.3, $T = D + \psi$ where $D : A \longrightarrow X$ is a Jordan derivation and $\psi : A \longrightarrow X$ is a Jordan multiplier. It follows from Johnson's theorem that D is a derivation and hence by [6, Theorem 2.8.63] D is zero. Therefore T is an n-Jordan multiplier by Lemma 1.3. □

Next we generalize Corollary 2.4 and give the affirmative answer to the preceding question.

Theorem 2.5 *Let A be a W^*-algebra and X be a unital left A-module. If $T : A \longrightarrow X$ is a continuous linear map satisfying* $(\mathbb{JM})$, *then T is a right n-Jordan multiplier.*

Proof Let p be an idempotent in A. Since $p(e_A - p) = (e_A - p)p = 0$, we have

$$pT(e_A - p) + (e_A - p)T(p) = 0.$$

So $T(p) = pT(p)$, for every idempotent $p \in A$. Let A_{sa} denote the set of self-adjoint elements of A and let $x \in A_{sa}$. Then by Lemma 1.7.5 and Proposition 1.3.1 of [11],

$$x = \sum_{k=1}^{n} \lambda_k p_k,$$

where $\{\lambda_k\}$ are real numbers and $\{p_k\}$ is an orthogonal family of projections in A, i.e., self-adjoint idempotents. Since $p_i p_j = p_j p_i = 0$ for $i \neq j$, condition $(\mathbb{JM})$ implies that $p_i T(p_j) + p_j T(p_i) = 0$ for all i, j with $i \neq j$. Hence

$$\begin{aligned} T(x^2) &= T\left(\sum_{k=1}^{n} \lambda_k^2 p_k^2\right) \\ &= \sum_{k=1}^{n} \lambda_k^2 T(p_k^2) \\ &= \left(\sum_{k=1}^{n} \lambda_k p_k\right)\left(\sum_{k=1}^{n} \lambda_k T(p_k)\right) = xT(x), \end{aligned}$$

for all $x \in A_{sa}$. Taking x, y self-adjoint elements in A, we get

$$\begin{aligned} xT(x) + yT(y) + T(xy + yx) &= T(x^2 + y^2 + xy + yx) \\ &= T\big((x+y)^2\big) \\ &= (x+y)T(x+y) \\ &= xT(x) + yT(y) + xT(y) + yT(x). \end{aligned}$$

Thus, $T(xy + yx) = xT(y) + yT(x)$ for all $x, y \in A_{sa}$. Now each arbitrary element $a \in A$, can be written as $a = x + iy$ for $x, y \in A_{sa}$. Therefore

$$\begin{aligned} T(a^2) &= T\big(x^2 - y^2 + i(xy + yx)\big) \\ &= T(x^2) - T(y^2) + iT(xy + yx) \\ &= xT(x) - yT(y) + i\big(xT(y) + yT(x)\big) \\ &= aT(a) \end{aligned}$$

Consequently, $T(a^2) = aT(a)$ for all $a \in A$. Now by Lemma 1.3, T is a right n-Jordan multiplier. □

Now we turns to the C^*-algebra case. First note that the linear span of projections are dense in a unital C^*-algebra of real rank zero [5], hence the conclusion of Theorem 2.5 also holds for such C^*-algebras.

It is well known that on the second dual space A^{**} of a Banach algebra A there are two multiplications, called the first and second Arens products which make A^{**} into a Banach algebra [6]. If these products coincide on A^{**}, then A is said to be Arens regular.

It is shown [6] that every C^*-algebra A is Arens regular and the second dual of each C^*-algebra is a W^*-algebra. Therefore by extending the continuous linear map $T : A \longrightarrow X$ to the second adjoint $T^{**} : A^{**} \longrightarrow X^{**}$ and applying Theorem 2.5, we get the following result.

Corollary 2.6 *Let A be a unital C^*-algebra and X be a unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying* $(\mathbb{JM})$, *then T is a right n-Jordan multiplier.*

Theorem 2.7 [8, *Corollary* 3.6] *Let A be Banach algebra, X be a Banach space and $\phi : A \times A \longrightarrow X$ be a continuous bilinear mapping such that*

$$a, b \in A, \quad ab = ba = 0 \implies \phi(a, b) = 0,$$

then

$$\phi(a, x) + \phi(x, a) = \phi(ax, e_A) + \phi(e_A, xa),$$

for all $a \in A$ and $x \in \mathfrak{J}(A)$. In particular, if A is generated by idempotents, then

$$\phi(a, b) + \phi(b, a) = \phi(ab, e_A) + \phi(e_A, ba), \qquad a, b \in A.$$

By using Theorem 2.7 we can obtain the following result.

Theorem 2.8 *Let A be a unital Banach algebra which is generated by idempotents and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying* $(\mathbb{JM})$. *Then there exist a Jordan derivation D and a Jordan multiplier ψ such that $T = D + \psi$.*

Corollary 2.9 *Let A be a commutative unital Banach algebra such that $A = \overline{\mathfrak{J}(A)}$ and X be a symmetric unital left A-module. Suppose that $T : A \longrightarrow X$ is a continuous linear map satisfying* $(\mathbb{JM})$. *Then T is an n-Jordan multiplier.*

Proof By Theorem 2.8, $T = D + \psi$ where $D : A \longrightarrow X$ is a Jordan derivation and $\psi : A \longrightarrow X$ is a Jordan multiplier. Since A is commutative and X is symmetric, D is actually a derivation. Now from Theorem 2.8.63 of [6] it follows that $D = 0$ and hence T is an n-Jordan multiplier by Lemma 1.3. □

Let $A = L^1(G)$ for a locally compact abelian group G. Then A is commutative and it is weakly amenable [6], but neither it is C^*-algebra nor generated by idempotents. Therefore Corollary 2.4 and Corollary 2.9 cannot be applied for it.

The next result shows that analogous of Corollary 2.4 is also true for group algebra.

Theorem 2.10 *Let $A = L^1(G)$ for a locally compact abelian group G. Suppose that X is a symmetric unital left A-module and $T : A \longrightarrow X$ is a continuous linear map satisfying* $(\mathbb{JM})$. *Then T is an n-Jordan multiplier.*

Proof Define a bilinear mapping $\phi : A \times A \longrightarrow X$ by

$$\phi(a, b) = aT(b) + bT(a),$$

for all $a, b \in A$. Then $ab = 0$ implies that $\phi(a, b) = 0$. Hence by [2, Theorem 2.11],

$$\phi(ab,c)=\phi(a,bc), \qquad a,b,c\in A. \tag{6}$$

On the other hand, $\phi(a,b)=\phi(b,a)$ for all $a,b\in A$. Thus, it follows from (6) that

$$\phi(c,ab)+\phi(ab,c)=\phi(a,bc)+\phi(cb,a), \tag{7}$$

for all $a,b,c\in A$. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity for A. Replacing a by (e_α) in (7) and using the continuity of ϕ, we get $\phi(c,b)+\phi(b,c)=\phi(e_\alpha,bc)+\phi(cb,e_\alpha)$ for all $b,c\in A$. Now by applying the same method of the proof of Theorem 2.3, there exist a Jordan derivation D and a Jordan multiplier ψ such that $T=D+\psi$. Since A is weakly amenable, $D=0$. Consequently, T is an n-Jordan multiplier by Lemma 1.3. □

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Declarations

Conflict of interest Authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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