The Journal of Analysis (2022) 30:965–987
https://doi.org/10.1007/s41478-022-00384-3

ORIGINAL RESEARCH PAPER

A strong convergence algorithm for solving pseudomonotone variational inequalities with a single projection

Chibueze C. Okeke[1] · Abdulmalik U. Bello[2,3] · Olawale K. Oyewole[4]

Received: 21 August 2021 / Accepted: 9 January 2022 / Published online: 25 January 2022

Abstract
This paper proposes a single projection method with the Bregman distance technique for solving pseudomonotone variational inequalities in real reflexive Banach space. The step-sizes, which varies from step to step, are found over each iteration by cheap computation without any linesearch. We prove strong convergence result under suitable conditions on the cost operator. We further provide an application of our main result and also report some numerical experiments to illustrate the performance and efficiency of our proposed method.

Keywords Variational inequalities · Pseudomonotone mapping · Self-adaptive step size · Single projection · Banach spaces

Mathematics Subject Classification 65k15 · 47J25 · 65J15 · 90C33

Communicated by Samy Ponnusamy.

✉ Chibueze C. Okeke
chibueze.okeke87@yahoo.com

Abdulmalik U. Bello
uabdulmalik@aust.edu.ng

Olawale K. Oyewole
oyewoleolawalekazeem@gmail.com

1 School of Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

2 Mathematics Institute African University of Science and Technology, Abuja, Nigeria

3 Department of Mathematics, Federal University Dutsin-Ma, Dutsin-Ma, Kastina State, Nigeria

4 DST-NRF Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS), Johannesburg, South Africa

1 Introduction

Numerous problems in science and engineering, optimization theory, nonlinear analysis, equilibrium theory and differential equations, lead to the study of variational inequality problem (VIP) in the sense of Stampacchia et al. in [19, 25]. This problem has been intensively investigated and developed after appearing in the mono graphic book [15, 26]. Let E be a real Banach space with norm $\|\cdot\|$ and represent the dual of E by E^* with $\langle x^*, x\rangle$ signifying the value of $x^* \in E^*$ at $x \in E$. Suppose C is nonempty, closed with $C \subset E$ and $\mathcal{A} : C \to E^*$ is a nonlinear mapping. The VIP is defined as finding $u \in C$ such that

$$\langle \mathcal{A}u, v - u\rangle \geq 0 \quad \forall v \in C. \tag{1.1}$$

We shall denote by $Sol(C, \mathcal{A})$, the solution set of VIP (1.1).

The projected gradient method [18] is known to solve VIP (1.1) when $\mathcal{A}$ is strongly monotone and Lipschitz continuous. This method fails to converge to any solution of VIP (1.1) when $\mathcal{A}$ is monotone. The Extragradient Method (EGM) [27] was introduced to solve VIP (1.1) when $\mathcal{A}$ is continuous and Lipschitz continuous:

$$\begin{cases} y_n = P_C(x_n - \rho\mathcal{A}x_n), \\ x_{n+1} = P_C(x_n - \rho\mathcal{A}y_n), \quad n \geq 0, \end{cases} \tag{1.2}$$

where P_C is the projection onto C. The weak convergence of EGM (1.2) is achieved when $\rho \in (0, 1/L)$, with L being the Lipschitz constant of $\mathcal{A}$ (see, [5–8, 10–12, 14, 21, 24, 33, 36, 40–43]). The major weakness in EGM (1.2) is that it involves two projection onto C per iteration and this requires solving a minimization problem twice per iteration during implementation, which can slow down the iterations. This motivated Tseng [37] to propose the following method;

$$\begin{cases} y_n = P_C(x_n - \rho\mathcal{A}x_n), \\ x_{n+1} = y_n - \rho(\mathcal{A}y_n - \mathcal{A}x_n), \quad n \geq 0 \end{cases} \tag{1.3}$$

which requires only one computation of P_C per iteration. The weak convergence of (1.3) was obtained in Hilbert spaces when either the step size $\rho \in (0, 1/L)$ or generated by a line search procedure. The requirement that the step size ρ in EGM (1.2) and (1.3) dependent on Lipschitz constant of the cost function is inefficient since the Lipschitz constant are difficult to estimate in most of the applications, and when they do, they are often too small and this in turn slows down the convergence of EGM (1.2) and (1.3).

It is very interesting to also study VIP in Banach spaces because several physical models and applications can be formulated as VIPs in real Banach spaces which are not Hilbert.

Recently, Jolaoso et al. [22] presented modified Bregman subgradient algorithms with line search technique for solving VIP with a pseudo-monotone operator without necessarily satisfying the Lipschitz condition. They introduced the following two algorithms:

Algorithm 1.1.

Step 0: *Given* $\gamma > 0$, $l \in (0,1)$ $\mu \in (0,1)$ *Let* $x_1 \in E$ *and set* $n = 1$.
Step 1: *Compute*

$$y_n = \Pi_C(\nabla f^*(\nabla f(x_n) - \alpha_n \mathcal{A} x_n)), \tag{1.4}$$

where $\alpha_n = \gamma l^{k_n}$, *with* k_n *being the smallest non-negative integer* k *satisfying*

$$\gamma l^k \|\mathcal{A} x_n - \mathcal{A} y_n\| \leq \mu \|x_n - y_n\|. \tag{1.5}$$

If $x_n = y_n$ *or* $\mathcal{A} y_n = 0$, *stop,* y_n *is a solution to the VIP. Else, do step 2.*
Step 2: *Compute*

$$x_{n+1} = \Pi_{T^n}(\nabla f^*(\nabla f(x_n) - \alpha_n \mathcal{A} y_n)), \tag{1.6}$$

where T^n *is the half-space defined by*

$$T^n = \{w \in E : \langle \nabla f(x_n) - \alpha_n x_n - \nabla f(y_n), w - y_n \rangle \leq 0\}. \tag{1.7}$$

Set $n := n + 1$ *and return to Step 1,*

---

and

Algorithm 1.2.

Step 0: *Given* $\gamma > 0$, $l \in (0,1)$ $\mu \in (0,1)$, $\{\delta_n\} \subset (0,1)$. *Let* $x_1, u \in E$ *and set* $n = 1$.
Step 1: *Compute*

$$y_n = \Pi_C(\nabla f^*(\nabla f(x_n) - \alpha_n \mathcal{A} x_n)), \tag{1.8}$$

where $\alpha_n = \gamma l^{k_n}$, *with* k_n *being the smallest non-negative integer* k *satisfying*

$$\gamma l^k \|\mathcal{A} x_n - \mathcal{A} y_n\| \leq \mu \|x_n - y_n\|. \tag{1.9}$$

If $x_n = y_n$ *or* $\mathcal{A} y_n = 0$, *stop,* y_n *is a solution to the VIP. Else, do step 2.*
Step 2: *Compute*

$$z_n = \Pi_{T^n}(\nabla f^*(\nabla f(x_n) - \alpha_n \mathcal{A} y_n)), \tag{1.10}$$

where T^n *is the half-space defined by*

$$T^n = \{w \in E : \langle \nabla f(x_n) - \alpha_n x_n - \nabla f(y_n), w - y_n \rangle \leq 0\}. \tag{1.11}$$

Step 3: *Compute*

$$x_{n+1} = \nabla f^*(\delta_n \nabla f(u) + (1 - \delta_n) \nabla f(z_n)). \tag{1.12}$$

Set $n := n + 1$ *and return to Step 1.*

Under suitable conditions of the given parameters, Jolaoso et al. [22] established the weak and strong convergence of Algorithm 1.1 and 1.2 respectively in reflexive Banach spaces. See also the recent paper by Reich et al. [31].

Following this direction, motivated and inspired by the presented results, we introduce a single projection method with self adaptive step size for solving the VIP (1.1) in a reflexive Banach space. The proposed algorithm uses variable step size from step to step which are updated over each iteration by a cheap computation. This step size allows the algorithm to work more easily without knowing previously the Lipschitz constant of operator $\mathcal{A}$. Unlike [22], we do not use any linesearch in our algorithm (a linesearch means that at each outer-iteration we use an inner-loop until some finite stopping criterion is satisfied, and this can be time consuming). We provide an application and some numerical experiment to illustrate the performance and efficiency of our proposed method.

2 Preliminaries

In this section, we give some definitions and preliminary results which will be used in our convergence analysis. Let C be a nonempty, closed and convex subset of a real Banach space E with the norm $\|\cdot\|$ and dual space E^*. We denote the weak and strong convergence of a sequence $\{x_n\} \subset E$ to a point $x^* \in E$ by $x_n \rightharpoonup x^*$ and $x_n \to x^*$ respectively.

Let $f : E \to (-\infty, +\infty]$ be a function satisfying the following:

(i) int (dom f) $\subset E$ is a uniformly convex set;
(ii) f is continuously differentiable on int (dom f);
(iii) f is strongly convex with strongly convexty constant $\varrho > 0$, i.e.

$$f(x) \geq f(y) - \langle \nabla f(y), x - y \rangle + \frac{\varrho}{2}\|x - y\|^2, \quad \forall x \in \text{dom} \quad \text{and} \quad y \in \text{ int (dom} f).$$

The subdifferential set of f at a point x denoted by ∂f is defined by

$$\partial f(x) := \{x^* \in E^* : f(x) - f(y) \leq \langle y - x, x^* \rangle, \quad y \in E\},$$

Every element $x^* \in \partial f(x)$ is called a subgradient of f at x. Since f is continuously differentiable, then $\partial f(x) = \{\nabla f(x)\}$, which is the gradient of f at x. The Fénchel conjugate of f is the convex functional $f^* : E^* \to (-\infty, +\infty]$ defined by $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in E\}$. Let E be a reflexive Banach space, the function f is said to be Legendre if and only if f satisfy the following conditions:

(R1) int (dom f) $\neq \emptyset$ and ∂f is single-valued on its domain;
(R2) int (dom f) $\neq \emptyset$ and ∂f^* is single-valued on its domain.

It is worth mentioning that the Bregman distance is not a metric in the usual sense because it does not satisfy the symmetric and triangular inequality properties. However, It posses the following interesting property called the three point identity: for $x \in \text{dom} f$ and $y, z \in$ int (dom f), we have Let $f : E \to \mathbb{R}$ be strictly convex and Gâteaux differentiable function. The Bregman distance $\phi_f : \text{dom} f \times$ int (dom f) $\to \mathbb{R}$ with respect to f is defined by

$$\begin{aligned} &\phi_f(x, y) = f(x) - f(y) - \langle x - y, \nabla f(y) \rangle, \quad \forall\, x \in \text{dom} f,\; y \in \text{ int (dom} f). \\ &\phi_f(x, y) + \phi_f(y, z) - \phi_f(x, z) \leq \langle \nabla f(z) - \nabla f(y), x - y \rangle. \end{aligned} \tag{2.1}$$

The Bregman function has been widely used by many authors for solving many optimization problems in the literature (see [30] and the references therein).

Remark 2.1 Practical important examples of the Bregman distance function can be found in [3]. If $f(x) = \frac{1}{2}\|x\|$, we have $\phi_f(x,y) = \frac{1}{2}\|x-y\|^2$ which is the Euclidean norm distance. Also, if $f(x) = -\sum_{j=1}^{m} x_j \log(x_j)$ which is Shannon's entropy for non-negative orthant $\mathbb{R}^m_{++} := \{x \in \mathbb{R}^m : x_j > 0\}$, we obtain the Kullback-Leibler cross entropy defined by

$$\phi_f(x,y) = \sum_{j=1}^{m}\left(x_j \log\left(\frac{x_j}{y_j}\right) - 1\right) + \sum_{j=1}^{m} y_i. \tag{2.2}$$

Also, if f is ϱ-strongly convex, then

$$\phi_f(x,y) \geq \frac{\varrho}{2}\|x-y\|^2, \quad \forall x \in \operatorname{dom} f,\ y \in \operatorname{int}(\operatorname{dom} f). \tag{2.3}$$

Definition 2.2 The Bregman projection (see e.g. [32]) with respect to f of $x \in \operatorname{int}(\operatorname{dom} f)$ onto a nonempty closed convex set $C \subset \operatorname{int}(\operatorname{dom} f)$ is the unique vector $Proj^f_C(x) \in C$ satisfying

$$Proj^f_C(x) := \inf\{\phi_f(x,y) : x \in C\}.$$

The Bregman projection is characterized by the inequality

$$z = Proj^f_C(x) \Longleftrightarrow \langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \forall y \in C. \tag{2.4}$$

Also

$$\phi_f(y, Proj^f_C(x)) + \phi_f(Proj^f_C(x), x) \leq \phi_f(y,x), \qquad \forall y \in C,\ x \in \operatorname{int}(\operatorname{dom} f). \tag{2.5}$$

Following [1, 13], we define the function $V_f : E \times E \to [0,\infty)$ associated with f by

$$V_f(x,y) = f(x) - \langle x, y\rangle + f^*(y), \quad \forall x, y \in E \tag{2.6}$$

V_f is non-negative and $V_f(x,y) = \phi_f(x, \nabla f(y))$ for all $x, y \in E$. Moreover, by the subdifferential inequality, it is easy to see that

$$V_f(x,y) + \langle z, \nabla f^*(y) - x\rangle \leq V_f(x, z+y) \tag{2.7}$$

for all $x, y, z \in E$. In addition, If $f : E \to \mathbb{R} \cup \{+\infty\}$ is proper lower semicontinous, then $f^* : E \to \mathbb{R} \cup \{+\infty\}$ is proper weak lower lower semicontinuous and convex. Hence, V_f is convex in second variable, i.e.,

$$\phi_f\left(z, \nabla f^*\left(\sum_{i=1}^{N} t_i \nabla f(x_i)\right)\right) \leq \sum_{i=1}^{N} t_i \phi_f(z, x_i), \tag{2.8}$$

where $\{x_i\} \subset E$ and $\{t_i\} \subset (0,1)$ with $\sum_{i=1}^{N} t_i = 1$.

Definition 2.3 Let $f : E \to \mathbb{R} \cup \{+\infty\}$ be a Gâteaux differentiable function. The function ϕ_f : dom $f \times$ int (dom f) $\to [0, +\infty)$ defined by

$$\phi_f(y, x) := f(y) - f(x) - \langle \nabla f(x), y - x \rangle \tag{2.9}$$

is the Bregman distance with respect to f. The Bregman distance does not satisfy the well-known properties of a metric, but it has the following important properties (see [28]): For any $w, x, y, z \in E$,

(i) three point identity:

$$\phi_f(y, z) + \phi_f(z, x) - \phi_f(y, x) = \langle \nabla f(z) - \nabla f(x), z - y \rangle; \tag{2.10}$$

(ii) four point identity: for

$$\phi_f(z, w) + \phi_f(y, x) - \phi_f(z, x) - \phi_f(y, w) = \langle \nabla f(x) - \nabla f(w), z - y \rangle. \tag{2.11}$$

Definition 2.4 [28, 29] The Minty Variational Inequality Problem (MVIP) is defined as finding a point $\bar{x} \in C$ such that

$$\langle \mathcal{A}y, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{2.12}$$

We denote by $M(C, \mathcal{A})$, the set of solution of (2). Some existence results for the MVIP has been presented in [28]. Also, the assumption that $M(C, \mathcal{A}) \neq \emptyset$ has already been used for solving $VI(C, \mathcal{A})$ in finite dimensional spaces (see e.g [36]). It is not difficult to prove that pseudomonotonicity implies property $M(C, \mathcal{A}) \neq \emptyset$, but the converse is not true. Indeed, let $\mathcal{A} : \mathbb{R} \to \mathbb{R}$ be defined by $\mathcal{A}(x) = \cos x$ with $C = [0, \frac{\pi}{2}]$. We have that $VI(C, \mathcal{A}) = \{0, \frac{\pi}{2}\}$ and $M(C, \mathcal{A}) = \{0\}$. But if we take $x = 0$ and $y = \frac{\pi}{2}$ in the definition of pseudomonotone (Assumption 3.1, (A1)), we see that $\mathcal{A}$ is not pseudomonotone.

Lemma 2.5 [29] *Consider the VIP* (1.1). *If the mapping* $h : [0, 1] \to E^*$ *defined as* $h(t) = \mathcal{A}(tx + (1 - t)y)$ *is continuous for all* $x, y \in C$ (*i.e.* h *is hemicontinuous*), *then* $M(C, \mathcal{A}) \subset VI(C, \mathcal{A})$. *Moreover, if* $\mathcal{A}$ *is pseudomonotone then* $VI(C, \mathcal{A})$ *is closed, convex and* $VI(C, \mathcal{A}) = M(C, \mathcal{A})$.

Lemma 2.6 [39] *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \sigma_n + \gamma_n, \quad n \geq 1.$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum \alpha_n = \infty$; (ii) $\limsup \sigma_n \leq 0$; (iii) $\gamma_n \geq 0$; $(n \geq 1)$ $\sum \gamma_n < \infty$.

Then $a_n \to 0$, *as* $n \to \infty$.

3 Main result

In this section, we give concise statement of our algorithm, discuss some of its elementary properties and its convergence analysis. In the sequel, we assume that the following hold.

Assumption 3.1 Let C be a nonempty closed convex subset of a real reflexive Banach space E. The operator $\mathcal{A} : C \to E^*$ satisfies the following:

(A1) $\mathcal{A}$ is pseudomonotone on C, i.e, for all $x, y \in E$, $\langle \mathcal{A}x, y - z \rangle \geq 0$ implies $\langle \mathcal{A}y, y - x \rangle \geq 0$;

(A2) $\mathcal{A}$ is L-Lipschitz continuous, i.e. there exists $L > 0$ such that $\|\mathcal{A}x - \mathcal{A}y\| \leq L\|x - y\|$, for all $x, y \in C$. However, the information of L need not to be known;

(A3) $\mathcal{A}$ is weakly sequentially continuous, i.e. for any $\{x_n\} \subset E$, we have $x_n \rightharpoonup x$ implies $\mathcal{A}x_n \rightharpoonup \mathcal{A}x$;

(A4) $Sol(C, \mathcal{A}) \neq \emptyset$;

(A5) In addition, the function $f : E \to [-\infty, +\infty]$ is proper, uniformly Fréchet differentiable on bounded subsets of E, strongly convex with constant $\varrho > 0$, strongly coercive and Legendre.

We now present our algorithm as follows.

Algorithm 3.2.

Initialization: *Chose* $x_1 \in E$ $\lambda_1 > 0$, $\{\delta_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ *and* $\delta \in (0, \varrho)$. *Set* $n = 1$.

Step 1: *Given nth iterates:* x_n *and* ρ_n, *compute*

$$y_n = Proj_C^f(\nabla f^*(\nabla f(x_n) - \rho_n \mathcal{A}x_n)). \tag{3.1}$$

If $x_n - y_n = 0$: *STOP. Else, do Step 2.*

Step 2:

$$z_n = \nabla f^*(\nabla(y_n) - \rho_n(\mathcal{A}(y_n) - \mathcal{A}(x_n)), \tag{3.2}$$

and update

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\delta\|x_n - y_n\|}{\|\mathcal{A}x_n - \mathcal{A}y_n\|}\right\}, & \text{if} \quad \mathcal{A}x_n \neq \mathcal{A}y_n, \\ \rho_n, & \text{if} \quad \mathcal{A}x_n = \mathcal{A}y_n. \end{cases} \tag{3.3}$$

Step 3: *Compute the* $(n+1) - th$ *iterate via*

$$x_{n+1} = \nabla f^* \left[\delta_n \nabla f(x_n) + (1 - \delta_n)(\alpha_n \nabla f(u) + (1 - \alpha_n)\nabla f(z_n)\right], \quad \forall n \geq 1. \tag{3.4}$$

Step 4: *Set* $n = n + 1$ *and GOTO Step 1.*

Remark 3.3

(1) The stepsize ρ_n in Algorithm 3.2 varies from step to step. This stepsize is updated at each iteration by a cheap computation. This stepsize rule allows Algorithm 3.2 to be implemented more easily where firstly the Lipschitz constant of $\mathcal{A}$ must not be the input parameter of the algorithm, i.e., this

constant is not necessary to be known, and secondly the stepsize is found without any line-search which can be time-consuming.

(2) The VIP is studied in a reflexive real Banach space which is more general than 2-uniformly convex real Banach space and real Hilbert space. This extends all the results in [9, 17, 33–35, 38] to mention but a few.

Lemma 3.4 *Assume that $\mathcal{A}$ is Lipschitz continuous on E. Then the sequence generated by* (3.3) is nonincreasing and

$$\lim_{n\to\infty}\rho_n = \rho \geq \min\left\{\rho_0, \frac{\delta}{L}\right\}.$$

Proof It is easy to prove this Lemma, hence we omit it. □

If at some iteration we have $x_n = y_n$ or $\mathcal{A}y_n = 0$ then Algorithm 3.2 terminates and $y_n \in Sol(C, \mathcal{A})$. From now on, we assume that $x_n \neq y_n$ and $\mathcal{A}y_n \neq 0$ for all n so that the algorithm does not terminate finitely.

Lemma 3.5 *Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be defined Algorithm* 3.2, *then for every $x^* \in Sol(C, \mathcal{A})$ the following inequalities holds for all $n \geq 1$.*

$$\phi_f(x^*, z_n) \leq \phi_f(x^*, x_n) - \left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(z_n, y_n) - \left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(y_n, x_n).$$

Proof Let $x^* \in Sol(C, \mathcal{A})$, then

$$\begin{aligned}
&\phi_f(x^*, z_n)\\
&\quad = \phi_f(x^*, \nabla f^*(\nabla f(y_n) - \rho_n(\mathcal{A}y_n - \mathcal{A}x_n))\\
&\quad = f(x^*) - \langle x^* - z_n, \nabla f(y_n) - \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle - f(z_n)\\
&\quad = f(x^*) + \langle z_n - x^*, \nabla f(y_n)\rangle + \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle - f(z_n)\\
&\quad = f(x^*) - \langle x^* - y_n, \nabla f(y_n)\rangle - f(y_n) + \langle x^* - y_n, \nabla f(y_n)\rangle + f(y_n)\\
&\qquad + \langle z_n - x^*, \nabla f(y_n)\rangle\\
&\qquad + \ \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle - f(z_n)\\
&\quad = \phi_f(x^*, y_n) + \langle z_n - y_n, \nabla f(y_n)\rangle + f(y_n) - f(z_n) + \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle\\
&\quad = \phi_f(x^*, y_n) - \phi_f(z_n, y_n) + \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle
\end{aligned} \tag{3.5}$$

Note that from (2.11), we get

$$\phi_f(x^*, y_n) - \phi_f(z_n, y_n) = \phi_f(x^*, x_n) - \phi_f(z_n, x_n) + \langle\nabla f(x_n) - \nabla f(y_n), x^* - z_n\rangle. \tag{3.6}$$

Hence from (3.5) and (3.6), we have

$$\phi_f(x^*, z_n) = \phi_f(x^*, x_n) - \phi_f(z_n, x_n) + \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle \\ + \langle \nabla f(x_n) - \nabla f(y_n), x^* - z_n\rangle. \tag{3.7}$$

Also from (2.10), we have

$$\phi_f(z_n, x_n) = \phi_f(z_n, y_n) + \phi_f(y_n, x_n) - \langle \nabla f(x_n) - \nabla f(y_n), z_n - y_n\rangle. \tag{3.8}$$

Substituting (3.8) into (3.7), we obtain

$$\begin{aligned}\phi_f(x^*, z_n) &= \phi_f(x^*, x_n) - \phi_f(z_n, y_n) - \phi_f(y_n, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), z_n - y_n\rangle \\ &\quad + \ \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle + \langle \nabla f(x_n) - \nabla f(y_n), x^* - z_n\rangle \\ &= \phi_f(x^*, x_n) - \phi_f(z_n, y_n) - \phi_f(y_n, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), x^* - y_n\rangle \\ &\quad + \ \langle x^* - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle \\ &= \phi_f(z^*, x_n) - \phi_f(z_n, y_n) - \phi_f(y_n, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), x^* - y_n\rangle \\ &\quad + \ \langle z_n - y_n + y_n - z, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n) \\ &= \phi_f(x^*, x_n) - \phi_f(z_n, y_n) - \phi_f(y_n, x_n) + \langle \nabla f(x_n) - \nabla f(y_n), x^* - y_n\rangle \\ &\quad + \ -\langle z_n - y_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle - \langle y_n - x^*, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle \\ &= \phi_f(x^*, x_n) - \phi_f(z_n, y_n) - \phi_f(y_n, x_n) - \langle z_n - y_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle \\ &\quad - \ \langle y_n - x^*, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n) - (\nabla f(y_n) - \nabla f(x_n))\rangle.\end{aligned} \tag{3.9}$$

Since $y_n = Proj_C^f(\nabla f^*(\nabla f(x_n) - \rho_n \mathcal{A}x_n))$, it follows from (2.4) that

$$\langle \nabla f(x_n) - \rho_n \mathcal{A}x_n - \nabla f(y_n), x^* - y_n\rangle \leq 0. \tag{3.10}$$

Also, since $x^* \in Sol(C, \mathcal{A})$ then $\langle \mathcal{A}x^*, y_n - x^*\rangle \geq 0$. From the pseudomonotononicity of $\mathcal{A}$, we obtain

$$\langle \mathcal{A}y_n, y_n - x^*\rangle \geq 0. \tag{3.11}$$

Combining (3.10) and (3.11), we get

$$\langle \rho_n(\mathcal{A}y_n - \mathcal{A}x_n) - (\nabla f(y_n) - \nabla f(x_n)), y_n - x^*\rangle \geq 0. \tag{3.12}$$

Hence from (3.9), we obtain

$$\phi_f(x^*, z_n) \leq \phi_f(x^*, x_n) - \phi_f(z_n, y_n) - \langle z_n - y_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle. \tag{3.13}$$

Now using (3.3) and Cauchy-Schwatz inequality, we have

$$\begin{aligned}\phi_f(x^*,z_n) &\le \phi_f(x^*,x_n) - \phi_f(z_n,y_n) - \phi_f(y_n,x_n) + \langle y_n - z_n, \rho_n(\mathcal{A}y_n - \mathcal{A}x_n)\rangle \\ &\le \phi_f(x^*,x_n) - \phi_f(z_n,y_n) - \phi_f(y_n,x_n) + \frac{\rho_n}{\rho_{n+1}}\rho_{n+1}\|y_n - z_n\|\|\mathcal{A}y_n - \mathcal{A}x_n\| \\ &\le \phi_f(x^*,x_n) - \phi_f(z_n,y_n) - \phi_f(y_n,x_n) + \frac{\rho_n}{\rho_{n+1}}\delta\|y_n - z_n\|\|y_n - x_n\| \\ &\le \phi_f(x^*,x_n) - \phi_f(z_n,y_n) - \phi_f(y_n,x_n) \\ &\quad + \frac{\rho_n}{\rho_{n+1}} \times \frac{\delta}{2}(\|z_n - y_n\|^2 + \|y_n - x_n\|^2).\end{aligned}$$

Then from (2.3), we obtain

$$\begin{aligned}\phi_f(x^*,z_n) &\le \phi_f(x^*,x_n) - \phi_f(z_n,y_n) - \phi_f(y_n,x_n) \\ &\quad + \frac{\rho_n}{\rho_{n+1}} \times \frac{\delta}{\varrho}(\phi_f(z_n,y_n) + \phi_f(y_n,x_n)) \\ &= \phi_f(x^*,x_n) - \left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(z_n,y_n) - \left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(y_n,x_n).\end{aligned} \tag{3.14}$$

□

Lemma 3.6 *The sequence $\{x_n\}$ generated by Algorithm 3.2, is bounded.*

Proof Note that since $\delta \in (0,\varrho)$ and $\varrho > 0$, we have

$$\lim_{n\to\infty}\left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right) = 1 - \frac{\delta}{\varrho} > 0.$$

Then, there exists $N > 0$ such that

$$1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho} > 0 \quad \forall n \ge N.$$

Thus, it from (3.14) that

$$\phi_f(x^*,z_n) \le \phi_f(x^*,x_n). \tag{3.15}$$

Furthermore, from (3.4) and (3.15), we have

$$\begin{aligned}\phi_f(x^*, x_{n+1}) &= \phi_f(x^*, \nabla f^*(\delta_n \nabla f(x_n) + (1-\delta_n)(\alpha_n \nabla f(u) + (1-\alpha_n)\nabla f(z_n)))\\ &\le \delta_n \phi_f(x^*, x_n) + (1-\delta_n)\phi_f(x^*, \alpha_n \nabla f(u) + (1-\alpha_n)\nabla f z_n)\\ &\le \delta_n \phi_f(x^*, x_n) + (1-\delta_n)[\alpha_n \phi_f(x^*, u) + (1-\alpha_n)\phi_f(x^*, z_n)]\\ &\le \delta_n \phi_f(x^*, x_n) + (1-\delta_n)[\alpha_n \phi_f(x^*, u) + (1-\alpha_n)\phi_f(x^*, x_n)]\\ &= \alpha_n(1-\delta_n)\phi_f(x^*, u) + (1-\alpha_n(1-\delta_n))\phi_f(x^*, x_n)\\ &\le \max\{\phi_f(x^*, u), \phi_f(x^*, x_n)\}\\ &\quad\vdots\\ &\le \max\{\phi_f(x^*, u), \phi_f(x^*, x_n)\}.\end{aligned} \tag{3.16}$$

This implies that $\{\phi_f(x^*, x_n)\}$ is bounded. Hence, $\{x_n\}$ is bounded. Consequently, we see that $\{\nabla f(x_n)\}$, $\{y_n\}$ and $\{z_n\}$ are bounded. □

Lemma 3.7 *The sequence* $\{x_n\}$ *generated by Algorithm* 3.2 *satisfies the following estimates*:

(i) $\mathcal{S}_{n+1} \le (1-\alpha_n(1-\delta_n))\mathcal{S}_n + \alpha_n(1-\delta_n)d_n$,
(ii) $-1 \le \limsup_{n\to\infty} d_n < +\infty$,

where $\mathcal{S}_n = \phi_f(x^*, x_n)$ and $d_n = \langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle$ for any $x^* \in Sol(C, \mathcal{A})$.

Proof Now, let $v_n = \nabla f^*[\alpha_n(1-\delta_n)\nabla f(u) + (1-\alpha_n(1-\delta_n))\nabla f(u_n)]$, $n \ge 1$ where

$$u_n = \nabla f^*\left[\frac{\delta_n}{1-\alpha_n(1-\delta_n)}\nabla f(x_n) + \frac{(1-\alpha_n)(1-\delta_n)}{1-\alpha_n(1-\delta_n)}\nabla f(z_n)\right],\ n \ge 1$$

$$\begin{aligned}&\phi_f(x^*, x_{n+1})\\ &\le \phi_f(x^*, \nabla f^*(\alpha_n(1-\delta_n)\nabla f(u) + \delta_n \nabla f(x_n) + (1-\alpha_n)(1-\alpha_n)J^p_{E_1}(z_n))\\ &= \phi_f(x^*, \nabla f^*(\alpha_n(1-\delta_n)\nabla f(u) + (1-\alpha_n)(1-\delta_n))\nabla f(u_n))\\ &\le V_f(x^*, \alpha_n(1-\delta_n)\nabla f(u) + (1-\alpha_n(1-\delta_n))\nabla f(u_n))\\ &\le V_f(x^*, \alpha_n(1-\delta_n)\nabla f(u)) + (1-\alpha_n(1-\delta_n))\nabla f(u_n)\\ &\quad - \alpha_n(1-\delta_n)(\nabla f(u) - \nabla f(x^*))\\ &\quad - \langle \nabla f^*(\alpha_n(1-\beta_n)\nabla f(u)) + (1-\alpha_n(1-\delta_n))\nabla f(u_n)) - x^*,\\ &\quad - \alpha_n(1-\delta_n)(\nabla f(u) - \nabla f(x^*))\rangle\\ &= \phi_f[x^*, \nabla f^*(\alpha_n(1-\delta_n)\nabla f(x^*) + (1-\alpha_n(1-\delta_n))\nabla f(u_n))]\\ &\quad + \alpha_n(1-\delta_n)\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle\\ &\le \alpha_n(1-\delta_n)\phi_f(x^*, x^*) + (1-\alpha_n(1-\delta_n))\phi_f(x^*, u_n)\\ &\quad + \alpha_n(1-\delta_n)\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle\\ &= (1-\alpha_n(1-\delta_n))\phi_f(x^*, u_n)\\ &\quad + \alpha_n(1-\beta_n)\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle\\ &\le (1-\alpha_n(1-\delta_n))\phi_f(x^*, x_n) + \alpha_n(1-\delta_n)\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle.\end{aligned} \tag{3.17}$$

This established (i).

Next we show (ii). Since $\{v_n\}$ is bounded, then we have

$$\sup_{n\geq 0} d_n \leq \sup\Big(||v_n - x^*||||\nabla f(u) - \nabla f(x^*)||\Big) < \infty.$$

This implies that $\limsup_{n\to\infty} d_n < \infty$. Next, we show that $\limsup_{n\to\infty} d_n \geq -1$. Assume the contrary, i.e. $\limsup_{n\to\infty} d_n < -1$. Then there exists $n_0 \in \mathbb{N}$ such that $d_n < -1$, for all $n \geq n_0$. Then for all $n \geq n_0$, we get from (i) that

$$\begin{aligned}\mathcal{S}_{n+1} &\leq (1-\alpha_n(1-\delta_n))\mathcal{S}_n + \alpha_n(1-\delta_n)d_n \\ &< (1-\alpha_n(1-\delta_n))\mathcal{S}_n - \alpha_n(1-\delta_n) \\ &= \mathcal{S}_n - \alpha_n(1-\delta_n)(\mathcal{S}_n+1) \leq \mathcal{S}_n - \alpha_n(1-\delta_n).\end{aligned}$$

Taking lim sup of the last inequality, we have

$$\limsup_{n\to\infty} \mathcal{S}_n \leq \mathcal{S}_{n_0} - \lim_{n\to\infty}\sum_{i=n_0}^{n} \alpha_i = -\infty.$$

This contradicts the fact that $\{\mathcal{S}_n\}$ is a non-negative real sequence. Therefore $\limsup_{n\to\infty} d_n \geq -1$. □

We now present our main convergence result.

Theorem 3.8 *Assume Conditions* $(A1)-(A5)$ *and suppose* $\{\alpha_n\}$, *and* $\{\delta_n\}$ *are sequences in* (0, 1) and satisfy the following conditions:

- (i) $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $0 < \liminf_{n\to\infty} \delta_n \leq \limsup_{n\to\infty} \delta_n < 1$.

Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges strongly to an element in $Sol(C, \mathcal{A})$.

Proof Let $x^* \in Sol(C, \mathcal{A})$ and $\mathcal{S}_n = \phi_f(x^*, x_n)$. we divide the proof into two cases.

Case 1: Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\mathcal{S}_n\}_{n=n_0}^{\infty}$ is non-increasing. Then $\{\mathcal{S}_n\}_{n=1}^{\infty}$ converges and

$$\lim_{n\to\infty} (\mathcal{S}_n - \mathcal{S}_{n+1}) = 0. \tag{3.18}$$

Thus, from Lemma 3.5 we have.

$$\left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(z_n, y_n) + \left(1 - \frac{\rho_n\delta}{\rho_{n+1}\varrho}\right)\phi_f(y_n, x_n) \leq \phi_f(x^*, x_n) - \phi_f(x^*, z_n). \tag{3.19}$$

Also, from (3.16), we get

$$
\begin{aligned}
&\phi_f(x^*, x_n) - \phi_f(x^*, z_n) \\
&\quad \le \frac{\alpha_n(1-\delta_n)}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, u) + \frac{(1-\alpha_n(1-\delta_n))}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, x_n) \\
&\quad\quad - \frac{1}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, x_{n+1}) \\
&\quad = \frac{\alpha_n(1-\delta_n)}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, u) - \frac{\alpha_n(1-\delta_n)}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, x_n) \\
&\quad\quad + \frac{1}{(1-\alpha_n)(1-\delta_n)} [\phi_f(x^*, x_n) - \phi_f(x^*, x_{n+1})] \\
&\quad \le \frac{\alpha_n(1-\delta_n)}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, u) + \frac{1}{(1-\alpha_n)(1-\delta_n)} [\phi_f(x^*, x_n) - \phi_f(x^*, x_{n+1})]
\end{aligned} \tag{3.20}
$$

From (3.19) and (3.20), we get

$$
\begin{aligned}
&\left(1 - \frac{\rho_n \delta}{\rho_{n+1} \varrho}\right) \phi_f(z_n, y_n) + \left(1 - \frac{\rho_n \delta}{\rho_{n+1} \varrho}\right) \phi_f(y_n, x_n) \\
&\quad \le \phi_f(x^*, x_n) - \phi_f(x^*, z_n) \\
&\quad \le \frac{\alpha_n(1-\delta_n)}{(1-\alpha_n)(1-\delta_n)} \phi_f(x^*, u) \\
&\quad\quad + \frac{1}{(1-\alpha_n)(1-\delta_n)} [\phi_f(x^*, x_n) - \phi_f(x^*, x_{n+1})]
\end{aligned} \tag{3.21}
$$

Passing limit as $n \to \infty$ in (3.21), since $\lim_{n\to\infty} \phi_f(x^*, x_n)$ exists and $\frac{\rho_n}{\rho_{n+1}} \to 1$, we have

$$\lim_{n\to\infty} \left(1 - \frac{\delta}{\varrho}\right) \phi_f(z_n, y_n) = 0.$$

Also, since $\delta \in (0, \varrho)$, then

$$\lim_{n\to\infty} \phi_f(y_n, z_n) = 0.$$

Thus

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{3.22}$$

Similarly from, (3.21), we obtain

$$\lim_{n\to\infty} \left(1 - \frac{\delta}{\varrho}\right) \phi_f(y_n, x_n) = 0.$$

Then

$$\lim_{n\to\infty} \phi_f(y_n, x_n) = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.23}$$

Combining (3.22) and (3.23), we have

$$\lim_{n\to\infty} \|z_n - x_n\| = 0. \tag{3.24}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup z$. We now show that $z \in Sol(C, \mathcal{A})$.

Since $y_{n_k} = Proj_C^f(\nabla f^*(\nabla f(x_{n_k}) - \rho_{n_k}\mathcal{A}x_{n_k}))$, from (2.4), we have

$$\langle \nabla f(x_{n_k}) - \rho_{n_k}\mathcal{A}x_{n_k} - \nabla f(y_{n_k}), x - y_{n_k}\rangle \leq 0 \quad \forall x \in C.$$

Hence

$$\begin{aligned}\langle \nabla f(x_{n_k}) - \nabla f(y_{n_k}), x - y_{n_k}\rangle &\leq \rho_{n_k}\langle \mathcal{A}x_{n_k}, x - y_{n_k}\rangle \\ &= \rho_{n_k}\langle \mathcal{A}x_{n_k}, x_{n_k} - y_{n_k}\rangle + \rho_{n_k}\langle \mathcal{A}x_{n_k}, x - x_{n_k}\rangle.\end{aligned}$$

This implies that

$$\langle \nabla f(x_{n_k}) - \nabla f(y_{n_k}), x - y_{n_k}\rangle + \rho_{n_k}\langle \mathcal{A}x_{n_k}, y_{n_k} - x_{n_k}\rangle \leq \rho_{n_k}\langle \mathcal{A}x_{n_k}, x - x_{n_k}\rangle. \tag{3.25}$$

Since $\|x_{n_k} - y_{n_k}\| \to 0$ and f is strongly coercive, then

$$\lim_{k\to\infty} \|\nabla f(x_{n_k}) - \nabla f(y_{n_k})\| = 0. \tag{3.26}$$

Next, fix $x \in C$, it follows from (3.25) and (3.26) and the fact that $\liminf_{k\to\infty} \rho_{n_k} > 0$, then

$$0 \leq \liminf_{k\to\infty} \langle \mathcal{A}x_{n_k}, x - x_{n_k}\rangle \quad \forall\, x \in C. \tag{3.27}$$

Let $\{\epsilon_k\}$ be a sequence of decreasing non-negative numbers such that $\epsilon_k \to 0$ as $k \to \infty$. For each ϵ_k, we denote by N_k the smallest positive integer such that

$$\langle \mathcal{A}x_{n_k}, x - x_{n_k}\rangle + \epsilon_k \geq 0 \quad \forall\, k \geq N_k,$$

where the existence of N_k follows from (3.27). Since $\{\epsilon_k\}$ is decreasing, then $\{N_k\}$ is increasing. Also, for some $t_{N_k} \in C$ assume $\langle \mathcal{A}x_{N_k}, t_{N_k}\rangle = 1$, for each k. Therefore,

$$\langle \mathcal{A}x_{N_k}, x + \epsilon_k t_{N_k} - x_{N_k}\rangle \geq 0.$$

Since $\mathcal{A}$ is pseudomonotone, then we have from (3.27) that

$$\langle A(x + \epsilon_k t_{N_k}), x + \epsilon_k t_{N_k} - x_{N_k}\rangle \geq 0. \tag{3.28}$$

Since $\{x_{n_k}\}$ converges weakly to z as $k \to \infty$ and $\mathcal{A}$ is weakly sequentially continuous, we have that $\{\mathcal{A}\}$ converges weakly to $\mathcal{A}z$. Suppose $\mathcal{A}z \neq 0$ (otherwise, $z \in Sol(C, \mathcal{A})$. Then by the sequentially weakly lower semicontinuous of the norm, we get

$$0 < \|\mathcal{A}z\| = \liminf_{k\to\infty} \\ \mathcal{A}x_{n_k}\|.$$

Since $\{x_{N_k}\} \subset \{x_{n_k}\}$ and $\epsilon_k \to 0$, we get

$$0 \leq \limsup_{k\to\infty} \|\epsilon_k t_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\epsilon_k}{\|\mathcal{A}x_{n_k}}\|\right)$$
$$\leq \frac{\limsup_{k\to\infty}\epsilon_k}{\liminf_{k\to\infty}\|\mathcal{A}x_{n_k}} \leq \frac{0}{\|\mathcal{A}z\|} = 0,$$

and this means $\lim_{k\to\infty}\|\epsilon_k t_{N_k}\| = 0$. Passing the limit $k\to\infty$ in (3.28), we get

$$\langle \mathcal{A}x, x - z\rangle \geq 0.$$

Therefore, from Lemma 2.5, we have $z \in Sol(C,\mathcal{A})$. Furthermore, from the definition of u_n and v_n, we get

$$\begin{aligned}\phi_f(u_n,x_n) &= \frac{\delta_n}{1-\alpha_n(1-\delta_n)}\phi_f(x_n,x_n) + \frac{(1-\alpha_n)(1-\delta_n)}{1-\alpha_n(1-\delta_n)}\phi_f(z_n,x_n)\\ &= \frac{(1-\alpha_n)(1-\delta_n)}{1-\alpha_n(1-\delta_n)}\phi_f(z_n,x_n) \to 0, \quad \text{as} \quad n\to\infty,\end{aligned}$$

and

$$\begin{aligned}\phi_f(v_n,u_n) &= \alpha_n(1-\delta_n)\phi_f(u,u_n) + (1-\alpha_n(1-\delta_n))\phi_f(u_n,u_n)\\ &= \alpha_n(1-\delta_n)\phi_f(u,u_n) \to 0, \quad \text{as} \quad n\to\infty.\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\|u_n - x_n\| = 0 = \lim_{n\to\infty}\|v_n - u_n\|.$$

Hence, we obtain

$$\lim_{n\to\infty}\|v_n - x_n\| = 0.$$

Since $x_{n_k} \rightharpoonup z$ and $\lim_{n\to\infty}\|v_n - x_n\| = 0$ we have that $v_{n_k} \rightharpoonup z$. Hence from (2.4), we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle &= \lim_{k\to\infty}\langle v_{n_k} - x^*, \nabla f(u) - \nabla f(x^*)\rangle\\ &= \langle z - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq 0.\end{aligned}$$

Hence

$$\limsup_{n\to\infty} d_n = \limsup_{n\to\infty}\langle v_n - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq 0. \tag{3.29}$$

Using Lemma 2.6 and Lemma 3.7(i), we obtain

$$\lim_{n\to\infty}\mathcal{S}_n = \lim_{n\to\infty}\phi_f(x^*,x_n) = 0.$$

This implies that $\|x_n - x^*\| \to 0$ as $n\to\infty$. Therefore $\{x_n\}$ converges strongly to x^*.

Case 2. Assume that $\{\mathcal{S}_n\}_{n=1}^{\infty}$ is not a monotonically decreasing sequences. Set

$\Gamma_n := \mathcal{S}_n, \ \forall n \geq 1$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \ \Gamma_k \leq \Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n) \to \infty$ as $n \to \infty$ and

$$0 \leq \Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0$$

Following a similar argument as in **Case 1** we immediately conclude that

$$\lim_{n\to\infty} \|v_{\tau(n)} - x_{\tau(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_{\tau(n)} - y_{\tau(n)}\| = 0.$$

Also from (3.29), we get that

$$\limsup_{n\to\infty} d_{\tau(n)} = \limsup_{n\to\infty} \langle v_{\tau(n)} - x^*, \nabla f(u) - \nabla f(x^*)\rangle \leq 0.$$

From (3.17) we have that

$$\mathcal{S}_{\tau(n)} \leq d_{\tau(n)}$$

which implies

$$\lim_{n\to\infty} \mathcal{S}_{\tau(n)} = 0$$

and

$$\lim_{n\to\infty} \mathcal{S}_{\tau(n)+1} = 0.$$

Therefore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is, $\tau(n) < n$), because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Hence, $\lim_{n\to\infty} \Gamma_n = 0$. This implies that $\lim_{n\to\infty} \phi_f(x^*, x_n) = 0$, thus $\|x_n - x^*\| \to 0$ as $n \to \infty$. Therefore, $\{x_n\}$ converges strongly to x^*. This completes the proof. □

Remark 3.9 If the operator is monotone and Lipschitz continuous, then we do not need it to be sequentially weakly continuous. This is because the sequential weakly continuity assumption was only used after (3.25). From the definition of y_n, we have

$$\langle \nabla f(x_{n_k}) - \rho_{n_k} \mathcal{A} x_{n_k} - \nabla f(y_{n_k}, x - y_{n_k})\rangle \leq 0, \quad \forall x \in C.$$

Thus from the monotonicity of $\mathcal{A}$,

$$\begin{aligned}0 &\leq \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}), x - y_{n_k} \rangle + \rho_{n_k} \langle \mathcal{A}x_{n_k}, x - y_{n_k} \rangle \\ &= \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}), x - y_{n_k} \rangle + \rho_{n_k} \langle \mathcal{A}x_{n_k}, z - x_{n_k} \rangle + \rho_{n_k} \langle \mathcal{A}x_{n_k}, x_{n_k} - y_{n_k} \rangle \\ &\leq \langle \nabla f(y_{n_k}) - \nabla f(x_{n_k}), x - y_{n_k} \rangle + \rho_{n_k} \langle \mathcal{A}x, x - x_{n_k} \rangle + \rho_{n_k} \langle \mathcal{A}x_{n_k}, x_{n_k} - y_{n_k} \rangle.\end{aligned}$$

Since ∇f is weakly - weakly continuous, passing to the limit in the last inequality as $k \to \infty$ and using relation (3.23), we obtain $\langle \mathcal{A}x, x - z \rangle \geq 0 \quad \forall z \in C$. Thus, from Lemma 2.5 we get that $z \in Sol(C, \mathcal{A})$. Hence the conclusion of Theorem 3.8 still hold.

4 Application

4.1 Application to computing dynamic user equilibria

In this section, we apply Algorithm 3.2 to compute dynamic user equilibria (see [16]). Let $\mathcal{P}$ be set of paths in the network. $\mathbb{W}$ be set of *O-D* pairs in the network, Q_{ij} be fixed *O-D* demand between $(i,j) \in \mathbb{W}$, $\mathcal{P}_{ij}$ be subset of paths that connect *O-D* (i, j), t be continuous time parameter in a fixed time horizon $[t_0, t_1]$, $h_p(t)$ be depature rate along path p at time t, $h(t)$ be complete vector of departure rates $h(t) = (h_p(t) : p \in \mathcal{P})$, $\Psi_p(t, h)$ be travel cost along path p with departure time t, under departure profile h, $v_{ij}(h)$ be minimum travel cost between *O-D* pair (i, j) for all paths and departure times.

Assume that $h_p(\cdot) \in L^2_+[t_0, t_1]$ and $h(\cdot) \in (L^2_+[t_0, t_1])^{|\mathcal{P}|}$. Define the effective delay operator $\Psi : (L^2_+[t_0, t_1])^{|\mathcal{P}|} \to (L^2_+[t_0, t_1])^{|\mathcal{P}|}$ as follows:

$$h(\cdot) = \{h_p(\cdot), p \in \mathcal{P}\} \mapsto \Psi(h) = \{\Psi_p(\cdot, h), p \in \mathcal{P}\}.$$

The travel demand satisfaction constraint satisfies

$$Q_{ij} = \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_1} h_p(t)dt, \quad \forall\ (i,j) \in \mathbb{W}.$$

Then, the set of feasible path departures vector can be expressed as

$$\Lambda = \{h \geq 0 : \sum_{p \in \mathcal{P}_{ij}} \int_{t_0}^{t_1} h_p(t)dt, \quad \forall\ (i,j) \in \mathbb{W}\} \subset (L^2_+[t_0, t_1])^{|\mathcal{P}|}.$$

Recall that a vector of departures $h^* \in \Lambda$ is a dynamic user equilibrium with simultaneous route and departure time choice if

$$h^*_p(y) > 0,\ p \in \mathcal{P}_{ij} \Rightarrow \Psi_p(t, h^*) = v_{ij}(h^*), \quad \text{for almost every } t \in [t_0, t_1]. \tag{4.1}$$

Note that (4.1) is equivalent to the following variational inequality ([16]):

$$\langle \Psi(h^*), h - h^* \rangle \geq 0, \quad \forall\ h \in \Lambda. \tag{4.2}$$

Based on Algorithm 3.2, we have the following algorithm.

Algorithm 4.1.

Initialization: *Chose* $x_1 \in E$ $\lambda_1 > 0$, $\{\delta_n\} \subset (0,1)$, $\{\alpha_n\} \subset (0,1)$ *and* $\delta \in (0, \varrho)$. *Set* $n = 1$.

Step 1: *Given nth iterates:* x_n *and* ρ_n, *compute*

$$y_n = Proj_C^f(\nabla f^*(\nabla f(x_n) - \rho_n \Psi x_n)). \tag{4.3}$$

If $x_n - y_n = 0$: *STOP. Else, do Step 2.*

Step 2:

$$z_n = \nabla f^*(\nabla(y_n) - \rho_n(\Psi(y_n) - \Psi(x_n)), \tag{4.4}$$

and update

$$\rho_{n+1} = \begin{cases} \min\left\{\rho_n, \frac{\delta\|x_n - y_n\|}{\|\Psi x_n - \Psi y_n\|}\right\}, & \textit{if} \quad \Psi x_n \neq \Psi y_n, \\ \rho_n, & \textit{if} \quad \Psi x_n = \Psi y_n. \end{cases} \tag{4.5}$$

Step 3: *Compute the* $(n+1) - th$ *iterate via*

$$x_{n+1} = \nabla f^* \left[\delta_n \nabla f(x_n) + (1 - \delta_n)(\alpha_n \nabla f(u) + (1 - \alpha_n) \nabla f(z_n)\right], \quad \forall n \geq 1. \tag{4.6}$$

Step 4: *Set* $n = n + 1$ *and GOTO Step 1.*

If the delay operator Ψ is Lipschitz continuous and pseudomonotone, then we can apply Algorithm 4.1 to compute dynamic user equilibria.

5 Numerical examples

In this section, we consider some examples to illustrate the convergence of the proposed algorithm and compare it with other algorithms. We also compare the convergence of Algorithm 3.2 for various examples of the Bregman distance.

Example 5.1 Let $E = \ell_2(\mathbb{R})$ where $\ell_2(\mathbb{R}) = \{x = (x_1, x_2, x_3, ...,), \; x_i \in \mathbb{R} : \sum_{i=1}^{\infty}\}$ with norm $\|x\|_{\ell_2} = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ and inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$, for all $x = (x_1, x_2, x_3, ...)$, $y = (y_1, y_2, y_3, ...) \in E$. Let $C = \{x = (x_1, x_2, x_3, ...) \in \ell_2 : \|x\| \leq 1\}$ and $\mathcal{A} : \ell_2 \to \ell_2$ be defined by

$$\mathcal{A}(x_1, x_2, x_3, ...) = (x_1 \exp(-x_1^2), 0, 0, ...).$$

It was shown in Example 2.1 of [4] that $\mathcal{A}$ is pseudomonotone, Lipschitz continuous and sequentially weakly continuous but not monotone in $\ell_2(\mathbb{R})$.

We now list some known Bregman distances. These distances are listed in the following forms:

(i) The function $f^{IS}(x) = -\sum_{i=1}^{m} \log x_i$ and the Itakura-Saito distance

$$\phi_f^{IS}(x, y) = \sum_{i=1}^{m} \left(\frac{x_i}{y_i} - \log\left(\frac{x_i}{y_i}\right) - 1\right).$$

(ii) The function $f^{KL}(x) = \sum_{i=1}^{m} x_i \log x_i$ and the Kullback-Leibler distance

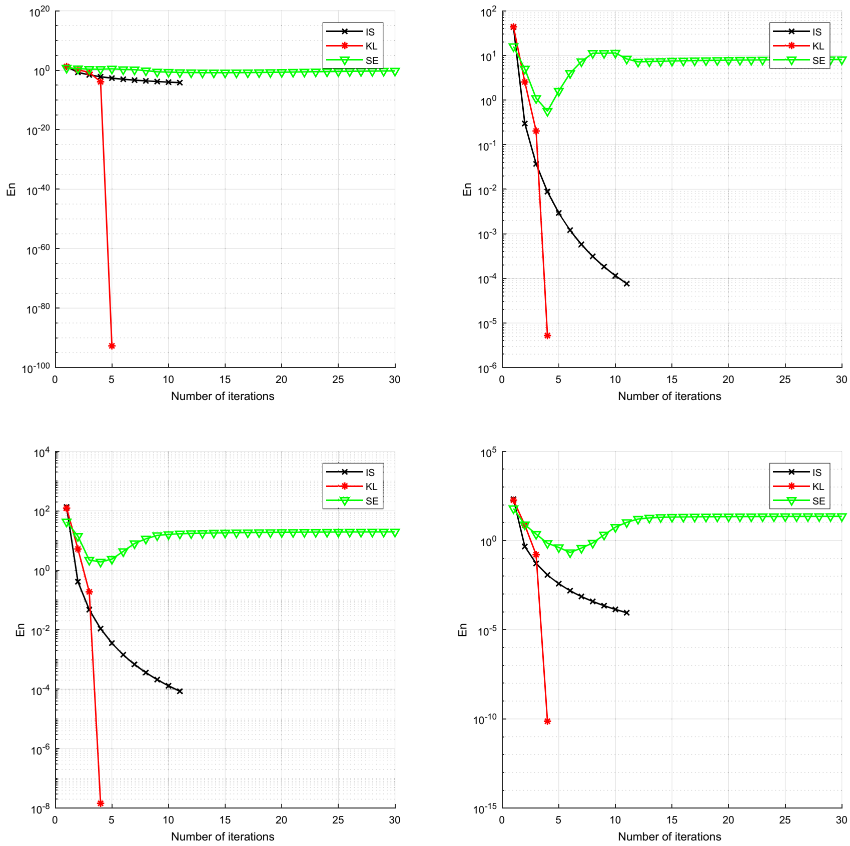

Fig. 1 Example 5.1. Top left: $m = 10$, Top right: $m = 20$, Bottom left: $m = 50$, Bottom right: $m = 80$

$$\phi_f^{KL}(x, y) = \sum_{i=1}^{m} \left(x_i \log\left(\frac{x_i}{y_i}\right) + y_i - x_i \right).$$

(iii) The function $f^{SE}(x) = \frac{1}{2}\|x\|^2$ and the squared Euclidean distance

$$\phi_f^{SE}(x, y) = \frac{1}{2}\|x - y\|^2.$$

The values $\nabla f(x)$ and $\nabla f^*(x) = (\nabla f)^{-1}(x)$ are computed explicitly. More precisely,

(i) $\nabla f^{IS}(x) = -(1/x_1, \ldots, 1/x_m)^T$ and $(\nabla f^{IS})^{-1}(x) = -(1/x_1, \ldots, 1/x_m)^T$.

(ii) $\nabla f^{KL}(x) = (1 + \log(x_1), \ldots, 1 + \log(x_m))^T$ and

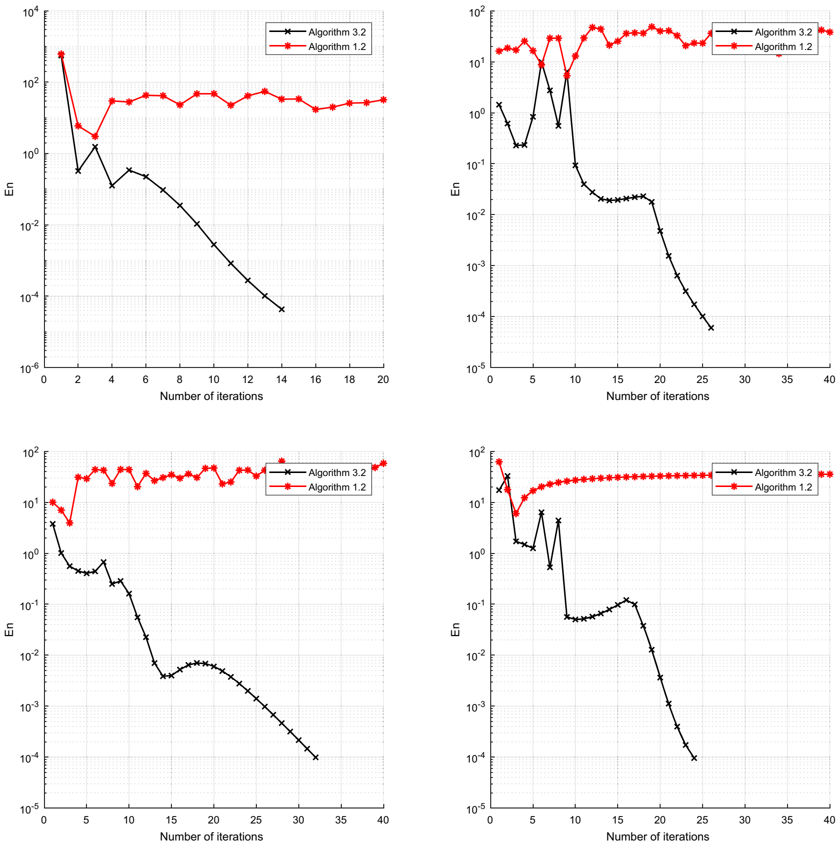

Fig. 2 Performance of Algorithm 3.2 compared with Algorithm 1.2

$$(\nabla f^{KL})^{-1}(x) = (\exp(x_1 - 1), ..., \exp(x_m - 1))^T.$$

(iii) $\nabla f^{SE}(x) = x$ and $(\nabla f^{SE})^{-1}(x) = x$.

The feasibility set of our problem VIP is of the form,

$$C = \{x = (x_1, x_2, ..., x_m)^T \in \mathbb{R}^m : \|x\| \leq 1 \quad \text{and} \quad x_i \geq a > 0, \quad i = 1, 2, ..., m\},$$

where $a < 1/\sqrt{m}$ (which ensures that $C \neq \emptyset$). For the experiment in Algorithm 3.2, we choose $\alpha_n = \frac{1}{n+1}$, $\delta_n = \frac{2n}{5n+4}$, $u = \mu = 0.5$ and $\rho = 3.5$.

Let $E_n = \|x_{n+1} - x_n\|^2 < 10^{-4}$, we consider this example for various types of Bregman distance with $m = 10, 20, 50, 80$. The results of this experiment are reported in Fig. 1.

Example 5.2 The following example was first considered in [20],

$$\min g(x) = \frac{x^T P x + a^T x + a_0}{b^T x + b_0}$$
$$\text{subject to}\, x \in X = \{x \in \mathbb{R}^5 : b^T x + b_0 > 0\},$$

where

$$P = \begin{pmatrix} 5 & -1 & 2 & 0 \\ -1 & 5 & -1 & 3 \\ 2 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, a = \begin{pmatrix} 1 \\ -2 \\ -2 \\ 1 \end{pmatrix} b = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, a_0 = -2, b_0 = 4.$$

Since P is symmetric and positive definite, g is pseudoconvex on X. We minimize g on $K = \{x \in \mathbb{R}^4 : 1 \leq x_i \leq 3\} \subset X$.

It is easy to see that

$$F(x) = \nabla g(x) = \frac{(b^T x + b_0)(2Px + a) - b(x^T P x + a^T x + a_0)}{(b^T x + b_0)^2}. \tag{5.1}$$

The following choices of parameters are made: $\alpha_n = \frac{1}{n+1}$, $\delta_n = \frac{2n}{5n+7}$, $\rho = 3.5$ and $u = \mu = 0.5$.

We terminate the iterations at $E_n = \|x_{n+1} - x_n\|^2 \leq \epsilon$ with $\epsilon = 10^{-4}$. The results are presented in Fig. 2 for various initial values of x_1.

Case 1: $x_1 = (-10, -10, -10, -10)'$;
Case 2: $x_1 = (1, 2, 3, 4)'$;
Case 3: $x_1 = (4, 4, 4, 4)'$;
Case 4 $x_1 = (5, 0, 0, 10)'$.

We compare the performance of our Algorithm 3.2 with Algorithm 1.2. For algorithm 1.2, we let $l = 0.001$ and $\gamma = 0.002$.

Acknowledgements The third author acknowledges with thanks the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to the CoE-MaSS.

References

1. Alber, Y.I. 1996. Metric and generalized projection operators in Banach spaces: properties and applications. In *Theory and applications of nonlinear operator of accretive and monotone type*, ed. A.G. Kartsatos, 15–50. New York: Marcel Dekker.
2. Alber, Y., and I. Ryazantseva. 2006. *Nonlinear ill-posed problems of monotone type*. Dordrecht: Springer.

3. Beck, A. 2017. *First -order methods in optimization*. Philadephia (PA): Society for Industrial and Applied Mathematics.
4. Bot, R.I., E.R. Csetnek, and P.T. Voung. 2020. The forward-backward-forward method from continuous and discrete perspective for pseudomonotone variational inequality inequalities in Hilbert spaces. *European Journal of Operational Research* 287: 49–60.
5. Ceng, L.C., N. Hadjisavas, and N.C. Weng. 2010. Strong convergence theorems by a hybrid extragradient-like approximation method for variational inequalities and fixed point problems. *Journal of Global Optimization* 46: 635–646.
6. Ceng, C.L., M. Teboulle, and J.C. Yao. 2010. Weak convergence of an iterative method for pseudomonotone variational inequalities and fixed-point problems. *Journal of Optimization Theory and Applications* 146: 19–31.
7. Ceng, C.L., A. Petrusel, and J.C. Yao. 2018. Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. *Fixed Point Theory* 19: 487–502.
8. Ceng, C.L., A. Petrusel, and J.C. Yao. 2019. Systems of variational inequalities with hierarchical variational inequality constraints for Lipschitzian pseudocontractions. *Fixed Point Theory* 20: 113–133.
9. Ceng, C.L., A. Petrusel, X. Qin, and J.C. Yao. 2021. Pseudomonotone variational inequalities and fixed points. *Fixed Point Theory* 22 (2): 543–558.
10. Censor, Y., A. Gibali, and S. Reich. 2012. Extensions of Korpelevich's extragradient method for variational inequality problems in Euclidean space. *Optimization* 61: 1119–1132.
11. Censor, Y., A. Gibali, and S. Reich. 2012. The subgradient extragradient method for solving variational inequalities in Hilbert spaces. *Journal of Optimization Theory and Applications* 148: 318–335.
12. Censor, Y., A. Gibali, and S. Reich. 2011. Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. *Optimization Methods and Software* 26: 827–845.
13. Censor, Y., and A. Lent. 1981. An iterative row-action method for interval convex programming. *Journal of Optimization Theory and Applications* 34: 321–353.
14. Denisov, S.V., V.V. Semenov, and L.M. Chabak. 2015. Convergence of the modified extragradient method for variational inequalities with non-Lipschitz operators. *Cybernetics and Systems Analysis* 51: 757–765.
15. Facchinei, F., and J.S. Pang. 2003. *Finite-dimensional variational inequalities and complementarity problems*. Berlin: Springer.
16. Friesz, T.L., and K. Han. 2019. The mathematical foundations of dynamic user equilibrium. *Transportation Research Part B: Methodological* 126: 309–328.
17. Gibali, A., and Y. Shehu. 2019. An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces. *Optimization* 68 (1): 13–32.
18. Goldstein, A.A. 1964. Convex programming in Hilbert spaces. *Bulletin of the American Mathematical Society* 70: 709–710.
19. Hartman, P., and G. Stampacchia. 1966. On some non-linear elliptic differential-functional equations. *Acta Mathematica* 115: 271–310.
20. Hu, X., and J. Wang. 2006. Solving pseudo-monotone variational inequalities and pseudo-convex optimization problems using the projection neural network. *IEEE Transactions on Neural Networks and Learning Systems* 17: 1487–1499.
21. Jolaoso, L.O., A. Taiwo, and O.T. Alakoye. 2019. A unified algorithm for solving variational inequality and fixed point problems with application to the split equality problem. *Computational and Applied Mathematics* 39: 38.
22. Jolaoso, L.O., and M. Aphane. 2020. Weak and strong convergence Bregman extragradient scheme for solving pseudo-monotone and non-Lipschitz variational inequalities. *Journal of Inequalities and Applications* 2020: 195.
23. Jolaoso, L.O., and Y. Shehu. 2021. Single Bregman projection method for solving variational inequalities in reflexive Banach spaces. *Applicable Analysis*. https://doi.org/10.1080/0036811.2020.1869947.
24. Kanzow, C., and Y. Shehu. 2018. Strong convergence of a double projection-type method for monotone variational inequalities in Hilbert spaces. *Journal of Fixed Point Theory and Applications*. https://doi.org/10.1007/s11784-018-0531-8.

25. Kinderlehrer, D., and G. Stampacchia. 1980. *An introduction to variational inequalities and their applications*. New York: Academic Press.
26. Konnov, I.V. 2007. *Equilibrium models and variational inequalities*. Amsterdam: Elsevier.
27. Korpelevich, G.M. 1976. The extragradient method for finding saddle points and other problems. *Ekon Mat Metody* 12: 747–756.
28. Lin, L.J., M.F. Yang, and Q.H. Ansari. 2005. Existence results for stampacchia and Minty type implicit variational inequality in multivalued maps. *Nonlinear Analysis: Theory, Methods and Applications* 61: 1–19.
29. Mashreghi, J., and M. Nasri. 2010. Forcing strong convergence of Korpelevich's method in Banach spaces with its application in game theory. *Nonlinear Analysis* 72: 2086–2099.
30. Reem, D., S. Reich, and A. De Pierro. 2019. Re-examination of Bregman functions and new properties of their divergences. *Optimization* 68: 279–348.
31. Reich, S., D.V. Thong, P. Cholamjiak, and L.V. Long. 2021. Inertial projection-type methods for solving pseudomonotone variational inequality problems in Hilbert space. *Numerical Algorithms* 88: 813–835.
32. Reich, S., and S. Sabach. 2009. A strong convergence theorem for proximal-type algorithm in reflexive Banach space. *Journal of Nonlinear and Convex Analysis* 10: 471–485.
33. Shehu, Y., Q.L. Dong, and D. Jiang. 2019. Single projection method for pseudomonotone variational inequality in Hilbert space. *Optimization* 68: 385–409.
34. Shehu, Y., O.S. Iyiola, and S. Reich. 2021. A modified inertial subgradient extragradient method for solving variational inequalities. *Optimization and Engineering*. https://doi.org/10.1007/s11081-020-09593-w.
35. Shehu, Y., X.-H. Li, and Q.-L. Dong. 2020. An efficient projection-type method for monotone variational inequalities in Hilbert spaces. *Numerical Algorithms* 84: 365–388.
36. Solodov, M.V., and B.F. Svaiter. 1999. A new projection method for variational inequality problems. *SIAM Journal on Control and Optimization* 37: 765–776.
37. Tseng, P.A. 2009. Modified forward-backward splitting method for maximal monotone mappings. *SIAM Journal on Control and Optimization* 38: 431–446.
38. Wairojjana, N., N. Pakkaranang, and N. Nuttapol. 2021. Pholasa, Strong convergence inertial projection algorithm with self-adaptive step size rule for pseudomonotone variational inequalities in Hilbert spaces. *Demonstration Mathematica* 54 (1): 110–128.
39. Xu, H.K. 1991. Inequalities in Banach spaces with applications. *Nonlinear Analysis* 16 (2): 1127–1138.
40. Yao, Y., M. Postolache, and J.C. Yao. 2019. Iterative algorithms for the generalized variational inequalities. *UPB Scientific Bulletin Series A* 81: 3–16.
41. Yao, Y., M. Postolache, and J.C. Yao. 2019. Iterative algorithms for pseudomonotone variational inequalities and fixed point problems of pseudocontrative operators. *Mathematics* 7: 1189.
42. Yao, Y., M. Postolache, and J.C. Yao. 2020. Strong convergence of an extragradient algorithm for variational inequality and fixed point problems. *UPB Scientific Bulletin Series A* 82 (3): 3–12.
43. Zhao, X., J.C. Yao, and Y. Yao. 2020. A proximal algorithm for solving split monotone inclusion. *University Politehnica of Bucharest Series A* 80 (3): 43–52.