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ORIGINAL RESEARCH PAPER

# Inequalities for the derivative of a polynomial with restricted zeros

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**Abstract**
For a polynomial $p(z)$ of degree $n$, it is known that

$$\max_{|z|=1}|p^{'}(z)| \leq \frac{n}{1+k}\max_{|z|=1}|p(z)|,$$

if $p(z) \neq 0$ in $|z|<k, k \geq 1$ and

$$\max_{|z|=1}|p^{'}(z)| \geq \frac{n}{1+k}\max_{|z|=1}|p(z)|,$$

if $p(z) \neq 0$ for $|z| > k, k \leq 1$. In this paper, we assume that there is a zero of multiplicity $s$, $s<n$ at a point inside $|z|=1$ and prove some generalizations and improvements of these inequalities.

**Keywords** Inequalities · Restricted zeros · Polynomials

**Mathematics Subject Classification** 30A10 · 30C10 · 30C15

## 1 Introduction

Let $\mathscr{P}_n$ be the class of polynomials $p(z) := \sum_{j=0}^{n} a_j z^j$ of degree at most $n$. For $p \in \mathscr{P}_n$ and a positive real number $k$, we write:

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$$\mathcal{D}_k := \{z : |z| = k\},\ \mathcal{D}_k^+ := \{z : |z| > k\},\ \mathcal{D}_k^- := \{z : |z| < k\},$$
$$M(p,k) := \max_{z\in\mathcal{D}_k}|p(z)| \text{ and } m(p,k) := \min_{z\in\mathcal{D}_k}|p(z)|.$$

If $p \in \mathscr{P}_n$ and $p^{'}$ is the derivative of $p$, then

$$M(p', 1) \leq nM(p, 1). \tag{1}$$

(1) is a famous sharp inequality due to Bernstein [2] (see also [6]). If we restrict ourselves to the class of polynomials $p \in \mathscr{P}_n$, such that $p(z) \neq 0$ for $z \in \mathcal{D}_1^-$, then inequality (1) can be sharpened. Infact, it was conjectured by Erdös and latter proved by Lax [3], that if $p(z)$ does not vanish in $\mathcal{D}_1^-$, then

$$M(p^{'}, 1) \leq \frac{n}{2}M(p, 1). \tag{2}$$

In case $p(z)$ is a polynomial of degree $n$ and does not vanish in $\mathcal{D}_1^+$, then it was shown by Turán [7] that

$$M(p^{'}, 1) \geq \frac{n}{2}M(p, 1). \tag{3}$$

For the polynomials $p \in \mathscr{P}_n$, with $p(z) \neq 0$, $z \in \mathcal{D}_k^-$, $k \geq 1$, Malik [4] proved

$$M(p^{'}, 1) \leq \frac{n}{1+k}M(p, 1), \tag{4}$$

where for the $n^{th}$ degree polynomial $p(z) \neq 0$, for $z \in \mathcal{D}_k^+$, $k \leq 1$, he obtained

$$M(p^{'}, 1) \geq \frac{n}{1+k}M(p, 1). \tag{5}$$

Aziz and Shah [1] generalized (5) and proved that if $p(z)$, the polynomial of degree $n$ has all its zeros in $\mathcal{D}_k \cup \mathcal{D}_k^-$, $k \leq 1$, with $s$-fold zeros at the origin, then

$$M(p^{'}, 1) \geq \frac{n+sk}{1+k}M(p, 1). \tag{6}$$

Nakprasit and Somsuwan [5] investigated $M(p^{'}, 1)$ in terms of $M(p, 1)$ for a polynomial $p \in \mathscr{P}_n$, having a zero of order $s$ at some point $z_0$, where $z_0 \in \mathcal{D}_1^-$ and proved:

**Theorem A** *If* $p(z) := (z - z_0)^s(a_0 + \sum_{v=\mu}^{n-s} a_v z^v)$, $1 \leq \mu \leq n-s$, $0 \leq s \leq n-1$, *is a polynomial of degree* $n$, *having a zero of order* $s$ *at* $z_0$, *where* $z_0 \in \mathcal{D}_1^-$ *and the remaining* $n-s$ *zeros are outside* $\mathcal{D}_k^-$, $k \geq 1$, *then*

$$M(p^{'}, 1) \leq \left\{\frac{s}{(1-|z_0|)} + \frac{A}{(1-|z_0|)^s}\right\}M(p, 1) - \frac{A}{(k+|z_0|)^s}m(p, k), \tag{7}$$

*where*

$$A = \frac{(1+|z_0|)^{s+1}(n-s)}{(1+k^{\mu})(1-|z_0|)}.$$

**Observation** In Theorem A, if we put $s = 0$, that is, if we assume, there is no zero inside $\mathcal{D}_k^-$, then

$$M(p',1) \le \frac{n(1+|z_0|)}{(1+k^{\mu})(1-|z_0|)}M(p,1) - \frac{n(1+|z_0|)}{(1+k^{\mu})(1-|z_0|)}m(p,k). \tag{8}$$

The presence of $z_0$ in the R.H.S of (8) as well as $\left(\frac{1+|z_0|}{1-|z_0|}\right) \ge 1$, shows that their attempt of obtaining the desired result is not only incomplete but incorrect.

In the light of Theorem A followed by the observation, we are in a position to prove the following results.

## 2 Main results

**Theorem 1** *If $p(z)$ is a polynomial of degree $n$ having no zeros in $\mathcal{D}_k^-$, $k > 1$, except a zero of multiplicity $s$, $0 \le s < n$ at $z_0$, where $|z_0| \le 1 - \frac{2s(k+1)}{n(k-1)+2s}$ then for $n > \frac{2sk}{k-1}$,*

$$M(p',1) \le \frac{1}{2}\left[n + \frac{2s(k+|z_0|)}{(k+1)(1-|z_0|)} - \frac{n(k-1)}{(k+1)}\frac{|p(z)|^2}{(M(p,1))^2}\right]M(p,1).$$

*The result is best possible for $z_0 = 0$ and equality holds for $p(z) := z^s(z+k)^{n-s}$, $0 \le s < n$, evaluated at $z = 1$.*

*In particular if $z_0 = 0$, then we have the following sharp result.*

**Corollary 1** *If $p(z)$ is a polynomial of degree $n$ having all zeros outside $\mathcal{D}_k^-$, $k > 1$, except a zero of multiplicity $s$, $0 \le s < n$ at origin, then*

$$M(p',1) \le \frac{1}{2}\left[n + \frac{2sk}{k+1} - \frac{n(k-1)}{k+1}\frac{|p(z)|^2}{(M(p,1))^2}\right]M(p,1),$$

*where $n > \frac{2sk}{k-1}$.*

*Theorem 1 reduces to the following result, by taking $s = 0$.*

**Corollary 2** *If $p(z)$ is a polynomial of degree $n$ having no zeros in $\mathcal{D}_k^-$, $k > 1$, then*

$$M(p',1) \le \frac{1}{2}\left[n - \frac{n(k-1)}{k+1}\frac{|p(z)|^2}{(M(p,1))^2}\right]M(p,1).$$

*Equality sign holds for the polynomial $p(z) := (z+k)^n$, evaluated at $z = 1$.*

**Theorem 2** *If $p(z)$ is a polynomial of degree n having no zeros in $\mathcal{D}_k^+$, $k \leq 1$, except a zero of multiplicity s, $0 \leq s < n$ at $z_0$, where $z_0 \in \mathcal{D}_1^-$, then for $z \in \mathcal{D}_1$*

$$|p^{'}(z)| \geq \frac{1}{1+k}\left\{n + s\left(\frac{k-|z_0|}{1+|z_0|}\right)\right\}|p(z)|.$$

*The result is best possible for $z_0 = 0$ and equality holds for $p(z) := z^s(z+k)^{n-s}$, $0 \leq s < n$, evaluated at $z = 1$.*

*For $k = 1$, we have the following result from Theorem 2.*

**Corollary 3** If $p(z)$ is a polynomial of degree n having no zeros in $\mathcal{D}_1^+$, except a zero of multiplicity $s$, $0 \leq s < n$ at $z_0$, where $z_0 \in \mathcal{D}_1^-$, then for $z \in \mathcal{D}_1$

$$|p^{'}(z)| \geq \frac{1}{2}\left\{n + s\left(\frac{1-|z_0|}{1+|z_0|}\right)\right\}|p(z)|.$$

***Remark 1*** In particular if $z_0 = 0$, then Theorem 2 reduces to inequality (6) due to Aziz and Shah [1].

## 3 Lemmas

For the proof of Theorem 1, we need the following Lemma due to Malik [4].

**Lemma 1** *If $p(z)$ is a polynomial of degree at most n and $p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, then for $|z|=1$*

$$|(p^{*'}(z)| + |p^{'}(z)| \leq nM(p,1). \tag{9}$$

*The result is best possible and equality is attained at $p(z) = \alpha z^n$, $\alpha$ being a complex number.*

## 4 Proofs of theorems

***Proof of Theorem 1*** Since $p(z)$ has all its zeros in $\mathcal{D}_k \cup \mathcal{D}_k^+$, except a zero of multiplicity $s$ at $z_0$, $z_0 \in \mathcal{D}_1^-$, $0 \leq s < n$, therefore

$$p(z) = (z - z_0)^s u(z),$$

where $u(z)$ is a polynomial of degree $n - s$ having all zeros in $\mathcal{D}_k \cup \mathcal{D}_k^+$. Therefore, if $z_1, z_2, \ldots, z_{n-s}$ be the zeros of $u(z)$, then $|z_j| \geq k$, $k > 1, j = 1, 2, \ldots, n-s$. Hence, we have

$$\frac{zp'(z)}{p(z)} = \frac{sz}{z - z_0} + \sum_{j=1}^{n-s} \frac{z}{z - z_j}.$$

This, in particular, gives

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) = \mathrm{Re}\left(\frac{sz}{z - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{z}{z - z_j}\right).$$

For the points $e^{i\theta}$, $0 \leq \theta < 2\pi$ which are not the zeros of $p(z)$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) &= \mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{e^{i\theta}}{e^{i\theta} - z_j}\right) \\ &= \mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{1}{1 - e^{-i\theta} z_j}\right).\end{aligned}$$

Using the fact that for $|w| \geq k > 1$,

$$\mathrm{Re}\left(\frac{1}{1 - w}\right) \leq \frac{1}{1 + k}$$

and

$$\mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) \leq \left|\frac{se^{i\theta}}{e^{i\theta} - z_0}\right| \leq \frac{s}{1 - |z_0|}$$

we get, for $0 \leq \theta < 2\pi$,

$$\mathrm{Re}\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) \leq \frac{s}{1 - |z_0|} + \frac{n - s}{1 + k},$$

Now, for $p^*(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$, it can be easily verified that

$$|(p^*(z))'| = |np(z) - zp'(z)|, \quad z \in \mathcal{D}_1$$

This gives, for $z \in \mathcal{D}_1$

$$\begin{aligned}\left|\frac{(p^*(z))'}{p(z)}\right|^2 &= \left|n - \frac{zp'(z)}{p(z)}\right|^2\\ &= n^2 + \left|\frac{zp'(z)}{p(z)}\right|^2 - 2n\mathrm{Re}\left(\frac{\mathrm{zp'(z)}}{\mathrm{p(z)}}\right)\\ &\geq n^2 + \left|\frac{zp'(z)}{p(z)}\right|^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right).\end{aligned}$$

That is

$$|(p^*(z))'|^2 \geq |zp'(z)|^2 + \left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|p(z)|^2.$$

Since

$$|z_0| \leq 1 - \frac{2s(k+1)}{n(k-1)+2s} \text{ and } n > \frac{2sk}{k-1},$$

it can be easily verified that

$$n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right) \geq 0. \tag{10}$$

From this we get, for $z \in \mathcal{D}_1$

$$|(p^*(z))'| \geq \left[|p'(z)|^2 + \left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|p(z)|^2\right]^{\frac{1}{2}}. \tag{11}$$

Inequality (11) together with Lemma 1, gives

$$|p'(z)| + \left[|p'(z)|^2 + \left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|p(z)|^2\right]^{\frac{1}{2}} \leq nM(p,1).$$

This gives, for $z \in \mathcal{D}_1$

$$|p'(z)|^2 + \left\{n^2 - 2n\left(\frac{s}{1-|z_0|} + \frac{n-s}{1+k}\right)\right\}|p(z)|^2 \leq \left(nM(p,1) - |p'(z)|\right)^2.$$

On simplifying we get, for $z \in \mathcal{D}_1$

$$\begin{aligned}|p'(z)| &\le \frac{1}{2}\left[n-\left\{\frac{n(k-1)}{(k+1)}-\frac{2s(k+|z_0|)}{(k+1)(1-|z_0|)}\right\}\frac{|p(z)|^2}{(M(p,1))^2}\right]M(p,1)\\ &\le \frac{1}{2}\left[n+\frac{2s(k+|z_0|)}{(k+1)(1-|z_0|)}-\frac{n(k-1)}{(k+1)}\frac{|p(z)|^2}{(M(p,1))^2}\right]M(p,1).\end{aligned}$$

From which the result follows.

***Proof of Theorem 2*** Since $p(z)$ has all its zeros in $\mathcal{D}_k \cup \mathcal{D}_k^-$, except a zero of multiplicity $s$ at $z_0$, $z_0 \in \mathcal{D}_1^-$, $0 \le s < n$, therefore

$$p(z) = (z - z_0)^s u(z),$$

where $u(z)$ is a polynomial of degree $n - s$ having all its zeros in $\mathcal{D}_k \cup \mathcal{D}_k^-$. Therefore, if $z_1, z_2, \ldots, z_{n-s}$ be the zeros of $u(z)$, then $|z_j| \le k$, $k \le 1$, $j = 1, 2, \ldots, n - s$. Hence, we have

$$\frac{zp'(z)}{p(z)} = \frac{sz}{z - z_0} + \sum_{j=1}^{n-s} \frac{z}{z - z_j}.$$

This, in particular, gives

$$\mathrm{Re}\left(\frac{zp'(z)}{p(z)}\right) = \mathrm{Re}\left(\frac{sz}{z - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{z}{z - z_j}\right).$$

Therefore, for the points $e^{i\theta}$, $0 \le \theta < 2\pi$ which are not the zeros of $p(z)$, we have

$$\begin{aligned}\mathrm{Re}\left(\frac{e^{i\theta}p'(e^{i\theta})}{p(e^{i\theta})}\right) &= \mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{e^{i\theta}}{e^{i\theta} - z_j}\right)\\ &= \mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) + \mathrm{Re}\left(\sum_{j=1}^{n-s} \frac{1}{1 - e^{-i\theta}z_j}\right).\end{aligned} \tag{12}$$

Using the facts that for $|w| \le k \le 1$ we have

$$\mathrm{Re}\left(\frac{1}{1 - w}\right) \ge \frac{1}{1 + k}$$

and for $z_0 \in \mathcal{D}_1^-$

$$\mathrm{Re}\left(\frac{se^{i\theta}}{e^{i\theta} - z_0}\right) \ge \frac{s}{1 + |z_0|}.$$

We get from (12)

$$|p^{'}(z)| \geq \left\{ \frac{s}{1+|z_0|} + \frac{n-s}{1+k} \right\} |p(z)|$$
$$= \frac{1}{1+k} \left\{ n + s\left( \frac{k-|z_0|}{1+|z_0|} \right) \right\} |p(z)|.$$

This proves the desired result.

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## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

## References

1. Aziz, A. and Shah, W. M. 2004. Inequalities for a polynomial and its derivative. *Mathematical Inequalities & Applications* 7: 379–391.
2. Bernstein, S.N. 1926. *Lecons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques une variable réelle*. Paris: Gauthier-Villars.
3. Lax, P.D. 1944. Proof of a conjecture of P. Erdös on the derivative of a polynomial. *Bulletin of the American Mathematical Society (NS)* 50: 509–513.
4. Malik, M.A. 1969. On the derivative of a polynomial. *Journal of the London Mathematical Society* 1: 57–60.
5. Nakprasit, K.M. and Somsuwan, J. 2017. An upper bound of a derivative for some class of polynomials. *Journal of Mathematical Inequalities* 11 (1): 143–150.
6. Schaeffer, A.C. 1941. Inequalities of A. Markoff and S. Bernstein for polynomials and related functions. *Bulletin of the American Mathematical Society (NS)* 47: 565–579.
7. Turán, P. 1939. Über die ableitung von polynomen. *Compositio Mathematica* 7: 89–95.