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ORIGINAL RESEARCH PAPER

Fixed point via implicit contraction mapping on quasi-partial b-metric space

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Abstract

The notion of non-linear contraction via implicit function was first introduced by Popa. Sub-sequentially, Aydi extended and proved fixed point results for α-implicit contraction in quasi b-metric space. In this paper, we have obtained some new fixed point results for the implicit contraction in the setting of quasi-partial b-metric space. The results are validated with the application based on them.

Keywords Fixed point · Implicit contraction · Non-linear contraction · Quasi-partial b-metric space

Mathematics Subject Classification 54H25 · 47H10

1 Introduction

Metric fixed point theory came into existence with the elegant result of contraction mapping principle given by Banach [6] in 1922. Researchers have generalized this result by refining the contraction condition and replacing the metric space with a generalized abstract space [8, 15, 21]. In 1997, Popa [17] introduced the concept of an implicit relation in contractive condition. In 2012, Berinde [7] obtained some constructive fixed point theorems for almost contractions satisfying an implicit relation. Several classical and common fixed point theorems which were unified via self-mappings satisfying implicit relation were proved in [1–5, 11–13, 16, 18–20, 22].

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In 2012, Karapinar [14] introduced the notion of quasi-partial metric space and discussed the existence of fixed points of self-mapping on this space. Gupta and Gautam [9, 10] further generalized the quasi-partial metric space to the class of quasi-partial b-metric space. The aim of this paper is to determine a fixed point satisfying an implicit relation in the setting of quasi-partial b-metric space. Some examples are also given to verify the validity of our results.

2 Preliminaries

We begin the section with some basic definition and concept.

Definition 1 [14] A quasi-partial metric on a non-empty set X is a function $q : X \times X \rightarrow R^{+}$, satisfying the following conditions:

(QPM_1) If $q(x,x) = q(x,y) = q(y,y)$, then $x = y$
(QPM_2) $q(x,x) \leq q(x,y)$
(QPM_3) $q(x,x) \leq q(y,x)$
(QPM_4) $q(x,y) + q(z,z) \leq q(x,z) + q(z,y)$ for all $x, y, z \in X$.

A quasi-partial metric space is a pair (X, q) such that X is a non-empty set and q is a quasi-partial metric on X.

Definition 2 [9] A quasi-partial b-metric on a non-empty set X is a mapping $qp_b :$ $X \times X \rightarrow R^{+}$ such that for some real number $s \geq 1$ and for all $x, y, z \in X$

(QPb_1) $qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$
(QPb_2) $qp_b(x,x) \leq qp_b(x,y)$
(QPb_3) $qp_b(x,x) \leq qp_b(y,x)$
(QPb_4) $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$.

A quasi-partial b-metric space is a pair (X, qp_b) such that X is a non-empty set and qp_b is a quasi-partial b-metric on X. The number s is called the coefficient of (X, qp_b).

Let qp_b be a quasi-partial b-metric on the set X. Then $d_{qp_b}(x,y) = qp_b(x,y) + qp_b(y,x) - qp_b(x,x) - qp_b(y,y)$ is a b-metric on X.

Example 1 Let $X = [0,1]$. Define $qp_b : X \times X \rightarrow R^{+}$ as $qp_b(x,y) = (x-y)^2 + x$. It can be shown here that (X, qp_b) is a quasi-partial b-metric space.

$qp_b(x,x) = qp_b(x,y) = qp_b(y,y) \Rightarrow x = y$ as $x = (x-y)^2 + x = y$ gives $x = y$.

Again $qp_b(x,x) \leq qp_b(x,y)$ as $x \leq (x-y)^2 + x$ and similarly $qp_b(x,x) \leq qp_b(y,x)$ as $x \leq (y-x)^2 + y$ for $0 < x < y$.

Also $qp_b(x,y) \leq s[qp_b(x,z) + qp_b(z,y)] - qp_b(z,z)$

As $(x-y)^2 + x + z \leq s[(x-z)^2 + x + (z-y)^2 + z]$ for fixed $s = 2$.

It can be observed that

$(x-y)^2 + x + z \leq (x - z + z + y)^2 + x + z \leq 2[(x-z)^2 + x + (z-y)^2 + z]$.

So (QPb_4) holds. Thus (X, qp_b) is a quasi-partial b-metric space with $s = 2$.

Example 2 Let $X = [1, \infty)$. Define $qp_b : X \times X \to R^+$ as $qp_b(x, y) = e^x + e^y$.
Then (X, qp_b) is a quasi-partial b-metric space.
Let $qp_b(x, x) = qp_b(x, y) = qp_b(y, y) \Rightarrow e^x + e^x = e^x + e^y = e^y + e^y \Rightarrow e^x = e^y$, which implies $x = y$.
Let $x, y \in X$. Without loss of generality, for $x \le y$ we have $2e^x \le e^x + e^y$.

Thus $qp_b(x, x) \le qp_b(x, y)$.
Similarly, $qp_b(x, x) \le qp_b(y, x)$.
For (QPb_4), we have
$qp_b(x, y) = e^x + e^y \ \le s[e^x + e^y]$ since $s \ge 1$, e^x, e^y, $e^z > 0$
$\le se^x + se^y + 2e^z(s - 1)$ since $(s - 1) \ge 0$,
$\le s[e^x + e^z + e^z + e^y] - 2e^z$
$qp_b(x, y) \le s[qp_b(x, z) + qp_b(z, y)] - qp_b(z, z)$.

Lemma 1 [9] *Let (X, qp_b) be a quasi-partial b-metric space. Then the following hold*:

- *If $qp_b(x, y) = 0$, then $x = y$.*
- *If $x = y$, then $qp_b(x, y) > 0$ and $qp_b(y, x) > 0$.*

Proof is similar as for the case of quasi-partial b-metric space [9].

Definition 3 [9] Let (X, qp_b) be a quasi-partial b-metric. Then

- A sequence $\{x_n\} \subset X$ converges to $x \in X$ if and only if $qp_b(x, x) = \lim_{n\to\infty} qp_b(x, x_n) = \lim_{n\to\infty} qp_b(x_n, x)$.
- A sequence $\{x_n\} \subset X$ is called a Cauchy sequence if and only if $\lim_{n,m\to\infty} qp_b(x_n, x_m)$ and $\lim_{n,m\to\infty} qp_b(x_m, x_n)$ exist (and are finite).
- The quasi-partial b-metric space (X, qp_b) is said to be complete if every Cauchy sequence $\{x_n\} \subset X$converges with respect to τ_{qp_b} to a point $x \in X$ such that $qp_b(x, x) = \lim_{n,m\to\infty} qp_b(x_n, x_m) = \lim_{m,n\to\infty} qp_b(x_m, x_n)$.

Lemma 2 [9] *Let (X, qp_b) be a quasi-partial b-metric space. The following statements are equivalent*:

- *(X, qp_b) is Cauchy.*
- *(X, d_{qp_b}) is Cauchy.*

Lemma 3 [9] *Let (X, qp_b) be a quasi-partial b-metric space. The following statements are equivalent*:

- *(X, qp_b) is complete.*
- *(X, d_{qp_b}) is complete.*

Definition 4 Let (X, qp_b) be a quasi-partial b-metric space and $T : X \to X$ be a given mapping. T is said to be sequentially continuous at $z \in X$ if for each sequence

$\{x_n\}$ in X converging to z, we have

$Tx_n \to Tz$, i.e., $\lim_{n\to\infty} qp_b(Tx_n, Tz) = qp_b(Tz, Tz)$.

T is said to be sequentially continuous on X if T is sequentially continuous at each $z \in X$.

Lemma 4 *Let (X, qp_b) be a quasi-partial b-metric space and $\{x_n\}$ be a convergent sequence in X to a point $z \in X$ such that $\lim_{n\to\infty} qp_b(x_n, z) = 0 = \lim_{n\to\infty} qp_b(z, x_n)$ and $qp_b(z, z) = 0$, $y \in X$, then*

- *z is unique and*
- $\frac{1}{s} qp_b(z, y) \le \lim_{n\to\infty} qp_b(x_n, y) \le sqp_b(z, y)$.

Proof Suppose that there exist $z' \in X$ such that $\lim_{n\to\infty} qp_b(x_n, z') = 0$.

Since $qp_b(z, z') \le s[qp_b(z, x_n) + qp_b(x_n, z')] - qp_b(x_n, x_n)$.

Letting $n \to \infty$, we obtain, $z = z'$

$\frac{1}{s} qp_b(z, y) \le \frac{1}{s}[qp_b(z, x_n) + qp_b(x_n, y)] - \frac{1}{s} qp_b(x_n, x_n)$

$\frac{1}{s} qp_b(z, y) \le \lim_{n\to\infty} qp_b(x_n, y)$.

Also $qp_b(x_n, y) \le s(qp_b(x_n, z) + qp_b(z, y)) - qp_b(z, z)$

$\lim_{n\to\infty} qp_b(x_n, y) \le sqp_b(z, y)$. □

3 Implicit relation

Here, we have defined Implicit relation in a different manner:

Definition 5 Let F_Q be the family of lower semi continuous functions $F : R^5 \to R^+$ such that

(F_1) : F is non-increasing in variable t_1 and t_5.

(F_2) : For all $u, v \ge 0$, $s \ge 1$, there exist $h \in [0, 1)$ such that $F(u, v, v, u, s(u + v)) \le 0$ implies $u \le hv$.

(F_3) : $F(t, t, 0, 0, t) > 0 \ \forall \ t > 0$.

Example 3 Let $F : R^5 \to R^+$. Define $F(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha max\{t_2, t_3, t_4, t_5\}$, where $\alpha \in [0, \frac{1}{2s})$. Then F satisfies an implicit relation.

(F_1) : F is non increasing in variable t_1 and t_5.

(F_2) : Let $u, v \ge 0$ such that

$F(u, v, v, u, s(u + v)) = u - \alpha max\{v, v, u, s(u + v)\} \le 0 \Rightarrow u - \alpha(s(u + v)) \le 0$, where $\alpha \in [0, \frac{1}{2s})$.

Thus $u \le hv$ with $h = \frac{s\alpha}{(1 - s\alpha)} < 1$.

(F_3): $F(t, t, 0, 0, t) = t(1 - \alpha) > 0 \ \forall \ t > 0$.

So $F \in F_Q$ satisfies an implicit relation with $\alpha \in [0, \dfrac{1}{2s})$.

Example 4 Let $F \in F_Q$. Define $F : R^5 \to R^+$ as $F(t_1, t_2, t_3, t_4, t_5) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5$ where $a_i \ge 0, i = 1, 2, 3, 4$. Also if we have

- $0 < a_1 + a_2 + a_3 + 2sa_4 < 1$,
- $0 < a_1 + a_4 < 1$,

then F satisfies an implicit relation.

(F_1) Here F is non increasing in variable t_1 and t_5.

(F_2) For all $u, v > 0$, we have
$F(u, v, v, u, s(u + v)) = u(1 - a_3 - sa_4) - v(a_1 + a_2 + sa_4)$.

Without loss of generality, if $F(u, v, v, u, s(u + v)) \leq 0$, then $u \leq hv$, where $h = \frac{a_1 + a_2 + sa_4}{1 - a_3 - sa_4}$.

By first assumption $0 < a_1 + a_2 + a_3 + 2sa_4 < 1$, we have $h \in [0, 1)$. Thus (F_2) is satisfied.

(F_3) By $0 < a_1 + a_4 < 1$, it can be observed that $F(t, t, 0, 0, t) = t(1 - a_1 - a_4) > 0$ for all $t > 0$.

4 Main result

Let us discuss the main result.

Theorem 1 *Let (X, qp_b) be a complete quasi-partial b-metric space and $T : X \to X$ is continuous self map for all $x \in X$. Suppose that*

$$F[qp_b(Tx, Ty), qp_b(x, y), qp_b(x, Tx), qp_b(y, Ty), (qp_b(x, Ty) + (qp_b(y, Tx))] \leq 0. \tag{1}$$

For some $F \in F_Q$ and if F satisfies $F(u, 0, v, v, 2su) \leq 0$ for all $u, v \geq 0$, there exist $\beta \in [0, \frac{1}{s})$ such that $u \leq \beta v$, then z is a unique fixed point of T. i.e., $Tz = z$ with $qp_b(z, z) = 0$.

Proof Let x_0 be an arbitrary point in X. Define $\{x_n\}$ in X by $x_n = Tx_{n-1}$ for all $n = 1, 2, 3, \ldots$. If there exist $n_0 \in N$ with $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T. Suppose that $x_n \neq x_{n+1}$, for all $n \in N$ by (1),

$F[qp_b(Tx_{n-1}, Tx_n), qp_b(x_{n-1}, x_n),$
$qp_b(x_{n-1}, Tx_{n-1}), qp_b(x_n, Tx_n), (qp_b(x_{n-1}, Tx_n) + (qp_b(x_n, Tx_{n-1}))] \leq 0$

$$\begin{aligned} F[qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_n), qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), (qp_b(x_{n-1}, x_{n+1}) \\ + (qp_b(x_n, x_n))] \leq 0. \end{aligned} \tag{2}$$

By (QPb_4),

$$qp_b(x_{n-1}, x_{n+1}) + (qp_b(x_n, x_n) \leq s[qp_b(x_{n-1}, x_n) + qp_b(x_n, x_{n+1})]. \tag{3}$$

By (3) and (F_1) we obtain,

$F[qp_b(x_n, x_{n+1}), qp_b(x_{n-1}, x_n),$
$qp_b(x_{n-1}, x_n), qp_b(x_n, x_{n+1}), sqp_b(x_{n-1}, x_n) + sqp_b(x_n, x_{n+1})] \leq 0.$

By (F_2), there exist $h \in [0, 1)$ such that
$qp_b(x_n, x_{n+1}) \leq hqp_b(x_{n-1}, x_n)$ which implies

$qp_b(x_n, x_{n+1}) \leq hqp_b(x_{n-1}, x_n) \leq \ldots \leq h^n qp_b(x_0, x_1)$.

Let $n, m \in N, m > n$

$$\begin{aligned} qp_b(x_n, x_m) & \leq sqp_b(x_n, x_{n+1}) + s^2 qp_b(x_{n+1}, x_{n+2}) + \ldots + s^{m-n-1} qp_b(x_{m-1}, x_m) \\ & \leq [sh^n + s^2 h^{n+1} + \ldots + s^{m-n-1} h^{m-1}] qp_b(x_0, x_1) \\ & \leq \textstyle\sum_{i=n}^{m-1} s^i h^i qp_b(x_0, x_1) \\ & \leq \textstyle\sum_{i=n}^{\infty} s^i h^i qp_b(x_0, x_1) \end{aligned}$$

$$qp_b(x_n, x_m) \to 0 \text{ as } n, m \to \infty. \tag{4}$$

This implies $\{x_n\}$ is a right Cauchy sequence. □

Similarly, by (2)

$F[qp_b(Tx_n, Tx_{n-1}), qp_b(x_n, x_{n-1}), qp_b(Tx_{n-1}, x_{n-1}), qp_b(Tx_n, x_n), (qp_b(Tx_n, x_{n-1}) + qp_b(x_n, Tx_{n-1}))] \leq 0$

$$\begin{aligned} & F[qp_b(x_{n+1}, x_n), qp_b(x_n, x_{n-1}), qp_b(x_n, x_{n-1}), qp_b(x_{n+1}, x_n), (qp_b(x_{n+1}, x_{n-1}) \\ & \quad + qp_b(x_n, x_n))] \leq 0. \end{aligned} \tag{5}$$

By (QPb_4),

$$qp_b(x_{n+1}, x_{n-1}) + (qp_b(x_n, x_n) \leq s[qp_b(x_{n+1}, x_n) + qp_b(x_n, x_{n-1})]. \tag{6}$$

By (6) and (F_1) we obtain,

$F[qp_b(x_{n+1}, x_n), qp_b(x_n, x_{n-1}), qp_b(x_n, x_{n-1}), qp_b(x_{n+1}, x_n), s(qp_b(x_{n+1}, x_n) + qp_b(x_n, x_{n-1}))] \leq 0$.

By (F_2), there exist $h \in [0, 1)$ such that

$qp_b(x_{n+1}, x_n) \leq hqp_b(x_n, x_{n-1}) \leq \ldots \leq h^n qp_b(x_1, x_0)$.

Let $n, m \in N, m < n$

$$\begin{aligned} qp_b(x_n, x_m) & \leq sqp_b(x_n, x_{n-1}) + s^2 qp_b(x_{n-1}, x_{n-2}) + \ldots + s^{m-n-1} qp_b(x_{m+1}, x_m) \\ & \leq [sh^{n-1} + s^2 h^{n-2} + \ldots + s^{m-n+1} h^m] qp_b(x_1, x_0) \\ & \leq \textstyle\sum_{i=m}^{n-1} s^i h^i qp_b(x_1, x_0) \\ & \leq \textstyle\sum_{i=m}^{\infty} s^i h^i qp_b(x_1, x_0) \end{aligned}$$

$$qp_b(x_n, x_m) \to 0 \text{ as } n, m \to \infty. \tag{7}$$

This implies $\{x_n\}$ is a left Cauchy sequence. Since (X, qp_b) is complete, $\{x_n\}$ converges to some point $z \in X$ with $qp_b(z, z) = 0$.

Therefore, $qp_b(z, z) = \lim_{n\to\infty} qp_b(x_n, z) = \lim_{n,m\to\infty} qp_b(x_n, x_m)$.

By (4) and (7) we get,

$qp_b(z, z) = \lim_{n\to\infty} qp_b(x_n, z) = \lim_{n,m\to\infty} qp_b(x_n, x_m) = 0$

$\lim_{n\to\infty} qp_b(x_{n+1}, z) = \lim_{n\to\infty} qp_b(x_n, z) = 0$

$\lim_{n\to\infty} qp_b(Tx_n, z) = \lim_{n\to\infty} qp_b(Tx_{n-1}, z) = 0$.

Using the sequential continuity, $x_n \to z$ in (X, qp_b).

$\lim_{n\to\infty} qp_b(x_{n+1}, Tz) = \lim_{n\to\infty} qp_b(Tx_n, Tz) = \lim_{n\to\infty} qp_b(Tz, Tz)$.

On the other side, $\lim_{n\to\infty} qp_b(x_n, z) = 0 = qp_b(z, z)$.

By Lemma 4,

$$\begin{aligned} &\frac{1}{s} qp_b(z, Tz) \le \lim_{n\to\infty} qp_b(x_{n+1}, Tz) \le sqp_b(z, Tz) \\ &\frac{1}{s} qp_b(z, Tz) \le qp_b(Tz, Tz) \le sqp_b(z, Tz). \end{aligned} \tag{8}$$

For $x = y = z$,

$F[qp_b(Tz, Tz), qp_b(z, z), qp_b(z, Tz), qp_b(z, Tz), (qp_b(z, Tz) + (qp_b(z, Tz))] \le 0$

$F[qp_b(Tz, Tz), 0, qp_b(z, Tz), qp_b(z, Tz), (qp_b(z, Tz) + (qp_b(z, Tz))] \le 0.$

By (8),

$F[qp_b(Tz, Tz), 0, qp_b(z, Tz), qp_b(z, Tz), 2s(qp_b(Tz, Tz)] \le 0.$

Since F satisfies $qp_b(Tz, Tz) \le \beta qp_b(z, Tz) \le \beta sqp_b(Tz, Tz)\ \forall\ \beta \in [0, \frac{1}{s})$

which holds unless $qp_b(Tz, Tz) = 0$, we deduce that $qp_b(z, Tz) = 0$.

Therefore, Tz = z. Hence z is a fixed point of T.

Suppose there exists another fixed point $z' \neq z$ of T such that $qp_b(z', z') = 0$.

By (1) we obtain,

$F[qp_b(Tz, Tz'), qp_b(z, z'), qp_b(z, Tz), qp_b(z', Tz'), qp_b(z, Tz') + qp_b(z', Tz)] \le 0,$

$F[qp_b(z, z'), qp_b(z, z'), qp_b(z, z), qp_b(z', z'), qp_b(z, z') + qp_b(z', z)] \le 0,$

$F[qp_b(z, z'), qp_b(z, z'), 0, 0, qp_b(z, z') + qp_b(z', z)] \le 0.$

Since F satisfies property (F_3), so it is a contradiction. Hence $z = z'$.

Example 5 Let $X = [0, 1]$. Define $qp_b : X \times X \to X$ as $qp_b(x, y) = (x - y)^2 + x$ with $s = 2$. Also if we have $F \in F_Q$ such that

$$F(t_1, t_2, t_3, t_4, t_5) = t_1 - \alpha(t_3 + t_4) - \beta t_5,$$

where $\alpha \in [0, 1)$, $\beta \in [0, 1)$.

Consider a self map $T : X \to X$ such that $Tx = x$ for all $x \in X$, where T is sequentially continuous on (X, qp_b).

Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ in (X, qp_b) as $n \to \infty$ and T is continuous on $(X, |.|)$ which implies $|Tx_n - Tx| \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} qp_b(Tx_n, Tx) \to qp_b(Tx, Tx)$

$F[qp_b(Tx, Ty), qp_b(x, y), qp_b(x, Tx), qp_b(y, Ty), (qp_b(x, Ty) + (qp_b(y, Tx))]$

$= qp_b(Tx, Ty) - \alpha(qp_b(x, Tx) + (qp_b(y, Ty)) - \beta(qp_b(x, Ty) + qp_b(y, Tx))]$

$= qp_b(x, y) - \alpha(qp_b(x, x) + (qp_b(y, y)) - \beta(qp_b(x, y) + (qp_b(y, x))$

$= (x - y)^2 + x - \alpha(x + y) - \beta((x - y)^2 + x + (y - x)^2 + y).$

For $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, it can be observed that

$(x - y)^2 + x - \alpha(x + y) - \beta((x - y)^2 + x + (y - x)^2 + y) \le 0.$

Then the conditions of Theorem 1 are satisfied and 0 is the unique fixed point of T as shown in Fig. 1.

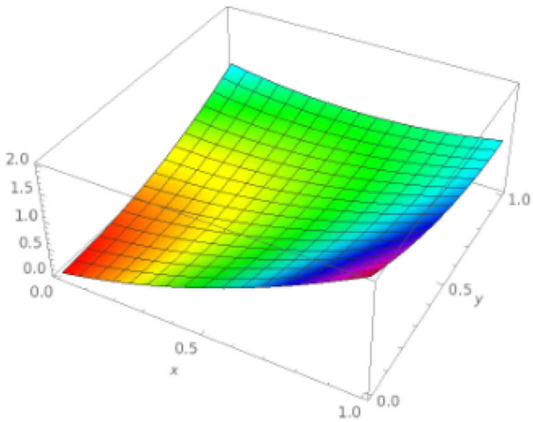

Fig. 1 Zero is the fixed point of T

Let us define Modified Implicit Relation here.

Definition 6 Let F_Q be the family of lower semi continuous functions $F : R^5 \to R^+$ such that

(F_1) : F is non-increasing in variable t_1 and t_5.
(F_2) : For all $u, v \geq 0$, $s \geq 1$, there exist $h \in [0, 1)$ such that $F(\frac{u}{s}, v, v, u, s(u + v)) \leq 0$ implies $u \leq hv$.
(F_3) : $F(t, t, 0, 0, t) > 0 \; \forall \; t > 0$.

Theorem 2 *Let (X, qp_b) be a complete quasi-partial b-metric space and $T : X \to X$ be a continuous function for all $x \in X$. Assume there exists $F \in F_Q$ such that*
$F[qp_b(Tx, Ty), qp_b(x, y), qp_b(x, Tx), qp_b(y, Ty), (qp_b(x, Ty) + (qp_b(y, Tx))] \leq 0.$
Then z is a unique fixed point of T with $qp_b(z, z) = 0$.

Proof Following the proof of Theorem 1, the sequence $\{x_n\}$ is Cauchy and converges to some $z \in X$ in (X, qp_b). We shall show that z = Tz. Taking $x = x_n$ and y = z in (1),
$F[qp_b(Tx_n, Tz), qp_b(x_n, z), qp_b(x_n, Tx_n), qp_b(z, Tz), (qp_b(x_n, Tz) + (qp_b (z, Tx_n))] \leq 0$
$F[qp_b(x_{n+1}, Tz), qp_b(x_n, z), qp_b(x_n, x_{n+1}), qp_b(z, Tz), (qp_b(x_n, Tz)+ (qp_b(z, x_{n+1}))] \leq 0$
$\frac{1}{s} qp_b(z, Tz) \leq \lim_{n \to \infty} qp_b(Tz, Tz) \leq sqp_b(z, Tz).$
Letting $n \to \infty$,
$F[\frac{1}{s} qp_b(z, Tz), 0, 0, qp_b(z, Tz), sqp_b(z, Tz) + 0)] \leq 0$. By ($F_2$), it follows that $qp_b(z, Tz) \leq 0$, which implies that $z = Tz$. □

Corollary 1 *Let (X, qp_b) be a complete quasi-partial b-metric space and $T : X \to X$ be a mapping such that*
$qp_b(Tx, Ty) \leq k \; max \; \{qp_b(x, y), qp_b(x, Tx), qp_b(y, Ty), qp_b((x, Ty) + qp_b(y, Tx))\},$ *where $k \in [0, \frac{1}{2s})$. Then there exists $z \in X$ such that z is a unique fixed point of T. i.e., $z = Tz$ with $qp_b(z, z) = 0$.*

Proof It is sufficient to take F as given in Example 3, i.e., $F(t_1, \ldots, t_5) = k \max \{t_1, \ldots, t_5\}$, where $k \in [0, \frac{1}{2s})$. □

Corollary 2 *Let* (X, qp_b) *be a complete quasi-partial b-metric space and* $T : X \to X$ *be a mapping such that*

$$qp_b(Tx, Ty) \leq a_1 qp_b(x, y) + a_2 qp_b(x, Tx) + a_3 qp_b(y, Ty) + a_4(qp_b((x, Ty) + qp_b(y, Tx)),$$

for all $x, y \in X$, *there exists* $z \in X$ *such that* z *is a unique fixed point of* T. *i.e.,* $z = Tz$ *with* $qp_b(z, z) = 0$.

Proof It is sufficient to take F as given in Example 4, i.e., $F(t_1, \ldots, t_5) = t_1 - a_1 t_2 - a_2 t_3 - a_3 t_4 - a_4 t_5$, where $a_i \geq 0, i = 1, 2, 3, 4$.

$0 < a_1 + a_2 + a_3 + 2sa_4 < 1, 0 < a_1 + a_4 < 1$.

Let us define partial order in modified implied relation. □

Definition 7 Consider F_Q be the family of lower semi-continuous function $F\ \mathbb{R}^5 \to \mathbb{R}$ such that

$F_1)$: F is non-increasing in variable t_1 and t_5 with respect to $\preceq$.
$F_2)$: For all $u \succeq 0, v \succeq 0, s > 1$, there exist $h \in [0, 1]$ s.t. $F[\frac{u}{s}, v, v, s(u + v)] \preceq 0$ implies $u \preceq hv$.
$F_3)$: $F(t, t, 0, 0, t) \succ 0\ \forall\ t > 0$.

Example 6 Let $\preceq$ be a partial order with respect to quasi–partial b–metric space space (X, qp_b) and $F : \mathbb{R}^5 \to \mathbb{R}^+$(where $F \in F_Q$) as $F(t_1, t_2, t_3, t_4, t_5) = t_1 - a_1 t_2 - a_2(t_3 + t_4) - a_3 t_5$.

- $a_1 s + 2a_2 s + 2a_3 s^2 < 1$,
- $a_1 + a_2 < 1$,

$F_1)$: F is non–increasing in variable t_1 and t_5.
$F_2)$: For all $u \succeq 0, v \succeq 0$,

$F\left(\frac{u}{s}, v, v, u, s(u + v)\right) \preceq 0$, then we have,
$u \preceq \frac{(a_1 + a_2 + a_3 s)}{(\frac{1}{s} - a_2 - a_3 s)} v$.

Thus by assumption $a_1 s + 2a_2 s + 2a_3 s^2 < 1$, (F_2) is satisfied.

$F_3)$: Since $a_1 + a_3 < 1$, $F(t, t, 0, 0, t) = t(1 - a_2 + a_3) \succ 0$ for all $t > 0$.

5 Application

In this paper, we have discussed metric fixed point theory in quasi-partial b-metric space to obtain solution of non-linear integral equation defined as

$$x(t) = \int_0^t K(t, s, x(s))\ ds \tag{9}$$

where $t \in M = [c, d]$ and $K : M \times M \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Let $X = C(M, \mathbb{R})$ with the usual supremum norm i.e., $||x||_\infty = max_{t \in M} |x(t)|$.

We define quasi-partial b-metric space $qp_b : X \times X \to \mathbb{R}^+$ as

$$qp_b(x, y) = \begin{cases} ||x - y||_\infty + ||x||_\infty & \text{for all } x, y \in X \text{ with } x \neq y \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

with $s = 2$.

Theorem 3 *Suppose the following conditions are satisfied*:

(i) *Assume that there exist a function* $P : [0, M] \times [0, M] \to [0, +\infty)$ *with* $P(t, \cdot) \in L'$ *norm for* $t \in [0, M]$,

$$0 \le K(t, s, y(s)) - K(t, s, x(s)) \le P(t, s)(y(s) - x(s))$$

Also,

$$|K(t, s, x(s))| \le P(t, s)|x(s)| \text{ for all } x, y \in X.$$

(ii) *There exist* $x_0 \in X$ *such that* $x_0(t) \le \int_0^t K(t, s, x_0(s))\ ds$ *for all* $t \in [0, M]$.

(iii) $Sup_{t \in M} P(t, s) = h \le \frac{1}{2}$ *then the integral Eq.* (9) *has a unique solution.*

Proof Consider the mapping $T : X \to X$ defined by

$$Tx(t) = \int_0^t K(t, s, x(s))\ ds \quad \forall x \in X.$$

Now, we shall show that T has a unique fixed point.

$$|Tx(t)| \le \int_0^t |K(t, s, x(s))\ ds| \le \int_0^t P(t, s)|x(s)|ds = h||x||_\infty$$

and

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^t |K(t,s,x(s)) - K(t,s,y(s))|ds \\
&\leq \int_0^t P(t,s)|x(s) - y(s)|ds \\
&= h||x-y||_\infty.
\end{aligned} \tag{11}
$$

Thus, $||Tx||_\infty \leq \lambda||x||_\infty$ and $||Tx - Ty||_\infty \leq h||x-y||_\infty$.

Hence,

$$
qp_b(Tx, Ty) \leq h\ qp_b(x,y) \tag{12}
$$

is satisfied for all $x, y \in X$ with $x \neq y$ which implies $Tx \neq Ty$. For $Tx = Ty$, Eq. (11) is trivial. Therefore by Corollary 2, if $F(t_1, t_2, t_3, t_4, t_5) = t_1 - a_1t_2 - a_2t_3 - a_3t_4 - a_4t_5$ with $a_1 = a_2 = a_3 = a_4 = 0$ then T has a fixed point. i.e., Eq. (12) has a solution.

Let us consider the space $X = C([0,1], (\mathbb{R}))$ by the quasi-partial b-metric space $qp_b : X \times X \to \mathbb{R}^+$ defined as

$$
qp_b(x,y) = \begin{cases} ||e^x + e^y||_\infty & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}
$$

For each $x, y \in X$. Note that (X, qp_b) is a complete quasi-partial b-metric space. From Eq. (11), we have

$$
\begin{aligned}
|Tx(t) - Ty(t)| &\leq \int_0^t \rho(t,s)|(x(s) - y(s)|\ ds \\
&\leq \int_0^t \rho(t,s)(|(x(s)| + |y(s)|)\ ds \\
&= \int_0^t \rho(t,s)[(x(s) + y(s)]\ ds \\
&\leq \int_0^t \rho(t,s)[e^{x(s)} + e^{y(s)}]\ ds \\
&= h||e^x + e^y||_\infty
\end{aligned}
$$

For all $t \in [0,1]$ and $x, y \in C([0,1], \mathbb{R})$ with $x \neq y$, we deduce

$$
qp_b(Tx, Ty) \leq hqp_b(x,y) \tag{13}
$$

For $Tx = Ty$, Eq. (13) is trivial. Therefore by corollary 2, if $F(t_1, t_2, t_3, t_4, t_5) = t_1 - a_1t_2 - a_2(t_3 + t_4) - a_3t_5$ with $a_1 = a_2 = a_3 = 0$, then Eq. (13) has a solution. □

6 Conclusion

In recent decades, one of the significant research work is restudying the differential and integral equations in the context of metric spaces. In the present study, the authors have investigated an implicit contraction mapping to obtain fixed point on quasi-partial b-metric space and have solved a non-linear differential equation by adopting the approach of fixed point theory. Determining the solution of more generalized integral equations will be an interesting work for future studies.

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Conflict of interest Both the authors declare that they have no conflicts of interest.

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