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ORIGINAL RESEARCH PAPER

Remarks on balls in metric spaces

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Abstract
In this article we discuss metric spaces in which closure of open balls are the corresponding closed balls, and interior of closed balls are the corresponding open balls. Moreover, we try to explore relationships between these two assertions.

Keywords Closure of open ball · Interior of closed ball · Metric convexity · External convexity · Strict convexity

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1 Introduction

Let (X, d) be a metric space and $B(x, r)$, $B[x, r]$ be respectively the open and the closed balls in X with center x and radius $r > 0$, that is, $B(x,r) = \{y \in X | d(x,y) < r\}$ and $B[x,r] = \{y \in X | d(x,y) \leq r\}$. We shall denote by $\bar{B}(x,r)$, the closure of $B(x, r)$ in X and $B^{\circ}[x,r]$ as the interior of $B[x, r]$ in X. Also, for any subset A of X, ∂A denotes the boundary of the set A in X. Throughout, the symbols $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{R}_0$ respectively denote the set of all natural numbers, the set of all real numbers, and the set of all nonnegative real numbers.

In [2], Artémiadis considered the following assertion:

Assertion 1 For each $x \in X$ and $r > 0$, $\bar{B}(x,r) = B[x,r]$.

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If X is a normed linear space, then Assertion 1 is always true. It was erroneously stated in [8, p. 199] that Assertion 1 is true in general. It can be easily seen that Assertion 1 does not hold if one takes X with at least two elements and d as the discrete metric on X. Here, for any $x \in X$, we have

$$B(x, 1) = \{x\}; \bar{B}(x, 1) = \{x\} \neq X = B[x, 1].$$

The following example shows that even for locally convex complete linear metric spaces, Assertion 1 fails to hold.

Examples 1.1 Consider the metric space $(\mathbb{R}, d)$ such that $d(x, y) = p(x - y)$ where p is the pseudo norm on $\mathbb{R}$ defined by

$$p(x) = \begin{cases} |x|, & \text{if } |x| \leq 1; \\ 1, & \text{if } |x| > 1. \end{cases}$$

We observe that $(\mathbb{R}, d)$ is a locally convex linear metric space. Here we have $B(0, 1) = (-1, 1)$, $\bar{B}(0, 1) = [-1, 1]$ but $B[0, 1] = \mathbb{R}$.

However, there are some necessary and sufficient conditions known in the literature for Assertion 1 to hold (see [2, 6, 10, 11]) which we discuss in this note.

Theorem A (*Wong* [11]) *In a metric space* (X, d), *Assertion* 1 *holds if and only if for any two distinct points* a, $b \in X$, *a as well as b are both limit points of the intersection* $B(a, r) \cap B(b, r)$ *where* $r = d(a, b)$.

Definition 1.1 (*λ-convexity*) The metric d of a metric space (X, d) is said to be λ-convex if for each $x, y \in X$, there exists $z \in X$ such that $d(x, y) = d(x, z)/\lambda = d(z, y)/(1 - \lambda)$ for some fixed $\lambda \in (0, 1)$.

Every normed linear space is λ-convex for each fixed $\lambda \in (0, 1)$. The following result of Wong [11] recovers Assertion 1 from λ-convexity.

Theorem B (*Wong* [11]) *Let* (X, d) *be a metric space. If* d *is* λ*-convex, then Assertion* 1 *holds.*

Definition 1.2 (*Weak convexity*) The metric d of a metric space (X, d) is said to be weakly convex [11] or convex [7] if for every pair of distinct points $x, y \in X$, there exists $z \in X$ such that $z \notin \{x, y\}$ and $d(x, z) + d(z, y) = d(x, y)$. If the metric d is convex, we say that (X, d) is weakly convex or metrically convex.

Theorem C (*Wong* [11]) *Let* (X, d) *be a complete metric space. If* d *is weakly convex, then Assertion* 1 *holds.*

Remarks

(1) Since every λ-convex metric is weakly convex, Theorem C in a way is generalization of Theorem B.
(2) Weak convexity is not necessary for Assertion 1 to hold. To see this, consider the set $X = \{(x, x^2) | x \in [0, 1]\} \subset \mathbb{R}^2$ with the Euclidean metric d. Here Assertion 1 holds in X but d is not weakly convex.

(3) If Assertion 1 holds in a metric space (X, d) and Y is an open or dense subset of X, then Assertion 1 holds in the metric space (Y, d). In particular, Assertion 1 holds in the convex metric space (Y, d) which is not complete, where $Y = (0,1) \cup (2,3)$ and d is the usual metric of real line. So completeness is not necessary for Assertion 1 to hold.

(4) Let $X = \{-1/2\} \cup (0,1]$ with the usual metric of $\mathbb{R}$ restricted to X. Then d is weakly convex but $(0,1] = \bar{B}(1/4, 3/4) \neq B[1/4, 3/4] = X$. Here we note that (X, d) is not complete.

In a short paper, Kiventidis [6] proved the following:

Theorem D (*Kiventidis* [6]) *Let* (X, d) *be a metric space. For any pair of distinct points* x *and* y *in* X, *if there exist two sequences* $\{s_n\}$ *and* $\{s'_n\}$ *of points of* X *such that*

1. $s_n \to x$ and $s'_n \to y$,
2. $\max\{d(s_n, x), d(s_n, y)\} < d(x, y)$ *and* $\max\{d(s'_n, x), d(s'_n, y)\} < d(x, y)$,

then Assertion 1 *holds in* X.

Remark 1.1 Theorem D is a generalization of Theorem C, since the hypothesis of Theorem D holds in weakly convex complete metric spaces. However, Wong's alternative proof of Theorem C is indispensable and is much simpler than the earlier proof given in Blumenthal [3, Theorem 14.1].

Definition 1.3 (*Strict convexity*) A linear metric space (X, d) is said to be strictly convex [1] if for any $r > 0$ and any two distinct points $x, y \in X$ such that $d(x, 0) \leq r$ and $d(y, 0) \leq r$, we have $d((x + y)/2, 0) < r$.

The closed ball $B[0, r]$ in a linear metric space is said to be strictly convex [10] if for any pair of distinct points x and y in $B[0, r]$ and $\lambda \in (0,1)$, the point $\{(1-\lambda)x + \lambda y\} \in B^\circ[0, r]$.

In [10], among several characterizations of strict convexity in linear metric spaces, Vasil'ev gave the following characterization in terms of Assertion 1:

Theorem E (*Vasil'ev* [10]) *A linear metric space* (X, d) *is strictly convex if and only if for any* $r > 0$, *the ball* $B[0, r]$ *is strictly convex and* $\bar{B}(0, r) = B[0, r]$.

2 Interior of a closed ball

Analogous to Assertion 1, we consider the following assertion which also fails to hold in general metric spaces.

Assertion 2 For each $x \in X$ and $r > 0$, $B^\circ[x, r] = B(x, r)$.

If X has at least two points and d is the discrete metric on X, then $B(x, 1) = \{x\}$ but $B^\circ[x, 1] = X$. So, Assertion 2 does not hold. Even in the locally convex complete linear metric space as discussed in Example 1.1, $B^\circ[0,1] = \mathbb{R} \neq B(0,1)$ and so Assertion 2 fails to hold.

It can be easily verified that Assertion 2 holds in normed linear spaces. Assertion 2 was also considered by Kiventidis [6].

Theorem F (*Kiventidis* [6]) *Let* (X, d) *be a metric space in which hypothesis of Theorem D is satisfied. Further, if for every pair of distinct points x and y in X, there exists a point* $z \in X$ *such that* $B(x, d(x,y)) \cap B(z, d(z,y)) = \emptyset$*, then Assertion 2 holds in X.*

Analogous to Theorem A, we have the following characterization of Assertion 2:

Theorem 2.1 *In a metric space* (X, d)*, Assertion 2 holds if and only if for every pair of distinct points x and y in X,* $x, y \notin (B[x,r] \cap B[y,r])^\circ$*, where* $r = d(x,y)$*.*

Proof If Assertion 2 holds in X and $x, y \in X$ such that $x \neq y$ and $d(x,y) = r$, then $y \notin B(x,r) = B^\circ[x,r]$ and $x \notin B(y,r) = B^\circ[y,r]$. So,

$$x, y \notin B^\circ[x,r] \cap B^\circ[y,r] = (B[x,r] \cap B[y,r])^\circ.$$

Conversely, if the hypothesis of the theorem holds, then for $a \in X$ and $s > 0$ we have $B(a,s) \subseteq B^\circ[a,s]$. Suppose there exists $b \in B^\circ[a,s]$ satisfying $d(a,b) = s$. Then $b \in B^\circ[a,s] \cap B^\circ[b,s] = (B[a,s] \cap B[b,s])^\circ$, which contradicts the hypothesis. Thus $d(a,b) < s$ for all $b \in B^\circ[a,s]$ which shows that $B^\circ[a,s] \subseteq B(a,s)$. This completes the proof. □

Examples 2.1 Consider the metric space $((0, 1), d)$ where d is the usual metric of real line. If $x, y \in (0,1)$ with $x < y$ and $r = d(x,y) = |x - y|$, then $B^\circ[x,r] = (x - r, x + r) \cap (0,1)$ which shows that $y \notin B^\circ[x,r]$. Similarly, $x \notin B^\circ[y,r]$. So, by Theorem 2.1, Assertion 2 holds in $((0, 1), d)$.

Yet another well known notion of convexity, called external convexity (see [3, Ch. II, p. 55]) turns out to be useful in the context of Assertion 2.

Definition 2.1 (*External convexity*) A metric space (X, d) is said to be externally convex if for every pair of distinct points x and y in X, there exists a point $z \notin \{x, y\}$ in X such that $d(x,y) + d(y,z) = d(x,z)$.

Examples 2.2 The metric space $(\mathbb{Z}, d)$ with the usual metric d of real line is an externally convex metric space, since if $m, n \in \mathbb{Z}$ with $m < n$, then $(2n - m) \in \mathbb{Z}$ satisfies

$$d(m,n) + d(n, 2n - m) = (n - m) + (2n - m - n) = d(m, 2n - m).$$

Definition 2.2 Let (X, d) be a metrically and externally convex metric space. For $x, y \in X$, let $\mathcal{C}_{x,y}$ denotes the collection of all $z \in X$ for which $d(y,z) - d(z,x) = d(x,y)$. Define the relation $\preceq$ on $\mathcal{C}_{x,y}$ by declaring that $z_1 \preceq z_2$ in $\mathcal{C}_{x,y}$ if there exists a sequence $\{u_m\}_{m \in \mathbb{N}}$ of points of X satisfying the following conditions:

(1) $d(x, u_m) + d(u_m, z_1) = d(x, z_1)$,
(2) $d(u_m, z_2) \to 0$ as $m \to \infty$.

Examples 2.3 The metric space $(\mathbb{Q}, d)$ is metrically and externally convex, where d is the usual metric of real line restricted to $\mathbb{Q}$. Here, we find that $\mathcal{C}_{0,1} = (-\infty, 0] \cap \mathbb{Q}$. Let $z_1, z_2 \in \mathcal{C}_{0,1}$ with $z_1 < z_2$. Define for each natural number m, $u_m = z_2 - (z_2 - z_1)/m$. Then $\{u_m\}$ is a sequence of rational numbers such that $d(0, u_m) + d(u_m, z_1) = |z_1| = d(0, z_1)$ and $d(u_m, z_2) = (z_2 - z_1)/m \to 0$ as $m \to \infty$. So, in view of Definition 2.2, we have $z_1 \preceq z_2$.

In view of Definition 2.2, we have the following two Lemmas 2.2 and 2.3, which will be used in the next theorem.

Lemma 2.2 *If $z_1 \preceq z_2$ in $\mathcal{C}_{x,y}$, then $d(x, z_2) + d(z_2, z_1) = d(x, z_1)$.*

Proof In view of condition (1) in the definition of $\preceq$, we have for all positive integers m that

$$\begin{aligned} d(x,y) \geq d(y,u_m) - d(u_m,x) &= d(y,u_m) + d(u_m,z_1) - d(x,z_1) \\ &= d(y,u_m) + d(u_m,z_1) - (d(y,z_1) - d(x,y)) \\ &\geq d(x,y), \end{aligned}$$

where the last inequality holds by triangle inequality, $d(y, u_m) + d(u_m, z_1) - d(y, z_1) \geq 0$. It then follows that $d(y, u_m) - d(u_m, x) = d(x, y)$ for all $m \in \mathbb{N}$. Consequently, $u_m \in \mathcal{C}_{x,y}$ for each m. Also, for any $\xi \in X$ and each positive integer m, we have from condition (3) in the definition of $\preceq$ that $|d(\xi, z_2) - d(\xi, u_m)| \leq d(z_2, u_m) \to 0$, which shows that $d(\xi, u_m) \to d(\xi, z_2)$. Using this in the condition (1) of the definition of $\preceq$, we get $d(x, z_2) + d(z_2, z_1) = d(x, z_1)$, as desired. □

Lemma 2.3 *The relation $\preceq$ is a partial order on the collection $\mathcal{C}_{x,y}$.*

Proof Clearly $\preceq$ is reflexive. If $z_1 \preceq z_2$ in $\mathcal{C}_{x,y}$, then antisymmetry of $\preceq$ is immediate from Lemma 2.2. Now let $z_1 \preceq z_2$ and $z_2 \preceq z_3$ in $\mathcal{C}_{x,y}$. Then there exist two sequences $\{u_m\}$ and $\{v_n\}$ in $\mathcal{C}_{x,y}$ such that $d(u_m, z_2) \to 0$ and $d(v_n, z_3) \to 0$. Since $d(x, v_n) + d(v_n, z_2) = d(x, z_2)$, this along with Lemma 2.2 gives

$$\begin{aligned} d(x,z_1) \leq d(x,v_n) + d(v_n,z_1) \leq\ & d(x,v_n) + d(v_n,z_2) + d(z_2,z_1) \\ =\ & d(x,z_2) + d(z_2,z_1) = d(x,z_1). \end{aligned}$$

So, $z_1 \preceq z_3$, which proves that $\preceq$ is transitive. This completes the proof. □

Recall that a directed set is a nonempty set $\mathcal{D}$ with a reflexive and transitive relation $\simeq$ such that for any a and b in $\mathcal{D}$, there exists $c \in \mathcal{D}$ for which $a \simeq c$ and $b \simeq c$. A net in a metric space (X, d) is a map from a directed set to (X, d). The net $f : \mathcal{D} \to (X, d)$ denoted by $\{f(a)\}_{a \in \mathcal{D}}$, is said to converge to a point $x \in X$, that is, $\{f(a)\}_{a \in \mathcal{D}} \to x$, if for every open set U in X containing the point x, there exists $\alpha \in \mathcal{D}$ such that $f(a) \in U$ for all a satisfying $\alpha \simeq a$.

Remark 2.1 In view of Lemma 2.3, we observe that the partially ordered set $\mathcal{C}_{x,y}$ is a directed set, since for any $z \in X$, we have $x \in \mathcal{C}_{x,y}$ for which $z \preceq x$. Then a chain $\mathcal{N}$ in $\mathcal{C}_{x,y}$ containing the point x is also a directed set and can be identified by the inclusion map $i_{\mathcal{N}} : \mathcal{N} \to X$ and vice-versa. We shall use the symbol $\mathcal{N}$ to denote the associated net $i_{\mathcal{N}}$.

Theorem 2.4 *If* (X, d) *is a metrically and externally convex complete metric space, then Assertion 2 holds in X.*

Proof Suppose that $x \in B^{\circ}[y, r]$ such that $d(x, y) = r$. Since $B^{\circ}[y, r]$ is open, there exists $\delta > 0$ such that $B(x, \delta) \subset B^{\circ}[y, r]$. Let $\mathcal{C}_{x,y}$ be the collection of all $z \in X$ satisfying $d(y, z) - d(z, x) = d(x, y)$. By Lemma 2.3, the collection $\mathcal{C}_{x,y}$ is partially ordered by $\preceq$. Also, for every $z \in \mathcal{C}_{x,y}$, $z \preceq x$ so that every chain in $\mathcal{C}_{x,y}$ has an upper bound in $\mathcal{C}_{x,y}$. By Zorn's Lemma, we have a maximal chain in the partially ordered set $\mathcal{C}_{x,y}$ which we denote by $\mathcal{N}$. By Lemma 2.2, the net $\{d(x, z)\}_{z \in \mathcal{N}}$ is monotonically decreasing and hence convergent. Let $\{d(x, z)\}_{z \in \mathcal{N}} \to a$ for some nonnegative real number a. So for every $\epsilon > 0$, there exists $\gamma \in \mathcal{N}$ such that

$$a - \epsilon/2 < d(x, z) < a + \epsilon/2, \gamma \preceq z. \tag{1}$$

In view of Lemma 2.2, we have $d(x, z) \leq d(x, z')$ for all $z' \preceq z$ in $\mathcal{N}$. Consequently, we have from (1) that $d(z, z') = d(x, z') - d(x, z) \leq a + \epsilon/2 - (a - \epsilon/2) = \epsilon, \gamma \preceq z$, which shows that the chain $\mathcal{N}$ is a Cauchy net in X. Since X is complete, there exists $\xi \in X$ such that $\mathcal{N} \to \xi$. By the continuity of d and the fact that $\mathcal{N} \subseteq \mathcal{C}_{x,y}$, we must have $\xi \in \mathcal{C}_{x,y}$. We claim that $\xi = x$. Suppose that $\xi \neq x$. Then $\xi \prec x$. Let $\{x_k\}$ be the sequence of points of X such that $x_1 = \xi$ and given x_m, by metric convexity of X, we choose $x_{m+1} \in X$ such that

$$x_{m+1} \notin \{x_m, x\}; d(x, x_{m+1}) + d(x_m, x_{m+1}) = d(x, x_m).$$

It follows inductively that $d(x, x_m) + d(x_m, \xi) = d(x, \xi)$ and the sequence $\{x_m\}$ is Cauchy, which by completeness of X converges to some point ξ^* of X so that $d(x_m, \xi^*) \to 0$. Since the sequence $\{d(x, x_m)\}$ is strictly decreasing, we have $d(x, \xi^*) < d(x, \xi)$. Consequently, from the fact that $d(x, \xi^*) + d(\xi^*, \xi) = d(x, \xi)$, we have $\xi \prec \xi^*$. Thus, $\mathcal{N} \cup \{\xi^*\}$ is a chain in $\mathcal{C}_{x,y}$ strictly containing $\mathcal{N}$. This contradicts the maximality of $\mathcal{N}$ and the claim holds.

Thus, the net $\mathcal{N}$ converges to x which allows existence of a point $z \in B(x, \delta) \cap \mathcal{N}$. But then $r = d(x, y) < d(z, x) + d(x, y) = d(y, z)$, which shows that $z \notin B^{\circ}[y, r]$. This contradicts the fact that $z \in B(x, \delta) \subset B^{\circ}[y, r]$. So, $x \notin B^{\circ}[y, r]$. Similarly, interchanging the role of x and y proves that $y \notin B^{\circ}[x, r]$. Now by Theorem 2.1, Assertion 2 holds in X. □

A result similar to Theorem 2.4 has been proved by Kiventidis [6].

Theorem G (*Kiventidis* [6]) *If a metric space* (X, d) *satisfies the hypothesis of Theorem D and is externally convex, then Assertion 2 holds in X.*

It may be remarked that deducing Theorem 2.4 from Theorem G amounts to deduce first that the hypothesis of Theorem D holds for weakly convex complete metric spaces, which in view of Remark 1.1 is relatively difficult.

Example Consider the usual metric d of real line defined by $d(x, y) = |x - y|$.

(1) Assertion 2 holds in the metric space $((0, \infty), d)$ which is metrically convex and externally convex but not complete. This example shows that completeness is not necessary for Theorem 2.4 to hold.
(2) If we take $X = \mathbb{R} \times \{0, 1\}$with the Euclidean metric of $\mathbb{R}^2$, then Assertion 2 holds in the metric space X which is neither metrically convex nor externally convex. So, even metric convexity and external convexity together are not necessary for Theorem 2.4 to hold.
(3) Assertion 2 fails to hold in the externally convex complete metric space $(\mathbb{Z}, d)$. Here we note that $(\mathbb{Z}, d)$ is not metrically convex.
(4) Assertion 2 fails to hold in the metrically convex metric space ([0, 1], d). Here we note that ([0, 1], d) is not externally convex.

In Theorem 2.4, we have used a weaker form of strong external convexity (see Freese et al. [4]) as introduced below.

Definition 2.3 (*Strong external convexity*) A metric space (X, d) is called strongly externally convex if for all distinct points x, y in X such that $d(x, y) = r$ and $s > r$, there exists a point z in X such that $d(x, y) + d(y, z) = d(x, z) = s$.

Khalil [5] also defines strong external convexity requiring the uniqueness of z.

A metrically convex metric space need not be strongly externally convex. This is evident from the following example.

Examples 2.4 Let $X = [0, 1]$ with the usual metric d of $\mathbb{R}$. Clearly (X, d) is metrically convex but it is not strongly externally convex, since for the two points 0 and 1 in X, there does not exist z in X, which is different from 0 and 1, and satisfies $|0 - 1| + |1 - z| = |0 - z|$.

Also not every strongly externally convex metric space is metrically convex as can be seen from the following example given in Khalil [5].

Examples 2.5 Consider the set $X = L_1 \cup L_2$, where $L_i = \{(x, i) | x \in \mathbb{R}\}$, $i = 1, 2$ with the metric d defined by

$$d((x_1, y_1), (x_2, y_2)) = \begin{cases} |x_1 - x_2|, & \text{if } y_1 = y_2; \\ 1 + |x_1| + |x_2|, & \text{if } y_1 \neq y_2. \end{cases}$$

The metric space (X, d) is strongly externally convex, since if $(x_1, y_1), (x_2, y_2) \in X$ such that $d((x_1, y_1), (x_2, y_2)) = r > 0$ and $s > r$, then we can take $(x_3, y_3) = (x_2 + s - r, y_2) \in X$ to obtain the following: $d((x_1, y_1), (x_2, y_2)) + d((x_2, y_2), (x_3, y_3)) = r + s - r = s$.

If we take $a = (0, 1)$ and $b = (0, 2)$, then for any point of the form $p = (x, y) \in X$, we have

$$d(a,p) + d(p,b) = 2|x| + 1.$$

Now if $p \notin \{a,b\}$, then $x \neq 0$. So, we have $d(a,p) + d(p,b) > 1$ whereas $d(a,b) = 1$, which shows that (X, d) is not metrically convex.

Theorem 2.5 *If (X, d) is a strongly externally convex metric space, then Assertion 2 holds in X.*

Proof Suppose the contrary that $y \in B^\circ[x,r]$ such that $d(x,y) = r$. Since $B^\circ[x,r]$ is open, there exists $\delta > 0$ such that $B(y,\delta) \subset B^\circ[x,r]$. By the strong external convexity of (X, d), we can choose $z \in X$ such that $d(x,y) + d(y,z) = d(x,z) = r + \delta/2$. Then $d(y,z) = \delta/2$ so that $z \in B(y,\delta) \subset B^\circ[x,r]$, which is a contradiction since $z \notin B^\circ[x,r]$. So, $y \notin B^\circ[x,r]$. Similarly, on interchanging the roles of x and y, we have $x \notin B^\circ[y,r]$. By Theorem 2.1, Assertion 2 holds in X. □

We have the following analogue of Theorem E.

Theorem 2.6 *A linear metric space (X, d) is strictly convex if and only if for any $r > 0$, the ball $B[0, r]$ is strictly convex and $B^\circ[0,r] = B(0,r)$.*

Proof Suppose that (X, d) is strictly convex. For each $u \in X$, let $f_u : \mathbb{R} \to \mathbb{R}$ be defined by $f_u(t) = d(tu, 0)$, $t \in \mathbb{R}$. Since X is strictly convex, $f_u|_{\mathbb{R}_0}$ is a strictly increasing continuous map [9]. Now if x and y are two distinct points of X such that $d(x,y) = r > 0$, then for all $t \in (0,1)$, we have $(1-t)x + ty \in B(x,r)$ by the convexity of balls in the strictly convex space X (see [9]). We take $u = x - y$ and use the continuity of f_{x-y} at 1 so that for $\epsilon > 0$ we can choose a point $t_0 \in (0,1)$ such that $f_{x-y}(1-t_0) < \epsilon$. Taking $z = (2-t_0)y - (1-t_0)x$, we have $d(y,z) = f_{x-y}(1-t_0) < \epsilon$. This shows that $z \in B(y,\epsilon)$. Also, since f_{x-y} is strictly increasing on the set of all nonnegative real numbers, we have $d(x,z) = f_{x-y}(2-t_0) > f_{x-y}(1) = d(x,y)$. We have shown that there is a point $z \in B(y,\epsilon)$ such that $d(x,z) > d(x,y)$. Consequently $y \notin B^\circ[x,r]$. Hence, $B^\circ[x,r] = B(x,r)$.

Conversely, suppose that the hypotheses hold. Suppose that $x, y \in X$ such that $d(x,0) \le r$ and $d(y,0) \le r$, $r > 0$, that is, $x, y \in B[0,r]$, then by strict convexity of $B[0, r]$, we have $(x+y)/2 \in B^\circ[0,r] = B(0,r)$. Therefore, $d((x+y)/2, 0) < r$ and hence (X, d) is strictly convex. □

We have another interesting characterization of Assertion 2 as follows:

Theorem 2.7 *Let (X, d) be a metric space. Then Assertion 2 holds in X if and only if for any $x \in X$ and $r > 0$, each $y \in B[x,r]$ satisfying $d(x,y) = r$ is a limit point of the set $X - B[x,r]$.*

Proof Let Assertion 2 holds, that is, $B^\circ[x,r] = B(x,r)$. Since $d(x,y) = r$, it follows that $y \notin B^\circ[x,r]$. So, for every open set U containing y, $U \cap (X - B[x,r]) \neq \emptyset$. Since $y \notin (X - B[x,r])$, it follows that y is a limit point of the set $X - B[x,r]$.

Conversely, suppose that the hypothesis holds. Assume the contrary, that is, $B^\circ[x,r] \neq B(x,r)$. Then there exists $y \in B^\circ[x,r] - B(x,r)$ satisfying $d(x,y) = r$. By the hypothesis, we have $X - B[x, r] \neq \emptyset$. Since $B^{\circ}[x, r]$ is an open set containing y and is disjoint from $X - B[x, r]$, y can't be a limit point of $X - B[x, r]$, a contradiction to the hypothesis. Hence $B^{\circ}[x, r] = B(x, r)$.

Remark 2.2 In view of Theorems 2.1 and 2.6, we observe that Assertions 1 and 2 hold in strictly convex linear metric spaces. This motivates us to investigate further the relationships between Assertion 1 and Assertion 2. It appears that perhaps the following is true.

$$\bar{B}(x, r) = B[x, r] \Leftrightarrow B^{\circ}[x, r] = B(x, r). \tag{2}$$

However, following examples show that Assertion 1 and Assertion 2 are independent of each other.

Example Let d be the usual metric of real line defined by $d(x, y) = |x - y|$.

(1) Assertion 1 as well as Assertion 2 hold in the metric space ((0, 1), d).
(2) Assertion 1 holds in the metric space ([0, 1], d). However, Assertion 2 fails to hold here since $B^{\circ}[0, 1] = [0, 1] \neq [0, 1) = B(0, 1)$.
(3) Assertion 2 holds in the metric space (X, d'), where $X = \{0, 1\} \times \mathbb{R}$ and d' is the Euclidean metric of $\mathbb{R}^2$. Here, Assertion 1 fails to hold, since $barB(0 \times 0, 1) = \{0\} \times [-1, 1]; B[0 \times 0, 1] = (\{0\} \times [-1, 1]) \cup \{1 \times 0\}$.
(4) Both Assertion 1 as well as Assertion 2 fail to hold in the metric space (Y, d), where $Y = \{\frac{1}{n} | n \in \mathbb{N}\} \cup \{0\}$. Here $B^{\circ}[0, 1] = Y - \{1\} = B(0, 1)$ but $\bar{B}(0, 1) = Y - \{1\} \neq Y = B[0, 1]$, which shows that Assertion 1 fails to hold. Also, $\bar{B}(1, 1) = Y = B[1, 1]$ but $B^{\circ}[1, 1] = Y \neq Y - \{0\} = B(1, 1)$ which shows that Assertion 2 fails to hold.

Under certain conditions, (2) can be procured from the following:

Theorem 2.8 *Let* (X, d) *be a metric space. For* $r > 0$ *and* $x \in X$, *any two of the following statements imply the remaining one*:

i. $\partial B(x, r) = \partial B[x, r]$
ii. $\bar{B}(x, r) = B[x, r]$
iii. $B^{\circ}[x, r])$=$B(x, r)$

Proof We first observe the following facts:

$$B[x, r] = B^{\circ}[x, r] \cup \partial B[x, r]; \bar{B}(x, r) = B(x, r) \cup \partial B(x, r). \tag{3}$$

(i) and (ii) $\Rightarrow$ (iii). Since for any subset A of a metric space, $\bar{A} = A^{\circ} \cup \partial A$ and $\partial A = \bar{A} - A^{\circ}$, we must have $A^{\circ} \cap \partial A = \emptyset$. Using this fact along with (i) and (ii) in (3) we have

$$B^\circ[x,r] = B[x,r] - \partial B[x,r] = \bar{B}(x,r) - \partial B(x,r) = B(x,r).$$

Thus (iii) holds.

(ii) and (iii) $\Rightarrow$ (i). As before, we use (ii) and (iii) in (3) to get

$$\partial B(x,r) = \bar{B}(x,r) - B(x,r) = B[x,r] - B(x,r) = B[x,r] - B^\circ[x,r] = \partial B[x,r]$$

so that (i) holds.

(iii) and (i) $\Rightarrow$ (ii). Using (iii) and (i) in (3) we have

$$\bar{B}(x,r) = B(x,r) \cup \partial B(x,r) = B^\circ[x,r] \cup \partial B[x,r] = B[x,r]$$

which proves (ii).

Remarks In view of Theorem 2.8 we obtain the following:

(a) (ii) and (iii) are equivalent if and only if (i) is true.
(b) (iii) and (i) are equivalent if and only if (ii) is true.
(c) (i) and (ii) are equivalent if and only if (iii) is true.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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