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ORIGINAL RESEARCH PAPER

Unitarily invariant norm inequalities for matrix means

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Abstract
The main target of this article is to present several unitarily invariant norm inequalities which are refinements of arithmetic-geometric mean, Heinz and Cauchy-Schwartz inequalities by convexity of some special functions.

Keywords Unitarily invariant norm inequality · Young inequality · Heinz inequality · Cauchy-Schwartz inequality

Mathematics Subject Classification Primary 39B82 · Secondary 44B20 · 46C05

1 Introduction

In this sequel, we use the standard notation M_n, M_n^+ and M_n^{++} for the algebra of all $n \times n$ complex matrices, the cone of positive (or positive semidefinite) matrix and that of strictly positive matrices in M_n, respectively. Matrices and their inequalities have attracted researchers working in functional analysis. These inequalities have

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been studied in different approaches among which unitarily invariant norms inequalities are most popular. Recall that a unitarily invariant norm is a norm $\|\cdot\|$ defined on M_n satisfying the property $\|UAV\| = \|A\|$ for all $A \in M_n$ and unitaries $U, V \in M_n$. The absolute value of a matrix $A = (a_{ij})$ is defined by $|A| = (A^*A)^{1/2}$. The motivation behind this work starts with some crucial inequalities which will be presented as follows.

The classical arithmetic-geometric mean inequality [1] states that for $A, B \in M_n^+$ and $X \in M_n$,

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \frac{1}{2}\|AX + XB\|. \tag{1}$$

Heinz inequality [1] is a refinement of inequality (1) which states that

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leq \left\|\frac{A^tXB^{1-t} + A^{1-t}XB^t}{2}\right\| \leq \left\|\frac{AX + XB}{2}\right\| \tag{2}$$

hold for $A, B \in M_n^+, X \in M_n$ and $0 \leq t \leq 1$.

A general form of Cauchy-Schwartz inequality [2] states that for $A, B \in M_n^+, X \in M_n$ and $r > 0$,

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \leq \||AX|^r\|\||XB|^r\|. \tag{3}$$

We remark that the above inequalities have been studied deeply in the literature. We refer the reader to [3–5] as samples of recent work treating such inequalities and their variants.

Motivated by Bhatia and Bourin [2, 6], here we define two functions f and h for a given unitarily invariant norm $\|\cdot\|$,

$$f(t) = \|A^tXB^{1-t}\|\|A^{1-t}XB^t\| \quad \text{and} \quad h(t) = \left\|\frac{A^tXB^{1-t} + A^{1-t}XB^t}{2}\right\|^2,$$

where $A, B \in M_n^+$ and $X \in M_n$. The above functions f and h are convex on [0,1] and attain their minimum at $t = \frac{1}{2}$. In this article, we utilize convexity of these functions to obtain refinements of arithmetic-geometric mean, Heinz and Cauchy-Schwartz inequalities. The following convex function inequalities are also essential to our results.

Hermite-Hadaward inequality [7] states that for every real-valued convex function g on the interval $[a, b]$, we have

$$g\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b g(t)dt \leq \frac{g(a) + g(b)}{2}.$$

In 2010, EL Farissi [8] refined Hermite-Hadaward inequality as follows

$$g\left(\frac{a+b}{2}\right) \leq l(\lambda) \leq \frac{1}{b-a}\int_a^b g(t)dt \leq L(\lambda) \leq \frac{g(a) + g(b)}{2}$$

for all $\lambda \in [0, 1]$, where

$$l(\lambda) = \lambda g\left(\frac{\lambda b + (2-\lambda)a}{2}\right) + (1-\lambda) g\left(\frac{(1+\lambda)b + (1-\lambda)a}{2}\right)$$

and

$$L(\lambda) = \frac{1}{2}(g(\lambda b + (1-\lambda)a) + \lambda g(a) + (1-\lambda)g(b)).$$

A few years later, Abbas and Mourad [9] got that

$$g\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b g(t)dt \le \frac{1}{4n}\left[(2n-1)g(a) + 2g\left(\frac{a+b}{2}\right) + (2n-1)g(b)\right] \le \frac{g(a)+g(b)}{2}.$$

The following lemma combining Farissi and Abbas' results will be essential for our main results. The main results in this paper, Theorems 1, 2 and 3, are obtained by applying some refinements of Hermite-Hadaward inequalities on the convex functions f and h using the same method from Kittaneh [10].

Lemma 1 *Let g be a real-valued convex function which is convex on the interval $[a,b]$. Then for any positive integer n, we have*

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &\le \frac{1}{n} g\left(\frac{b+(2n-1)a}{2n}\right) + \left(1-\frac{1}{n}\right) g\left(\frac{(n+1)b+(n-1)a}{2n}\right) \\ &\le \frac{1}{b-a}\int_a^b g(t)dt \\ &\le \frac{1}{4n}\left[(2n-1)g(a) + 2g\left(\frac{a+b}{2}\right) + (2n-1)g(b)\right] \\ &\le \frac{g(a)+g(b)}{2}. \end{aligned}$$

Recently, Chen, Chen and Gao [11] obtained the following refinements of Hermite-Hadaward inequality.

Lemma 2 *Let $m, n : [a, b] \to [0, +\infty)$ be convex functions and meet $[m(a) - m(b)] \cdot [n(a) - n(b)] \le 0$. Then for all $\lambda \in [0, 1]$, we have*

$$\frac{1}{b-a}\int_a^b m(t)n(t)dt \le L'(\lambda) \le \frac{1}{3}M(a,b) + \frac{1}{6}N(a,b)$$

and

$$2m\left(\frac{a+b}{2}\right) n\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a,b) - \frac{1}{3}N(a,b) \le l'(\lambda) \le \frac{1}{b-a}\int_a^b m(t)n(t)dt$$

where

$$M(a,b) = m(a)n(a) + m(b)n(b), N(a,b) = m(a)n(b) + m(b)n(a),$$
$$L'(\lambda) = \frac{\lambda}{3}m(a)n(a) + \frac{1-\lambda}{3}m(b)n(b) + \frac{\lambda}{3}m(\lambda b + (1-\lambda)a)n(\lambda b + (1-\lambda)a) + \frac{\lambda}{6}m(\lambda b + (1-\lambda)a)[\lambda n(a) + (1-\lambda)n(b)] + \frac{\lambda}{6}n(\lambda b + (1-\lambda)a)[\lambda n(b) + (1-\lambda)n(a)]$$

and

$$l'(\lambda) = 2\lambda m\left(\frac{(2-\lambda)a + \lambda b}{2}\right)n\left(\frac{(2-\lambda)a + \lambda b}{2}\right) - \frac{1+3\lambda-3\lambda^2}{6}M(a,b) - \frac{2-3\lambda+3\lambda^2}{6}N(a,b) + 2(1-\lambda)m\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right)n\left(\frac{(1-\lambda)a + (1+\lambda)b}{2}\right).$$

The organization of this article will be as follows. In the following, we mainly present some unitarily invariant norm inequalities for matrix means which are refinements of arithmetic-geometric mean, Heinz and Cauchy-Schwartz inequalities utilizing Lemmas 1 and 2.

2 Unitarily invariant norm inequalities

Now we are in a position to begin our main results.

Applying Lemma 1 to the convex function $h(t)$ on the interval $[\mu, 1-\mu]$ when $0 \le \mu < \frac{1}{2}$ and on the interval $[1-\mu, \mu]$ when $\frac{1}{2} < \mu \le 1$, we obtain the following refinement of arithmetic-geometric mean and Heinz inequalities.

Theorem 1 *If $A, B \in M_n^+$, $X \in M_n$ and $0 \le \mu \le 1$, then for unitarily invariant norm $\|\cdot\|$,*

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 \le \frac{1}{n}\left\|\frac{A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}} + A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}}{2}\right\|^2$$
$$\le \frac{1}{|2\mu-1|}\left|\int_{\mu}^{1-\mu}\left\|\frac{A^tXB^{1-t} + A^{1-t}XB^t}{2}\right\|^2 dt\right| \quad \left(\mu \ne \frac{1}{2}\right)$$
$$\le \frac{1}{2n}\left[(2n-1)\left\|\frac{A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}}{2}\right\|^2 + \|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]$$
$$\le \left\|\frac{A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}}{2}\right\|^2$$

hold for any positive integer n.

Proof Assume that $A, B \in M_n^+$, $X \in M_n$ and $0 \le \mu < \frac{1}{2}$, then it follows by Lemma 1 that

$$
\begin{aligned}
h\left(\frac{\mu+1-\mu}{2}\right) &\le \frac{1}{n}h\left(\frac{1-\mu+(2n-1)\mu}{2n}\right)+\left(1-\frac{1}{n}\right)h\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)\\
&\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} h(t)dt \quad \left(\mu\neq\frac{1}{2}\right)\\
&\le \frac{1}{4n}\left[(2n-1)h(\mu)+2h\left(\frac{\mu+1-\mu}{2}\right)+(2n-1)h(1-\mu)\right]\\
&\le \frac{h(\mu)+h(1-\mu)}{2},
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
h\left(\frac{1}{2}\right) &\le \frac{1}{n}h\left(\frac{1-\mu+(2n-1)\mu}{2n}\right)+\left(1-\frac{1}{n}\right)h\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)\\
&\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} h(t)dt \quad \left(\mu\neq\frac{1}{2}\right)\\
&\le \frac{1}{4n}\left[(2n-1)h(\mu)+2h\left(\frac{1}{2}\right)+(2n-1)h(1-\mu)\right]\\
&\le \frac{h(\mu)+h(1-\mu)}{2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 &\le \frac{1}{n}\left\|\frac{A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}+A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}}{2}\right\|^2\\
&\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu}\left\|\frac{A^tXB^{1-t}+A^{1-t}XB^t}{2}\right\|^2 dt \quad \left(\mu\neq\frac{1}{2}\right)\\
&\le \frac{1}{2n}\left[(2n-1)\left\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\right\|^2+\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]\\
&\le \left\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\right\|^2.
\end{aligned}
\tag{4}
$$

On the other hand, if $\frac{1}{2}<\mu\le 1$, then it follows by symmetry (i.e., by applying the above inequality (4) to $1-\mu$) that

$$\begin{aligned}\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 &\le \frac{1}{n}\|\frac{A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}+A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}}{2}\|^2\\ &\le \frac{1}{2\mu-1}\int_{1-\mu}^{\mu}\|\frac{A^tXB^{1-t}+A^{1-t}XB^t}{2}\|^2dt \quad \left(\mu\neq\frac{1}{2}\right)\\ &\le \frac{1}{2n}\left[(2n-1)\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2+\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]\\ &\le \|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2.\end{aligned} \tag{5}$$

We complete the proof of Theorem 1 by combining the inequalities (4) and (5). □

Following the same logic of Theorem 1 and applying Lemma 1 to the function $f(t)$ on the interval $[\mu, 1-\mu]$ when $0\le\mu<\frac{1}{2}$ and on the interval $[1-\mu,\mu]$ when $\frac{1}{2}<\mu\le 1$, we have the following refinement of Cauchy-Schwartz inequality.

Theorem 2 *If $A,B\in M_n^+$, $X\in M_n$ and $0\le\mu\le 1$, then for unitarily invariant norm $\|\cdot\|$,*

$$\begin{aligned}\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 &\le \frac{1}{n}\|A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}\|\|A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}\|\\ &\le \frac{1}{|1-2\mu|}\left|\int_{\mu}^{1-\mu}\|A^tXB^{1-t}\|\|A^{1-t}XB^t\|dt\right| \quad \left(\mu\neq\frac{1}{2}\right)\\ &\le \frac{1}{2n}\left[(2n-1)\|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|+\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]\\ &\le \|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|\end{aligned}$$

hold for any positive integer n.

Proof Assume that $A,B\in M_n^+$, $X\in M_n$ and $0\le\mu<\frac{1}{2}$, then it follows by Lemma 1 that

$$\begin{aligned}f\left(\frac{\mu+1-\mu}{2}\right) &\le \frac{1}{n}f\left(\frac{1-\mu+(2n-1)\mu}{2n}\right)+\left(1-\frac{1}{n}\right)f\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)\\ &\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu}f(t)dt \quad \left(\mu\neq\frac{1}{2}\right)\\ &\le \frac{1}{4n}\left[(2n-1)f(\mu)+2f\left(\frac{\mu+1-\mu}{2}\right)+(2n-1)f(1-\mu)\right]\\ &\le \frac{f(\mu)+f(1-\mu)}{2},\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
f\left(\frac{1}{2}\right) &\le \frac{1}{n}f\left(\frac{1-\mu+(2n-1)\mu}{2n}\right)+\left(1-\frac{1}{n}\right)f\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)\\
&\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} f(t)dt \quad \left(\mu\ne\frac{1}{2}\right)\\
&\le \frac{1}{4n}\left[(2n-1)f(\mu)+2f\left(\frac{1}{2}\right)+(2n-1)f(1-\mu)\right]\\
&\le \frac{f(\mu)+f(1-\mu)}{2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 &\le \frac{1}{n}\|A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}\|\|A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}\|\\
&\le \frac{1}{1-2\mu}\int_{\mu}^{1-\mu}\|A^tXB^{1-t}\|\|A^{1-t}XB^t\|dt \quad \left(\mu\ne\frac{1}{2}\right) \qquad (6)\\
&\le \frac{1}{2n}\left[(2n-1)\|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|+\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]\\
&\le \|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|.
\end{aligned}
$$

On the other hand, if $\frac{1}{2}<\mu\le 1$, then it follows by symmetry (i.e., by applying the above inequality (6) to $1-\mu$) that

$$
\begin{aligned}
\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2 &\le \frac{1}{n}\|A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}\|\|A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}\|\\
&\le \frac{1}{2\mu-1}\int_{1-\mu}^{\mu}\|A^tXB^{1-t}\|\|A^{1-t}XB^t\|dt \quad \left(\mu\ne\frac{1}{2}\right) \qquad (7)\\
&\le \frac{1}{2n}\left[(2n-1)\|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|+\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^2\right]\\
&\le \|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|.
\end{aligned}
$$

We complete the proof of Theorem 2 by combining the inequalities (6) and (7). □

Next, for every positive real number r, we consider the function

$$\phi(t)=\||A^tXB^{1-t}|^r\|\cdot\||A^{1-t}XB^t|^r\|$$

which is convex on [0,1] and attains its minimum at $t=\frac{1}{2}$ obtained by Hiai and Zhan [12].

Applying Lemma 1 to the function $\phi(t)$ on the interval $[\mu,1-\mu]$ when $0\le\mu<\frac{1}{2}$ and on the interval $[1-\mu,\mu]$ when $\frac{1}{2}<\mu\le 1$, then we have the following refinement of general Cauchy-Schwartz inequality.

Theorem 3 *If* $A, B \in M_n^+$, $X \in M_n$ *and* $0 \le \mu \le 1$, *then for unitarily invariant norm* $\|\cdot\|$,

$$\||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \le \frac{1}{n}\||A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}|^r\| \cdot \||A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}|^r\|$$

$$\le \frac{1}{|1-2\mu|}\left|\int_\mu^{1-\mu} \||A^tXB^{1-t}|^r\| \cdot \||A^{1-t}XB^t|^r\|dt\right| \quad \left(\mu \ne \frac{1}{2}\right)$$

$$\le \frac{1}{2n}\left[(2n-1)\||A^\mu XB^{1-\mu}|^r\| \cdot \||A^{1-\mu}XB^\mu|^r\| + \||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2\right]$$

$$\le \||A^\mu XB^{1-\mu}|^r\| \cdot \||A^{1-\mu}XB^\mu|^r\|$$

hold for any positive integer n.

Proof Assume that $A, B \in M_n^+$, $X \in M_n$ and $0 \le \mu < \frac{1}{2}$, then it follows by Lemma 1 that

$$\phi\left(\frac{\mu+1-\mu}{2}\right) \le \frac{1}{n}\phi\left(\frac{1-\mu+(2n-1)\mu}{2n}\right) + \left(1-\frac{1}{n}\right)\phi\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)$$

$$\le \frac{1}{1-2\mu}\int_\mu^{1-\mu}\phi(t)dt \quad \left(\mu \ne \frac{1}{2}\right)$$

$$\le \frac{1}{4n}\left[(2n-1)\phi(\mu) + 2\phi\left(\frac{\mu+1-\mu}{2}\right) + (2n-1)\phi(1-\mu)\right]$$

$$\le \frac{\phi(\mu)+\phi(1-\mu)}{2},$$

which is equivalent to

$$\phi\left(\frac{1}{2}\right) \le \frac{1}{n}\phi\left(\frac{1-\mu+(2n-1)\mu}{2n}\right) + \left(1-\frac{1}{n}\right)\phi\left(\frac{(n+1)(1-\mu)+(n-1)\mu}{2n}\right)$$

$$\le \frac{1}{1-2\mu}\int_\mu^{1-\mu}\phi(t)dt \quad \left(\mu \ne \frac{1}{2}\right)$$

$$\le \frac{1}{4n}\left[(2n-1)\phi(\mu) + 2\phi(\frac{1}{2}) + (2n-1)\phi(1-\mu)\right]$$

$$\le \frac{\phi(\mu)+\phi(1-\mu)}{2}.$$

Hence,

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \leq \frac{1}{n}|||A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}|^r\| \cdot |||A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}|^r\|$$

$$\begin{aligned} &\leq \frac{1}{1-2\mu}\int_{\mu}^{1-\mu} |||A^tXB^{1-t}|^r\| \cdot |||A^{1-t}XB^t|^r\|dt \quad \left(\mu \neq \frac{1}{2}\right) \quad (8)\\ &\leq \frac{1}{2n}\left[(2n-1)|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\| + |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2\right]\\ &\leq |||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|. \end{aligned}$$

On the other hand, if $\frac{1}{2} < \mu \leq 1$, then it follows by symmetry (i.e., by applying the above inequality (8) to $1-\mu$) that

$$|||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2 \leq \frac{1}{n}|||A^{\frac{1+(2n-2)\mu}{2n}}XB^{1-\frac{1+(2n-2)\mu}{2n}}|^r\| \cdot |||A^{1-\frac{1+(2n-2)\mu}{2n}}XB^{\frac{1+(2n-2)\mu}{2n}}|^r\|$$

$$\begin{aligned} &\leq \frac{1}{2\mu-1}\int_{1-\mu}^{\mu} |||A^tXB^{1-t}|^r\| \cdot |||A^{1-t}XB^t|^r\|dt \quad \left(\mu \neq \frac{1}{2}\right) \quad (9)\\ &\leq \frac{1}{2n}\left[(2n-1)|||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\| + |||A^{\frac{1}{2}}XB^{\frac{1}{2}}|^r\|^2\right]\\ &\leq |||A^{\mu}XB^{1-\mu}|^r\| \cdot |||A^{1-\mu}XB^{\mu}|^r\|. \end{aligned}$$

We complete the proof of Theorem 3 by combining the inequalities (8) and (9). □

In view of the fact that the functions $f(t)$ and $h(t)$ are symmetric, we have

$$\begin{aligned} &|f(\mu) - f(1-\mu)| \cdot |h(\mu) - h(1-\mu)|\\ &= |||A^{\mu}XB^{1-\mu}|||A^{1-\mu}XB^{\mu}\| - \|A^{1-\mu}XB^{\mu}|||A^{\mu}XB^{1-\mu}|||\cdot\\ &= 0. \end{aligned}$$

We can have the following result by applying Lemma 2 to function

$$f(t) \cdot h(t) = \|A^tXB^{1-t}\|\|A^{1-t}XB^t\|\|\frac{A^{\mu}XB^{1-\mu} + A^{1-\mu}XB^{\mu}}{2}\|^2.$$

Corollary 1 *For* $0 \leq \mu \leq 1$ *and all* $\lambda \in [0,1]$, *we have*

$$
\begin{aligned}
&2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|^4-\|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2\\
&\le l'(\lambda)\\
&\le \frac{1}{|1-2\mu|}|\int_{\mu}^{1-\mu}\|A^{t}XB^{1-t}\|\|A^{1-t}XB^{t}\|\|\frac{A^{t}XB^{1-t}+A^{1-t}XB^{t}}{2}\|^2dt| \quad \left(\mu\neq\frac{1}{2}\right)\\
&\le L'(\lambda)\\
&\le \|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2,
\end{aligned}
$$

where

$$
\begin{aligned}
L'(\lambda)=&\frac{1}{3}\|A^{1-\mu}XB^{\mu}\|\|A^{\mu}XB^{1-\mu}\|\|\frac{A^{1-\mu}XB^{\mu}+A^{\mu}XB^{1-\mu}}{2}\|^2\\
&+\frac{\lambda}{3}\|A^{\lambda+\mu-2\lambda\mu}XB^{1-(\lambda+\mu-2\lambda\mu)}\|\|A^{1-(\lambda+\mu-2\lambda\mu)}XB^{\lambda+\mu-2\lambda\mu}\|\cdot\\
&\|\frac{A^{\lambda+\mu-2\lambda\mu}XB^{1-(\lambda+\mu-2\lambda\mu)}+A^{1-(\lambda+\mu-2\lambda\mu)}XB^{\lambda+\mu-2\lambda\mu}}{2}\|^2\\
&+\frac{\lambda}{6}\|A^{\lambda+\mu-2\lambda\mu}XB^{1-(\lambda+\mu-2\lambda\mu)}\|\|A^{1-(\lambda+\mu-2\lambda\mu)}XB^{\lambda+\mu-2\lambda\mu}\|\cdot\\
&\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2\\
&+\frac{\lambda}{6}\|\frac{A^{\lambda+\mu-2\lambda\mu}XB^{1-(\lambda+\mu-2\lambda\mu)}+A^{1-(\lambda+\mu-2\lambda\mu)}XB^{\lambda+\mu-2\lambda\mu}}{2}\|^2\cdot\\
&\|\frac{A^{1-\mu}XB^{\mu}+A^{\mu}XB^{1-\mu}}{2}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
l'(\lambda)=&2\lambda\|A^{\frac{\lambda+2\mu-2\lambda\mu}{2}}XB^{1-\frac{\lambda+2\mu-2\lambda\mu}{2}}\|\|A^{1-\frac{\lambda+2\mu-2\lambda\mu}{2}}XB^{\frac{\lambda+2\mu-2\lambda\mu}{2}}\|\cdot\\
&\|\frac{A^{\frac{\lambda+2\mu-2\lambda\mu}{2}}XB^{1-\frac{\lambda+2\mu-2\lambda\mu}{2}}+A^{1-\frac{\lambda+2\mu-2\lambda\mu}{2}}XB^{\frac{\lambda+2\mu-2\lambda\mu}{2}}}{2}\|^2\\
&-\|A^{\mu}XB^{1-\mu}\|\|A^{1-\mu}XB^{\mu}\|\|\frac{A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}}{2}\|^2\\
&+2(1-\lambda)\|A^{\frac{\lambda+1-2\lambda\mu}{2}}XB^{1-\frac{\lambda+1-2\lambda\mu}{2}}\|\|A^{1-\frac{\lambda+1-2\lambda\mu}{2}}XB^{\frac{\lambda+1-2\lambda\mu}{2}}\|\cdot\\
&\|\frac{A^{\frac{\lambda+1-2\lambda\mu}{2}}XB^{1-\frac{\lambda+1-2\lambda\mu}{2}}+A^{1-\frac{\lambda+1-2\lambda\mu}{2}}XB^{\frac{\lambda+1-2\lambda\mu}{2}}}{2}\|^2.
\end{aligned}
$$

Here we remark that $|f(\mu)-f(1-\mu)|\cdot|\phi(\mu)-\phi(1-\mu)|=0$ and $|h(\mu)-h(1-\mu)|\cdot|\phi(\mu)-\phi(1-\mu)|=0$ for $0\le\mu\le1$. Hence, results similar to Corollary 1 can be obtained by using $f\cdot\phi$ and $h\cdot\phi$.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Ethical approval This article does not contain any studies with animals performed by any of the authors.

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