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ORIGINAL RESEARCH PAPER

Normal family of meromorphic functions concerning limited the numbers of zeros

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Abstract
Let $k, n \in \mathbb{N}, l \in \mathbb{N}\backslash\{1\}, m \in \mathbb{N} \cup \{0\}$, and let $a(z)(\not\equiv 0)$ be a holomorphic function, all zeros of $a(z)$ have multiplicities at most m. Let $\mathcal{F}$ be a family of meromorphic functions in D. If for each $f \in \mathcal{F}$, the zeros of f have multiplicity at least $k+m$, and for $f \in \mathcal{F}, f^l(f^{(k)})^n - a(z)$ has at most one zero in D, then $\mathcal{F}$ is normal in D.

Keywords Meromorphic function · Normal families · Zero numbers

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1 Introduction and main results

Let f be a meromorphic function in $\mathbb{C}$ and we shall use the usual notations and classical results of Nevanlinna's theory, such as $m(r,f), N(r,f), \overline{N}(r,f), T(r,f), \ldots$.

Let D be a domain in $\mathbb{C}$ and $\mathcal{F}$ be a family of meromorphic functions in D. A family $\mathcal{F}$ is said to be normal in D, in the sense of Montel, if each sequence f_n has a subsequence f_{n_k} that converges spherically locally uniformly in D to a meromorphic function or to the constant ∞.

The following well-known normal conjecture was proposed by Hayman in 1967.

Theorem A [1] *Let $n \in \mathbb{N}$, and $a \in \mathbb{C}\backslash\{0\}$. let $\mathcal{F}$ be a family of meromorphic function in D. If $f^n f' \neq a$, for each $f \in \mathcal{F}$, then $\mathcal{F}$ is normal in D.*

This normal conjecture was showed by Yang and Zhang [2] *(for $n \geq 5$), Gu* [3] *(for $n = 4, 3$), Pang* [4] *(for $n \geq 2$) and Chen and Fang* [5] *(for $n = 1$).*

For the related results, see Zhang [6], *Meng and Hu* [7], *Deng et al.*[8].

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Ding et al. [9] *studied the general case of* $f^l(f^{(k)})^n$ *and and proved the following theorem.*

Theorem B *Let* $k, l \in \mathbb{N}, n \in \mathbb{N}\backslash\{1\}, a \in \mathbb{C}\backslash\{0\}$. *Let* $\mathcal{F}$ *be a family of meromorphic functions in D. If for each* $f \in \mathcal{F}$, *the zeros of f have multiplicity at least* $max\{k, 2\}$, *and for* $f, g \in \mathcal{F}, f^l(f^{(k)})^n$ *and* $g^l(g^{(k)})^n$ *share a, then* $\mathcal{F}$ *is normal in D.*

Recently, Meng et al. [10] *considered the case of sharing a holomorphic function and and proved the following result.*

Theorem C *Let* $k, l \in \mathbb{N}, n \in \mathbb{N}\backslash\{1\}, m \in \mathbb{N} \cup \{0\}$, *and let* $a(z)(\not\equiv 0)$ *be a holomorphic function, all zeros of* $a(z)$ *have multiplicities at most m, which is divisible by* $n + l$. *Let* $\mathcal{F}$ *be a family of meromorphic functions in D. If for each* $f \in \mathcal{F}$, *the zeros of f have multiplicities at least* $k + m + 1$ *and all poles of f are of multiplicity at least* $m + 1$, *and for* $f, g \in \mathcal{F}, f^l(f^{(k)})^n$ *and* $g^l(g^{(k)})^n$ *share* $a(z)$, *then* $\mathcal{F}$ *is normal in D.*

By Theorem C, the following question arises naturally:

Question 1.1 Is it possible to omit the conditions: (1)“ m is divisible by $n + l$” and (2)“all poles of f have multiplicity at least $m + 1$” ?

In this paper, we study this problem and obtain the following result.

Theorem 1.1 *Let* $k, n \in \mathbb{N}, l \in \mathbb{N}\backslash\{1\}, m \in \mathbb{N} \cup \{0\}$, *and let* $a(z)(\not\equiv 0)$ *be a holomorphic function, all zeros of* $a(z)$ *have multiplicities at most m. Let* $\mathcal{F}$ *be a family of meromorphic functions in D. If for each* $f \in \mathcal{F}$, *the zeros of f have multiplicity at least* $k + m$, *and for* $f \in \mathcal{F}, f^l(f^{(k)})^n - a(z)$ *has at most one zero in D, then* $\mathcal{F}$ *is normal in D.*

Now we give some examples to show that the conditions in our results are necessary.

Example 1.1 Let $D = \{z : |z| < 1\}$ and $a(z) \equiv 0$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = e^{jz}, z \in D, j = 1, 2. \ldots$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z) - a(z) \neq 0$ in D, however $\mathcal{F}$ is not normal at $z = 0$. This shows that $a(z) \not\equiv 0$ is necessary in Theorem 1.1.

Example 1.2 Let $D = \{z : |z| < 1\}$ and $a(z) = \frac{1}{z^{l+kn+n}}$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = \frac{1}{jz}, z \in D, j = 1, 2 \cdots, j^{l+n} \neq [(-1)^k k!]^n.$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z) - a(z) \neq 0$ in D, however $\mathcal{F}$ is not normal at $z = 0$. This shows that Theorem 1.1 is not valid if $a(z)$ is a meromorphic function in D.

Example 1.3 Let $D = \{z : |z| < 1\}$, $a(z) = a$. Let $\mathcal{F} = \{f_j(z)\}$, where

$$f_j(z) = jz^{k-1}, z \in D, j = 1, 2 \cdots.$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z)-a$, which has no zero in D , however $\mathcal{F}$ is not normal at $z=0$. This shows that the condition " all zeros of f have multiplicity at least $k+m$ " in Theorem 1.1 is sharp.

Example 1.4 Let $D=\{z:|z|<1\}$, $a(z)=a$. Let $\mathcal{F}=\{f_j(z)\}$, where

$$f_j(z)=jz^k, z\in D, j=1,2.\ldots$$

Then $f_j^l(z)\left(f_j^{(k)}\right)^n(z)-a=j^{l+n}(k!)^n z^{lk}-a$, which has at least $l\geq 2$ distinct zeros in D, however $\mathcal{F}$ is not normal at $z=0$. This shows that the condition "$f^l(f^{(k)})^n-a(z)$ has at most one zero" in Theorem 1.1 is necessary.

2 Some lemmas

Lemma 2.1 [11] *Let $\mathcal{F}$ be a family of functions meromorphic in the unit disc Δ, all of whose zeros have multiplicity at least k. Then if $\mathcal{F}$ is not normal in any neighbourhood of $z_0\in\Delta$, there exist, for each α, $0\leq\alpha<k$,*

(i) *points z_n, $z_n\to z_0$, $z_0\in\Delta$;*
(ii) *functions $f_n\in\mathcal{F}$; and*
(iii) *positive numbers $\rho_n\to 0^+$, such that $g_n(\xi)=\rho_n^{-\alpha}f_n(z_n+\rho_n\xi)\to g(\xi)$ spherically uniformly on compact subsets of $\mathbb{C}$, where g is a non-constant meromorphic function, all of whose zeros have multiplicity at least k.*

Lemma 2.2 [12] *Let $k,n\in\mathbb{N}$, $l\in\mathbb{N}\backslash\{1\}$, $a\in\mathbb{C}\backslash\{0\}$, and let $f(z)$ be a non-constant meromorphic with all zeros that have multiplicity at least k. Then $f^l(z)(f^{(k)})^n(z)-a$ has at least two distinct zeros.*

Using the idea of Chang [13], *we get the following lemma.*

Lemma 2.3 *Let $k,l,n,m\in\mathbb{N}$, let $q(z)$ be a polynomial of degree m, and let $f(z)$ be a non-constant rational function with $f(z)\neq 0$. Then $f^l(z)(f^{(k)})^n(z)-q(z)$ has at least $l+kn+n$ distinct zeros.*

The proof of Lemma 2.3 is almost exactly the same with Lemma 11 *in Deng etc.* [14], *here, we omit the details.*

Lemma 2.4 [15] *Let $f_j(j=1,2)$ be two nonconstant meromorphic functions, then*

$$N(r,f_1f_2)-N\left(r,\frac{1}{f_1f_2}\right)=N(r,f_1)+N(r,f_2)-N\left(r,\frac{1}{f_1}\right)-N\left(r,\frac{1}{f_2}\right).$$

Lemma 2.5 *Let $k,m,n\in\mathbb{N}$, $l\in\mathbb{N}\backslash\{1\}$, let $q(z)$ be a polynomial of degree m, and let $f(z)$ be a non-constant meromorphic function in $\mathbb{C}$, the zeros of $f(z)$ have multiplicities at least $k+m$.Then $(f(z))^l(f^{(k)})^n(z)-q(z)$ has at least two distinct zeros.*

Proof Since

$$\frac{1}{f^{l+n}}=\left(\frac{f^{(k)}}{f}\right)^n\cdot\frac{1}{q}-\frac{f^l\left(f^{(k)}\right)^n-q}{qf^{l+n}}$$
$$=\frac{f^l\left(f^{(k)}\right)^n}{qf^{l+n}}-\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}}\cdot\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}.$$

Noticing that $m(r,\frac{f^{(k)}}{f})=S(r,f), m(r,\frac{1}{q})=O(1)$, and $m(r,q)=mlogr+O(1)$. By Nevanlinna's Fundamental Theorem, we get

$$\begin{aligned}
(l+n)m\left(r,\frac{1}{f}\right)&=m\left(r,\frac{1}{f^{l+n}}\right)\\
&=m\left(r,\frac{f^l\left(f^{(k)}\right)^n}{qf^{l+n}}-\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}}\cdot\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)\\
&\le m\left(r,\frac{1}{q}\right)+nm\left(r,\frac{f^{(k)}}{f}\right)+m\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}}\right)\\
&\quad+m\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+O(1)\\
&\le m\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{qf^{l+n}}\right)\\
&\quad+m\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+S(r,f)\\
&\le T\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)\\
&\quad-N\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+S(r,f)\\
&\le T\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n-q}\right)\\
&\quad-N\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+S(r,f)\\
&=m\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n-q}\right)+N\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n-q}\right)\\
&\quad-N\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+S(r,f)\\
&=m\left(r,q\cdot\frac{\left[\frac{f^l\left(f^{(k)}\right)^n-q}{q}\right]'}{\frac{f^l\left(f^{(k)}\right)^n-q}{q}}\right)+N\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n-q}\right)\\
&\quad-N\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)\\
&\quad+S(r,f).\\
&\le m\left(r,\frac{\left[\frac{f^l\left(f^{(k)}\right)^n-q}{q}\right]'}{\frac{f^l\left(f^{(k)}\right)^n-q}{q}}\right)+N\left(r,\frac{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}{f^l\left(f^{(k)}\right)^n-q}\right)\\
&\quad-N\left(r,\frac{f^l\left(f^{(k)}\right)^n-q}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)\\
&\quad+m(r,q)+S(r,f).
\end{aligned}\tag{2.1}$$

By Lemma 2.4 applied to (2.1), we can get

$$(l+n)m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{\left[\frac{f^l\left(f^{(k)}\right)^n-q}{q}\right]'}{\frac{f^l\left(f^{(k)}\right)^n-q}{q}}\right)+N\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)$$
$$-N\left(r,f^l\left(f^{(k)}\right)^n-q\right)+N\left(r,\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]\right)$$
$$-N\left(r,\frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+m\log r+S(r,f).$$

This is

$$(l+n)m\left(r,\frac{1}{f}\right)\le\overline{N}(r,f)+N\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)$$
$$-N\left(r,\frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+m\log r+S(r,f). \tag{2.2}$$

We add $(l+n)N\left(r,\frac{1}{f}\right)$ to both sides in (2.2), then

$$(l+n)T\left(r,\frac{1}{f}\right)\le(l+n)N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+N\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)$$
$$-N\left(r,\frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)+m\log r+S(r,f). \tag{2.3}$$

Let ξ be a zero of f with multiplicity $t(\ge k+m)$, then ξ is a zero of $[f^l(f^{(k)})^n]'q-[f^l(f^{(k)})^n]q'$ with multiplicity at least $(l+n)t-kn-1$. Noticing that

$$\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]=\left[f^l\left(f^{(k)}\right)^n-q\right]'q-q'\left[f^l\left(f^{(k)}\right)^n-q\right],$$

which implies

$$N\left(r,\frac{1}{\left[f^l\left(f^{(k)}\right)^n\right]'q-q'\left[f^l\left(f^{(k)}\right)^n\right]}\right)\ge N\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)$$
$$-\overline{N}\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right).$$

Therefore, from (2.3), we get

$$(l+n)T(r,f)\le(kn+1)\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)+m\log r+S(r,f)$$
$$\le\frac{kn+1}{k+m}N\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f^l\left(f^{(k)}\right)^n-q}\right)+m\log r+S(r,f).$$

i.e.,

$$MT(r,f) \le \overline{N}\left(r, \frac{1}{f^l (f^{(k)})^n - q}\right) + m \log r + S(r,f), \tag{2.4}$$

where

$$M = l + n - 1 - \frac{kn+1}{k+m} = l - 1 + \frac{mn-1}{k+m}.$$

Suppose that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at most one zero.

Next, we consider two cases.

Case 1: $n \ge 2$. By the assumptions,

$$M \ge 1 + \frac{1}{k+m}.$$

From (2.4), we get

$$T(r,f) < MT(r,f) \le (m+1)\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree $< m+1$. Since the zeros of $f(z)$ have multiplicities at least $k + m \ge m + 1$, then we get $f(z) \ne 0$. Thus, by Lemma 2.3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l + kn + n \ge 6$ distinct zeros, which is a contradiction.

Case 2: $n = 1$. Then $M = l - \frac{k+1}{k+m}$.

Subcase 2.1: $m \ge 2$. By the assumptions, $M > 1$ and from (2.4), we get

$$T(r,f) < (m+1)\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree $< m+1$. Since the zeros of $f(z)$ have multiplicities at least $k + m \ge m + 1$, then we get $f(z) \ne 0$. Thus, by Lemma 2.3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l + k + 1 \ge 4$ distinct zeros, which is a contradiction.

Subcase 2.2: $m = 1$. From (2.4), we get

$$(l-1)T(r,f) \le \overline{N}\left(r, \frac{1}{f^l f^{(k)} - q}\right) + \log r + S(r,f),$$

Subcase 2.2.1: $f^l(z)f^{(k)}(z) - q(z) \ne 0$. From (2.4), we get

$$T(r,f) \le (l-1)T(r,f) \le \log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 1. Since the zeros of $f(z)$ have multiplicities at least $k + 1 \ge 2$, then we get $f(z) \ne 0$. Thus, by Lemma 2.3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l + k + 1 \ge 4$ distinct zeros, which is a contradiction.

Subcase 2.2.2: $f^l(z)f^{(k)}(z) - q(z) = 0$. By the assumptions, we get $f^l(z)f^{(k)}(z) - q(z)$ has only one zero. Then, from (2.4), we obtain

$$(l-1)T(r,f) \leq 2\log r + S(r,f).$$

Subcase 2.2.2.1: $l \geq 3$, from (2.4), we obtain

$$T(r,f) \leq \log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 1. Since the zeros of $f(z)$ have multiplicities at least $k+1 \geq 2$, then we get $f(z) \neq 0$. Thus, by Lemma 2.3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 5$ distinct zeros, which is a contradiction.

Subcase 2.2.2.2: $l = 2$, from (2.4), we obtain

$$T(r,f) \leq 2\log r + S(r,f).$$

It follows that $f(z)$ is a rational function of degree ≤ 2.

Subcase 2.2.2.2.1: $k \geq 2$. Since the zeros of $f(z)$ have multiplicities at least $k+1 \geq 3$, then we get $f(z) \neq 0$. Thus, by Lemma 2.3, we obtain that $f^l(z)(f^{(k)})^n(z) - q(z)$ has at least $l+k+1 \geq 5$ distinct zeros, which is a contradiction.

Subcase 2.2.2.2.2: $k = 1$. Then we get $f(z) \neq 0$ or $f(z)$ has only one zero with multiplicity 2.

The former case can be ruled out from Lemma 2.3. Hence $f(z)$ has the following forms:

$$\text{(i) } f(z) = A(z-z_0)^2; \text{(ii) } f(z) = \frac{A(z-z_0)^2}{(z-z_1)};$$
$$\text{(iii) } f(z) = \frac{A(z-z_0)^2}{(z-z_1)^2}; \text{(iv) } f(z) = \frac{A(z-z_0)^2}{(z-z_1)(z-z_2)},$$

where A, z_0 are nonzero constants, and z_1, z_2 are distinct constants. Clearly, $z_0 \neq z_1, z_0 \neq z_2$, and $T(r,f) = 2\log r + O(1)$.

(i) $f(z) = A(z-z_0)^2$. Obviously, $\overline{N}\left(r, \frac{1}{f}\right) \leq \frac{1}{2}T(r,f) + O(1)$. From (2.4), we obtain

$$3T(r,f) \leq 2\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + 2\log r + S(r,f).$$

Then

$$T(r,f) \leq \log r + S(r,f),$$

a contradiction.

(ii) $f(z)=\frac{A(z-z_0)^2}{(z-z_1)}$. Then, $\overline{N}\left(r,\frac{1}{f}\right)\leq\frac{1}{2}T(r,f)+O(1),\overline{N}(r,f)=\log r$. From (2.4), we obtain

$$3T(r,f)\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+2\log r+S(r,f).$$

Then

$$T(r,f)\leq\frac{4}{3}\log r+S(r,f),$$

which is a contradiction.

(iii) $f(z)=\frac{A(z-z_0)^2}{(z-z_1)^2}$. Then, $\overline{N}\left(r,\frac{1}{f}\right)\leq\frac{1}{2}T(r,f)+O(1),\overline{N}(r,f)\leq\frac{1}{2}T(r,f)+O(1)$. From (2.4), we obtain

$$3T(r,f)\leq 2\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}(r,f)+2\log r+S(r,f).$$

Then

$$T(r,f)\leq\frac{7}{6}\log r+S(r,f),$$

we also get a contradiction.

(iv) $f(z)=\frac{A(z-z_0)^2}{(z-z_1)(z-z_2)}$. Then

$$f^2(z)f'(z)=\frac{A^3(z-z_0)^5[(2z_0-(z_1+z_2))z+2z_1z_2-z_0(z_1+z_2)]}{(z-z_1)^4(z-z_2)^4}. \tag{2.5}$$

Since $q(z)=Bz+C$, where $B\neq 0,C$ are constants, and $f^l(z)f^{(k)}(z)-q(z)$ has only one zero. Then we have

$$f^2(z)f'(z)=Bz+C+\frac{d(z-\zeta)^t}{(z-z_1)^4(z-z_2)^4}. \tag{2.6}$$

Obviously, By calculation, we get $d=-B,t=9$, and $\zeta\neq z_0$.

Differentiating (2.5)–(2.6) two times separately, we obtain

$$[f^2(z)f'(z)]''=\frac{(z-z_0)^3g(z)}{(z-z_1)^6(z-z_2)^6},$$

where $g(z)$ is a polynomial of degree ≤ 5, and

$$[f^2(z)f'(z)]''=\frac{(z-\zeta)^7h(z)}{(z-z_1)^6(z-z_2)^6},$$

where $h(z)$ is a polynomial of degree ≤ 4.

Since $z_0\neq\zeta$, then $(z-\zeta)^7$ is a factor of $g(z)$. Thus $g(z)$ is a polynomial of degree ≥ 7, which is impossible. □

Lemma 2.6 *Let $k, n \in \mathbb{N}, l \in \mathbb{N}\backslash\{1\}$, and let $\mathcal{F} = \{f_m\}$ be a sequence of meromorphic functions, $g_m(z)$ be a sequence of holomorphic functions in D such that $g_m(z) \longrightarrow g(z)$, where $g(z)(\neq 0)$ be a holomorphic function. If all zeros of function $f_m(z)$ have multiplicity at least k, and $f_m^l(z)(f_n^{(k)}(z))^n - g_n(z)$ has at most one zero, then $\mathcal{F}$ is normal in D.*

Proof Suppose that $\mathcal{F}$ is not normal at $z_0 \in D$. By Lemma 2.1, there exists $z_m \to z_0$, $\rho_m \to 0^+$, and $f_m \in \mathcal{F}$ such that

$$h_m(\xi) = \frac{f_m(z_m + \rho_m \xi)}{\rho_m^{\frac{kn}{l+n}}} \longrightarrow h(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $h(\xi)$ is a non-constant meromorphic function in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $h(\xi)$ have multiplicity at least k.

For each $\xi \in \mathbb{C}/\{h^{-1}(\infty)\}$, we have

$$\begin{aligned} h_m^l(\xi)(h_m^{(k)}(\xi))^n - g_m(z_m + \rho_m\xi) &= f_m^l(z_m + \rho_m\xi)(f_m^{(k)})^n(z_m + \rho_m\xi) \\ &- g_m(z_m + \rho_m\xi) \longrightarrow h^l(\xi)(h^{(k)})^n(\xi) - g(z_0). \end{aligned}$$

Obviously, $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0) \not\equiv 0$.

Suppose that $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0) \equiv 0$, then $h(\xi) \neq 0$ since $g(z_0) \neq 0$. It follows that

$$\frac{1}{h^{l+n}(\xi)} \equiv \frac{1}{g(z_0)} \left[\frac{h^{(k)}(\xi)}{h(\xi)}\right]^n.$$

Thus

$$(l+n)m\left(r, \frac{1}{h}\right) = m\left(r, \frac{1}{g(z_0)} \left[\frac{h^{(k)}(\xi)}{h(\xi)}\right]^n\right) = S(r, h).$$

Then $T(r, h) = S(r, h)$ since $h \neq 0$. we can deduce that $h(\xi)$ is a constant, a contradiction.

We claim that $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$ has at most one zero, Suppose this is not the case, and $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$ has two distinct zeros ξ_1, and ξ_2. We choose a positive number δ small enough such that $D_1 \cap D_2 = \emptyset$ and $h^l(\xi)(h^{(k)})^n(\xi) - g(z_0)$ has no other zeros in $D_1 \cup D_2$ except for ξ_1 and ξ_2, where $D_1 = \{\xi : |\xi - \xi_1| < \delta\}$ and $D_2 = \{\xi : |\xi - \xi_2| < \delta\}$.

By Hurwitz's theorem, for sufficiently large m, there exist points $\xi_{1,m} \to \xi_1$ and $\xi_{2,m} \to \xi_2$ such that

$$f_m^l(z_m + \rho_m\xi_{1,m})(f_m^{(k)})^n(z_m + \rho_m\xi_{1,m}) - g_m(z_m + \rho_m\xi_{1,m}) = 0,$$

and

$$f_m^l(z_m + \rho_m\xi_{2,m})(f_m^{(k)})^n(z_m + \rho_m\xi_{2,m}) - g_m(z_m + \rho_m\xi_{2,m}) = 0$$

Since $f_m^l(z)(f_m^{(k)}(z))^n - g_m(z)$ has at most one zero in D, then

$$z_m + \rho_m \xi_{1,m} = z_m + \rho_m \xi_{2,m},$$

this is

$$\xi_{1,m} = \xi_{2,m} = \frac{z_0 - z_m}{\rho_m},$$

which contradicts the fact $D_1 \cap D_2 = \emptyset$. The claim is proved.

From Lemma 2.2, we get $h^l(z)(h^{(k)})^n(z) - g(z_0)$ has at least two distinct zeros, a contradiction. Therefore $\mathcal{F}$ is normal in D. □

3 Proof of Theorem

Proof of Theorem 1.1 Suppose that $\mathcal{F}$ is not normal at z_0. From Lemma 2.6, we obtain $a(z_0) = 0$. Without loss of generality, we assume that $z_0 = 0$ and $a(z) = z^t b(z)$, where $1 \le t \le m$, $b(0) = 1$. Then by Lemma 2.1, there exists $z_j \longrightarrow 0$, $f_j \in \mathcal{F}$ and $\rho_j \longrightarrow 0^+$ such that

$$g_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{kn+t}{l+n}}} \longrightarrow g(\xi)$$

locally uniformly on compact subsets of $\mathbb{C}$, where $g(\xi)$ is a non-constant meromorphic functions in $\mathbb{C}$. By Hurwitz's theorem, all zeros of $g(\xi)$ have multiplicity at least $k + m$.

Next, we discuss two cases.

Case 1. Let $\frac{z_n}{\rho_n} \to \alpha, \alpha \in \mathbb{C}$.

For each $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$, It can be easily calculated that

$$\begin{aligned} g_j^l(\xi)(g_j^{(k)}(\xi))^n - \left(\xi + \frac{z_j}{\rho_j}\right)^t b\big(z_j + \rho_j \xi\big) \\ = \frac{f_j^l\big(z_j + \rho_j \xi\big)(f_j^{(k)}\big(z_j + \rho_j \xi\big))^n - a\big(z_j + \rho_j \xi\big)}{\rho_j^t} \longrightarrow g^l(\xi)(g^{(k)}(\xi))^n - (\xi + \alpha)^t. \end{aligned}$$

Since for sufficiently large j, $f_j^l\big(z_j + \rho_j \xi\big)(f_j^{(k)}\big(z_j + \rho_j \xi\big))^n - a\big(z_j + \rho_j \xi\big)$ has one zero, from the proof Lemma 2.6, we can deduce that $g^l(\xi)(g^{(k)}(\xi))^n - (\xi + \alpha)^t$ has at most one distinct zero.

By Lemma 2.5, $g^l(\xi)(g^{(k)}(\xi))^n - (\xi + \alpha)^t$ have at least two distinct zeros. Thus $g(\xi)$ is a constant, we can get a contradiction.

Case 2. Let $\frac{z_n}{\rho_n} \to \infty$.

Set

$$F_j(\xi) = \frac{f_j(z_j + \rho_j \xi)}{\rho_j^{\frac{kn+t}{l+n}}}.$$

It follows that

$$F_j^l(\xi)(F_j^{(k)}(\xi))^n - (1+\xi)^t b(z_j + z_j\xi) = \frac{f_j^l(z_j + z_j\xi)(f_j^{(k)}(z_j + z_j\xi))^n - a(z_j + z_j\xi)}{z_j^t}.$$

As the same argument as in Lemma 2.6, we can deduce that $F_j^l(\xi)(F_j^{(k)}(\xi))^n - (1+\xi)^t b(z_j + z_j\xi)$ has at most one zero in $\Delta = \{\xi : |\xi| < 1\}$.

Since all zeros of F_j have multiplicity at least $k+m$, and $(1+\xi)^t b(z_j + z_j\xi) \to (1+\xi)^t \neq 0$ for $\xi \in \Delta$. Then by Lemma 2.6, $\{F_n\}$ is normal in Δ.

Therefore, there exists a subsequence of $\{F_n(z)\}$(we still express it as $\{F_n(z)\}$) such that $\{F_n(z)\}$ converges spherically locally uniformly to a meromorphic function $F(z)$ or ∞.

If $F(0) \neq \infty$, then, for each $\xi \in \mathbb{C}/\{g^{-1}(\infty)\}$, we have

$$\begin{aligned} g^{(k+m-1)}(\xi) &= \lim_{j\to\infty} g_j^{(k+m-1)}(\xi) = \lim_{j\to\infty} \frac{f_j^{(k+m-1)}\left(z_j + \rho_j\xi\right)}{\rho_j^{\frac{kn+t}{l+n}-(k+m-1)}} \\ &= \lim_{j\to\infty}\left(\frac{\rho_j}{z_j}\right)^{k+m-1-\frac{kn+t}{l+n}} F_j^{(k+m-1)}\left(\frac{\rho_j}{z_j}\xi\right) = 0. \end{aligned}$$

Hence $g^{(k+m-1)} \equiv 0$. It follows that g is a polynomial of degree $\leq k+m-1$. Note that all zeros of g have multiplicity at least $k+m$, then we get that g is a constant, which is a contradiction.

If $F(0) = \infty$, then, for each $\xi \in \mathbb{C}/\{g^{-1}(0)\}$, we get

$$\frac{1}{F_j\left(\frac{\rho_j}{z_j}\xi\right)} = \frac{z_j^{\frac{kn+t}{l+n}}}{f_j\left(z_j + \rho_j\xi\right)} \to \frac{1}{F(0)} = 0,$$

It follows that we have

$$\frac{1}{g(\xi)} = \lim_{j\to\infty} \frac{\rho_j^{\frac{kn+t}{l+n}}}{f_j\left(z_j + \rho_j\xi\right)} = \lim_{j\to\infty}\left(\frac{\rho_j}{z_j}\right)^{\frac{kn+t}{l+n}} \frac{z_j^{\frac{kn+t}{l+n}}}{f_j\left(z_j + \rho_j\xi\right)} = 0.$$

Thus $g(\xi) = \infty$, which contradicts that $g(\xi)$ is a non-constant meromorphic function.

Therefore $\mathcal{F}$ is normal at $z_0 = 0$. Hence $\mathcal{F}$ is normal in D.

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no conflict of interest.

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