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ORIGINAL RESEARCH PAPER

Common fixed points for generalized $(\alpha - \psi)$-Meir–Keeler–Khan mappings in metric spaces

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Abstract

In this article, we prove a common fixed point results for two pairs of weakly compatible self-mappings in a complete metric space satisfying (α, ψ)-Meir–Keeler–Khan type contractive condition. We present an example to illustrate main result. Some other results and consequences are also given. These results generalize some classical results in the current literature.

Keywords Common fixed point · Generalized (α, ψ)-Meir–Keeler–Khan type contractions · Weakly compatible mappings · Complete metric space · α-Admissible mapping

Mathematics Subject Classification 47H10 · 54H25

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1 Introduction

The theory of fixed points takes an important place in the transition from classical analysis to modern analysis. One of the most remarkable works on fixed point of functions defined in metric spaces was done by Banach [6]. This classical principle has been generalized by several authors in different directions (see [1, 2, 4, 7–10, 13–15, 19, 23–25]). A classical generalization was given by Meir and Keeler [15]. They studied the fixed point of the class of mappings satisfying the condition that for each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that $\varepsilon \leq d(x, \jmath) < \varepsilon + \delta(\varepsilon)$ implies $d(fx, f\jmath) < \varepsilon$ for any $x, \jmath \in M$. Subsequently, many authors extended and improved this condition and established fixed point results (see [5, 11, 12, 16–18, 20, 21]).

Jungck and Rhoades [10] introduced the notion of weakly compatible mapping and showed that compatible mappings are weakly compatible but converse does not hold in general. In this paper, we study and establish the fixed point results for four mappings based on Meir–Keeler–Khan type contraction in complete metric spaces via α-admissible weakly compatible mappings. Our results extend the results proved by Redjel et al. [20]. Moreover, we present some consequences of our new results. In the sequel, the following definitions will be used.

Let Ψ be the family of nondecreasing functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < +\infty$, for all $t > 0$, where ψ^n is the $n-th$ iterate of ψ.

Lemma 1.1 [3] *Let* $\psi \in \Psi$. *Then*

1. $\psi(t) < t$, *for all* $t > 0$,
2. $\psi(0) = 0$.

Definition 1.2 Let S and f be two self-maps on M. If $Sx = fx$, for some $x \in M$, then x is called coincidence point of S and f.

Definition 1.3 [10] Let S and f be two self-mappings defined on a set M. S and f are said to be weakly compatible if they commute at coincidence points. That is, if $Sx = fx$, for some $x \in M$, then $Sfx = fSx$.

On the other hand, Samet et al. [22] introduced the notions of $\alpha - \psi$ contractive mapping using $\alpha-$ admissible mapping in a metric space and proved a fixed point results for α, ψ contractive mappings in a complete metric space.

Definition 1.4 [22] Let $f : M \to M$ and $\alpha : M \times M \to [0, \infty)$ be two mappings. The mapping f is said to be an α-admissible if the following condition satisfied:

$$\text{for all } x, \jmath \in M, \ \alpha(x, \jmath) \geq 1 \text{ implies } \alpha(fx, f\jmath) \geq 1. \tag{1.1}$$

Recently, Patel et al. [17] introduced criteria of α-admissible for four self-mappings as follows:

Definition 1.5 [17] Let $T, \Im, S, f : M \to M$ be four self-mappings of a non-empty set M and let $\alpha : T(M) \cup \Im(M) \times T(M) \cup \Im(M) \to [0, \infty)$ be a mapping. A pair

(S,f) is called an α-admissible with respect to T and $\Im$, if for all $x, \jmath \in M$, $\alpha(Tx, \Im\jmath) \geq 1$ or $\alpha(\Im x, T\jmath) \geq 1$, implies

$$\alpha(Sx, f\jmath) \geq 1 \text{ and } \alpha(fx, S\jmath) \geq 1. \tag{1.2}$$

Fisher [8] proved the following revised version the result given by Khan [13].

Theorem 1.6 [8] *Let f be a self map on a complete metric space* (M,d) *satisfying the following:*

$$d(fx,f\jmath) \leq \mu \frac{d(x,fx)d(x,f\jmath) + d(\jmath,f\jmath)d(\jmath,fx)}{d(x,f\jmath) + d(\jmath,fx)}, \ \mu \in \left]0,1\right[\tag{1.3}$$

if $d(x,f\jmath) + d(\jmath,fx) \neq 0$ *and* $d(fx,f\jmath) = 0$, *if* $d(x,f\jmath) + d(\jmath,fx) = 0$. *Then f has a unique fixed point* $t \in M$. *Moreover, for every* $t_0 \in M$, *the sequence* $\{f^n t_0\}$ *converges to t.*

Definition 1.7 [20] Let (M, d) be a metric space and $f : M \to M$ be a self mapping. f is called (α, ψ)-Meir–Keeler–Khan mapping, if there exist $\psi \in \Psi$ and $\alpha : M \times M \to [0, \infty)$ satisfying the following condition: For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that

$$\varepsilon \leq \psi\left(\frac{d(x,f(x))d(x,f(\jmath)) + d(\jmath,f(\jmath))d(\jmath,f(x))}{d(x,f(\jmath)) + d(\jmath,f(x))}\right) < \varepsilon + \delta(\varepsilon)$$

implies

$$\alpha(x,\jmath)d(f(x),f(\jmath)) < \varepsilon. \tag{1.4}$$

2 Main results

In this section, we introducing the class of common fixed point result for two pairs of weakly compatible self mappings in complete metric spaces satisfies (α, ψ)-Meir–Keeler–Khan type contractive via α-admissible mappings.

Definition 2.1 Let (M, d) be a complete metric space. The self-mappings $T, \Im, S, f : M \to M$ are said to be (α, ψ)-Meir–Keeler–Khan type, if there exists $\psi \in \Psi$ and $\alpha : T(M) \cup \Im(M) \times T(M) \cup \Im(M) \to [0, \infty)$ satisfying the following condition:For each $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that,

$$\varepsilon \leq \psi\left(\frac{d(Tx, Sx)d(Tx,f\jmath) + d(\Im\jmath,f\jmath)d(\Im\jmath, Sx)}{d(Tx,f\jmath) + d(\Im\jmath, Sx)}\right) < \varepsilon + \delta(\varepsilon)$$

implies

$$\alpha(Tx, \Im\jmath)d(Sx,f\jmath) < \varepsilon. \tag{2.1}$$

Remark 2.2 It is easy to see that if $T, \Im, S, f : M \to M$ are (α, ψ)-Meir–Keeler–Khan type mappings, then

$$\alpha(Tx, \Im \jmath)d(Sx, f\jmath) \leq \psi\left(\frac{d(Tx, Sx)d(Tx, f\jmath) + d(\Im\jmath, f\jmath)d(\Im\jmath, Sx)}{d(Tx, f\jmath) + d(\Im\jmath, Sx)}\right), \tag{2.2}$$

for all $x, \jmath \in M$.

Theorem 2.3 *Let (M, d) be a complete metric space and $T, \Im, S, f : M \to M$ be an (α, ψ)-Meir–Keeler–Khan type mappings such that $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$. Assume that:*

1. *The pair (S, f) is α-admissible with respect to T and $\Im$ (shortly $\alpha_{T,\Im}$-admissible);*
2. *There exists $x_0 \in M$ such that $\alpha(Tx_0, Sx_0) \geq 1$;*
3. *One of $T, \Im, S$ and f is continuous.*
4. *(S, T) and $(f, \Im)$ are weakly compatible pairs of self-mappings.*

Then $T, \Im, S$ and f have a common fixed point $z \in M$.

Proof By assumption (2), there exists $x_0 \in M$ such that $\alpha(Tx_0, Sx_0) \geq 1$. Define the sequences $\{x_n\}$ and $\{\jmath_n\}$ in M such that

$$\jmath_{2n} = Sx_{2n} = \Im x_{2n+1} \quad \text{and} \quad \jmath_{2n+1} = fx_{2n+1} = Tx_{2n+2}. \tag{2.3}$$

This can be done, since $f(M) \subseteq T(M)$ and $S(M) \subseteq \Im(M)$. Since (S, f) is $\alpha_{T,\Im}$-admissible, we have

$$\alpha(Tx_0, Sx_0) = \alpha(Tx_0, \Im x_1) \geq 1$$
$$\text{implies } \alpha(Sx_0, fx_1) \geq 1 \quad \text{and} \quad \alpha(fx_0, Sx_1) \geq 1,$$

which gives

$$\alpha(\Im x_1, Tx_2) \geq 1 = \alpha(\jmath_0, \jmath_1) \geq 1.$$

Again by (1), we have

$$\alpha(\Im x_1, fx_1) = \alpha(\Im x_1, Tx_2) \geq 1 \text{ implies } \alpha(fx_1, Sx_2) \geq 1 \quad \text{and} \quad \alpha(Sx_1, fx_2) \geq 1,$$

which gives,

$$\alpha(Tx_2, \Im x_3) = \alpha(\jmath_1, \jmath_2) \geq 1.$$

Inductively, we obtain

$$\alpha(\jmath_{2n}, \jmath_{2n+1}) \geq 1, \; n = 0, 1, 2, \ldots. \tag{2.4}$$

That is $\alpha(Tx_{2n}, \Im x_{2n+1}) \geq 1$ and $\alpha(\Im x_{2n+1}, Tx_{2n+2}) \geq 1$. By (2.2) and (2.4), we have

$$
\begin{aligned}
d(\jmath_{2n}, \jmath_{2n+1}) &= d(Sx_{2n}, fx_{2n+1}) \leq \alpha(Tx_{2n}, \Im x_{2n+1}) d(Sx_{2n}, fx_{2n+1}) \\
&\leq \psi\left(\frac{d(Tx_{2n}, Sx_{2n}) d(Tx_{2n}, fx_{2n+1}) + d(\Im x_{2n+1}, fx_{2n+1}) d(\Im x_{2n+1}, Sx_{2n})}{d(Tx_{2n}, fx_{2n+1}) + d(\Im x_{2n+1}, Sx_{2n})}\right) \\
&\leq \psi\left(\frac{d(fx_{2n-1}, Sx_{2n}) d(fx_{2n-1}, fx_{2n+1}) + d(Sx_{2n}, fx_{2n+1}) d(Sx_{2n}, Sx_{2n})}{d(fx_{2n-1}, fx_{2n+1}) + d(Sx_{2n}, Sx_{2n})}\right) \\
&\leq \psi\left(\frac{d(fx_{2n-1}, Sx_{2n}) d(fx_{2n-1}, fx_{2n+1})}{d(fx_{2n-1}, fx_{2n+1})}\right) \\
&\leq \psi d(fx_{2n-1}, Sx_{2n}) \\
&\leq \psi d(\jmath_{2n-1}, \jmath_{2n}), \text{ for all } n \in \mathbb{N}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
d(\jmath_{2n-1}, \jmath_{2n}) &= d(fx_{2n-1}, Sx_{2n}) \leq \alpha(\Im x_{2n-1}, Tx_{2n}) d(fx_{2n-1}, Sx_{2n}) \\
&\leq \psi\left(\frac{d(\Im x_{2n-1}, fx_{2n-1}) d(\Im x_{2n-1}, Sx_{2n}) + d(Tx_{2n}, Sx_{2n}) d(Tx_{2n}, fx_{2n-1})}{d(\Im x_{2n-1}, Sx_{2n}) + d(Tx_{2n}, fx_{2n-1})}\right) \\
&\leq \psi\left(\frac{d(Sx_{2n-2}, fx_{2n-1}) d(Sx_{2n-2}, Sx_{2n}) + d(fx_{2n-1}, Sx_{2n}) d(fx_{2n-1}, fx_{2n-1})}{d(Sx_{2n-2}, Sx_{2n}) + d(fx_{2n-1}, fx_{2n-1})}\right) \\
&\leq \psi d(Sx_{2n-2}, fx_{2n-1}) \leq \psi(\jmath_{2n-2}, \jmath_{2n-1}).
\end{aligned}
$$

That is

$$d(\jmath_{2n}, \jmath_{2n+1}) \leq \psi d(\jmath_{2n-1}, \jmath_{2n}) \leq \psi^2 d(\jmath_{2n-2}, \jmath_{2n-1}).$$

Continuing in this manner, we obtain

$$d(\jmath_{2n}, \jmath_{2n+1}) \leq \psi^{2n} d(\jmath_0, \jmath_1).$$

We can write above inequality as

$$d(\jmath_n, \jmath_{n+1}) \leq \psi^n d(\jmath_0, \jmath_1).$$

Now, we show that $\{\jmath_n\}$ is a Cauchy sequence. By the properties of the function ψ, for any $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$ such that $\sum_{n \geq n(\epsilon)} \psi^n(d(\jmath_0, \jmath_1)) < \varepsilon$. Let $n, m \in \mathbb{N}$ with $n > m > n(\varepsilon)$, using the triangle inequality, we get

$$
\begin{aligned}
d(\jmath_m, \jmath_n) \leq \sum_{k=m}^{n-1} d(\jmath_k, \jmath_{k+1}) &\leq \sum_{n=m} \psi^k(d(\jmath_0, \jmath_1)) \\
&\leq \sum_{k=n(\epsilon)} \psi^k(d(\jmath_0, \jmath_1)) < \varepsilon.
\end{aligned}
$$

We deduce that $\{\jmath_n\}$ is a Cauchy sequence in a complete metric space (M, d). There is exists $z \in M$ such that $\lim_{n\to\infty} \jmath_n = z$ and sequentially, Sx_{2n}, $\Im x_{2n+1}$, fx_{2n+1}, $Tx_{2n+2} \to z$, as $n \to \infty$. By assumption (3)

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} \Im x_{2n+1} = \lim_{n\to\infty} fx_{2n+1} = \lim_{n\to\infty} Tx_{2n+2} = z.$$

Since $f(M) \subseteq T(M)$, there exists $u \in M$ such that $z = Tu$. By (2.2) and (2.4), we have

$$\begin{aligned} d(Su,z) &\le d(Su,fx_{2n+1}) + d(fx_{2n+1},z) \\ &\le \alpha(Tu,\Im x_{2n+1})d(Su,fx_{2n+1}) + d(fx_{2n+1},z) \\ &\le \psi\left(\frac{d(Tu,Su)d(Tu,fx_{2n+1}) + d(\Im x_{2n+1},fx_{2n+1})d(\Im x_{2n+1},Su)}{d(Tu,fx_{2n+1}) + d(\Im x_{2n+1},Su)}\right) \\ &\quad + d(fx_{2n+1},z) \\ &\le \psi\left(\frac{d(z,Su)d(z,fx_{2n+1}) + d(Sx_{2n},fx_{2n+1})d(Sx_{2n},Su)}{d(z,fx_{2n+1}) + d(Sx_{2n},Su)}\right) \\ &\quad + d(fx_{2n+1},z). \end{aligned}$$

Letting $\lim_{n\to\infty}$ in above inequality, we get

$$d(Su,z) \le \psi\left(\frac{d(z,Su)d(z,z) + d(z,z)d(z,Su)}{d(z,z) + d(z,Su)}\right) + d(z,z) = 0.$$

That is $Su = z$. Thus $Tu = Su = z$. Therefore u is a coincidence point of T and S. Since the pair of mappings S and T are weakly compatible, we have

$$\begin{aligned} STu &= TSu, \\ Sz &= Tz. \end{aligned}$$

Since $S(M) \subseteq \Im(M)$, there exists a point $v \in M$ such that $z = \Im v$. By (2.2) and (2.4), we have

$$\begin{aligned} d(z,fv) = d(Su,fv) &\le \alpha(Tu,\Im v)d(Su,fv) \\ &\le \psi\left(\frac{d(Tu,Su)d(Tu,fv) + d(\Im v,fv)d(\Im v,Su)}{d(Tu,fv) + d(\Im v,Su)}\right) \\ &\le \psi\left(\frac{d(z,z)d(z,fv) + d(z,fv)d(z,z)}{d(z,fv) + d(z,z)}\right) \le \psi(0). \end{aligned}$$

That is $d(z,fv) = 0$. Thus, $z = fv$. Therefore $fv = \Im v = z$. So v is coincident point of $\Im$ and f. Since, the pair of maps $\Im$ and f are weakly compatible

$$\begin{aligned} \Im fv &= f\Im v, \\ \Im z &= fz. \end{aligned}$$

Now, we show that z is a fixed point of S. By (2.2) and (2.4), we get

$$\begin{aligned}
d(Sz,z) = d(Sz,fv) &\le \alpha(Tz,\Im v)d(Sz,fv) \\
&\le \psi\left(\frac{d(Tz,Sz)d(Tz,fv)+d(\Im v,fv)d(\Im v,Sz)}{d(Tz,fv)+d(\Im v,Sz)}\right) \\
&\le \psi\left(\frac{d(Sz,Sz)d(Sz,z)+d(z,z)d(z,Sz)}{d(Sz,z)+d(z,Sz)}\right) \\
d(Sz,z) &\le 0.
\end{aligned}$$

So, $d(Sz,z)=0$. Thus, $Sz=z$. Hence,

$$Sz = Tz = z.$$

Now, we show that z is a fixed point of f. By using (2.2) and (2.4), we get

$$\begin{aligned}
d(z,fz) = d(Sz,fz) &\le \alpha(Tz,\Im z)d(Sz,fz) \\
&\le \psi\left(\frac{d(Tz,Sz)d(Tz,fz)+d(\Im z,fz)d(\Im z,Sz)}{d(Tz,fz)+d(\Im z,Sz)}\right) \\
&\le \psi\left(\frac{d(z,z)d(z,fz)+d(fz,fz)d(fz,z)}{d(z,fz)+d(fz,z)}\right) = \psi(0) = 0
\end{aligned}$$

Thus, $d(z,fz)=0$. That is, $z=fz$. Therefore, $fz=\Im z=z$. Thus, $Sz=Tz=fz=\Im z=z$.

Hence, z is a common fixed point of $T,\Im,S$ and f. □

Theorem 2.4 *Let $(M,\ d)$ be a complete metric space and $T,\Im,S,f: M\to M$ be an (α,ψ)-Meir–Keeler–Khan type mappings such that $f(M)\subseteq T(M)$ and $S(M)\subseteq \Im(M)$. Assume that:*

1. *The pair (S,f) is α-admissible with respect to T and $\Im$ (shortly $\alpha_{T,\Im}$-admissible);*
2. *There exists $x_0\in M$ such that $\alpha(Tx_0,Sx_0)\ge 1$;*
3. *If $\{\jmath_n\}$ is a sequence in M such that $\alpha(\jmath_n,\jmath_{n+1})\ge 1$ for all $n\in\mathbb{N}$ and $\jmath_n\to z\in M$ as $n\to\infty$, then $\alpha(\jmath_n,z)\ge 1$, for all $n\in\mathbb{N}$.*

Then $T,\Im,S$ and f have a common fixed point $z\in M$ provided $(S,\ T)$ and $(f,\Im)$ are weakly compatible pairs of self-mappings.

Proof Following the proof of Theorem 2.3, we obtain the sequence $\{\jmath_n\}$ in M defined by:

$$\jmath_{2n} = Sx_{2n} = \Im x_{2n+1} \quad \text{and} \quad \jmath_{2n+1} = fx_{2n+1} = Tx_{2n+2},$$

for all $n\ge 0$, which converges to some $z\in M$. Sequentially,

$$Sx_{2n}, \Im x_{2n+1},\ fx_{2n+1},\ Tx_{2n+2} \to z,$$

as $n\to\infty$. Since $f(M)\subseteq T(M)$, there exists $u\in M$ such that $z=Tu$. By (3) and (2.4), we have

$$\begin{aligned} d(Su, z) = d(Su, fx_{2n+1}) &\leq \alpha(Tu, \Im x_{2n+1}) d(Su, fx_{2n+1}) \\ &\leq \psi\left(\frac{d(Tu, Su)d(Tu, fx_{2n+1}) + d(\Im x_{2n+1}, fx_{2n+1})d(\Im x_{2n+1}, Su)}{d(Tu, fx_{2n+1}) + d(\Im x_{2n+1}, Su)}\right) \\ &\leq \psi\left(\frac{d(z, Su)d(z, fx_{2n+1}) + d(Sx_{2n}, fx_{2n+1})d(Sx_{2n}, Su)}{d(z, fx_{2n+1}) + d(Sx_{2n}, Su)}\right). \end{aligned}$$

Letting $\lim_{n\to\infty}$ in above inequality we end up with

$$d(Su, z) \leq \psi\left(\frac{d(z, Su)d(z, z) + d(z, z)d(z, Su)}{d(z, z) + d(z, Su)}\right) \leq 0.$$

Thus $Su = z$, so, $Tu = Su = z$, Therefore u is a coincidence point of T and S. Since the pair of mappings S and T are weakly compatible, we have

$$STu = TSu,$$
$$Sz = Tz.$$

Similarly, as $S(M) \subseteq \Im(M)$, we obtain $d(z, fv) = 0$. Thus, $z = fv$. Therefore $fv = \Im v = z$. So v is coincident point of $\Im$ and f. Since, the pair of maps $(\Im, f)$ are weakly compatible so,

$$\Im fv = f\Im v,$$
$$\Im z = fz.$$

We can easily show that z is fixed point of S and f and the proof is completed. □

For the uniqueness of the fixed point of a generalized (α, ψ)-Meir–Keeler–Khan type contractive mapping, we will consider the following hypothesis:

(H) For all common fixed points x and $\jmath$ of $T, \Im, S$ and f, there exists $v \in M$ such that $\alpha(x, v) \geq 1$ and $\alpha(\jmath, v) \geq 1$.

Theorem 2.5 *Adding the condition (H) to the statement of Theorem 2.3 or 2.4, we obtain the uniqueness of the common fixed point of S, T, f and $\Im$.*

Proof The existence of a fixed point is proved in Theorem 2.3 (respectively Theorem 2.4). To prove a uniqueness assume that $\hat{w}$ is another common fixed point of $T, \Im, S$ and f such that $z \neq \hat{w}$. By condition (H), there exists $v \in M$ such that $\alpha(Tz, v) \geq 1$ and $\alpha(\Im\hat{w}, v) \geq 1$. Define a sequence $\{v_n\}$ in M by

$$v_0 = Sv_0 = \Im v_1, \quad v_{2n} = Sv_{2n} = \Im v_{2n+1}$$

and

$$v_1 = fv_1 = Tv_2, \quad v_{2n+1} = fv_{2n+1} = Tv_{2n+2},$$

for all $n \geq 0$. Since the pair (S, f) is $\alpha_{T,\Im}$ -admissible, we obtain

$$\alpha(z, v_{2n}) \geq 1 \quad \text{and} \quad \alpha(\hat{w}, v_{2n}) \geq 1, \quad \text{for all } n.$$

Now, by Remark 2.2, we have

$$\begin{aligned} d(z, v_{2n+1}) = d(Sz, fv_{2n+1}) &\leq \alpha(Tz, \Im v_{2n+1}) d(Sz, fv_{2n+1}), \\ &\leq \psi\left(\frac{d(Tz, Sz)d(Tz, fv_{2n+1}) + d(\Im v_{2n+1}, fv_{2n+1})d(\Im v_{2n+1}, Sz)}{d(Tz, fv_{2n+1}) + d(\Im v_{2n+1}, Sz)}\right), \\ &\leq \psi\left(\frac{d(Sz, Sz)d(Sz, fv_{2n+1}) + d(\Im v_{2n+1}, fv_{2n+1})d(\Im v_{2n+1}, Sz)}{d(Sz, fv_{2n+1}) + d(\Im v_{2n+1}, Sz)}\right). \end{aligned}$$

By triangle inequality, we have

$$\begin{aligned} d(\Im v_{2n+1}, fv_{2n+1}) &\leq d(Sz, fv_{2n+1}) + d(\Im v_{2n+1}, Sz), \\ &\leq \psi\left(\frac{d(\Im v_{2n+1}, fv_{2n+1})d(\Im v_{2n+1}, Sz)}{d(Sz, fv_{2n+1}) + d(\Im v_{2n+1}, Sz)}\right), \\ &\leq \psi d(\Im v_{2n+1}, Sz), \\ &\leq \psi d(z, v_{2n}). \end{aligned}$$

Iteratively, this inequality implies

$$d(z, v_{2n+1}) \leq \psi^{2n+1}(d(z, v_0)), \quad \text{for all } n.$$

Putting $n \to \infty$, in above inequality, we obtain

$$\lim_{n\to\infty} d(v_{2n}, z) = 0. \tag{2.5}$$

$$\lim_{n\to\infty} d(v_{2n}, \hat{w}) = 0. \tag{2.6}$$

From (2.5), (2.6) we get $z = \hat{w}$. □

Now, we give an example to support Theorem 2.3.

Example 2.6 Let $M = [2, 20]$ and (M, d) be usual metric space. Define T, $\Im$, S and f as follows:

$$S(x) = 2, \quad \text{for all } x.$$

$$f(x) = \begin{cases} 2, & \text{if } x \in [2, 5) \cup [6, 20], \\ x + 1, & \text{if } x \in [5, 6). \end{cases}$$

$$T(x) = \begin{cases} x, & \text{if } x \in [2, 7], \\ 7, & \text{if } x \in (7, 20]. \end{cases}$$

$$\Im(x) = \begin{cases} 2, & \text{if } x = 2, \\ 3, & \text{if } x \in (2, 5) \cup [6, 20], \\ x + 3, & \text{if } x \in [5, 6). \end{cases}$$

Note that $f(M) \subseteq T(M)$, and $S(M) \subseteq \Im(M)$, we note $Sx = Tx$ for which $x = 2$ implies $STx = TSx$ and $fx = \Im x$ implies $f\Im x = \Im fx$, thus the pairs $\{S, T\}$ and $\{f, \Im\}$

are weakly compatible. Consider $\varepsilon = \frac{3}{4}$ and suppose that $\psi(t) = \frac{3t}{4}$, then $T, \Im, S$ and f satisfy the (α, ψ)-Meir–Keeler–Khan contractive condition with the mapping $\alpha : T(M) \cup \Im(M) \times T(M) \cup \Im(M) \to [0, \infty)$ defined by

$$\alpha(u, v) = \begin{cases} 1, & \text{if } u, v \in [2, 5) \cup [9, 20], \\ \frac{1}{10}, & \text{otherwise.} \end{cases}$$

Clearly $x = 2$ is our unique common fixed point. Indeed, hypothesis (2) is satisfied with $x_0 = 2 \in M$ with $\alpha(2, 2) \geq 1$. Then, all the conditions of Theorem 2.3 are satisfied.

Corollary 2.7 [20] *Let (M, d) be a complete metric space and let $f : M \to M$ be an (α, ψ) -Meir–Keeler–Khan mapping. Assume that:*

1. *f is an α-admissible mapping.*
2. *There exists $x_0 \in M$ such that $\alpha(x_0, f(x_0)) \geq 1$;*
3. *f is continuous.*

Then f has a fixed point in M, that is, there exists $u \in M$ such that $f(u) = u$.

Proof Immediately by taking $S = f = \Im = T$ in the Theorem 2.3. □

Corollary 2.8 [20] *Let (M, d) be a complete metric space and let $f : M \to M$ be an (α, ψ)-Meir–Keeler–Khan mapping. Assume that:*

1. *f is an α-admissible.*
2. *There exists $x_0 \in M$ such that $\alpha(x_0, f(x_0)) \geq 1$.*
3. *If $\{x_n\}$ is a sequence in M such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in M$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$, for all $n \in \mathbb{N}$. Then there exists $u \in M$ such that $f(u) = u$.*

In the Theorem 2.4, if we take $\psi(t) = \lambda t$, where $\lambda \in]0, 1[$ and $\alpha(Tx, \Im y) = 1$, for all $x, y \in M$, we obtain the following result.

Corollary 2.9 *Let (M, d) be a complete metric space and $T, \Im, S, f : M \to M$ be the mappings satisfies the following condition:*

For $\varepsilon > 0$, there exists $\delta' > 0$ such that,

$$\varepsilon \leq \lambda \left(\frac{d(Tx, Sx)d(Tx, fy) + d(\Im y, fy)d(\Im y, Sx)}{d(Tx, fy) + d(\Im y, Sx)} \right) < \varepsilon + \delta' \quad \text{implies } d(Sx, fy) < \varepsilon. \tag{2.7}$$

Then $T, \Im, S$ and f have a unique common fixed point $z \in M$. Moreover, for all x_0 the sequence $\{fx_0\}$ converge to z.

Proof Let $\mu \in]0, 1[$ and choose $\lambda_0 \in]0, 1[$ with $\lambda_0 > \mu$. Fix $\varepsilon > 0$. If we take $\delta' =$

$\varepsilon\left(\frac{1}{\mu}-\frac{1}{\lambda_0}\right)$. Assume that

$$\frac{1}{\lambda_0}\varepsilon \leq \frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)} < \frac{1}{\lambda_0}\varepsilon+\delta',$$

From (1.3), it follows that

$$\begin{aligned} d(Sx,f\jmath) &< \mu\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)} \\ &< \mu\left(\frac{1}{\lambda_0}\varepsilon+\delta'\right) \\ &= \mu\left(\frac{1}{\lambda_0}\varepsilon+\varepsilon\left(\frac{1}{\mu}-\frac{1}{\lambda_0}\right)\right)=\varepsilon. \end{aligned}$$

Hence (2.7) is satisfied which makes Theorem 1.6 an immediate consequence of Corollary 2.9. □

3 Consequences

In this section, following the idea of Samet [21], we will show that Corollary 2.9 allows us to obtain an integral version of Fisher's result. Our first new result is the following:

Theorem 3.1 *Let (M,d) be a complete metric space and $T,\Im,S,f: M\to M$, and let $\lambda\in\left]0,1\right[$. Assume that there exists a function ρ from $\left[0,\infty\right[$ into it self satisfying the following conditions;*

1. *$\rho(0)=0$ and $\rho(t)>0$ for all $t>0$;*
2. *ρ is nondecreasing and right continuous;*
3. *for every $\varepsilon>0$, there exists $\delta'>0$ such that*

$$\frac{1}{\lambda}\varepsilon < \rho\left(\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}\right) < \frac{1}{\lambda}\varepsilon+\delta'$$
$$\text{implies } \rho\left(\frac{1}{\lambda}d(Sx,f\jmath)\right) < \frac{1}{\lambda}\varepsilon, \quad \text{for all } x,\jmath\in M.$$

Then inequality (2.7) is hold.

Proof Fix $\varepsilon>0$, since $\rho\left(\frac{1}{\lambda}\varepsilon\right)>0$, by (3) there exists $\beta>0$ such that

$$\begin{aligned} \rho\left(\frac{1}{\lambda}\varepsilon\right) < \rho\left(\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta \\ \text{implies } \rho\left(\frac{1}{\lambda}(d(Sx,f\jmath))\right) < \rho\left(\frac{1}{\lambda}\varepsilon\right). \end{aligned} \tag{3.1}$$

From the right continuity of ρ there exists $\delta'>0$ such that;

$$\rho\left(\frac{1}{\lambda}\varepsilon+\delta'\right)<\rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta.$$

For all $x, \jmath \in M$, such that

$$\frac{1}{\lambda}\varepsilon<\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}<\frac{1}{\lambda}\varepsilon+\delta'.$$

Since ρ is nondecreasing, we have

$$\rho\left(\frac{1}{\lambda}\varepsilon\right)<\rho\left(\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}\right)$$
$$<\rho\left(\frac{1}{\lambda}\varepsilon+\delta'\right)<\rho\left(\frac{1}{\lambda}\varepsilon\right)+\beta.$$

Then by (3.1), we have

$$\rho\left(\frac{1}{\lambda}d(Sx,f\jmath)\right)<\rho\left(\frac{1}{\lambda}\varepsilon\right),$$

which implies that

$$d(Sx,f\jmath)<\varepsilon.$$

Then (2.7) is satisfied.

We denote by Ξ the set of all mappings $g : [0,+\infty[\to[0,+\infty[$ satisfying:

1. g continuous and nondecreasing;
2. $g(0)=0$ and $g(t)>0$ for all $t>0$.

□

Corollary 3.2 *Let (M,d) be a complete metric space and let $T,\Im,S,f : M\to M$ be the mappings, let $g\in\Xi$ be such that for $\varepsilon>0$ there exist $\delta'>0$, with*

$$\frac{1}{\lambda}\varepsilon<g\left(\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}\right)<\frac{1}{\lambda}\varepsilon+\delta'$$
$$\text{implies } g\left(\frac{1}{\lambda}d(Sx,f\jmath)\right)<\frac{1}{\lambda}\varepsilon.$$

Then (2.7) is satisfied.

Proof Since every continuous function $g : [0,+\infty[\to[0,+\infty[$ is right continuous, the proof follows immediately from Theorem 3.1. □

Corollary 3.3 *Let (M, d) be a complete metric space and let T, $\Im$, S and f be four mappings from M into itself. Let φ be a locally integrable function from $[0,+\infty[$ into itself such that*

$$\int_0^t \varphi(u)du > 0, \quad \text{for all } t > 0.$$

Assume that for each $\varepsilon > 0$, *there exists* $\delta' > 0$ *such that*

$$\frac{1}{\lambda}\varepsilon \leq \int_0^{\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}} \varphi(t)dt < \frac{1}{\lambda}\varepsilon + \delta',$$

implies that

$$\int_0^{\frac{1}{\lambda}d(Sx,f\jmath)} \varphi(t)dt < \frac{1}{\lambda}\varepsilon. \tag{3.2}$$

Then (2.7) *is satisfied. Now we are able to obtain an integral version of Khan result.*

Corollary 3.4 *Let* (M, d) *be a complete metric space and* $T, \Im, S, f : M \to M$ *be self mappings. Let* φ *be locally integrable function from* $[0, +\infty[$ *into it self such that* $\int_0^t \varphi(u)du > 0$, *for all* $t > 0$ *and let* $\lambda \in]0, 1[$. *Assume that* $T, \Im, S$ *and* f *satisfies the following condition. For all* $x, \jmath \in M$,

$$\int_0^{\frac{1}{\lambda}d(Sx,fx)} \varphi(t)dt \leq \mu' \int_0^{\frac{d(Tx,Sx)d(Tx,f\jmath)+d(\Im\jmath,f\jmath)d(\Im\jmath,Sx)}{d(Tx,f\jmath)+d(\Im\jmath,Sx)}} \varphi(t)dt,$$

where, $\mu' \in]0, 1[$. *Then* $T, \Im, S$ *and* f *have an unique common fixed point. Moreover, for any* $x \in M$, *the sequence* $\{\jmath^n(x)\}$ *converges to* z.

Proof Let $\varepsilon > 0$. it is easy to observe that (3.2) is satisfied. Take $\delta' = \frac{\varepsilon}{\lambda}\left(\frac{1}{\mu'} - 1\right)$, then (2.7) is satisfied, which proved the existence and uniqueness of a common fixed point. □

Compliance with ethical standards

Conflict of interest All authors declare that they have no conflict of interest.

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