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ORIGINAL RESEARCH PAPER

Coupled fixed point theorems with rational expressions in partially ordered metric spaces

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Abstract
The purpose of this paper is to establish some coupled fixed point theorems using a contractive condition of rational type with monotone property in the frame work of partially ordered metric space. The existence and uniqueness of the result is also presented for the coupled fixed point to the mappings. The result presented over here generalize and extend several well-know results in the literature.

Keywords Partially ordered metric spaces · Rational contractions · Coupled fixed point · Monotone property

Mathematics Subject Classification 47H10 · 55M20 · 54H25 · 26A42

1 Introduction

In mathematics fixed point theory is one of the famous and traditional theories in solving large number of applications. The Banach contraction principle is one of the most important result and plays a central role in finding a unique solutions of the results in analysis. It is a very popular tool for solving the existence problems in many different fields of mathematics. Several authors have extended the Banach contraction principle in the literature [1–6] using either ordinary or rational contraction conditions or by imposing some weaker conditions on different spaces

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like rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi-metric spaces, probabilistic metric spaces, D-metric spaces, G-metric spaces, F-metric spaces, cone metric spaces, and so on.

The extension of Banach contraction principle over a partially ordered sets are studied by Wolk [7] and Monjardet [8] for obtaining fixed points under certain conditions of the mapping. The existence of fixed points in partially ordered metric spaces with some applications to linear and nonlinear matrix equations can be seen from Ran and Reurings [9]. Later the results on fixed point theorems in partially order sets and applications to ordinary differential equations are investigated by Nieto et al. [10–12]. The vast information on the existence and uniqueness of fixed points in partially ordered metric spaces can be seen from the work of authors [10–34].

The main aim of this paper is to prove the existence and uniqueness of some coupled fixed point results for a rational type contraction mapping together with monotone property in metric space endowed with a partial order. The presented result is an extensions of, the result of Singh and Chatterjee [35] contraction for taking two mappings in partial order metric space.

2 Preliminaries

Definition 1 Let $(X, \leq)$ be a partially ordered set. A self-mapping $f : X \to X$ is said to be strictly increasing if $f(x) < f(y), \forall x, y \in X$ with $x < y$ and is also said to be strictly decreasing if $f(x) > f(y), \forall x, y \in X$ with $x < y$.

Definition 2 Let $(X, \leq)$ be a partially ordered set and f is self mapping defined over X is said to be strict mixed monotone property if $f(x, y)$ is strictly increasing in x and as well as strictly decreasing in y.

$$\text{i.e for any } x_1, x_2 \in X \text{ with } x_1 < x_2 \Rightarrow f(x_1, y) < f(x_2, y) \text{ and also}$$
$$\text{for any } y_1, y_2 \in X, \text{with } y_1 < y_2 \Rightarrow f(x, y_1) > f(x, y_2).$$

Definition 3 Let $(X, \leq)$ be a partially ordered set and $f : X \times X \to X$ be a mapping. A point $(x, y) \in X \times X$ is said to be coupled fixed point to f if $f(x, y) = x$ and $f(y, x) = y$.

Definition 4 The triple $(X, d, \leq)$ is called partially ordered metric spaces (POMS) if $(X, \leq)$ is a partially ordered set and (X, d) is a metric space.

Definition 5 If (X, d) is a complete metric space, then triple $(X, d, \leq)$ is called a partially ordered complete metric spaces (POCMS).

Definition 6 A partially ordered metric space $(X, d, \leq)$ is called ordered complete (OC) if for each convergent sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \subset X$, the following condition holds:

- if $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \to x$ implies $x_n \leq x, \forall n \in N$ that is, $x = \sup\{x_n\}$.
- if $\{y_n\}$ is a non-increasing sequence in X such that $y_n \to y$ implies $y \leq y_n, \forall n \in N$ that is, $y = \inf\{y_n\}$.

3 Main results

In this section, we prove some coupled fixed point theorems in the context of ordered metric spaces.

Theorem 1 *Let $(X, d, \leq)$ be an ordered complete partially ordered metric space. Suppose that a continuous self mapping $f : X \times X$ is a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned} d(f(x,y),f(\mu,\upsilon)) \leq \alpha & \frac{d(x,f(x,y))[1 + d(\mu,f(\mu,\upsilon))]}{1 + d(x,\mu)} \\ & + \beta[d(x,f(x,y)) + d(\mu,f(\mu,\upsilon))] \\ & + \gamma d(x,\mu) \end{aligned} \tag{1}$$

where $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$, and if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$, then f has coupled fixed point $(x, y) \in X \times X$ (i.e there exists $x, y \in X$ such that $f(x, y) = x$ and $f(y, x) = y$).

Proof Let $x_0, y_0 \in X$ such that $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$. Now define two sequences $\{x_n\}, \{y_n\}$ in X as

$$x_{n+1} = f(x_n, y_n) \quad \text{and} \quad y_{n+1} = f(y_n, x_n), \quad \text{for all}\, n \geq 0. \tag{2}$$

Now, we have to show that for all $n \geq 0$,

$$x_n < x_{n+1} \tag{3}$$

and

$$y_n > y_{n+1} \tag{4}$$

for this, we use the method of mathematical induction. Suppose $n = 0$, since $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$, and from (2), we have $x_0 < f(x_0, y_0) = x_1$ and $y_0 > f(y_0, x_0) = y_1$ and hence the inequalities (3) and (4) holds for $n = 0$. Suppose that the inequalities (3) and (4) holds for all $n > 0$ and by using the strict mixed monotone property of f, we get

$$x_{n+1} = f(x_n, y_n) < f(x_{n+1}, y_n) < f(x_{n+1}, y_{n+1}) = x_{n+2} \tag{5}$$

and

$$y_{n+1} = f(y_n, x_n) > f(y_{n+1}, x_n) > f(y_{n+1}, x_{n+1}) = y_{n+2} \tag{6}$$

and hence the inequalities (3) and (4) holds for all $n \geq 0$ and we obtain that

$$x_0 < x_1 < x_2 < x_3 < \cdots < x_n < x_{n+1} < \cdots \tag{7}$$

and

$$y_0 > y_1 > y_2 > y_3 > \cdots > y_n > y_{n+1} > \cdots \tag{8}$$

From hypothesis, we have as $x_n < x_{n+1}$, $y_n > y_{n+1}$ and from (2),

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n, y_n), f(x_{n-1}, y_{n-1})) \\ &\leq \alpha \frac{d(x_n, f(x_n, y_n))[1 + d(x_{n-1}, f(x_{n-1}, y_{n-1}))]}{1 + d(x_n, x_{n-1})} \\ &\quad + \beta[d(x_n, f(x_n, y_n)) + d(x_{n-1}, f(x_{n-1}, y_{n-1}))] + \gamma d(x_n, x_{n-1}) \end{aligned}$$

which implies that

$$\begin{aligned} d(x_{n+1}, x_n) \leq \alpha \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_n, x_{n-1})} \\ + \beta[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + \gamma d(x_n, x_{n-1}) \end{aligned}$$

Finally, we arrive at

$$d(x_{n+1}, x_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(x_n, x_{n-1}) \tag{9}$$

Similarly by following above, we get

$$d(y_{n+1}, y_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) d(y_n, y_{n-1}) \tag{10}$$

So, from (9) and (10), we have

$$d(x_{n+1}, x_n) + d(y_{n+1}, y_n) \leq \left(\frac{\beta + \gamma}{1 - \alpha - \beta}\right) [d(x_n, x_{n-1}) + d(y_n, y_{n-1})]$$

Now let us define a sequence $\{S_n\} = \{d(x_{n+1}, x_n) + d(y_{n+1}, y_n)\}$, by induction we get

$$0 \leq S_n \leq kS_{n-1} \leq k^2 S_{n-2} \leq k^3 S_{n-3} \leq \cdots \leq k^n S_0$$

where $k = \frac{\beta+\gamma}{1-\alpha-\beta} < 1$ and so,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} [d(x_n, x_{n-1}) + d(y_n, y_{n-1})] = 0$$

from this we get $\lim_{n\to\infty} d(x_n, x_{n-1}) = 0$ and $\lim_{n\to\infty} d(y_n, y_{n-1}) = 0$. Using triangular inequality for $m \geq n$, we have

$$d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_{n+1}, x_n)$$

and

$$d(y_m, y_n) \leq d(y_m, y_{m-1}) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_{n+1}, y_n)$$

and hence

$$\begin{aligned} d(x_m, x_n) + d(y_m, y_n) &\leq S_{m-1} + S_{m-2} + \cdots + S_n \\ &\leq \left(k^{m-1} + k^{m-2} + \cdots + k^n\right) S_0 \\ &\leq \frac{k^n}{1-k} S_0 \end{aligned}$$

as $n \to \infty$, $d(x_m, x_n) + d(y_m, y_n) \to 0$, which shows that both the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. So, by completeness of X, there exists a point $(x, y) \in X \times X$ such that $x_n \to x$ and $y_n \to y$. Again by the continuity of f, we have

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n, y_n) = f(\lim_{n\to\infty} x_n, \lim_{n\to\infty} y_n) = f(x, y)$$

and

$$y = \lim_{n\to\infty} y_{n+1} = \lim_{n\to\infty} f(y_n, x_n) = f(\lim_{n\to\infty} y_n, \lim_{n\to\infty} x_n) = f(y, x)$$

and hence we have $x = f(x, y)$ and $y = f(y, x)$. Since $\{x_n\}$ is an increasing sequence in X and converges to a point x in X as it is a Cauchy sequence, then $x = \sup\{x_n\}$, i.e $x_n \leq x, \forall n \in N$ and suppose there exists no number $n_0 \in N$ such that $x_{n_0} = x$, because of $x = x_{n_0} = x_{n_0+1} = x$. By the strict monotone increasing nature of f over the first variable, we have

$$f(x_n, y_n) < f(x, y_n) \tag{11}$$

Similarly, since $\{y_n\}$ is a decreasing, Cauchy sequence in a complete metric space X and is converging to a point y in X. Then we have $y = \inf\{y_n\}$, i.e $y_n \geq y, \forall n \in N$ and by the strict monotone decreasing nature of f, we have

$$f(x, y_n) < f(x, y) \tag{12}$$

From Eqs. (11) and (12), we obtain

$$f(x_n, y_n) < f(x, y) \Rightarrow x_{n+1} < f(x, y), \quad \forall n \in N. \tag{13}$$

Since $x_n < x_{n+1} < f(x, y), \forall n \in N$ and $x = \sup\{x_n\}$ then we have $x \leq f(x, y)$. Now let $z_0 = x$ and $z_{n+1} = f(z_n, y_n)$ then the sequence $\{z_n\}$ is a non decreasing sequence as $z_0 = f(z_0, y_0)$ and converges to a point say z in X, from which we have $z = \sup\{z_n\}$.

Since for all $n \in N, x_n \leq x = z_0 \leq f(z_0, y_0) \leq z_n \leq z$ then from (1), we have

$$\begin{aligned}d(x_{n+1},z_{n+1}) &= d(f(x_n,y_n),f(z_n,y_n))\\ &\leq \alpha\,\frac{d(x_n,f(x_n,y_n))[1+d(z_n,f(z_n,y_n))]}{1+d(x_n,z_n)}\\ &\quad+\beta[d(x_n,f(x_n,y_n))+d(z_n,f(z_n,y_n))]+\gamma d(x_n,z_n)\end{aligned}$$

On taking limit as $n\to\infty$ to the above inequality, we get

$$d(x,z)\leq\gamma\, d(x,z)$$

but $\gamma<1$, we get $d(x,z)=0$. Hence $x=z=\sup\{x_n\}$ which intern implies that $x\leq f(x,y)\leq x$ gives that $x=f(x,y)$. Again by following the above argument, we get $y=f(y,x)$. So, f has a coupled fixed point in $X\times X$. □

For the existence and uniqueness of a coupled fixed point of f over a complete partial ordered metric space X, we furnish the following partial order relation.

$$(\mu,\upsilon)\leq(x,y)\Leftrightarrow x\geq\mu,\ y\leq\upsilon,\quad \text{for any } (x,y),(\mu,\upsilon)\in X\times X.$$

Theorem 2 *Along the hypothesis stated in Theorem 1 and suppose that for every* $(x,y),(r,s)\in X\times X$, *there exists* $(u,v)\in X\times X$ *such that* $(f(u,v),f(v,u))$ *is comparable to* $(f(x,y),f(y,x))$ *and* $(f(r,s),f(s,r))$ *then f has a unique coupled fixed point in* $X\times X$, *i.e there exists a unique point* $(x,y)\in X\times X$ *such that* $x=f(x,y)$ *and* $y=f(y,x)$.

Proof As we know from Theorem 1, the set of coupled fixed points of f is non empty. Suppose that (x,y) and (r,s) are two coupled fixed points of the mapping f, then $x=f(x,y),y=f(y,x),r=f(r,s)$ and $s=f(s,r)$. Now we have to show that $x=r,y=s$ for the uniqueness of a coupled fixed point of f.

From hypothesis we have there exists $(u,v)\in X\times X$ such that $(f(u,v),f(v,u))$ is comparable to $(f(x,y),f(y,x))$ and $(f(r,s),f(s,r))$. Put $u=u_0$ and $v=v_0$ and let $u_1,v_1\in X$ then $u_1=f(u_0,v_0)$ and $v_1=f(v_0,u_0)$. Similarly by induction from Theorem 1, we can define two sequences $\{u_n\}$ and $\{v_n\}$ from $u_{n+1}=f(u_n,v_n)$ and $v_{n+1}=f(v_n,u_n)$ for all $n\in N$. Like above, define the sequences $\{x_n\}$, $\{y_n\}$, $\{r_n\}$ and $\{s_n\}$ by setting $x=x_0$, $y=y_0$, $r=r_0$ and $s=s_0$. So by Theorem 1, we have that $x_n\to x=f(x,y)$, $y_n\to y=f(y,x)$, $r_n\to r=f(r,s)$ and $s_n\to s=f(s,r)$ for all $n\geq 1$. But $(f(x,y),f(y,x))=(x,y)$ and $(f(u_0,v_0),f(v_0,u_0))=(u_1,v_1)$ are comparable and hence we have $x\geq u_1$ and $y_{\leq}v_1$. Next to show that (x,y) and (u_n,v_n) are comparable, i.e to show that $x\geq u_n$ and $y_{\leq}v_n$ for all $n\in N$. Suppose the inequalities holds for some $n\geq 0$, then from the nature of strict fixed monotone property of f, we have $u_{n+1}=f(u_n,v_n)\leq f(x,y)=x$ and $v_{n+1}=f(v_n,u_n)\geq f(y,x)=y$ and hence $x\geq u_n$ and $y_{\leq}v_n$ for all $n\in N$.

Then from Theorem 1, we get

$$\begin{aligned} d(x,u_{n+1}) &= d(f(x,y),f(u_n,v_n)) \\ &\leq \alpha \frac{d(x,f(x,y))[1+d(u_n,f(u_n,v_n))]}{1+d(x,u_n)} \\ &\quad + \beta[d(x,f(x,y))+d(u_n,f(u_n,v_n))] + \gamma d(x,u_n) \end{aligned}$$

which implies that

$$d(x,u_{n+1}) \leq \left(\frac{\beta+\gamma}{1-\beta}\right) d(x,u_n).$$

Similarly, we can get

$$d(y,v_{n+1}) \leq \left(\frac{\beta+\gamma}{1-\beta}\right) d(y,v_n).$$

Suppose $D = \frac{\beta+\gamma}{1-\beta} < 1$ then from above equations, we get

$$\begin{aligned} d(x,u_{n+1}) + d(y,v_{n+1}) &\leq D[d(x,u_n)+d(y,v_n)] \\ &\leq D^2[d(x,u_{n-1})+d(y,v_{n-1})] \\ &\ldots \\ &\leq D^n[d(x,u_0)+d(y,v_0)] \end{aligned}$$

Taking limit as $n \to \infty$ to the above equation, we get $\lim_{n\to\infty} d(x,u_{n+1}) + d(y,v_{n+1}) = 0$. From this we have $\lim_{n\to\infty} d(x,u_{n+1}) = 0$ and $\lim_{n\to\infty} d(y,v_{n+1}) = 0$.

Similarly, we can also prove that $\lim_{n\to\infty} d(r,u_n) = 0$ and $\lim_{n\to\infty} d(s,v_n) = 0$. Finally, we arrive at

$$d(x,r) \leq d(x,u_n) + d(u_n,r) \quad \text{and} \quad d(y,s) \leq d(y,v_n) + d(v_n,s).$$

On taking $n \to \infty$ to the above inequalities, we obtain $d(x,r) = 0 = d(y,s)$. Hence $x = r$ and $y = s$, this shows the uniqueness of f. This completes the proof. □

Theorem 3 *Along the hypothesis stated in Theorem 1 and if x_0, y_0 are comparable then f has a coupled fixed point in $X \times X$, i.e there exists a point $(x,y) \in X \times X$ such that $x = f(x,y)$ and $y = f(y,x)$.*

Proof Suppose (x, y) is a coupled fixed point of f, then from Theorem 1, we can get two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x$ and $y_n \to y$.

Suppose $x_0 \leq y_0$, we have to claim that $x_n \leq y_n, \forall\, n \geq 0$. From the strict monotone property of f, we have $x_{n+1} = f(x_n,y_n) \leq f(y_n,x_n) = y_{n+1}$. So, from the contraction condition of Theorem 1, we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) &= d(f(x_n, y_n), f(y_n, x_n)) \\ &\leq \alpha \frac{d(x_n, f(x_n, y_n))[1 + d(y_n, f(y_n, x_n))]}{1 + d(x_n, y_n)} \\ &\quad + \beta[d(x_n, f(x_n, y_n)) + d(y_n, f(y_n, x_n))] + \gamma d(x_n, y_n) \end{aligned}$$

On taking limit as $n \to \infty$, we get

$$d(x, y) \leq \gamma\, d(x, y)$$

which is contradiction since $\gamma < 1$ and hence $d(x, y) = 0$. So, $f(x, y) = x = y = f(y, x)$. Similarly, we can also show that $f(x, y) = x = y = f(y, x)$ for considering $y_0 \leq x_0$. Therefore (x, y) is a coupled fixed point of f in $X \times X$. □

4 Applications

In this section, we state some applications of the main result for a self mapping involving the contractions of integral type.

Let us consider the set of all functions χ defined on $[0, \infty)$ satisfying the following conditions:

1. Each χ is Lebesgue integrable mapping on each compact subset of $[0, \infty)$.
2. For any $\epsilon > 0$, we have $\int_0^{\epsilon} \chi > 0$.

Theorem 4 *Let $(X, d, \leq)$ be an ordered complete partially ordered metric space. Suppose that a continuous self mapping $f : X \times X$ is a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned} \int_0^{(d(f(x,y), f(\mu,\upsilon))} \varphi(t)dt \leq \alpha \int_0^{\frac{d(x, f(x,y))[1+d(\mu, f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt \\ + \beta \int_0^{d(x,f(x,y))+d(\mu,f(\mu,\upsilon))} \varphi(t)dt \\ + \gamma \int_0^{d(x,\mu)} \varphi(t)dt \end{aligned} \tag{14}$$

for all $x, y, \mu, \upsilon \in X$ with $x \geq \mu$ and $y \leq \upsilon$, $\varphi(t)$ is a function satisfies the above conditions defined on $[0, \infty)$ and $\alpha, \beta, \gamma \in [0, 1)$ with $0 \leq \alpha + 2\beta + \gamma < 1$, and if there exists two points $x_0, y_0 \in X$ with $x_0 < f(x_0, y_0)$ and $y_0 > f(y_0, x_0)$, then f has coupled fixed point $(x, y) \in X \times X$ (i.e there exists $x, y \in X$ such that $f(x, y) = x$ and $f(y, x) = y$).

Similarly, we get the following coupled fixed point results in partially ordered metric spaces, by taking $\beta = 0$, $\alpha = 0$, $\gamma = 0$ and $\alpha = \beta = 0$ in the above Theorems of Sect. 3.

Theorem 5 *Let* $(X, d, \leq)$ *be an ordered complete partially ordered metric space. Suppose that a continuous self mapping* $f : X \times X$ *is a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt + \gamma \int_0^{d(x,\mu)} \varphi(t)dt \tag{15}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \geq \mu$ *and* $y \leq \upsilon$, $\varphi(t)$ *is a function satisfies the above conditions defined on* $[0, \infty)$ *and* $\alpha, \gamma \in [0, 1)$ *with* $0 \leq \alpha + \gamma < 1$, *and if there exists two points* $x_0, y_0 \in X$ *with* $x_0 < f(x_0, y_0)$ *and* $y_0 > f(y_0, x_0)$, *then f has coupled fixed point* $(x, y) \in X \times X$.

Theorem 6 *Let* $(X, d, \leq)$ *be an ordered complete partially ordered metric space. Suppose that a continuous self mapping* $f : X \times X$ *is a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \beta \int_0^{d(x,f(x,y))+d(\mu,f(\mu,\upsilon))} \varphi(t)dt + \gamma \int_0^{d(x,\mu)} \varphi(t)dt \tag{16}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \geq \mu$ *and* $y \leq \upsilon$, $\varphi(t)$ *is a function satisfies the above conditions defined on* $[0, \infty)$ *and* $\beta, \gamma \in [0, 1)$ *with* $0 \leq 2\beta + \gamma < 1$, *and if there exists two points* $x_0, y_0 \in X$ *with* $x_0 < f(x_0, y_0)$ *and* $y_0 > f(y_0, x_0)$, *then f has coupled fixed point* $(x, y) \in X \times X$.

Theorem 7 *Let* $(X, d, \leq)$ *be an ordered complete partially ordered metric space. Suppose that a continuous self mapping* $f : X \times X$ *is a strict mixed monotone property on X satisfying the following condition*

$$\begin{aligned}\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \alpha \int_0^{\frac{d(x,f(x,y))[1+d(\mu,f(\mu,\upsilon))]}{1+d(x,\mu)}} \varphi(t)dt \\ + \beta \int_0^{d(x,f(x,y))+d(\mu,f(\mu,\upsilon))} \varphi(t)dt\end{aligned} \tag{17}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \geq \mu$ *and* $y \leq \upsilon$, $\varphi(t)$ *is a function satisfies the above conditions defined on* $[0, \infty)$ *and* $\alpha, \beta \in [0, 1)$ *with* $0 \leq \alpha + \beta < 1$, *and if there exists two points* $x_0, y_0 \in X$ *with* $x_0 < f(x_0, y_0)$ *and* $y_0 > f(y_0, x_0)$, *then f has coupled fixed point* $(x, y) \in X \times X$.

Theorem 8 *Let* $(X, d, \leq)$ *be an ordered complete partially ordered metric space. Suppose that a continuous self mapping* $f : X \times X$ *is a strict mixed monotone property on X satisfying the following condition*

$$\int_0^{(d(f(x,y),f(\mu,\upsilon))} \varphi(t)dt \leq \gamma \int_0^{d(x,\mu)} \varphi(t)dt \tag{18}$$

for all $x, y, \mu, \upsilon \in X$ *with* $x \geq \mu$ *and* $y \leq \upsilon$, $\varphi(t)$ *is a function satisfies the above conditions defined on* $[0, \infty)$ *and* $\gamma \in [0, 1)$ *with* $0 \leq \gamma < 1$, *and if there exists two* *points* $x_0, y_0 \in X$ *with* $x_0 < f(x_0, y_0)$ *and* $y_0 > f(y_0, x_0)$, *then f has coupled fixed point* $(x, y) \in X \times X$.

5 Conclusions

We have proved a coupled fixed point for two mappings over a partially order metric spaces satisfying certain rational contraction condition along with monotone property. The existence and uniqueness of the result is presented in this article. This article generalized and extended many existed results in the literature.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

Research involving human participants and/or animals This article does not contain any studies with human participants or animals performed by any of the authors.

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