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ORIGINAL RESEARCH PAPER

Inequalities for the polar derivative of a polynomial

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Abstract

Let $P(z)$ be a polynomial of degree n having all its zeros in $|z| \leq 1$, then according to Turan (Compositio Mathematica 7:89–95, 2004)

$$\max_{|Z|=1} |P'(z)| \geq \frac{n}{2} \max_{|Z|=1} |P(z)|.$$

In this paper, we shall use polar derivative and establish a generalisation and an extension of this result. Our results also generalize variety of other results.

Keywords Polynomial · Polar derivative · Inequalities

Mathematics Subject Classification 30A10 · 30C15

1 Introduction

Let $\mathcal{P}_n$ denote the class of all complex polynomials of degree at most n. Let $B = \{z; |z| = 1\}$ denotes the unit disk and B_- and B_+ denote the regions inside and outside the disk B respectively. If $P \in \mathcal{P}_n$, then according to the well known result of Bernstein [4]

$$\max_{z \in B} |P'(z)| \leq n \max_{z \in B} |P(z)|. \tag{1}$$

Inequality (1) is best possible and equality holds for the polynomial $P(z) = \lambda z^n$,

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where λ is a complex number. If we restrict ourselves to the class of polynomials having no zeros in $B \cup B_{-}$, then it was conjectured by Erdös and later on proved by Lax [6] that

$$\max_{z\in B} |P'(z)| \leq \frac{n}{2} \max_{z\in B} |P(z)|, \tag{2}$$

and if P has no zero in $B \cup B_{+}$, then it was proved by Turan [8] that

$$\max_{z\in B} |P'(z)| \geq \frac{n}{2} \max_{z\in B} |P(z)|. \tag{3}$$

The inequalities (2) and (3) are also best possible and equality holds for polynomials which have all zeros on B.

If $P(z)$ is a polynomial of degree n and α a complex number, then the polar derivative of $P(z)$ with respect to α, denoted by $D_\alpha P(z)$ is defined by

$$D_\alpha P(z) = nP(z) + (\alpha - z)P'(z).$$

Clearly $D_\alpha P(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z).$$

As an extension of (1), Aziz and Shah [3] used polar derivative and established that if $P(z)$ is a polynomial of degree n, then for every real or complex number α with $|\alpha| > 1$ and for $z \in B$,

$$|D_\alpha P(z)| \leq n|\alpha| \max_{z\in B} |P(z)|. \tag{4}$$

Aziz [1] extended inequality (2) to the polar derivative and proved that if p is a polynomial of degree n having all zero in $z \in B \cup B_{+}$ then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$

$$\max_{z\in B} |D_\alpha P(z)| \leq \frac{n(|\alpha|+1)}{2} \max_{z\in B} |P(z)|. \tag{5}$$

If we divide the two sides of (4) and (5) by $|\alpha|$ and let $|\alpha| \to \infty$, we get inequalities (1) and (2) respectively.

Shah [7] extended (3) to the polar derivative and proved the following result:

Theorem 1.1 *If $P \in \mathcal{P}_n$ and has all zeros in $z \in B \cup B_{-}$, then for $|\alpha| \geq 1$*

$$\max_{z\in B} |D_\alpha P(z)| \geq \frac{n(|\alpha|-1)}{2} \max_{z\in B} |P(z)|. \tag{6}$$

Theorem (1.1) generalizes (3) and to obtain (3), divide both sides of Theorem (1.1) by $|\alpha|$ and let $|\alpha| \to \infty$.

2 Main results

In this paper we obtain some more general results. First we prove the following generalization of Theorem (6).

Theorem 2.1 *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $B \cup B_-$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ and $z \in B$,*

$$|D_\alpha P(z)| \geq \frac{(|\alpha|-1)}{2}\left[n + \frac{\sqrt{|c_n|}-\sqrt{|c_0|}}{\sqrt{|c_n|}}\right]|P(z)|. \tag{7}$$

The result is sharp and equality holds for the polynomial $P(z) = c_n z^n + c_0$ with $|c_0| = |c_n| \neq 0$.

Remark 2.1 Since $P(z)$ has all its zeros in $B \cup B_-$, therefore $|c_n| \geq |c_0|$, it follows that Theorem 2.1 is an improvement of inequality (6)

Remark 2.2 If we divide the two sides of Theorem 2.1 by $|\alpha|$ and let $|\alpha| \to \infty$, we get a result due to Dubinin [5].

Theorem 2.2 *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $B \cup B_-$, then for $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$, $0 \leq l < 1$ and $z \in B$,*

$$\begin{aligned}\max_{z \in B}|D_\alpha P(z)| \geq{}& \frac{n}{2}\left\{(|\alpha|-1)\max_{z\in B}|P(z)| + (|\alpha|+1)lm\right\}\\ &+\frac{(|\alpha|-1)}{2}\left\{\frac{\sqrt{|c_n|-lm}-\sqrt{|c_0|}}{\sqrt{|c_n|-lm}}\right\}\left\{\max_{z\in B}|P(z)|-lm\right\},\end{aligned} \tag{8}$$

where $m = \min_{z \in B}|P(z)|$.

Dividing both sides of (8) by $|\alpha|$ and let $|\alpha| \to \infty$, we get the following result:

Corollary 2.1 *If $P \in \mathcal{P}_n$ and $P(z) = \sum_{j=0}^{n} c_j z^j$ has all its zeros in $B \cup B_-$, then for $0 \leq l < 1$ and $z \in B$,*

$$\begin{aligned}|P'(z)| \geq{}& \frac{1}{2}\left\{n + \frac{\sqrt{|c_n|-lm}-\sqrt{|c_0|}}{\sqrt{|c_n|-lm}}\right\}\max_{z\in B}|P(z)|\\ &+\frac{1}{2}\left\{n - \frac{\sqrt{|c_n|-lm}-\sqrt{|c_0|}}{\sqrt{|c_n|-lm}}\right\}lm.\end{aligned} \tag{9}$$

3 Lemmas

For the proof of above Theorems, we need the following lemmas.

Lemma 3.1 *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $B \cup B_-$ and $Q(z) = z^n\overline{P}(\frac{1}{\bar{z}})$, then for $z \in B$,*

$$|Q'(z)| \leq |P'(z)|.$$

Lemma 3.1 is a special case of a result due to Aziz and Rather [2].
We also need the following result which is due to Dubinin [5].

Lemma 3.2 *If $P \in \mathcal{P}_n$ and $P(z)$ has all zeros in $B \cup B_-$, then*

$$Re\frac{zP'(z)}{P(z)} \geq \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n|}}. \tag{10}$$

Inequality (10) is sharp and equality holds for polynomials which have all zeros on B.

4 Proofs of the theorems

Proof of Theorem (2.1) If $Q(z) = z^n\overline{P}\left(\frac{1}{\bar{z}}\right)$, it can be easily seen that $|Q'(z)| = |nP(z) - zP'(z)|$ for $z \in B$. Also $P(z)$ has all its zeros in $z \in B \cup B_-$ so by Lemma 3.1, we have

$$\begin{aligned} |P'(z)| &\geq |Q'(z)| \\ &= |nP(z) - zP'(z)| \quad \text{for} \quad z \in B. \end{aligned} \tag{11}$$

Now for every complex α with $|\alpha| \geq 1$, we have for $z \in B$,

$$\begin{aligned} |D_\alpha P(z)| &= |nP(z) + (\alpha - z)P'(z)| \\ &\geq |\alpha||P'(z)| - |nP(z) - zP'(z)|. \end{aligned}$$

This gives with the help of (11) that

$$|D_\alpha P(z)| \geq (|\alpha| - 1)P'(z). \tag{12}$$

By Lemma 3.2, we have for each z on B at which $P(z)$ does not vanish,

$$Re\frac{zP'(z)}{P(z)} \geq \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n|}}.$$

This gives

$$\left|\frac{P'(z)}{P(z)}\right| \geq Re\frac{zP'(z)}{P(z)} \geq \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n|}}. \tag{13}$$

Combining (12) and (13), we get for $z \in B$,

$$|D_\alpha P(z)| \geq (|\alpha| - 1)\left[\frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n|}}\right]|P(z)|. \tag{14}$$

That is

$$|D_\alpha(P(z))| \geq \frac{(|\alpha| - 1)}{2}\left[n + \frac{\sqrt{|c_n|} - \sqrt{|c_0|}}{\sqrt{|c_n|}}\right]|P(z)|. \tag{15}$$

This completes proof of Theorem 2.1.

Proof of Theorem (2.2) Since $P \in P_n$ and by hypothesis $P(z)$ has all its zeros in $B \cup B_-$, if $P(z)$ has a zero on B, then $m = \min_{|z|=1} |P(z)| = 0$ and the result follows from Theorem 2.1. So, assume that all the zeros of $P(z)$ lie in B_- so that $m > 0$. Now $m \leq |P(z)|$ for $z \in B$.

If λ is any complex number such that $|\lambda| < 1$, then $|m\lambda z^n| < |P(z)|$ for $z \in B$. Since all zeros of $P(z)$ lie in B_-, it follows by Rouche's Theorem that all the zeros of $F(z) = P(z) - \lambda m z^n$ also lie in B_-.

Let $G(z) = z^n \overline{F}(\frac{1}{\bar{z}})$, it can be easily seen that

$$|G'(z)| = |nF(z) - zF'(z)| \quad for \quad z \in B.$$

Also $F(z)$ has all its zeros in $z \in B_-$, so by Lemma 3.1 ,we have

$$\begin{aligned} |F'(z)| &\geq |G'(z)| \\ &= |nF(z) - zF'(z)| \quad for \quad z \in B. \end{aligned} \tag{16}$$

Now for every complex α with $|\alpha| \geq 1$, we have for $z \in B$,

$$\begin{aligned} |D_\alpha F(z)| &= |nF(z) + (\alpha - z)F'(z)| \\ &\geq |\alpha||F'(z)| - |nF(z) - zF'(z)|. \end{aligned}$$

This gives with the help of (16) that

$$|D_\alpha F(z)| \geq (|\alpha| - 1)F'(z). \tag{17}$$

Since the polynomial $F(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_{n-1}z^{n-1} + (c_n - \lambda m)z^n$ does not vanish in $|z| < 1$, we have by Lemma 3.2

$$Re\frac{zF'(z)}{F(z)} \geq \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n - \lambda m|}}.$$

This gives

$$\left|\frac{F'(z)}{F(z)}\right| \geq Re\frac{zF'(z)}{F(z)} \geq \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n - \lambda m|}}. \tag{18}$$

Combining (17) and (18), we get for $|z| = 1$,

$$|D_\alpha F(z)| \geq (|\alpha| - 1)\left[\frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n - \lambda m|}}\right]|F(z)|. \tag{19}$$

That is

$$|D_\alpha(P(z) - \lambda m z^n)| \geq \frac{(|\alpha| - 1)}{2}\left[n + \frac{\sqrt{|c_n - \lambda m|} - \sqrt{|c_0|}}{\sqrt{|c_n - \lambda m|}}\right]|P(z) - \lambda m z^n|. \tag{20}$$

$$|D_\alpha P(z) - \lambda m n \alpha z^{n-1}| \geq \frac{(|\alpha| - 1)}{2}\left[n + \frac{\sqrt{|c_n| - |\lambda| m} - \sqrt{|c_0|}}{\sqrt{|c_n| - |\lambda| m}}\right]|P(z) - \lambda m z^n|. \tag{21}$$

It follows by a simple consequence of Laguerre Theorem on the polar derivative of a polynomial that for every α with $|\alpha| \geq 1$,the polynomial

$$D_\alpha(P(z) - \lambda m z^n) = D_\alpha P(z) - \lambda m n \alpha z^{n-1} \tag{22}$$

has all its zeros in B_-. Thus, we have

$$|D_\alpha(P(z)| \geq \lambda m n |\alpha| |z|^{n-1} \quad for \quad |z| \geq 1. \tag{23}$$

Now choosing the argument of λ suitably in the left hand side of (21) such that

$$|D_\alpha P(z) - \lambda m n \alpha z^{n-1}| = |D_\alpha P(z)| - mn|\lambda||\alpha||z|^{n-1}$$

which is possible by (23), we get for $z \in B$

$$|D_\alpha P(z)| - mn|\alpha||\lambda| \geq \frac{(|\alpha| - 1)}{2}\left[n + \frac{\sqrt{|c_n| - |\lambda| m} - \sqrt{|c_0|}}{\sqrt{|c_n| - |\lambda| m}}\right]\{|P(z)| - |\lambda| m\}. \tag{24}$$

From (24), one can easily obtain for $z \in B$ and for any $\alpha \in \mathbb{C}$ with $|\alpha| \geq 1$ that

$$\begin{aligned}\max_{z\in B}|D_\alpha P(z)| \geq \frac{n}{2}\left\{(|\alpha| - 1)\max_{z\in B}|P(z)| + (|\alpha| + 1) l m\right\} \\ + \frac{(|\alpha| - 1)}{2}\left\{\frac{\sqrt{|c_n| - lm} - \sqrt{|c_0|}}{\sqrt{|c_n| - lm}}\right\}\left\{\max_{z\in B}|P(z)| - lm\right\},\end{aligned} \tag{25}$$

where $0 \leq l < 1$. That completes proof of Theorem 2.2.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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